Electronic Journal of Qualitative Theory of Differential Equations
2025, No. 14, 1–27; https://doi.org/10.14232/ejqtde.2025.1.14 www.math.u-szeged.hu/ejqtde/

Some results concerning a system of two max difference equations

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Received 2 February 2025, appeared 30 April 2025
Communicated by Stevo Stević

Abstract. In this paper we investigate the behaviour of the solutions of the following close-to-cyclic system of two difference equations with maximum:

$$x_{n+1} = \max\left\{A, \frac{y_n^p}{x_{n-1}^q}\right\},$$
$$y_{n+1} = \max\left\{B, \frac{x_n^p}{y_{n-1}^q}\right\}$$

where $n = 0, 1, \dots$, the coefficients A, B, are positive real numbers, the exponents p, q are positive real numbers such that $p > q + 1$, and the initial values x_{-1}, x_0, y_{-1}, y_0, are positive real numbers.

Keywords: difference equations with maximum, close-to-cyclic system, equilibrium, solution eventually equal to equilibrium, periodic solutions.

2020 Mathematics Subject Classification: 39A10.

1 Introduction

Difference equations and systems of difference equations have gained much attention during the last decades due to their wide applications. These equations can be used as mathematical models to describe the biological, population and generally physical as well as economic phenomena (several models can be found, for example, in [11] and [29]). Despite that difference equations have, usually, simple forms, it is extremely difficult to fully describe precisely the global behavior of their solutions (see, for instance, [1]–[15], [17]–[21], [23]–[60] and the related references therein). For some differential equations with maximum see, for example, [16, 22].

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In addition, max operators are used in the study of problems concerning automatic control (see [16], [22] and the references cited therein). Therefore, there exist many papers concerning max difference equations, that is, max-type difference equations or difference equations with maximum (see, e.g., [1,4–7,9,10,12–15,17,18,21,23–27,35–45,47–51,56,58–60] and the related references therein).

At the beginning, difference equations with maximum containing linear terms were studied and are special cases of the difference equation

$$x_{n+1} = \max\left\{\frac{A_n^{(0)}}{x_n}, \frac{A_n^{(1)}}{x_{n-1}}, \dots, \frac{A_n^{(k)}}{x_{n-k}}\right\}, \qquad n = 0, 1, \dots,$$

see for example [4,5,7,10,14,15,21,23,57–60].

In 2003 S. Stević started a systematic study of the difference equations containing powers of the terms; see [32] where the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}^p}{x_n^p}, \qquad n = 0, 1, \dots,$$

was studied (for some later results see [3]). The equation with $p = 1$ or $p = -1$, and related ones had been considerably investigated before it (see, for example, [11,28,30,31] and the references therein). For some later generalizations of the equation in these cases see [2].

The investigation in [32] have been continued by him in a series of papers. In [34] beside the equation

$$x_{n+1} = A + \frac{x_n^p}{x_{n-1}^r}, \qquad n = 0, 1, \dots,$$

(for some generalizations of the equation see, for instance, [20,33,36]), it was also studied its max-type counterpart

$$x_{n+1} = \max\left\{A, \frac{x_n^p}{x_{n-1}^r}\right\}, \qquad n = 0, 1, \dots,$$

where the parameters A, p, q and the initial values x_{-1} and x_0 are positive numbers. This is one of the first papers where a difference equation with maximum containing powers of the terms was studied. The case $p = r$, was also studied in [35], where, among other results, it was proved that if $A, p \in (0, 1)$ all positive solutions converge to one. This result was later generalized in [9]. For some other results on convergence of solutions to related difference equations with maximum see also [12,13,37,38].

A complete picture regarding the boundedness character of positive solutions to the following difference equation

$$x_n = \max\left\{A, \frac{x_{n-1}^p}{x_{n-k}^p}\right\}, \qquad n = 0, 1, \dots,$$

where $k \geq 2$, and the parameters A, p and the initial values are positive numbers, was given in [36]. The results on the boundedness therein were later generalized in [41]. For a quite general result in this direction see [49].

A study of a general difference equation with maximum, which includes above mentioned ones, was suggested in [40] (see difference equation (1) therein). Periodicity of quite general classes of difference equation with maximum was studied in [40,42,43,48,50]. Some results are devoted to solvability of max type equations such as [44] (for some recent results on the topic see, for instance, [46,52,54,55] and the references therein).

