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Existence and uniqueness of positive periodic solutions of a certain functional differential equation

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Abstract. In our work, we prove the existence and uniqueness of positive 1-periodic solutions of a first order differential equation that contains a piecewise constant term. Such equations admit periodic solutions and hence they can be used for numerical modelling of temperature changes in a space surrounded by environment with different temperature. For some types of functional equations, it is possible to use the comparison principle method of differential equations to obtain existence, uniqueness, and asymptotic stability results.

Keywords: functional differential equations, periodic solutions, existence, uniqueness.

2020 Mathematics Subject Classification: 34K13, 39B22.

1 Introduction

In our paper, we analyze positive 1-periodic solutions of functional differential equation

$$x'(t) = f(x(t)) + x(\lfloor t \rfloor)g(t - \lfloor t \rfloor), \qquad t \geq 0 \tag{1.1}$$

where $\lfloor . \rfloor : [0, \infty) \to \mathbb{N} \cup \{0\}$ is the standard floor function, i.e. $\lfloor t \rfloor = n$ if $t \in [n, n+1)$ for some $n \in \mathbb{N} \cup \{0\}$. We will assume that the function f is negative for positive x, and equals zero at zero, and g is positive over the interval $[0, 1]$ (see conditions 1) and 2) below for more precise assumptions).

The problem (1.1) represents a numeric model that describes temperature changes in a space bounded by outside environment. A solution of problem represents temperature of the space cooled by colder environment; the negative function f represents cooling up to zero. The space is simultaneously warmed by a heating agent inside; the positive function g represents heating. Moreover, the function g is multiplied by the temperature values at discrete points so that the actual temperature is affected by previous temperature.

In this work, we use the comparison principle for differential equations to prove the existence, uniqueness and asymptotic stability (see Theorem 3.6) of positive 1-periodic solution of

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the equation (1.1) under certain assumptions on the functions f and g. In some other works (e.g. [6]), authors analyzed an explicit form of solutions of a similar problem. In [5], a special type of Gronwall lemma was used. For more results and methods used regarding analyzing functional equations of this type, we refer to [4] and references therein. Next, linear functional equations similar to (1.1) have been studied by many authors. The floor function $\lfloor . \rfloor$ that appears in (1.1) was used in the automatic control theory in the frame of so-called sampled-data approach; see e.g. [10], [11]. In [3], [8], and [9], formulas of representation of solutions were found using the idea to the direction of the Floquet theory; the idea of such approach is to reduce equations in linear case to equations with finite-dimensional fundamental systems. In some other works (e.g. [6]), authors analyzed an explicit form of solutions. General theory of functional differential equations proposed in the books [1] and [2] was built namely on the idea of finite-dimensional fundamental systems. We also note that in [13], such delays are used in semi-discretization methods for the study of stability analysis of linear time-periodic and time-delay systems.

2 Preliminaries

Throughout the whole paper, for the functions f, g in the equation (1.1), we will assume the following conditions:

1) $f : [0, \infty) \to (-\infty, 0]$ is continuously differentiable, $f(0) = 0$ and $f(x) < 0$ for $x > 0$,

2) $g : [0, 1] \to (0, \infty)$ is continuous.

It is important to note that the derivative of solution x of the equation (1.1) need not exist at $t = n$ for $n \in \mathbb{N}$. In such case, x is not continuously differentiable at these points. However, one-sided derivatives of x at $t = n$ do exist, since in (1.1), one can pass to the limit for $t \to n^+$ or $t \to n^-$. Due to these facts, we state the following definition of solutions of (1.1).

We say that a continuous function $x : [0, \infty) \to \mathbb{R}$ is a solution of the equation (1.1) if for every $n \in \mathbb{N} \cup \{0\}$, the function x is continuously differentiable over $(n, n+1)$, and the equation (1.1) is satisfied for $t \in (n, n+1)$. Necessarily, there exist finite one-sided derivatives of x at points n and $n+1$.

