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Critical points approaches to fourth-order elastic beam equations with variable coefficients having combined effects of concave and convex nonlinearities

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Abstract. This study investigates the theoretical and practical aspects of fourth-order elastic beam equations characterized by variable coefficients and influenced by combined nonlinearities. The equations under consideration arise naturally in the context of high-order differential problems encountered in material science, particularly in the study of phase transitions. The research focuses on establishing the existence and multiplicity of solutions for these equations, which involve both concave and convex nonlinearities. Using advanced mathematical techniques, such as variational methods and critical point theory, the study provides rigorous proofs for the existence of solutions under specific conditions. A central result is the application of Bonanno's local minimum theorem, which ensures the existence of at least one solution. Moreover, the research shows that by imposing additional algebraic conditions, particularly the classical Ambrosetti–Rabinowitz condition, the presence of two distinct solutions can be guaranteed. Beyond this, critical point theorems from Averna and Bonanno are employed to demonstrate scenarios where three solutions are possible.

Keywords: multiple solutions, fourth-order equation, critical point, variational methods.

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1 Introduction

Elastic beam equations, particularly those of fourth order, represent a cornerstone of modern theoretical and applied mechanics. These equations are invaluable in modeling the behavior of elastic materials and structures under various forces, enabling scientists and engineers to predict deformation, stability, and stress distribution with high precision. Fourth-order elastic

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equations are especially important because they incorporate effects that lower-order models neglect, such as bending and torsion, and account for variability in material properties and external forces. The significance of fourth-order differential equations extends across multiple disciplines. In structural engineering, these equations are essential for analyzing the deflection and stability of beams and plates, forming the theoretical basis for the design of buildings, bridges, and other critical infrastructures. The pioneering work of Timoshenko and Gere [21] introduced a comprehensive framework for understanding elastic stability, which remains relevant for modern engineering practices. In materials science, fourth-order equations play a central role in studying phase transitions phenomena where materials shift between states, such as from liquid to solid, under controlled conditions. This is particularly important for developing advanced materials in industries such as aerospace and automotive, where performance and reliability are critical. These equations' ability to model real-world conditions, including variable coefficients and nonlinear behaviors, makes them indispensable for tackling complex scientific and engineering challenges. Despite their utility, solving fourth-order elastic equations is a mathematically challenging task. These equations often involve nonlinearities, such as concave and convex effects, which significantly complicate the solution process. Nonlinearities arise in real-world scenarios where material responses to stress are non-proportional to the applied force. For example, when a beam experiences large deformations, the relationship between load and deflection becomes nonlinear. Variable coefficients add another layer of complexity, representing changes in material properties or external conditions over space or time. Traditional analytical methods frequently fall short in addressing these issues, necessitating the use of advanced techniques such as variational methods and critical point theory. These tools provide a rigorous framework for proving the existence and uniqueness of solutions to higher-order differential equations. A key aspect of this study is the application of variational methods to solve fourth-order elastic equations. These methods involve reformulating differential equations into problems of finding critical points of associated functionals. This approach is particularly advantageous for addressing the nonlinearities and boundary conditions inherent in these equations. Bonanno's local minimum theorem [3] and the Ambrosetti-Rabinowitz condition [1] are instrumental in this context, as they provide conditions under which solutions exist. The Palais–Smale condition, a cornerstone of variational calculus, ensures that functionals possess the compactness properties needed to identify critical points. Furthermore, Sobolev spaces offer a robust mathematical framework for handling boundary conditions and variable coefficients with precision. Research by Averna and Bonanno [2] has demonstrated the power of variational methods in proving not only the existence but also the multiplicity of solutions. These results highlight the ability of fourth-order elastic equations to model a range of physical phenomena, from stable configurations to unstable states. The practical implications of this research are vast, spanning both theoretical and applied domains. In materials science, understanding the multiplicity of solutions allows for the prediction of material behavior under different conditions. For instance, during a phase transition, multiple solutions may correspond to distinct physical states, such as stable and unstable configurations. This insight is critical for designing materials with tailored properties, such as high-performance alloys or composites. In structural engineering, these equations inform the design of beams, plates, and other structural elements to ensure safety and reliability. For example, aerospace engineers rely on these models to optimize the design of lightweight yet robust materials for aircraft and spacecraft. By predicting behaviors such as buckling, vibration, and resonance, fourth-order elastic equations provide the tools needed to address the demands of modern engineering.

In this paper, we are going to establish the existence result for the following problem

$$\begin{cases} (p(t)z'')'' - (q(t)z')' + r(t)z = \lambda h(t,z(t)) + \mu g(t,z(t)), & t \in [0,1], \\ z(a) = z(b) = 0, \\ z''(a) = z''(b) = 0 \end{cases} \tag{$P^{h,g}$}$$

where $\lambda > 0$, $\mu \geq 0$, h, g are continuous functions, $p \in C^2[0,1]$, $q \in C^1[0,1]$, $r \in C[0,1]$ are regular functions with $p = \operatorname{ess\,inf}_{[0,1]} p > 0$. We aim to advance the understanding of fourth-order elastic equations by addressing both their theoretical and practical challenges. Specifically, we seek to: 1. Establish the existence of solutions using variational methods and critical point theory. 2. Explore conditions that lead to the multiplicity of solutions, highlighting the influence of nonlinearities and variable coefficients.

The remainder of this paper is organized as follows. Section 2 introduces the necessary mathematical preliminaries, including definitions, theorems, and frameworks that form the foundation of the analysis. Section 3 examines the existence of a single solution, employing foundational techniques such as the Palais–Smale condition. Section 4 extends the discussion to explore conditions under which multiple solutions arise, with particular attention to nonlinearities and coefficient variations. Section 5 addresses special cases, such as when specific parameters are set to zero, providing additional insights into the behavior of these equations. Finally, the study concludes with a discussion of the implications of the findings and potential directions for future research.

For additional information, we recommend consulting sources such as [7, 9, 11, 13, 14, 22]. For example, Bonanno et al. in [9] applied multiple critical points theorems to demonstrate the existence of two non-trivial solutions for equation ($P^{h,g}$), in the case $\mu = 0$.

2 Preliminaries

For relevant notations and foundational results, we direct the reader to references [17, 19]. Let E be represent a real Banach space. A functional $I : \mathrm{E} \to \mathbb{R}$, which is continuously Gâteaux differentiable, is said to satisfy the *Palais–Smale condition* (shortened as (PS)-condition) if every sequence $\{z_n\}$ such that $\{I(z_n)\}$ is bounded and $\lim_{n\longrightarrow\infty} \|I'(z_n)\|_{\mathrm{E}^*} = 0$, has a convergent subsequence. This condition is crucial in the framework of variational methods, as it ensures that critical points of λ, correspond to solutions of the associated boundary value problem.

Now, consider two continuously Gâteaux differentiable functions $\Phi, \Psi : \mathrm{E} \longrightarrow \mathbb{R}$ and the functional

$$I = \Phi - \Psi.$$

Let $s_1, s_2 \in [-\infty, \infty]$ with $s_1 < s_2$. The functional I is said to satisfy the *Palais–Smale condition with bounds s_1 and s_2* (denoted ${}^{[s_1]}(\mathrm{PS})^{[s_2]}$-condition) if any sequence $\{z_n\}$ such that $\{I(z_n)\}$ is bounded, $\lim_{n\longrightarrow\infty} \|I'(z_n)\|_{\mathrm{E}^*} = 0$ and $s_1 < \Phi(z_n) < s_2$ for each $n \in \mathbb{N}$, has a convergent subsequence.

When $s_1 = -\infty$ and $s_2 = \infty$, this condition reduces to the classical (PS)-condition. If $s_1 = -\infty$ and $s_2 \in \mathbb{R}$, it is referred to as the $(\mathrm{PS})^{[s_2]}$-condition. Similarly, when $s_1 \in \mathbb{R}$ and $s_2 = \infty$, it is denoted as ${}^{[s_1]}(\mathrm{PS})$-condition. Indeed, if Φ and Ψ be two continuously Gâteaux differentiable functionals defined on a real Banach space E and fix $s \in \mathbb{R}$. The functional $I = \Phi - \Psi$ is said to verify the Palais–Smale condition cut off upper at r (in short $(\mathrm{PS})^{[s]}$) if any sequence $\{z_n\}_{n\in\mathbb{N}}$ in E such that $\{I(z_n)\}$ is bounded, $\lim_{n\longrightarrow\infty} \|I'(z_n)\|_{\mathrm{E}^*} = 0$ and

$\Phi(z_n) < s$ for each $n \in \mathbb{N}$, has a convergent subsequence. Furthermore, if I satisfies the ${}^{[s_1]}(\mathrm{PS})^{[s_2]}$-condition, it automatically satisfies the ${}^{[\varrho_1]}(\mathrm{PS})^{[\varrho_2]}$-condition for all $\varrho_1, \varrho_2 \in [-\infty, \infty]$ such that $s_1 \leq \varrho_1 < \varrho_2 \leq s_2$.