In [8], studying cyclic systems of difference equations was initiated. The study was continued, for instance, in [19,46,47,49,52,55–57]. Paper [8] essentially suggested investigating all cyclic systems of difference equations, so also those corresponding to above-mentioned difference equations.

In [45] investigated the following max-type system

$$x_{n+1} = \max\left\{\alpha, \frac{y_n^p}{x_{n-1}^p}\right\}, \quad y_{n+1} = \max\left\{\alpha, \frac{x_n^p}{y_{n-1}^p}\right\}, \qquad n = 0, 1, \dots,$$

where the parameters α, p and the initial values are positive numbers, whereas its generalization

$$x_{n+1} = \max\left\{\alpha, \frac{y_n^p}{x_{n-1}^q}\right\}, \quad y_{n+1} = \max\left\{\alpha, \frac{x_n^p}{y_{n-1}^q}\right\}, \qquad n = 0, 1, \dots,$$

was studied in [51]. The corresponding cyclic system was studied in [56].

In [6] the authors studied the periodic character of the solutions of the system of difference equations with maximum

$$x_{n+1} = \max\left\{A, \frac{y_n}{x_{n-1}}\right\},$$
$$y_{n+1} = \max\left\{B, \frac{x_n}{y_{n-1}}\right\}$$

where A, B are positive constants and the initial values x_{-1}, x_0, y_{-1}, y_0, are positive real numbers.

Motivated by above mentioned studies, in this paper we continue the investigation of close-to-cyclic systems of difference equations by studying the behaviour of the solutions of the following close-to-cyclic system of difference equations with maximum:

$$x_{n+1} = \max\left\{A, \frac{y_n^p}{x_{n-1}^q}\right\}, \qquad y_{n+1} = \max\left\{B, \frac{x_n^p}{y_{n-1}^q}\right\} \tag{1.1}$$

where $n = 0, 1, \dots$, the coefficients A, B, are positive real numbers, the exponents p, q are positive real numbers such that $p > q + 1$, and the initial values x_{-1}, x_0, y_{-1}, y_0, are positive real numbers.

2 Main results

In the first lemma we study the existence of the positive equilibriums of (1.1).

Lemma 2.1. *Consider the system of difference equations (1.1), where the coefficients A, B, the exponents p, q, and the initial values x_{-1}, x_0, y_{-1}, y_0, are positive real numbers. Assume that*

$$p > 1 + q, \; q > 0, \tag{2.1}$$

then, the following statements are true:

(i) *If*

$$0 < A^{\frac{p}{q+1}} < B < A^{\frac{q+1}{p}} < 1, \tag{2.2}$$

then, system (1.1) *has two equilibria*

$$(\bar{x},\ \bar{y}) = (A, B), \quad (\bar{x},\ \bar{y}) = (1, 1).$$

(ii) *If*

$$0 < B \le A^{\frac{p}{q+1}} \le 1, \tag{2.3}$$

then, system (1.1) *has*

(iia) *two equilibria*

$$(\bar{x},\ \bar{y}) = (1, 1), \quad (\bar{x}, \bar{y}) = (A,\ A^{\frac{p}{q+1}}),$$

if $A \neq 1$, *and*

(iib) *a unique equilibrium*

$$(\bar{x},\ \bar{y}) = (1, 1),$$

if $A = 1$.

(iii) *If*

$$0 < A^{\frac{q+1}{p}} \le B \le 1, \tag{2.4}$$

then, system (1.1) *has*

(iiia) *two equilibria*

$$(\bar{x},\ \bar{y}) = (1, 1), \quad (\bar{x}, \bar{y}) = (B^{\frac{p}{q+1}}, B),$$

if $B \neq 1$, *and*

(iiib) *a unique equilibrium*

$$(\bar{x},\ \bar{y}) = (1, 1),$$

if $B = 1$.