In order to study $1-$periodic solutions of (1.1), we will consider the problem

$$x'(t) = f(x(t)) + x(0)g(t), \qquad t \in [0,1], \tag{2.1}$$

$$x(0) = x(1), \tag{2.2}$$

where $x'(0)$ and $x'(1)$ are the corresponding one-sided derivatives. Obviously, for every initial condition $x(0) > 0$, there exists a unique positive solution x of the equation (2.1) defined over the whole interval $[0, 1]$. The equation (1.1) has a 1-periodic solution if and only if there exists a solution of problem (2.1)–(2.2).

Let g satisfy the condition 2). We introduce some notation we will use throughout the whole paper. Denote $g_1 = \min_{t\in[0,1]} g(t)$, $g_2 = \max_{t\in[0,1]} g(t)$, and $g_3 = \int_0^1 g(t)\,\mathrm{d}t$. Thus, it holds $g_2 \geq g_3 \geq g_1 > 0$ and if g is nonconstant over $[0, 1]$, then strict inequalities are valid.

3 Results

In this section, we present our results. In the proofs, we use standard methods of analysis of solutions of ODE such as the comparison method.

Lemma 3.1. *Let x be a solution of the equation (2.1), $x(0) > 0$, and g is differentiable and nonincreasing over the interval $[0, 1]$. Then the function x attains its minimum either at $t = 0$ or $t = 1$.*

Proof. Observe that since $x(0) > 0$ and $g > 0$ over $[0, 1]$, the corresponding solution x of (2.1) remains positive over $[0, 1]$ (this is not obviously true for every continuous function g; see Remark 3.3). Also, if there is a continuously differentiable function y such that

$$y'(t) > f(y(t)) + y(0)g(t), \qquad t \in (0, 1]$$

with $y(0) = x(0)$, then necessarily $y(t) > x(t)$ for $t \in (0, 1]$. These assertions can be proven by using standard methods; see e.g. [12].

First, we assume that $g' < 0$ over $[0, 1]$ and proceed by a contradiction. Let $t_0 \in (0, 1)$ be the point of minimum of x. Hence $x'(t_0) = 0$ and differentiating (2.1) with respect to t, we obtain

$$x''(t) = f'(x(t))x'(t) + x(0)g'(t). \tag{3.1}$$

This means that $x''(t_0) < 0$ and so t_0 cannot be a point of minimum.

Now, we assume that g is differentiable and nonincreasing over $[0, 1]$. Denote $g_n(t) = g(t) - \frac{t}{n}$ for $n \in \mathbb{N}$ large enough so that $g_n > 0$ over $[0, 1]$ (say $n \geq n_0$). Let x_n be the corresponding solution of

$$x_n'(t) = f(x_n(t)) + x(0)g_n(t)$$

that satisfies $x_n(0) = x(0)$. Then x_n is bounded by zero from below and by x from above, and the sequence $\{x_n\}_{n \geq n_0}$, x_n converges uniformly to x over $[0, 1]$. Since $g_n' < 0$, we have $x_n(t) \geq \min\{x_n(0), x_n(1)\}$ by the previous part of the proof. Hence $x(t) \geq \min\{x(0), x(1)\}$. □

Remark 3.2. Using arguments from the proof of Lemma 3.1, one can prove that if there holds (2.2) and $g' < 0$ over $[0, 1]$, then x attains its maximum at a unique point $t_0 \in (0, 1)$.

Remark 3.3. It is well known that if g is not positive, then a solution x that starts at some $x(0) > 0$ may cross the zero solution. Necessary and sufficient conditions of this phenomenon for linear delay differential equations were obtained in [7]. For convenience of readers, we can consider the equation

$$x'(t) = -x(t) + x(0)(-2t - 1).$$

Observe that the function $x(t) = \frac{1}{2} - t$ is a solution that crosses zero.