Notably, if I adheres to the classical (PS)-condition, it also satisfies ${}^{[\varrho_1]}(\mathrm{PS})^{[\varrho_2]}$-condition for any $\varrho_1, \varrho_2 \in [-\infty, \infty]$ with $\varrho_1 < \varrho_2$.

The following four theorems will be utilized in proving the results of this paper.

Theorem 2.1 ([5, Theorem 2.3]). *Let* E *be a real Banach space and let* $\Phi, \Psi : \mathrm{E} \longrightarrow \mathbb{R}$ *be two continuously Gâteaux differentiable functions such that* $\inf_{z \in \mathrm{E}} \Phi(z) = \Phi(0) = \Psi(0) = 0$. *Assume that there exist* $s > 0$ *and* $\bar{z} \in \mathrm{E}$, *with* $0 < \Phi(\bar{z}) < s$, *such that*

(a_1) $\dfrac{\sup_{\Phi(z) \leq s} \Psi(z)}{s} < \dfrac{\Psi(\bar{z})}{\Phi(\bar{z})}$,

(a_2) *for each* $\lambda \in \left(\dfrac{\Phi(\bar{z})}{\Psi(\bar{z})}, \dfrac{s}{\sup_{\Phi(z) \leq s} \Psi(z)} \right)$ *the functional* $I_\lambda = \Phi - \lambda \Psi$ *satisfies* $(\mathrm{PS})^{[s]}$*-condition.*

Then, for each

$$\lambda \in \Lambda_s = \left(\frac{\Phi(\bar{z})}{\Psi(\bar{z})}, \frac{s}{\sup_{\Phi(z) \leq s} \Psi(z)} \right)$$

there exists $z_{0,\lambda} \in \Phi^{-1}(0, s)$ *such that* $I'_\lambda(z_{0,\lambda}) = 0$ *and* $I_\lambda(z_{0,\lambda}) \leq I_\lambda(z)$ *for each* $z \in \Phi^{-1}(0, s)$.

Theorem 2.2. *[5, Theorem 3.2] Let* E *be a real Banach space,* $\Phi, \Psi : \mathrm{E} \longrightarrow \mathbb{R}$ *be two continuously Gâteaux differentiable functionals such that* Φ *is bounded from below and* $\Phi(0) = \Psi(0) = 0$. *Fix* $s > 0$ *and assume that, for each*

$$\lambda \in \left(0, \frac{s}{\sup_{z \in \Phi^{-1}(-\infty, s)} \Psi(z)} \right),$$

the functional $I_\lambda = \Phi - \lambda \Psi$ *satisfies* (PS)*-condition and it is unbounded from below. Then, for each*

$$\lambda \in \left(0, \frac{s}{\sup_{z \in \Phi^{-1}(-\infty, s)} \Psi(z)} \right),$$

the functional I_λ *admits two distinct critical points.*

Theorem 2.3 ([2, Theorem A]). *Let* E *be a reflexive real Banach space,* $\Phi : \mathrm{E} \longrightarrow \mathbb{R}$ *a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on* E^*, *and* $\Psi : \mathrm{E} \longrightarrow \mathbb{R}$ *a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

(b_1) $\lim_{\|z\| \longrightarrow \infty} (\Phi(z) + \lambda \Psi(z)) = \infty$ *for each* $\lambda \in [0, \infty)$,

(b_2) *there is* $s \in \mathbb{R}$ *such that*

$$\inf_{\mathrm{E}} \Phi < s$$

and

$$\varphi_1(s) < \varphi_2(s)$$

where

$$\varphi_1(s) = \inf_{z \in \Phi^{-1}(-\infty, s)} \frac{\Psi(z) - \inf_{\overline{\Phi^{-1}(-\infty, s)}^{w}} \Psi}{s - \Phi(z)},$$

$$\varphi_2(s) = \inf_{z\in\Phi^{-1}(-\infty,s)} \sup_{v\in\Phi^{-1}[s,\infty)} \frac{\Psi(z)-\Psi(v)}{\Phi(v)-\Phi(z)},$$

and $\overline{\Phi^{-1}(-\infty,s)}^{w}$ *is the closure of* $\Phi^{-1}(-\infty,s)$ *in the weak topology.*

Then, for each $\lambda \in \left(\frac{1}{\varphi_2(s)}, \frac{1}{\varphi_1(s)}\right)$, *the functional* $\Phi + \lambda\Psi$ *has at least three critical points in* E.

It is important to note that $\varphi_1(s)$ in Theorem 2.3 may equal 0. In such cases, and in similar situations both here and later, we will interpret $\frac{1}{0}$ as ∞.

We will also utilize the following two theorems related to critical points.

Theorem 2.4 ([4, Theorem 1.1]). *Let* E *be a reflexive real Banach space, and let* $\Phi, \Psi : \mathrm{E} \longrightarrow \mathbb{R}$ *be two sequentially weakly lower semicontinuous and Gâteaux differentiable functions. Assume that* Φ *is (strongly) continuous and satisfies*

$$\lim_{\|z\|\longrightarrow\infty} \Phi(z) = \infty.$$

Assume also that there exist two constants s_1 *and* s_2 *such that*

(c_1) $\inf_{\mathrm{E}} \Phi < s_1 < s_2$,

(c_2) $\varphi_1(s_1) < \varphi_2^*(s_1, s_2)$,

(c_3) $\varphi_1(s_2) < \varphi_2^*(s_1, s_2)$, *where* φ_1 *is defined as in Theorem 2.3 and*

$$\varphi_2^*(s_1,s_2) = \inf_{z\in\Phi^{-1}(-\infty,s_1)} \sup_{v\in\Phi^{-1}[s_1,s_2)} \frac{\Psi(z)-\Psi(v)}{\Phi(v)-\Phi(z)}.$$

Then, for each

$$\lambda \in \left(\frac{1}{\varphi_2^*(s_1,s_2)}, \min\left\{\frac{1}{\varphi_1(s_1)}, \frac{1}{\varphi_1(s_2)}\right\}\right),$$

the functional $\Phi + \lambda\Psi$ *admits at least two critical points which lie in* $\Phi^{-1}(-\infty, s_1]$ *and* $\Phi^{-1}[s_1, s_2)$, *respectively.*

It is worth noting that Theorems 2.3 and 2.4 are based on Ricceri's variational principle [20]. For further details, readers are encouraged to consult [6], where Theorems 2.1 and 2.2 were employed to prove the existence of at least one and two solutions for elliptic Dirichlet problems with variable exponents. Similarly, in [10], Theorems 2.3 and 2.4 were utilized to establish the existence of at least two and three solutions for a boundary value problem on the half-line. Additionally, reference [12, 15, 16] demonstrates how Theorems 2.1 through 2.4 were applied to guarantee the existence of solutions for boundary value problems.

In this section, we present fundamental notations and supporting results to incorporate equation ($P^{h,g}$) into a variational framework. Let E denote the Sobolev space $W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$, equipped with the norm

$$\|z\| = \left(\|z''\|_2^2 + \|z'\|_2^2 + \|z\|_2^2\right)^{\frac{1}{2}} \tag{2.1}$$

for every $z \in \mathrm{E}$ where $\|\cdot\|_2$ is the usual norm in $L^2[a,b]$. It is well known that $\|\cdot\|$ is induced by the inner product

$$\int_a^b \left(z''(t)v''(t) + z'(t)v'(t) + z(t)v(t)\right) \mathrm{d}t$$

for every $z, v \in \mathrm{E}$.

We highlight the following Poincaré-type inequalities, which can be found in the works of [18].