(iv) *If*

$$A > 1,\ B > 0 \quad or \quad B > 1,\ A > 0 \tag{2.5}$$

then, system (1.1) *has no equilibria.*

Proof. We consider the system of algebraic equations

$$x = \max\Big\{A, \frac{y^p}{x^q}\Big\}, \quad y = \max\Big\{B, \frac{x^p}{y^q}\Big\}. \tag{2.6}$$

Then, one of the following relations is satisfied

$$(i)\ x = A,\ y = B, \quad (ii)\ x = A,\ y = \frac{x^p}{y^q}, \quad (iii)\ x = \frac{y^p}{x^q},\ y = B,$$
$$(iv)\ x = \frac{y^p}{x^q},\ y = \frac{x^p}{y^q},$$

and these relations are equivalent to

$$(i)\ x = A,\ y = B, \quad (ii)\ x = A,\ y = A^{\frac{p}{q+1}}, \quad (iii)\ x = B^{\frac{p}{q+1}},\ y = B,$$

and, since $x, y, p, q > 0$ and $p \neq q+1$,

$$(iv)\ x = 1,\ y = 1.$$

(i). From (2.1) and (2.2), it is obvious that, $0 < A < 1$.

Let $x = A,\ y = B$. From (2.1), (2.2), and since $0 < A \leq 1$, we get

$$\frac{y^p}{x^q} = \frac{B^p}{A^q} < A,$$

and

$$\frac{x^p}{y^q} = \frac{A^p}{B^q} < B.$$

Hence, $(\bar{x}, \bar{y}) = (A, B)$, is an equilibrium of (1.1).

Now, suppose that $x = A,\ y = A^{\frac{p}{q+1}}$, then from (2.6) we get

$$y = A^{\frac{p}{q+1}} \geq B,$$

which contradicts with (2.2). Hence, $(\bar{x}, \bar{y}) = (A, A^{\frac{p}{q+1}})$, is not an equilibrium of (1.1).

To continue, we assume that, $x = B^{\frac{p}{q+1}},\ y = B$, then from (2.6) we have

$$x = B^{\frac{p}{q+1}} \geq A,$$

which contradicts with (2.2), since $p,\ q > 0$. So, $(B^{\frac{p}{q+1}},\ B)$, is not an equilibrium of (1.1).

Finally, since $0 < A, B < 1$, from (2.6), we have that, $(\bar{x}, \bar{y}) = (1, 1)$ is an equilibrium of system (1.1).

(ii). From (2.1) and (2.3), it is obvious that, $0 < A \leq 1$.

Let $x = A,\ y = B$. Then, from (2.6), we get $B \geq \frac{A^p}{B^q}$, and so, $B \geq A^{\frac{p}{q+1}}$, which contradicts with (2.3), except if $B = A^{\frac{p}{q+1}}$, and then, $x = A,\ y = A^{\frac{p}{q+1}}$, case that considered below. Hence, (A, B), $B \neq A^{\frac{p}{q+1}}$, is not an equilibrium of (1.1).

Now, suppose that $x = A,\ y = A^{\frac{p}{q+1}}$. From (2.1), (2.3), and since $0 < A \leq 1$, we get

$$\frac{y^p}{x^q} = \frac{A^{\frac{p^2}{q+1}}}{A^q} \leq A,$$

and

$$\frac{x^p}{y^q} = \frac{A^p}{A^{\frac{qp}{q+1}}} = A^{\frac{p}{q+1}} \geq B.$$

Hence, $(\bar{x}, \bar{y}) = (A, A^{\frac{p}{q+1}})$, is an equilibrium of (1.1). Obviously, if $A = 1$, the equilibrium of (1.1) is $(\bar{x}, \bar{y}) = (1, 1)$.

To continue, we assume that, $x = B^{\frac{p}{q+1}},\ y = B$. Then, from (2.1), (2.3) and (2.6), we have

$$B^{\frac{p}{q+1}} \geq A \geq B^{\frac{q+1}{p}}. \tag{2.7}$$

If $0 < B < 1$, relation (2.7) can not be true, since $p > q+1 > 0$. So, $(B^{\frac{p}{q+1}},\ B)$, is not an equilibrium of (1.1), if $0 < B < 1$. On the other hand, relation (2.7) is true, if $A = B = 1$, and then, $(\bar{x}, \bar{y}) = (1, 1)$ is an equilibrium of (1.1).

Finally, since $0 < A, B \leq 1$, from (2.6), we have that, $(\bar{x}, \bar{y}) = (1, 1)$ is an equilibrium of system (1.1).

From all the above, it is obvious that if $A = 1$ and $0 < B \leq 1$, then, $(\bar{x}, \bar{y}) = (1, 1)$ is the unique equilibrium of (1.1). This completes the proof of (ii).

(iii). Using (2.4) and (2.6), and arguing as in (ii), we can easily prove (iii).