Lemma 3.4. *Let $f(x) = -\eta x$ where $\eta > 0$. Then there exists a unique $\eta^* = \eta^*(g) > 0$ such that for every positive solution x of (2.1), the following assertions are true:*

i) If $\eta > \eta^*$, then $x(1) < x(0)$.

ii) If $\eta < \eta^*$, then $x(1) > x(0)$.

iii) If $\eta = \eta^*$, then $x(1) = x(0)$.

Proof. Let x be a positive solution of the equation

$$x'(t) = -\eta x(t) + x(0)g(t), \qquad t \in [0, 1] \tag{3.2}$$

that starts at some $x(0) > 0$. Multiplying the equation with $e^{\eta t}$ and integrating over $[0, 1]$, we obtain

$$x(1) = x(0)e^{-\eta}\left(1 + \int_0^1 e^{\eta t} g(t)\, \mathrm{d}t\right).$$

Now, we discuss when there holds

$$e^{-\eta}\left(1+\int_0^1 e^{\eta t}g(t)\,\mathrm{d}t\right)=1 \tag{3.3}$$

or equivalently, $F(\eta):=\int_0^1 e^{\eta t}(g(t)-\eta)\,\mathrm{d}t=0$.

Let us prove that there is a unique solution of the equation $F(\eta)=0$. Note that F is a continuous function and $F(g_1)>0>F(g_2)$. Hence there exists at least one solution $\eta^*\in(g_1,g_2)$ of (3.3) and obviously, there is no solution beyond (g_1,g_2). Thus, every positive solution x of (3.2) for $\eta=\eta^*$ satisfies $x(0)=x(1)$. If there exists a root $\eta_1>\eta^*$ of (3.3) and y is a positive solution of problem

$$y'(t)=-\eta_1 y(t)+y(0)g(t),$$

then $y'(t)<-\eta^* y(t)+y(0)g(t)$. We can choose x, a solution of (3.2) with $\eta=\eta^*$ such that $x(0)=y(0)$, then $y(1)<x(1)=x(0)=y(0)$ what is a contradiction. If $\eta_1<\eta^*$, then a similar argument can be used. We conclude that $\eta_1\neq\eta^*$ cannot be a root of (3.3).

We turn to prove *i*). If $\eta>\eta^*$, then $F(\eta)<0$, hence $e^{-\eta}\left(1+\int_0^1 e^{\eta t}g(t)\,\mathrm{d}t\right)<1$ and $x(1)<x(0)$. The assertions *ii*) and *iii*) can be proven similarly. □

Remark 3.5. From the proof of Lemma 3.4, it is clear that $\eta^*\in(g_1,g_2)$. If moreover, g is nonconstant, nonincreasing and differentiable over $[0,1]$, then $\eta^*\in(g_1,g_3)$. Indeed, integrating by parts, we get

$$F(g_3)=\int_0^1 e^{g_3 t}(g(t)-g_3)\,\mathrm{d}t=[e^{g_3 t}\,(G(t)-g_3 t)]_0^1-g_3\int_0^1 e^{g_3 t}(G(t)-g_3 t)\,\mathrm{d}t,$$

where $G(t)=\int_0^t g(s)\,\mathrm{d}s$. Note that $G(0)=0$ and $G(1)=g_3$. Since G is a function concave over $[0,1]$ and not equal to the line that connects the points $[0,0]$ and $[1,g_3]$, it holds $G(t)>g_3 t$ for $t\in(0,1)$. We conclude that $F(g_3)<0$ and since $F(g_2)<0$, then necessarily, $\eta^*\in(g_1,g_3)$.

Theorem 3.6. *Let f,g satisfy the conditions 1), 2), and let η^* be as in Lemma 3.4. Assume that*

$$\lim_{x\to 0^+}\frac{f(x)}{x}>-\eta^*>\lim_{x\to\infty}\frac{f(x)}{x}. \tag{3.4}$$

Then there exists a positive solution x of problem (2.1)–(2.2), hence a 1-periodic solution of (1.1).