Proposition 2.5. *For every $z \in \mathrm{E}$, if $k = \frac{1}{\pi^2}$, one has*

$$\|z^{(i)}\|_2^2 \leq k^{j-i}\|z^{(j)}\|_2^2, \quad i = 0,1, \ \ j = 1,2 \ \textit{with } i < j. \tag{2.2}$$

Now, defining p^- as previously mentioned, and letting $q^- = \operatorname{ess\,inf}_{[0,1]} q$ and $r^- = \operatorname{ess\,inf}_{[0,1]} r$, we will examine the following set of conditions based on the signs of these quantities:

(H_1) $p^- > 0, q^- \geq 0, r^- \geq 0,$

(H_2) $p^- > 0, q^- < 0, r^- \geq 0$ and $p^- + q^-k > 0,$

(H_3) $p^- > 0, q^- \geq 0, r^- < 0$ and $p^- + r^-k > 0,$

(H_4) $p^- > 0, q^- < 0, r^- < 0$ and $p^- + q^-k + r^-k > 0.$

Additionally, take into account the following condition:

(H) $\min\{p^- + q^-k, \ p^- + r^-k, \ p^- + q^-k + r^-k\} > 0.$

Put

$$\sigma = \min\{p^-, \ p^- + q^-k, \ p^- + r^-k, \ p^- + q^-k + r^-k\}.$$

Clearly, assuming the condition (H) implies that $\sigma > 0$. Furthermore, a simple calculation reveals the following result.

Proposition 2.6. *Condition (H) is satisfied if and only if at least one of the conditions (H_1) through (H_4) is met.*

We will now introduce a useful norm, which is equivalent to $\|\cdot\|$ and still ensures that E remains a Hilbert space. Therefore, for the fixed values of p, q and r mentioned earlier, we define the function $N : \mathrm{E} \to \mathbb{R}$ as follows

$$N(z) = \int_0^1 (p(t)|z''(t)|^2 + q(t)|z'(t)|^2 + r(t)|z(t)|^2)\mathrm{d}t$$

holds for any $z \in \mathrm{E}$. We have the following proposition, which will be helpful in confirming that $\sqrt{N(\cdot)}$ is a norm equivalent to the standard one.

Proposition 2.7 ([9, Proposition 2.3])**.** *Assume (H) holds. Then, there exits $m > 0$ such that*

$$N(z) \geq m\|z\|^2 \tag{2.3}$$

for any $z \in \mathrm{E}$, with $m = \frac{\sigma}{1+k+k^2}$. Moreover, one has

$$N(z) \geq \sigma\|z''\|_2^2 \tag{2.4}$$

for any $z \in \mathrm{E}$.

Proposition 2.8 ([9, Proposition 2.4])**.** *Assume that condition (H) is satisfied and define*

$$\|\cdot\|_{\mathrm{E}} = \sqrt{N(\cdot)}$$

for any $z \in \mathrm{E}$. Then, $\|\cdot\|_{\mathrm{E}}$ is a norm equivalent to the usual one defined in (2.2) and $(\mathrm{E}, \|\cdot\|_{\mathrm{E}})$ is a Hilbert space.

Now, assuming again (H), put

$$\delta = \sqrt{\sigma} = \left(\min\{p^-,\ p^- + q^- k,\ p^- + r^- k,\ p^- + q^- k + r^- k\}\right)^{\frac{1}{2}}.$$

The constant δ is well-defined, given that $\sigma > 0$ holds under condition (H).

The following proposition will be useful in the next section.

Proposition 2.9 ([9, Proposition 2.5]). *Assume that (H) holds. One has*

$$\|z\|_\infty \leq \frac{1}{2\pi\delta}\|z\|_{\mathrm{E}}$$

for any $z \in \mathrm{E}$.

Definition 2.10. A function $z \in \mathrm{E}$ is called a weak solution of problem ($P^{h,g}$), if

$$\int_0^1 \left(p(t)z''(t)v''(t) + q(t)z'(t)v'(t) + r(t)z(t)v(t)\right) \mathrm{d}t \\ - \lambda \int_0^1 h(t, z(t))\mathrm{d}t - \mu \int_0^1 g(t, z(t))\mathrm{d}t = 0$$

holds for any $v \in \mathrm{E}$.

Put

$$H(t, m) = \int_0^m h(t, x)\mathrm{d}x \quad \text{for any } (t, m) \in [0, 1] \times \mathbb{R}$$

and

$$G(t, m) = \int_0^m g(t, x)\mathrm{d}x \quad \text{for any } (t, m) \in [0, 1] \times \mathbb{R}.$$

We define the functionals Φ and Ψ for each $z \in \mathrm{E}$, as follows

$$\Phi(z) = \frac{1}{2}\|z\|^2_{\mathrm{E}} \tag{2.5}$$

and

$$\Psi(z) = \int_0^1 H(t, z(t))\mathrm{d}t + \frac{\mu}{\lambda}\int_0^1 G(t, z(t))\mathrm{d}t \tag{2.6}$$

and we put

$$I_\lambda(z) = \Phi(z) - \lambda\Psi(z)$$

for every $z \in \mathrm{E}$.

Proposition 2.11 ([9, Proposition 2.6]). *Function $z \in \mathrm{E}$ is a generalized solution of ($P^{h,g}$) if only if $z \in \mathrm{E}$ is a critical point of I_λ.*

We need the following Proposition for existence our main results.

Proposition 2.12. *Let $S : \mathrm{E} \longrightarrow \mathrm{E}^*$ be the operator defined by*

$$S(z)(v) = \int_0^1 \left(p(t)z''(t)v''(t) + q(t)z'(t)v'(t) + r(t)z(t)v(t)\right) \mathrm{d}t$$

for every $z, v \in \mathrm{E}$. Then, S admits a continuous inverse on E^*.

Proof. It is obvious that

$$S(z)(z) = \int_0^1 \left(p(t)|z''(t)|^2 + q(t)|z'(t)|^2 + r(t)|z(t)|^2)\right) \mathrm{d}t = \|z\|_{\mathrm{E}}.$$

This follows that S is coercive. Owing to our assumptions on the data, one has

$$\langle S(z) - S(v), z - v\rangle \geq C\|z - v\|_{\mathrm{E}}^2 > 0$$

for some $C > 0$, for every $z, v \in \mathrm{E}$, which means that S is strictly monotone. Moreover, since E is reflexive, for $z_n \longrightarrow z$ strongly in E as $n \to +\infty$, one has $S(z_n) \to S(z)$ weakly in E^* as $n \to \infty$. Hence, S is demicontinuous, so by [23, Theorem 26.A(d)], the inverse operator S^{-1} of S exists and it is continuous. Indeed, let e_n be a sequence of E^* such that $e_n \to e$ strongly in E^* as $n \to \infty$. Let z_n and u in E such that $S^{-1}(e_n) = z_n$ and $S^{-1}(e) = z$. Taking into account that S is coercive, one has that the sequence z_n is bounded in the reflexive space E. For a suitable subsequence, we have $z_n \to \hat{z}$ weakly in E as $n \to \infty$, which concludes

$$\langle S(z_n) - S(z), z_n - \hat{z}\rangle = \langle e_n - e, z_n - \hat{z}\rangle = 0.$$

Note that if $z_n \to \hat{z}$ weakly in E as $n \to +\infty$ and $S(z_n) \to S(\hat{z})$ strongly in E^* as $n \to +\infty$, one has $z_n \to \hat{z}$ strongly in E as $n \to +\infty$, and since S is continuous, we have $z_n \to \hat{z}$ weakly in E as $n \to +\infty$ and $S(z_n) \to S(\hat{z}) = S(z)$ strongly in E^* as $n \to +\infty$. Hence, taking into account that S is an injection, we have $z = \hat{z}$. □

3 Existence of one solution

In this section, we address the existence of a solution for the problem ($P^{h,g}$).

First, put

$$B = \frac{1}{2}\left(\frac{4096}{27}p^- + \frac{64}{9}q^- + \frac{13}{20}r^-\right)$$

and

$$D = \frac{1}{2}\left(\frac{4096}{27}p^+ + \frac{64}{9}q^+ + \frac{13}{20}r^+\right)$$

where p^+, q^+ and r^+ are the $\operatorname{ess\,sup}$ in $[0, 1]$ of the functions p, q and r respectively. For ease of reference, we define

$$G^\theta = \int_0^1 \max_{|m|\leq\theta} G(t, m)\mathrm{d}t \quad \text{for each } \theta > 0$$

and

$$G_\sigma = \inf_{t\in[0,1]} G(t, \sigma) \quad \text{for each } \sigma > 0.$$

If g is sign-changing, then clearly $G^\theta \geq 0$ and $G_\sigma \leq 0$.