(iv). From (2.1) and (2.6), we get

$$x \geq y^{\frac{p}{q+1}}, \quad y \geq x^{\frac{p}{q+1}},$$

and so,

$$x \geq x^{\left(\frac{p}{q+1}\right)^2}. \tag{2.8}$$

If $x > 1$, then, from (2.8), $q + 1 \geq p$, which contradicts with (2.1). Hence, $0 < x \leq 1$, and similarly, $0 < y \leq 1$. Therefore, in order to have solutions for (2.6), it is necessary $0 < A, B \leq 1$, since (2.1) holds. So, system (1.1) has no equilibria, if (2.1) and (2.5) holds. This completes the proof of the lemma. □

In the following proposition we study the asymptotic behavior of the positive solutions of (1.1). We need the following lemma on a product-type difference equation, for which related results and methods can be found in the literature (see, for instance, [32, Remark 1], [41, Theorem 1], [54] and [55]).

Lemma 2.2. *Consider the difference equation of the form*

$$z_{n+1} = \frac{z_n^a}{z_{n-1}^b}, \qquad n \geq 0, \tag{2.9}$$

where z_{-1}, z_0 are positive real numbers, and a, b are real numbers, such that

$$a > 1 + b, \; b > 0. \tag{2.10}$$

Then, the following statements are true:

(i) *If*

$$\frac{z_0^{\lambda_1}}{z_{-1}^b} > 1, \tag{2.11}$$

where

$$\lambda_1 = \frac{a + \sqrt{a^2 - 4b}}{2},$$

then

$$\lim_{n\to\infty} z_n = \infty. \tag{2.12}$$

(ii) *If*

$$\frac{z_0^{\lambda_1}}{z_{-1}^b} = 1, \tag{2.13}$$

then

$$\lim_{n\to\infty} z_n = 1. \tag{2.14}$$

(iii) *If*

$$\frac{z_0^{\lambda_1}}{z_{-1}^{b}} < 1, \tag{2.15}$$

then

$$\lim_{n\to\infty} z_n = 0. \tag{2.16}$$

Proof. We prove that the solution z_n of (2.9) is given by

$$z_n = \frac{z_0^{f_n}}{z_{-1}^{g_n}}, \qquad n \geq 1, \tag{2.17}$$

where f_n satisfies the initial value problem

$$f_{n+1} = af_n - bf_{n-1}, \quad n = 2, 3, \ldots, \quad f_1 = a, \quad f_2 = a^2 - b, \tag{2.18}$$

and g_n satisfies the initial value problem

$$g_{n+1} = ag_n - bg_{n-1}, \quad n = 2, 3, \ldots, \quad g_1 = b, \quad g_2 = ab. \tag{2.19}$$

From (2.9), (2.18) and (2.19), we have

$$z_1 = \frac{z_0^a}{z_{-1}^b} = \frac{z_0^{f_1}}{z_{-1}^{g_1}}, \tag{2.20}$$

and

$$z_2 = \frac{z_1^a}{z_0^b} = \frac{z_0^{af_1-b}}{z_{-1}^{ag_1}} = \frac{z_0^{a^2-b}}{z_{-1}^{ab}} = \frac{z_0^{f_2}}{z_{-1}^{g_2}}. \tag{2.21}$$

In addition, from (2.9), (2.18), (2.19), (2.20) and (2.21), we get

$$z_3 = \frac{z_2^a}{z_1^b} = \frac{z_0^{af_2}}{z_{-1}^{ag_2}}\frac{z_0^{-bf_1}}{z_{-1}^{-bg_1}} = \frac{z_0^{f_3}}{z_{-1}^{g_3}}. \tag{2.22}$$

From (2.9), (2.18), (2.19), (2.21) and (2.22), and working inductively, we can easily prove (2.17).

From relations (2.10), (2.18) and (2.19), we get

$$f_n = c_1\lambda_1^n + c_2\lambda_2^n, \qquad n = 1, 2, \ldots, \tag{2.23}$$

where

$$\begin{aligned} \lambda_1 &= \frac{a + \sqrt{a^2 - 4b}}{2}, \qquad & \lambda_2 &= \frac{a - \sqrt{a^2 - 4b}}{2}, \\ c_1 &= \frac{\lambda_1}{\sqrt{a^2 - 4b}}, \qquad & c_2 &= \frac{-\lambda_2}{\sqrt{a^2 - 4b}}, \end{aligned} \tag{2.24}$$

and

$$g_n = d_1\lambda_1^n + d_2\lambda_2^n, \qquad n = 1, 2, \ldots, \tag{2.25}$$

where

$$d_1 = \frac{b}{\sqrt{a^2 - 4b}}, \qquad d_2 = -\frac{b}{\sqrt{a^2 - 4b}}. \tag{2.26}$$