Moreover, if the function $\frac{f(x)}{x}$ is decreasing on interval $(0,\infty)$, then the positive 1-periodic solution x is unique and for every solution y of (1.1) such that $y(0)>0$, it holds

$$\lim_{t\to\infty}(x(t)-y(t))=0. \tag{3.5}$$

Proof. Note that if x,y are two solutions of (2.1) and $x(0)>y(0)>0$, then $x(t)>y(t)$ for $t\in[0,1]$. Indeed, it holds $x'(t)-y'(t)=a(t,x,y)(x(t)-y(t))+(x(0)-y(0))g(t)$ for $a(t,x,y)=\int_0^1 f'(sx(t)+(1-s)y(t))\,\mathrm{d}s$. Thus,

$$x(t)-y(t)=(x(0)-y(0))\left(e^{\int_0^t a(s)\,\mathrm{d}s}+\int_0^t e^{\int_s^t a(v)\,\mathrm{d}v}g(s)\,\mathrm{d}s\right)$$

and hence, $x>y$ over $[0,1]$ due to the condition 2).

Observe that if there exist two positive solutions $x_{-,+}$ of (2.1) such that $x_-<x_+$ over $[0,1]$, $x_-(0)<x_-(1)$, and $x_+(0)>x_+(1)$, then there is a solution of (2.1)–(2.2) that lies between x_-

and x_+. In fact, if we define a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_1(0) = x_-(1)$ and $x_{n+1}(0) = x_n(1)$ for $n \in \mathbb{N}$, then $x_-(t) < x_1(t) < x_2(t) < \ldots < x_+(t)$ for $t \in [0,1]$, since two distinct positive solutions of (2.1) do not intersect each other. The sequence of functions $\{x_n\}_{n\in\mathbb{N}}$ is increasing with respect to n and bounded by x_+ from above hence convergent to some solution of (2.1)-(2.2). In the following, our goal will be to find such functions x_- and x_+.

Due to the assumption $\lim_{x\to 0^+} \frac{f(x)}{x} > -\eta^*$, there holds $f(x) > -\eta x$ for some $\eta < \eta^*$ close to η^* and for all $x \in (0,\varepsilon)$ where $\varepsilon = \varepsilon(\eta) > 0$ is sufficiently small. If $x(0) \in \big(0, \frac{\varepsilon}{2}\min\{1, \frac{1}{g_2}\}\big)$ and x is the corresponding solution of (2.1), then $x' < x(0)g_2 \le \frac{\varepsilon}{2}$. Thus, the function x is bounded by ε. If y is the solution of (2.1) with $f(x) = -\eta x$ and $y(0) = x(0)$, then we can similarly prove that $y(t) \in (0,\varepsilon)$ for $t \in [0,1]$ and so $y < x$ over $(0,1]$. Due to Lemma 3.4, it holds $y(1) > y(0)$ therefore $x(1) > x(0)$ and we can denote $x_- := x$.

Now, we turn to find a solution x_+. Observe that if the initial condition $x(0) > 0$ is arbitrarily large the the value $\min_{t\in[0,1]} x(t)$ is also arbitrarily large where x is the corresponding solution of (2.1). Assume by contradiction that there is a sequence $\{x_n(0)\}_{n\in\mathbb{N}}$ of initial values such that $x_n(0) \to \infty$ as $n \to \infty$ and $\min_{t\in[0,1]} x_n(t)$ is bounded with respect to n. Let $t_n \in [0,1]$ be such that $x_n(t_n) = \min_{t\in[0,1]} x_n(t)$. Clearly, $t_n > 0$ for all n sufficiently large, since $x_n(0)$ is unbounded, and so $x_n'(t_n) \le 0$. On the other hand, the sequence $\{f(x_n(t_n))\}_{n\in\mathbb{N}}$ is bounded from below by some negative constant K and we conclude that

$$0 \ge x_n'(t_n) = f(x_n(t_n)) + x_n(0)g(t_n) \ge K + x_n(0)g_1.$$

This is not true if n is large enough.