To achieve our objective, we will fix two positive constants θ and σ, and set

$$\underline{\delta}_{\lambda,g} = \min\left\{\frac{2\delta^2\pi^2\theta^2 - \lambda\int_0^1 \max_{|m|\leq\theta} H(t, m)\mathrm{d}t}{G^\theta}, \frac{D\sigma^2 - \lambda\int_{\frac{3}{8}}^{\frac{5}{8}} H(t, \sigma)\mathrm{d}t}{G_\sigma}\right\}$$

and

$$\overline{\delta}_{\lambda,g} = \min\left\{ \underline{\delta}_{\lambda,g}, \frac{1}{\max\left\{0, \frac{1}{2\delta^2\pi^2} \limsup_{|m|\longrightarrow\infty} \frac{G(t,m)}{|m|^2}\right\}} \right\}, \tag{3.1}$$

where we read $\epsilon/0 = +\infty$, so that, for instance, $\overline{\delta}_{\lambda,g} = +\infty$ when

$$\limsup_{|m|\longrightarrow\infty} \frac{G(t,m)}{|m|^2} \leq 0$$

and $G_\sigma = G^\theta = 0$.

Theorem 3.1. *Assume that there exist two positive constants θ and σ with the property*

$$\sqrt{\frac{D}{2}}\frac{\sigma}{\pi\delta} < \theta$$

such that

(A_1) $h(t,v) \geq 0$ *for each* $(t,v) \in \left([0,\frac{3}{8}) \cup (\frac{5}{8},1]\right) \times \mathbb{R}$,

(A_2) $\frac{\int_0^1 \max_{|m|\leq\theta} H(t,m)\mathrm{d}t}{\theta^2} < \frac{2\delta^2\pi^2}{D} \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} H(t,\sigma)\mathrm{d}t}{\sigma^2}$,

(A_3) $\min_{t\in[0,1]} \limsup_{|m|\longrightarrow\infty} \frac{H(t,m)}{|m|^2} \in (-\infty,0]$.

Then, for each

$$\lambda \in \Lambda = \left(\frac{D\sigma^2}{\int_{\frac{3}{8}}^{\frac{5}{8}} H(t,\sigma)\mathrm{d}t}, \frac{2\delta^2\pi^2\theta^2}{\int_0^1 \max_{|m|\leq\theta} H(t,m)\mathrm{d}t} \right)$$

and for each function $g : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfying the condition

$$\limsup_{|m|\longrightarrow\infty} \frac{G(t,m)}{|m|^2} \in (-\infty,0], \tag{3.2}$$

there exists $\overline{\delta}_{\lambda,g} > 0$ given by (3.1) such that for each $\mu \in [0,\overline{\delta}_{\lambda,g})$, the problem $(P^{h,g})$ admits at least one solution z_λ in E *such that*

$$|z_\lambda(t)| < \theta.$$

Proof. The objective is to apply Theorem 2.1 to address the problem $(P^{h,g})$. To do this, we consider the functionals Φ and Ψ as given in 2.5 and 2.6, respectively. We aim to show that these functionals satisfy the necessary conditions outlined in Theorem 2.1. It is well-known that both functionals are well-defined Gâteaux differentiable, and one has

$$\Psi'(z)(v) = \int_0^1 h(t,z(t))v(t)\,\mathrm{d}t + \frac{\mu}{\lambda}\int_0^1 g(t,z(t))v(t)\,\mathrm{d}t$$

and

$$\Phi'(z)(v) = \int_0^1 \left(p(t)z''(t)v''(t) + q(t)z'(t)v'(t) + r(t)z(t)v(t)\right)\mathrm{d}t$$

for every $z, v \in \mathrm{E}$. Furthermore, Φ and Ψ are C^1-functions. By utilizing the definition of Φ, it follows that

$$\lim_{\|z\|_{\mathrm{E}} \to +\infty} \Phi(z) = +\infty,$$

which implies that Φ is coercive. Furthermore, proposition 2.12 guarantees that Φ has a continuous inverse on E^*. Therefore, we can conclude that the regularity condition on Φ and Ψ, required in Theorem 2.1 are satisfied. The critical points of the functional I_λ in E correspond precisely to the generalized solutions of the considered problem ($P^{h,g}$). To establish the existence of a critical point I_λ in E, we verify that the regularity conditions on Φ and Ψ, as outlined in Theorem 2.1, are indeed fulfilled. It is crucial to note that the operator I_λ is a $C^1(\mathrm{E}, \mathbb{R})$ functional on E, and the critical points of I_λ correspond to weak solutions of the problem ($P^{h,g}$). Furthermore, for $\lambda > 0$, the functional I_λ is coercive. Indeed, since $\mu < \delta_\lambda$ we can fix t such that

$$\limsup_{|m| \longrightarrow \infty} \frac{G(t,m)}{|m|^2} \in (-\infty, 0]$$

and $\mu t < 2\pi^2\delta^2$. Consequently, there exists a positive constant ι such that

$$G(t,m) \le tm^2 + \iota$$

for each $(t,m) \in [0,1] \times \mathbb{R}$. Now, we fix

$$0 < \varepsilon < \frac{4\pi^2\delta^2}{\lambda}\left(\frac{1}{2} - \mu t \frac{1}{4\pi^2\delta^2}\right).$$

Based on the assumption (A_3) there is a bounded function ϕ_ε such that

$$H(t,m) \le \varepsilon m^2 + \phi$$

for each $(t,m) \in [0,1] \times \mathbb{R}$. It follows that, for each $z \in \mathrm{E}$, we have

$$\begin{aligned}
\Phi(z) - \lambda\Psi(z) &= \frac{1}{2}\|z\|_{\mathrm{E}}^2 - \lambda\left(\int_0^1 H(t,z(t))\mathrm{d}t + \frac{\mu}{\lambda}\int_0^1 G(t,z(t))\mathrm{d}t\right) \\
&\ge \frac{1}{2}\|z\|_0^2 - \lambda\left(\varepsilon\frac{1}{4\pi^2\delta^2}\|z\|_{\mathrm{E}}^2 + \phi\right) - \mu\left(t\frac{1}{4\pi^2\delta^2}\|z\|_{\mathrm{E}}^2 + \iota\right) \\
&\ge \left(\frac{1}{2} - \lambda\varepsilon\frac{1}{4\pi^2\delta^2} - \mu t\frac{1}{4\pi^2\delta^2}\right)\|z\|_{\mathrm{E}}^2 - \lambda\phi - \mu\iota
\end{aligned}$$

and thus

$$\lim_{\|z\|_{\mathrm{E}} \longrightarrow \infty} \Phi(z) - \lambda\Psi(z) = \infty,$$

which means the functional $I_\lambda = \Phi - \lambda\Psi$ is coercive. Thus, by [3, Proposition 2.1] the functional $I_\lambda = \Phi - \lambda\Psi$ verifies $(\mathrm{PS})^{[s]}$-condition for each $s > 0$ and so the condition (a_2) of Theorem 2.1 is verified. Fix $\lambda \in (0, \lambda^*)$, thus

$$\frac{\int_{\frac{3}{8}}^{\frac{5}{8}} H(t,\sigma)\mathrm{d}t + \frac{\mu}{\lambda}G_\sigma}{D\sigma^2} > \frac{1}{\lambda}.$$

Put $s = 2\delta^2\pi^2\theta^2$ and

$$w_\sigma(t) = \begin{cases} -\dfrac{64\sigma}{9}\left(t^2 - \dfrac{3}{4}t\right), & \text{if } t \in [0, \frac{3}{8}), \\ \sigma, & \text{if } t \in [\frac{3}{8}, \frac{5}{8}] \\ -\dfrac{64\sigma}{9}\left(t^2 - \dfrac{5}{4}t + \dfrac{1}{4}\right), & \text{if } t \in (\frac{5}{8}, 1]. \end{cases}$$

Clearly, $w_\sigma \in \mathrm{E}$. Obviously, one has

$$\begin{aligned}
&\left(\frac{4096}{27}p^- + \frac{64}{9}q^- + \frac{13}{20}r^-\right)\sigma^2 \\
&\quad \le \int_0^{\frac{3}{8}} p(t)\frac{16384}{81}\sigma^2 \mathrm{d}t + \int_{\frac{5}{8}}^1 p(t)\frac{16384}{81}\sigma^2 \mathrm{d}t + \int_0^{\frac{3}{8}} q(t)\left(-\frac{64\sigma}{9}\left(2t-\frac{3}{4}\right)\right)^2 \mathrm{d}t \\
&\qquad + \int_{\frac{5}{8}}^1 q(t)\left(-\frac{64\sigma}{9}\left(2t-\frac{5}{4}\right)\right)^2 \mathrm{d}t + \int_0^{\frac{3}{8}} r(t)\left(-\frac{64\sigma}{9}\left(t^2-\frac{3}{4}t\right)\right)^2 \mathrm{d}t \\
&\qquad + \int_{\frac{5}{8}}^1 r(t)\left(-\frac{64\sigma}{9}\left(t^2-\frac{5}{4}t+\frac{1}{4}\right)\right)^2 \mathrm{d}t + \int_{\frac{3}{8}}^{\frac{5}{8}} r(t)\sigma^2 \mathrm{d}t \\
&\quad = \int_0^1 (p(t)|z''(t)|^2 + q(t)|z'(t)|^2 + r(t)|z(t)|^2)\mathrm{d}t \\
&\quad = \|w_\sigma\|_{\mathrm{E}}^2 \le \left(\frac{4096}{27}p^+ + \frac{64}{9}q^+ + \frac{13}{20}r^+\right)\sigma^2.
\end{aligned}$$