From (2.23) and (2.24), we get

$$f_n = \frac{1}{\sqrt{a^2-4b}}\lambda_1^{n+1} - \frac{1}{\sqrt{a^2-4b}}\lambda_2^{n+1}. \tag{2.27}$$

So, from relations (2.17), (2.25), (2.26) and (2.27), we have

$$\begin{aligned} z_n = \frac{z_0^{f_n}}{z_{-1}^{g_n}} &= \frac{z_0^{\frac{1}{\sqrt{a^2-4b}}\lambda_1^{n+1} - \frac{1}{\sqrt{a^2-4b}}\lambda_2^{n+1}}}{z_{-1}^{\frac{b}{\sqrt{a^2-4b}}\lambda_1^{n} - \frac{b}{\sqrt{a^2-4b}}\lambda_2^{n}}} \\ &= \left(\frac{z_0^{\lambda_1}}{z_{-1}^{b}}\right)^{\frac{1}{\sqrt{a^2-4b}}\lambda_1^n} \left(\frac{z_0^{\lambda_2}}{z_{-1}^{b}}\right)^{-\frac{1}{\sqrt{a^2-4b}}\lambda_2^n}. \end{aligned} \tag{2.28}$$

(i). First, suppose that (2.11) is satisfied. Since, from (2.10),

$$\lambda_1 > 1, \qquad 0 < \lambda_2 < 1, \tag{2.29}$$

then, from (2.28), we get (2.12).

(ii). Now, suppose that (2.13) hold. Then, from (2.28), we get

$$z_n = \left(\frac{z_0^{\lambda_2}}{z_{-1}^{b}}\right)^{-\frac{1}{\sqrt{a^2-4b}}\lambda_2^n},$$

and since $0 < \lambda_2 < 1$, we get (2.14).

(iii). Finally, suppose that (2.15) is satisfied. Then, from (2.28) and (2.29), we get (2.16). This completes the proof of the lemma. □

Lemma 2.3. *Consider the system of difference equations (1.1), where relations (2.1) and (2.2) hold, and the initial values x_{-1}, x_0, y_{-1}, y_0, are positive real numbers. Then, the following statements are true:*

I. *If there exists a positive integer $m \geq 2$, such that*

$$x_m \leq B^{\frac{q+1}{p}}, \tag{2.30}$$

then

$$x_{m+2k} = A, \quad y_{m+2k-1} = B, \quad k \geq 1. \tag{2.31}$$

II. *Suppose that*

$$x_{2n} > B^{\frac{q+1}{p}}, \quad \textit{for any } n \geq 1. \tag{2.32}$$

(a). *If*

$$\frac{x_2^{\lambda}}{x_0^{q^2}} > 1, \qquad \lambda = \frac{p^2-2q+\sqrt{(p^2-2q)^2-4q^2}}{2}, \tag{2.33}$$

then

$$\lim_{n\to\infty} x_{2n} = \infty, \qquad \lim_{n\to\infty} y_{2n+1} = \infty. \tag{2.34}$$

(b). *If*

$$\frac{x_2^{\lambda}}{x_0^{q^2}} = 1, \tag{2.35}$$

then

$$\lim_{n\to\infty} x_{2n} = 1, \qquad \lim_{n\to\infty} y_{2n+1} = 1. \tag{2.36}$$

III. *Suppose that*

$$x_{2n+1} > B^{\frac{q+1}{p}}, \quad \text{for any } n \geq 1. \tag{2.37}$$

(a). *If*

$$\frac{x_3^{\lambda}}{x_1^{q^2}} > 1, \tag{2.38}$$

then

$$\lim_{n\to\infty} x_{2n+1} = \infty, \qquad \lim_{n\to\infty} y_{2n} = \infty. \tag{2.39}$$

(b). *If*

$$\frac{x_3^{\lambda}}{x_1^{q^2}} = 1, \tag{2.40}$$

then

$$\lim_{n\to\infty} x_{2n+1} = 1, \qquad \lim_{n\to\infty} y_{2n} = 1. \tag{2.41}$$