The assumption (3.4) implies that $f(x) < -\eta^* x$ for all $x \in (x_0,\infty)$ for some x_0 large enough. We choose $x(0) > x_0$ large enough so that $x(t) > x_0$ for $t \in [0,1]$ where x is the corresponding solution of (2.1). Let y be the solution of equation

$$y'(t) = -\eta^* y(t) + y(0)g(t)$$

with $y(0) = x(0)$. Lemma 3.4 implies that $y(1) = y(0)$ and y lies entirely in (x_0,∞) (increasing $x(0)$ if necessary). Consequently, $y > x$ over $(0,1]$, $x(1) < y(1) = x(0)$, and we can choose $x_+ = x$.

Let x be a fixed positive solution of problem (2.1)–(2.2). We prove that x is unique. Let $\alpha \in (0,1)$ and denote $z(t) = \alpha x(t)$ for $t \in [0,1]$. Since the function $\frac{f(x)}{x}$ is decreasing on $(0,\infty)$, it holds

$$\frac{f(\alpha\xi)}{\alpha\xi} > \frac{f(\xi)}{\xi} \quad \text{for } \xi > 0$$

and so $\alpha f(x(t)) < f(\alpha x(t))$. Thus, we obtain

$$z'(t) = \alpha\big(f(x(t)) + x(0)g(t)\big) < f(z(t)) + z(0)g(t).$$

If y is the solution of (2.1) equipped with $y(0) = z(0) = \alpha x(0)$, then $y(t) > z(t)$ over $(0,1]$. Hence $y(1) > \alpha x(1) = \alpha x(0) = y(0)$. Thus, any solution of (2.1) that lies between x and the zero solution cannot be extended to a 1-periodic solution of (1.1).

If $\alpha > 1$, then similar arguments imply that $y(1) < y(0)$ where y is the solution of (2.1) satisfying $y(0) = \alpha x(0)$. Hence the solution x is the unique positive 1-periodic solution.

The assertion (3.5) is a consequence of above arguments. Let y be as in the previous part of the proof. If $y(0) < x(0)$, then the sequence $\{y(n)\}_{n\in\mathbb{N}}$ is increasing and necessarily converges to $x(0)$ so there holds (3.5). Otherwise y would converge to a 1-periodic solution different from x which is not possible. The case $y(0) > x(0)$ is similar. $\square$

Remark 3.7. If $f(x) > -\eta^* x$ or $f(x) < -\eta^* x$ for all $x > 0$, then one can use arguments as in the proof of Theorem 3.6 and Lemma 3.4 to prove that there is no positive solution of problem (2.1)–(2.2).

Theorem 3.8. *Let $x_0 > 0$ and assume that one of the cases is true:*

i) It holds either $f(x_0) > -g_1 x_0$ or $f(x_0) < -g_2 x_0$.

ii) It holds $f(x_0) < -g_3 x_0$, f, g are decreasing.

Then there is no solution of the problem (2.1)–(2.2) with $x(0) = x_0$.

Proof. Let $i)$ be valid. Assume first that $f(x_0) > -g_1 x_0$ and there is a solution x of (2.1)–(2.2) satisfying $x(0) = x_0$. Let y be the solution of equation

$$y' = f(y) + x_0 g_1, \qquad y(0) = x_0.$$

Since the equation is autonomous, y is either strict monotone or constant. From $g \geq g_1$, we have $x \geq y$ over $[0, 1]$. On the other hand, $f(y(0)) = f(x_0) > -x_0 g_1$ and so y is increasing on some neighbourhood of 0, hence increasing on $[0, 1]$. Hence $x(1) > x_0 = x(0)$. The case $f(x_0) < -g_2 x_0$ is analogous.