Then, we have $\Phi(0) = \Psi(0) = 0$ and

$$\frac{1}{2}\left(\frac{4096}{27}p^- + \frac{64}{9}q^- + \frac{13}{20}r^-\right)\sigma^2 \le \Phi(w_\sigma) = \frac{1}{2}\|w_\sigma\|_{\mathrm{E}}^2 \le \frac{1}{2}\left(\frac{4096}{27}p^+ + \frac{64}{9}q^+ + \frac{13}{20}r^+\right)\sigma^2.$$

By using condition (A_1), we have

$$\begin{aligned}
\Psi(w_\sigma) &= \int_0^1 H(t, w_\sigma(t))\mathrm{d}t + \frac{\mu}{\lambda}\int_0^1 G(t, w_\sigma(t))\mathrm{d}t \\
&\ge \int_{\frac{3}{8}}^{\frac{5}{8}} H(t,\sigma)\mathrm{d}t + \frac{\mu}{\lambda}G_\sigma.
\end{aligned}$$

Thus, by the assumption

$$\sqrt{\frac{D}{2}}\frac{\sigma}{\pi\delta} < \theta,$$

we have $0 < \Phi(w_\sigma) < s$. Owing to Proposition 2.9, one has

$$\|z\|_\infty \le \frac{1}{2\pi\delta}\|z\|_{\mathrm{E}} \le \frac{1}{2\pi\delta}\sqrt{2s} = \theta,$$

hence, we have

$$\sup_{\Phi(z)<s} \Psi(z) \le \int_0^1 \max_{|m|\le\theta} H(t,m)\mathrm{d}t + \frac{\mu}{\lambda}G^\theta.$$

Therefore, we have

$$\begin{aligned}
&\frac{\sup_{z\in\Phi^{-1}(-\infty,s]}\Psi(z)}{s} \\
&\quad = \frac{\sup_{z\in\Phi^{-1}(-\infty,s]}\left(\int_0^1 H(t,z(t))\mathrm{d}t + \frac{\mu}{\lambda}\int_0^1 G(t,z(t))\mathrm{d}t\right)}{s} \\
&\quad \le \frac{\int_0^1 \max_{|m|\le\theta} H(t,m)\mathrm{d}t + \frac{\mu}{\lambda}G^\theta}{2\delta^2\pi^2\theta^2}
\end{aligned} \tag{3.3}$$

and

$$\frac{\Psi(w_\sigma)}{\Phi(w_\sigma)} \geq \frac{\int_0^1 H(t,w(t))\mathrm{d}t + \frac{\mu}{\lambda}\int_0^1 G(t,w(t))\mathrm{d}t}{D\sigma^2} \geq \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} H(t,\sigma)\mathrm{d}t + \frac{\mu}{\lambda}G_\sigma}{D\sigma^2}. \tag{3.4}$$

Since

$$\mu < \frac{2\delta^2\pi^2\theta^2 - \lambda\int_0^1 \max_{|m|\leq\theta} H(t,m)\mathrm{d}t}{G^\theta},$$

we have

$$\frac{\int_0^1 \max_{|m|\leq\theta} H(t,m)\mathrm{d}t + \frac{\mu}{\lambda}G^\theta}{2\delta^2\pi^2\theta^2} < \frac{1}{\lambda}.$$

Furthermore,

$$\mu < \frac{D\sigma^2 - \lambda\int_{\frac{3}{8}}^{\frac{5}{8}} H(t,\sigma)\mathrm{d}t}{G_\sigma},$$

this means

$$\frac{\int_{\frac{3}{8}}^{\frac{5}{8}} H(t,\sigma)\mathrm{d}t + \frac{\mu}{\lambda}G_\sigma}{D\sigma^2} > \frac{1}{\lambda}.$$

Then,

$$\frac{1}{2\delta^2\pi^2}\frac{\int_0^1 \max_{|m|\leq\theta} H(t,m)\mathrm{d}t + \frac{\mu}{\lambda}G^\theta}{\theta^2} < \frac{1}{\lambda} < \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} H(t,\sigma)\mathrm{d}t + \frac{\mu}{\lambda}G_\sigma}{D\sigma^2}. \tag{3.5}$$

Hence, from (3.3)–(3.5), the condition (a_1) of Theorem 2.1 is fulfilled. Since

$$\lambda \in \left(\frac{\Phi(w_\sigma)}{\Psi(w_\sigma)}, \frac{s}{\sup_{\Phi(z)\leq s}\Psi(z)}\right),$$

Theorem 2.1 with $\bar{z} = w$ guarantees the existence of a local minimum point z_λ for the functional I_λ such that $0 < \Phi(z_\lambda) < s$ and so z_λ is a nontrivial solution of the problem $(P^{h,g})$ such that

$$|z_\lambda(t)| < \theta. \qquad \square$$

Remark 3.2. We note that the preceding theorem also holds for a Carathéodory function h alongside $p \in W^{2,1}([0,1])$, $q \in W^{1,1}([0,1])$ and $r \in L^\infty([0,1])$. It is evident that, in this context, the solutions are generalized (see [8]).

We will now illustrate Theorem 3.1 by providing the following example.

Example 3.3. We consider the following problem

$$\begin{cases} z^{(4)} - z'' + z = \lambda h(z(t)) + \mu g(z(t)), & t \in [0,1], \\ z(0) = z(1) = 0, \\ z''(0) = z''(1) = 0 \end{cases} \tag{3.6}$$

where

$$h(m) = \begin{cases} 5m^4, & \text{for any } m \in (-\infty, 1), \\ \dfrac{5}{m}, & \text{for any } m \in [1, +\infty) \end{cases}$$

and

$$g(m) = \begin{cases} 2m, & \text{for any } m \in (-\infty, 1), \\ \dfrac{2}{m}, & \text{for any } m \in [1, +\infty). \end{cases}$$

Based on the expressions for h and g, we can conclude that

$$H(m) = \begin{cases} m^5, & \text{for each } m \in (-\infty, 1), \\ 5\ln(m) + 1, & \text{for each } m \in [1, +\infty) \end{cases}$$

and

$$G(m) = \begin{cases} m^2, & \text{for each } m \in (-\infty, 1), \\ 2\ln(m) + 1, & \text{for each } m \in [1, +\infty). \end{cases}$$

Through straightforward calculations, we derive $\delta = 1$ and $D = \frac{86111}{1080}$. Hence, $\lim_{|m| \longrightarrow \infty} \frac{H(m)}{|m|^2} = 0$, thus (A_3) is holds. Choose $\theta = 10^{-2}$, and $\sigma = 1$. Since

$$\frac{\int_0^1 \max_{|m| \leq \theta} H(m) \mathrm{d}t}{\theta^2} = \frac{1}{10^6} < \frac{540\pi^2}{86111} = \frac{2\delta^2\pi^2}{D} \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} H(\sigma) \mathrm{d}t}{\sigma^2},$$

therefore, if condition (A_2) is satisfied, all the requirements of Theorem 3.1 are met. Consequently, it follows that for each

$$\lambda \in \left(\frac{86111}{270}, 2\pi^2 \times 10^6 \right)$$

and since

$$\limsup_{|m| \longrightarrow \infty} \frac{G(,m)}{|m|^2} \in (-\infty, 0]$$

it follows that for each

$$\mu \in \left[0, \min \left\{ \frac{2\pi^2 \times 10^{-4} - \lambda 10^{-10}}{10^{-4}}, \frac{86111}{1080} - \frac{\lambda}{4} \right\} \right)$$

the problem (3.6) admits at least one solution z_λ in E such that

$$|z_\lambda(t)| < 10^{-2}.$$

4 Existence of two solutions

Applying Theorem 2.2, we obtain the following results:

Theorem 4.1. *Assume that there exist two positive constants θ and σ with the property*

$$\sqrt{\frac{D}{2}} \frac{\sigma}{\pi\delta} < \theta$$

and

(A_4) *there exist $\nu > 2$ and $R > 0$ such that*

$$0 < \nu H(t, m) \leq m h(t, m) \tag{4.1}$$

for each $|m| \geq R$ and for each $t \in [0, 1]$.

Then, for each

$$\lambda \in \left(0, \frac{2\delta^2\pi^2\theta^2}{\int_0^1 \max_{|m|\leq\theta} H(t,m)\mathrm{d}t}\right),$$

and for each function $g : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfying the condition (A_4), there exists $\delta_\lambda > 0$ for each $\mu \in [0, \delta_\lambda[$, the problem $(P^{h,g})$ admits at least two solutions z_1 and z_2 in E *such that*

$$|z_1(t)| < \theta.$$

Proof. Our objective is to utilize Theorem 2.2 in the context of the space E, and the functionals Φ and Ψ as outlined in the proof of Theorem 3.1. Additionally, the functional I_λ adheres to the (PS)-condition. Indeed, assume that $\{z_k\}_{k\in\mathbb{N}} \subset \mathrm{E}$ such that $\{I_\lambda(z_k)\}_{k\in\mathbb{N}}$ is bounded and $I'_\alpha(z_k) \longrightarrow 0$ as $k \longrightarrow +\infty$. Then, there exists a positive constant C_0 such that $|I_\lambda(z_k)| \leq C_0$ for each $k \in \mathbb{N}$. Therefore, we infer to deduce from the definition of I'_λ and the assumption (A_4), for k large enough, we get

$$\begin{aligned}
C_0 + C_1\|z_n\|_{\mathrm{E}} \geq \nu I_\lambda(z_n) - I'_\lambda(z_n)(z_n) &= \left(\frac{\nu}{2} - 1\right)\|z_n\|_{\mathrm{E}}^2 \\
&\quad - \lambda \int_0^1 (\nu H(t, z_n(t)) - h(t, z_n(t))\, z_n(t)\mathrm{d}t \\
&\quad - \mu \int_0^1 (\nu G(t, z_n(t)) - g(t, z_n(t))\, z_n(t)\mathrm{d}t \\
&\geq \left(\frac{\nu}{2} - 1\right)\|z_n\|_{\mathrm{E}}^2
\end{aligned}$$

for some $C_1 > 0$. Since $\nu > 2$, this implies that $\{z_k\}_{k\in\mathbb{N}}$ is bounded. Consequently, since E is a reflexive Banach space, we have, up to a subsequence, $z_k \rightharpoonup z$ in E, $z_k \to z$ in $L^2([0,1])$ and $z_k \to z$ a.e. on $[0,1]$. By $I'_\lambda(z_k) \to 0$ and $z_k \rightharpoonup z$ in E, we obtain

$$\left(I'_\lambda(z_k) - I'_\lambda(z)\right)(z_k - z) \to 0.$$

From the continuity of h and g, we have

$$\int_0^1 \left(h(t, z_k(t)) - h(t, z(t))\right)(z_k(t) - z(t))\mathrm{d}t \to 0$$

and

$$\int_0^1 \left(g(t, z_k(t)) - g(t, z(t))\right)(z_k(t) - z(t))\mathrm{d}t \to 0.$$

Thus

$$0 = \lim_{k\to+\infty} \left(I'_\lambda(z_k) - I'_\lambda(z)\right)(z_k - z) \geq \lim_{k\to+\infty} \|z_k - z\|_{\mathrm{E}}^2.$$

So $\|z_k - z\|_{\mathrm{E}} \to 0$ as $k \to +\infty$, which implies that $\{z_k\}$ converges strongly to u in E. Therefore, I_λ satisfies the Palais–Smale condition. Additionally, by incorporating the condition (4.1), there are constants $a_1, a_2, a_3, a_4 > 0$ such that

$$H(t,z) \geq a_1|z|^\nu - a_2 \quad \text{and} \quad G(t,z) \geq a_3|z|^\nu - a_4$$

for each $(t,z) \in [0,1] \times \mathbb{R}$. Now, by selecting any $z \in \mathrm{E} \setminus \{0\}$, we have

$$\begin{aligned}
I_\lambda(z) &= (\Phi + \lambda\Psi)(z) \\
&= \frac{1}{2}\|z\|_{\mathrm{E}}^2 - \lambda \int_0^1 H(t, z(t))\mathrm{d}t - \mu \int_0^1 G(t, z(t))\mathrm{d}t \\
&\leq \frac{1}{2}\|z\|_{\mathrm{E}}^2 - \lambda a_1 \frac{1}{2^\nu \pi^\nu \delta^\nu}\|z\|_{\mathrm{E}}^\nu - \mu a_3 \frac{1}{2^\nu \pi^\nu \delta^\nu}\|z\|_{\mathrm{E}}^\nu + \lambda a_2 + \mu a_4.
\end{aligned}$$

Given that $\nu > 2$, this condition ensures that I_λ is unbounded from below. Consequently, all the requirements of Theorem 2.2 are satisfied. Thus, for each

$$\lambda \in \left(0, \frac{2\delta^2\pi^2\theta^2}{\int_0^1 \max_{|m|\leq\theta} H(t,m)\mathrm{d}t}\right),$$

the functional I_λ admits two distinct critical points that are solutions of the problem ($P^{h,g}$). □

Remark 4.2. In Theorem 2.1, if either $h(t,0) \neq 0$ for some $t \in [0,1]$ or $g(t,0) \neq 0$ for some $t \in [0,1]$ or if both conditions are satisfied, then Theorem 4.1 guarantees the existence of two nontrivial solutions for the problem ($P^{h,g}$). if neither $h(t,0) \neq 0$ and $g(t,0) \neq 0$ is satisfied for any $t \in [0,1]$, the second solution z_2 to the problem ($P^{h,g}$) may be trivial. However, there is still at least one nontrivial solution.

Remark 4.3. Using similar arguments as presented in the proof of [5, Theorem 3.5], the nontrivial nature of the second solution guaranteed by Theorem 4.1 can also be established in the scenario where $h(t,0) = 0$ for each $t \in [0,1]$. This requires an additional condition at zero: namely, there exists a non-empty open set $D \subseteq [0,1]$ and a subset $B \subset D$ with positive Lebesgue measure such that

$$\limsup_{m\longrightarrow 0^+} \frac{\operatorname{ess\,inf}_{t\in B} H(t,m)}{|m|^2} = \infty \quad \text{and} \quad \liminf_{m\longrightarrow 0^+} \frac{\operatorname{ess\,inf}_{t\in D} H(t,m)}{|m|^2} > -\infty.$$

5 Another multiplicity result for the case $\mu = 0$

In this section, we discuss the existence of at least two and three solutions for the problem ($P^{h,g}$) when $\mu = 0$. To do this, we define

$$F^c = \int_0^1 \max_{|m|\leq c} H(t,m)\mathrm{d}t$$

and

$$F_c = \left(\frac{1}{4}\right) \inf_{t\in[0,1]} H(t,c)$$

for each $c > 0$.

Theorem 5.1. *Assume that there exist two positive constants $\bar{\theta}$ and $\bar{\sigma}$ such that*

$$\sqrt{\frac{D}{2}}\frac{\bar{\sigma}}{\pi\delta} < \bar{\theta} \tag{5.1}$$

and suppose that the assumptions (A_1) and (A_3) in Theorem 3.1 hold. Moreover, assume that

(A_5) $\frac{F^{\bar{\theta}}}{\bar{\theta}^2} < \frac{2\delta^2\pi^2}{D}\frac{F_{\bar{\sigma}}-F^{\bar{\theta}}}{\bar{\sigma}^2}$.

Then, for each

$$\lambda \in \left(\frac{D\bar{\sigma}^2}{F_{\bar{\sigma}} - F^{\bar{\theta}}}, \frac{2\delta^2\pi^2\bar{\theta}^2}{F^{\bar{\theta}}}\right),$$

the problem ($P^{h,g}$) in the case $\mu = 0$ admits at least three solutions in E.