Assume that $ii)$ is valid. Lemma 3.1 implies that if x is a solution of (2.1)–(2.2), then $x(t) \geq x_0$ for $t \in (0, 1)$, hence $f(x(t)) \leq f(x_0)$. Integrating the equation (2.1) over $[0, 1]$, it holds

$$0 = \int_0^1 f(x(t))\,\mathrm{d}t + x_0 \int_0^1 g(t)\,\mathrm{d}t \leq f(x_0) + x_0 g_3,$$

a contradiction. □

4 Example

Consider the equation

$$x'(t) = -\frac{x(t)}{3} - x^2(t) + x(\lfloor t \rfloor)e^{-(t-\lfloor t \rfloor)}, \qquad t \geq 0. \tag{4.1}$$

Note that the functions $f(x) = -\frac{x}{3} - x^2$, $x \geq 0$, and $g(t) = e^{-t}$, $t \in [0, 1]$ satisfy the conditions 1), 2). One can easily check that all of the assumptions of Theorems 3.6 and 3.8 are true so that there is a unique positive 1-periodic solution x. Due to Remark 3.5, η^* lies in the interval $(e^{-1}, 1 - e^{-1})$. Theorem 3.8 provides us a bound for the starting value $x(0) := x_0$. From the part $i)$, we come to inequality $-\frac{x_0}{3} - x_0^2 > -e^{-1}x_0$, or equivalently, $x_0 > e^{-1} - \frac{1}{3}$. We can use the part $ii)$ and we obtain $-\frac{x_0}{3} - x_0^2 < -(1 - e^{-1})x_0$, or, $x_0 > \frac{2}{3} - e^{-1}$. Hence for $x(0) \notin [e^{-1} - \frac{1}{3}, \frac{2}{3} - e^{-1}]$, then there is no positive 1-periodic solution of (4.1).

The equation (3.3) can be solved numerically and we find that $\eta^* \approx 0.601$. In the Figure 4.1, we see two solutions $x_{-,+}$ of (4.1) over the interval $[0, 100]$. For x_-, we choose $x_-(0) = 0.03 < e^{-1} - \frac{1}{3}$ and for x_+, $x_-(0) = 0.3 > \frac{2}{3} - e^{-1}$. From the proof of Theorem 3.6, we know that the values $x_-(n)$ and $x_+(n)$ for integer n converge to a value that is the initial condition for the periodic solution. Numerical computations show that $x_-(100) \approx 0.256$ and $x_+(100) - x_-(100) \approx 1.64 \times 10^{-9}$. This means that $x_-(100)$ approximates the initial value of the 1-periodic solution with error 1.64×10^{-9}. As observed in Remark 3.2, the graphs of both solutions consist of smooth parts of solutions of (1.1) defined on intervals $[n-1, n]$

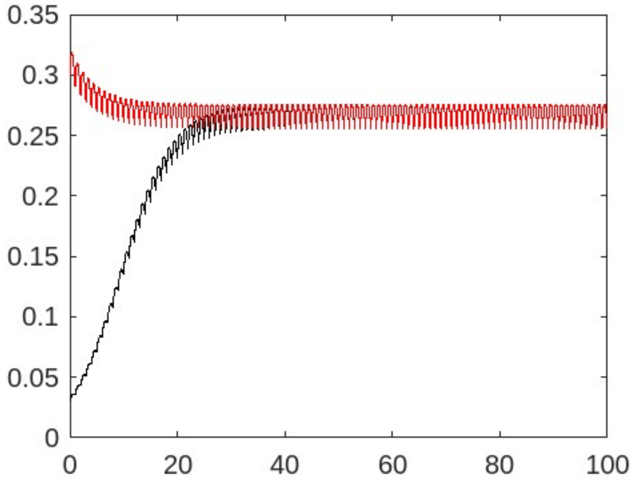

Figure 4.1: Comparison of solutions x_- (black line) and x_+ (red line).

for $n = 1, \dots, 100$ that attain local maximum at exactly one interior point on each interval $[n-1, n]$ and attain local minima at points $n-1$ and n.

The problem (4.1) describes temperature changes in a space bounded by outside cold environment for particular functions f and g. We see that for such f, g, the optimal initial temperature is approximately equal to 0.256 and repeats regularly at every positive integer point. If the initial temperature is higher or lower, then, after some time, it becomes close to optimal due to the asymptotic stability of the periodic solution in sense of (3.5).

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