Proof. Put $I_\lambda = \Phi + \lambda\Psi$, where

$$\Phi(z) = \frac{1}{2}\|z\|_{\mathrm{E}}^2 \tag{5.2}$$

and

$$\Psi(z) = -\int_0^1 H(t, z(t))\mathrm{d}t$$

for each $z \in \mathrm{E}$. Standard arguments demonstrate that the functionals Φ and Ψ are Gâteaux differentiable. The Gâteaux derivatives at a point $z \in \mathrm{E}$ are expressed as follows

$$\Phi'(z)(v) = \int_0^1 z'(t)v'(t)\mathrm{d}t + \int_0^1 \delta(t)(z(t))v(t)\mathrm{d}t$$

and

$$\Psi'(z)(v) = -\int_0^1 h(t, z(t))v(t)\mathrm{d}t$$

for each $z, v \in \mathrm{E}$, respectively. Therefore, a critical point of the functional $\Phi + \lambda\Psi$ corresponds to a solution of equation $(P^{h,g})$ when $\mu = 0$. Our goal is to apply Theorem 2.3 to the functionals Φ and Ψ. The Functional Φ is sequentially weakly lower semicontinuous, and, as discussed in Section 2, Φ is continuously Gâteaux differentiable Moreover, Proposition 2.12 indicates that its Gâteaux derivative has a continuous inverse on E^*. The functional $\Psi : \mathrm{E} \longrightarrow \mathbb{R}$ is well defined, continuously Gâteaux differentiable, and its Gâteaux derivative is compact. Thus, we need to verify that Φ and Ψ satisfy conditions (c_1) and (c_2) as stated in Theorem 2.3. We will now fix $0 < \varepsilon < \frac{2\pi^2\delta^2}{\lambda}$. From the assumption (A_3) there is a function $\phi_\varepsilon : [0,1] \longrightarrow \mathbb{R}$ with $\phi_\epsilon(t) < \infty$ for each $t \in [0,1]$ such that

$$H(t, \xi) \le \varepsilon\xi^2 + \phi_\varepsilon(t)$$

for each $(t, \xi) \in [0,1]$. It can be concluded that for each $z \in \mathrm{E}$

$$\begin{aligned}
\Phi(z) + \lambda\Psi(z) &= \frac{1}{2}\|z\|_{\mathrm{E}}^2 - \lambda\int_0^1 H(t, z(t))\mathrm{d}t \\
&\le \frac{1}{2}\|z\|_{\mathrm{E}}^2 - \lambda\varepsilon\frac{1}{4\pi^2\delta^2}\|z\|_{\mathrm{E}}^2 - \lambda\int_0^1 \phi_\varepsilon(t)dt \\
&= \left(\frac{1}{2} - \lambda\varepsilon\frac{1}{4\pi^2\delta^2}\right)\|z\|_{\mathrm{E}}^2 - \lambda\int_0^1 \phi_\varepsilon(t)dt
\end{aligned}$$

and thus

$$\lim_{\|z\|_{\mathrm{E}} \longrightarrow \infty} (\Phi(z) + \lambda\Psi(z)) = \infty,$$

which means the functional $I_\lambda = \Phi + \lambda\Psi$ is coercive. Now, it remains to show that (c_2) of Theorem 2.3 is fulfilled. Let $\bar{s} = 2\delta^2\pi^2\bar{\theta}^2$ and

$$\varpi_{\bar{\sigma}}(t) = \begin{cases} -\dfrac{64\bar{\sigma}}{9}\left(t^2 - \dfrac{3}{4}t\right), & \text{if } t \in \left[0, \dfrac{3}{8}\right), \\ \bar{\sigma}, & \text{if } t \in \left[\dfrac{3}{8}, \dfrac{5}{8}\right] \\ -\dfrac{64\bar{\sigma}}{9}\left(t^2 - \dfrac{5}{4}t + \dfrac{1}{4}\right), & \text{if } t \in \left(\dfrac{5}{8}, 1\right]. \end{cases}$$

Clearly $w \in$ E. Therefore, we can conclude that

$$B\bar{\sigma}^2 \leq \Phi(\bar{w}_{\bar{\sigma}}) \leq D\bar{\sigma}^2.$$

Thus by (5.1), $\Phi(w_\sigma) > \bar{s}$. Moreover

$$\Psi(\bar{w}_{\bar{\sigma}}) = -\int_0^1 H(t, \bar{w}_{\bar{\sigma}})\mathrm{d}t \leq -\int_{\frac{3}{8}}^{\frac{5}{8}} H(t, \bar{\sigma})\mathrm{d}t = -F_{\bar{\sigma}}.$$

For each $z \in$ E such that $\Phi(z) < \bar{s}$, we obtain

$$\sup_{t\in[0,1]} |z(t)| \leq \bar{\theta}.$$

Thus,

$$\sup_{z\in\Phi^{-1}(-\infty,\bar{s})} \Psi(x) \leq \int_0^1 \max_{|m|\leq\bar{\theta}} H(t,m)\mathrm{d}t = F^{\bar{\theta}}. \tag{5.3}$$

Through straightforward calculations and based on the definition of $\varphi(\bar{s})$, we find that since $\Phi(0) = \Psi(0) = 0$ and $\overline{\Phi^{-1}(-\infty,\bar{s})}^{w} = \Phi^{-1}(-\infty,\bar{s})$, it follows that By simple calculations and from the definition of $\varphi(\bar{s})$, since $\Phi(0) = \Psi(0) = 0$ and $\overline{\Phi^{-1}(-\infty,\bar{s})}^{w} = \Phi^{-1}(-\infty,\bar{s})$, one has

$$\begin{aligned}\varphi_1(\bar{s}) &= \inf_{z\in\Phi^{-1}(]-\infty,\bar{s}[)} \frac{\Psi(z) - \inf_{\overline{\Phi^{-1}(-\infty,\bar{s})}^{w}} \Psi}{\bar{s} - \Phi(z)} \leq \frac{-\inf_{\overline{\Phi^{-1}(-\infty,\bar{s})}^{w}} \Psi}{\bar{s}} \\ &\leq \frac{\int_0^1 \max_{|m|\leq\bar{\theta}} H(t,m)\mathrm{d}t}{2\delta^2\pi^2\bar{\theta}^2} = \frac{F^{\bar{\theta}}}{2\delta^2\pi^2\bar{\theta}^2}.\end{aligned}$$

Alternatively, according to (5.3), one can conclude that

$$\begin{aligned}\varphi_2(\bar{s}) &= \inf_{z\in\Phi^{-1}(-\infty,\bar{s})} \sup_{v\in\Phi^{-1}[\bar{s},\infty)} \frac{\Psi(z) - \Psi(v)}{\Phi(z) - \Phi(v)} \geq \inf_{z\in\Phi^{-1}(-\infty,\bar{s})} \frac{\Psi(z) - \Psi(\bar{w}_{\bar{\sigma}})}{\Phi(\bar{w}_{\bar{\sigma}}) - \Phi(z)} \\ &\geq \frac{\inf_{z\in\Phi^{-1}(-\infty,\bar{s})} \Psi(z) - \Psi(\bar{w}_{\bar{\sigma}})}{\Phi(\bar{w}_{\bar{\sigma}}) - \Phi(z)} \\ &\geq \frac{-\int_0^1 \max_{|m|\leq\bar{\theta}} H(t,m)\mathrm{d}t + \int_{\frac{3}{8}}^{\frac{5}{8}} H(t,\bar{\sigma})\mathrm{d}t}{\Phi(w_\sigma) - \Phi(z)} \\ &\geq \frac{F_{\bar{\sigma}} - F^{\bar{\theta}}}{D\sigma^2}.\end{aligned}$$

Consequently, from (A_5), we derive

$$\varphi_1(\bar{s}) < \varphi_2(\bar{s}).$$

Thus, based on Theorem 2.3, while also considering that

$$\frac{1}{\varphi_2(\bar{s})} \leq \frac{D\sigma^2}{F_{\bar{\sigma}} - F^{\bar{\theta}}}$$

and

$$\frac{1}{\varphi_1(\bar{s})} \geq \frac{2\delta^2\pi^2\bar{\theta}^2}{F^{\bar{\theta}}},$$

we arrive at the intended conclusion. □

Remark 5.2. When the assumption (A_5) of Theorem 5.1 is satisfied, straightforward calculations demonstrate that the condition

(A_6) $\frac{F^{\bar{\theta}}}{2\delta^2\pi^2\bar{\theta}^2} < \frac{F_{\bar{\sigma}}}{D\bar{\sigma}^2}$

implies (A_5) of Theorem 5.1. Therefore, if the conditions (5.1) and (A_5) are satisfied, then for each

$$\lambda \in \left(\frac{D\bar{\sigma}^2}{F_{\bar{\sigma}}}, \frac{2\delta^2\pi^2\bar{\theta}^2}{F^{\bar{\theta}}}\right),$$

the problem $(P^{h,g})$ in the case $\mu = 0$ admits at least three solutions.

We now apply Theorem 2.4, which will later be used to establish the existence of multiple solutions for the problem $(P^{h,g})$ when $\mu = 0$, without relying on assumption (A_3).

Theorem 5.3. *Assume that there exist three positive constants $\bar{\theta}_1$, $\bar{\sigma}$ and $\bar{\theta}_2$ with*

$$\bar{\theta}_1 < \sqrt{\frac{B}{2}}\frac{\bar{\sigma}}{\pi\delta} \tag{5.4}$$

and

$$\sqrt{\frac{D}{2}}\frac{\bar{\sigma}}{\pi\delta} < \bar{\theta}_2 \tag{5.5}$$

in a manner that satisfies the assumption (A_4) in Theorem 2.3 and

(A_7) $\dfrac{1}{2\delta^2\pi^2}\max\left\{\frac{F^{\bar{\theta}_1}}{\bar{\theta}_1^2}, \frac{F^{\bar{\theta}_2}}{\bar{\theta}_2^2}\right\} < \frac{F_{\bar{\sigma}}}{D\bar{\sigma}^2}.$

Then, for each

$$\lambda \in \Lambda = \left(\frac{D\bar{\sigma}^2}{F_{\bar{\sigma}}}, \min\left\{\frac{2\delta^2\pi^2\bar{\theta}_1^2}{F^{\bar{\theta}_1}}, \frac{2\delta^2\pi^2\bar{\theta}_2^2}{F^{\bar{\theta}_2}}\right\}\right),$$

the problem $(P^{h,g})$ in the case $\mu = 0$ admits at least two solutions $z_{1,\lambda}$ and $z_{2,\lambda}$ such that $|z_{1,\lambda}(t)| < \bar{\theta}_1$ and $|z_{2,\lambda}(t)| < \bar{\theta}_2$.

Proof. Put

$$\overline{h}(t,\epsilon) = \begin{cases} h(t,-\bar{\theta}_2) & \text{if } [0,1]\times(-\infty,\bar{\theta}_2), \\ h(t,\epsilon) & \text{if } [0,1]\times[-\bar{\theta}_2,\bar{\theta}_2], \\ h(t,\bar{\theta}_2) & \text{if } [0,1]\times(\bar{\theta}_2,\infty). \end{cases}$$

Clearly, $\overline{h} : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. Now put $\overline{H}(t,m) = \int_0^m \overline{h}(t,\epsilon)\mathrm{d}\epsilon$ for each $(t,m) \in [0,1]\times\mathbb{R}$ and take E and Φ as given in (5.2), and

$$\Psi(z) = -\int_0^1 \overline{H}(t,z(t))\mathrm{d}t$$

for each $z \in \mathrm{E}$. Our goal is to apply Theorem 2.4 to Φ and Ψ. It is well known that $\lim_{\|z\|_{\mathrm{E}}\longrightarrow\infty}\Phi(z) = \infty$ and Ψ is a differentiable functional whose differential at the point $z \in \mathrm{E}$ is

$$\Psi'(z)(v) = -\int_0^1 \overline{h}(t,z(t))v(t)\mathrm{d}t$$

for any $v \in \mathrm{E}$ as well as it is sequentially weakly lower semicontinuous. Furthermore $\Psi' : \mathrm{E} \longrightarrow \mathrm{E}^*$ is a compact operator. Therefore, it suffices to demonstrate that Φ and Ψ meet the criteria (c_1), (c_2) and (c_3) outlined in Theorem 2.4. Let

$$\bar{s}_1 = 2\delta^2\pi^2\bar{\theta}_1^2, \quad \bar{s}_2 = 2\delta^2\pi^2\bar{\theta}_2^2$$

and $w \in \mathrm{E}$ in a manner similar to the proof of Theorem 2.4, we note that under the assumptions (5.4) and (5.5), it follows that $\bar{s}_1 < \Phi(w_\sigma) < \bar{s}_2$ and $\inf_{\mathrm{E}} \Phi < \bar{s}_1 < \bar{s}_2$. Furthermore, by applying the reasoning used in the proof of Theorem 5.1 and considering Remark 5.2, we arrive at

$$\varphi_1(\bar{s}_1) \le \frac{\int_0^1 \max_{|m|\le\bar{\theta}_1} \overline{H}(t,m)\mathrm{d}t}{2\delta^2\pi^2\bar{\theta}_1^2} = \frac{F^{\bar{\theta}_1}}{2\delta^2\pi^2\bar{\theta}_1^2},$$

$$\varphi_1(\bar{s}_2) \le \frac{\int_0^1 \max_{|m|\le\bar{\theta}_2} \overline{H}(t,m)\mathrm{d}t}{2\delta^2\pi^2\bar{\theta}_2^2} = \frac{F^{\bar{\theta}_2}}{2\delta^2\pi^2\bar{\theta}_2^2}$$

and

$$\varphi_2^*(\bar{s}_1,\bar{s}_2) \ge \frac{F_{\bar{\sigma}}}{D\bar{\sigma}^2}.$$

Therefore, based on (A_7), the conditions (c_2) and (c_3) of Theorem 2.4 are satisfied. As a result, we can conclude from Theorem 2.4 that for each $\lambda \in \Lambda$, the problem

$$\begin{cases} -z'' + \alpha(t)z' + \delta(t)z = \lambda\bar{h}(t,z(t)) + \mu\bar{g}(t,z(t)), & t \in [0,1], \\ z(a) = z(b) = 0 \end{cases}$$

admits at least two solutions $z_{1,\lambda}$ and $z_{2,\lambda}$ such that $|z_{1,\lambda}(t)| < \bar{\theta}_1$ and $|z_{2,\lambda}(t)| < \bar{\theta}_2$. Noting that these solutions also satisfy the problem $(P^{h,g})$ when $\mu = 0$, we can draw the conclusion. □

In this section, we will provide some observations regarding our results.

Remark 5.4. In Theorems 5.1 and 5.3, we examined the critical points of the functional I_λ, which is naturally linked to the problem $(P^{h,g})$ when $\mu = 0$. It is important to note that, in general, I_λ can be unbounded from below in the space E. For instance, consider the scenario where $h(t) = 1 + |t|^{v-2}t$ for each $t \in \mathbb{R}$ with $v > 2$. In this case, for any fixed $z \in \mathrm{E}\setminus\{0\}$ and $\iota \in \mathbb{R}$, we can derive

$$\begin{aligned} I_\lambda(\iota z) &= \frac{1}{2}\|\iota z\|_{\mathrm{E}}^2 - \lambda\int_0^1 H(\iota z(t))\mathrm{d}t \\ &\le \frac{\iota^2}{2}\|z\|_{\mathrm{E}}^2 - \lambda\frac{1}{2\pi\delta}\|z\|_{\mathrm{E}} - \lambda\frac{1}{2^v\pi^v\delta^v}\|z\|_{\mathrm{E}}^v \longrightarrow -\infty \end{aligned}$$

as $\iota \longrightarrow \infty$. Thus, direct minimization cannot be used to locate the critical points of I_λ.

Remark 5.5. If h is non-negative, Theorem 5.3 provides a bifurcation result where the pair $(0,0)$ is included in the closure of the set:

$$\left\{(z_\lambda,\lambda) \in \mathrm{E}\times(0,\infty) : z_\lambda \text{ is a non-trivial solution of } (P^{h,g}),\ \ \mu = 0\right\} \subset \mathrm{E}\times\mathbb{R}.$$

Indeed, if λ goes to zero, by Theorem 5.3 we have that $\bar{\theta}_i \longrightarrow 0$, $i = 1, 2$ and since $|z_{i,\lambda}(t)| < \bar{\theta}_i$, $i = 1, 2$, there exist two sequences $\{t_j\}$ in E and $\{\lambda_j\}$ in $\mathbb{R}^+$ (here $t_j = z_{\lambda_j}$) such that

$$\lambda_j \longrightarrow 0^+ \quad \text{and} \quad \|t_j\|_{\mathrm{E}} \longrightarrow 0,$$

as $j \longrightarrow \infty$. Moreover, since f is nonnegative, $\Psi(z) < 0$ for each $z \in \mathbb{R}$ and thus

$$(0, \lambda^*) \ni \lambda \mapsto I_\lambda(z_\lambda)$$

is strictly decreasing. Hence, for each $\lambda_1, \lambda_2 \in (0, \lambda^*)$, with $\lambda_1 \neq \lambda_2$, solutions z_{λ_1} and z_{λ_2} ensured by Theorem 2.4 are different.

Remark 5.6. As noted in [18, Remark 3.10], if h is non-negative, the solutions guaranteed by Theorems 5.1 and 5.3 are also non-negative.

6 Ethical statement

Author contributions: All authors contributed equally to this work.

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Declarations

Conflict of interest: The authors declare that they have no conflict of interest.

Consent to participate: Not applicable.

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