Proof. From (2.2) we have

$$\text{(i) } 0 < A < 1, \quad 0 < B < 1, \quad \text{(ii) } A > B^{\frac{p}{q+1}}, \quad \text{(iii) } B > A^{\frac{p}{q+1}}, \tag{2.42}$$

I. From (1.1) and (2.30), we get

$$\frac{x_m^p}{y_{m-1}^q} \leq \frac{B^{q+1}}{B^q} = B,$$

and so, from (1.1),

$$y_{m+1} = B. \tag{2.43}$$

Then, from (1.1), (i) and (ii) of (2.42) and (2.43), we get

$$\frac{y_{m+1}^p}{x_m^q} \leq \frac{B^p}{A^q} < A,$$

and so, from (1.1),

$$x_{m+2} = A. \tag{2.44}$$

In addition, from (iii) of (2.42) and (2.44), it follows that

$$x_{m+2} < B^{\frac{q+1}{p}}. \tag{2.45}$$

Using (2.45) and working as above we have

$$y_{m+3} = B, \qquad x_{m+4} = A,$$

and so, working inductively, we get (2.31).

II. From (1.1), (iii) of (2.42) and (2.32), we have

$$x_{2n} = \frac{y_{2n-1}^p}{x_{2n-2}^q}, \qquad n \geq 1. \tag{2.46}$$

From (2.1), (i) of (2.42), (2.32) and (2.46), we have, for $n \geq 2$,

$$y_{2n-1} = x_{2n}^{1/p} x_{2n-2}^{q/p} > B^{\frac{q+1}{p}\frac{1}{p}} B^{\frac{q+1}{p}\frac{q}{p}} = B^{(\frac{q+1}{p})^2} > B,$$

and so, from (1.1),

$$y_{2n-1} = \frac{x_{2n-2}^p}{y_{2n-3}^q}, \qquad n \geq 2. \tag{2.47}$$

Relations (2.46) and (2.47) imply that

$$x_{2n} = \frac{x_{2n-2}^{p^2-q}}{y_{2n-3}^{pq}}, \qquad n \geq 2,$$

and then, since from (2.46),

$$y_{2n-3}^p = x_{2n-2} x_{2n-4}^q, \qquad n \geq 2,$$

we get the following product-type difference equation (see, e.g., [54,55])

$$x_{2n} = \frac{x_{2n-2}^a}{x_{2n-4}^b}, \quad n \geq 2, \quad a = p^2 - 2q, \quad b = q^2, \tag{2.48}$$

with interlacing indices (see [53])

Since (2.1) holds, we can easily prove that (2.10) is true. In (2.48), we set

$$x_{2n-4} = z_{n-1}, \tag{2.49}$$

(see [53]), and we get the equation

$$z_{n+1} = \frac{z_n^a}{z_{n-1}^b}, \qquad n \geq 2. \tag{2.50}$$

We consider the difference equation

$$w_{n+1} = \frac{w_n^a}{w_{n-1}^b}, \qquad n \geq 0. \tag{2.51}$$

Let w_n be the solution of (2.51), such that

$$w_{-1} = z_1, \qquad w_0 = z_2, \tag{2.52}$$

II(a). Using (2.33), (2.49) and (2.52) we have

$$\frac{w_0^\lambda}{w_{-1}^{q^2}} = \frac{z_2^\lambda}{z_1^{q^2}} = \frac{x_2^\lambda}{x_0^{q^2}} > 1. \tag{2.53}$$

Since for equation (2.51), relations (2.10) and (2.53) hold, from (i) of Lemma 2.2, we get

$$\lim_{n\to\infty} w_n = \infty,$$

and using (2.50), (2.51) and (2.52), we have

$$\lim_{n\to\infty} z_n = \infty,$$

and so, from (2.49),

$$\lim_{n\to\infty} x_{2n} = \infty.$$

Then, from (2.46), we get

$$\lim_{n\to\infty} y_{2n+1} = \infty.$$

Hence, (2.34) is true.

II(b). Since (2.35), (2.49) and (2.52) hold, we have

$$\frac{w_0^{\lambda}}{w_{-1}^{q^2}} = \frac{z_2^{\lambda}}{z_1^{q^2}} = \frac{x_2^{\lambda}}{x_0^{q^2}} = 1. \tag{2.54}$$

Since for equation (2.51), relations (2.10) and (2.54) hold, from (ii) of Lemma 2.2, we get

$$\lim_{n\to\infty} w_n = 1,$$

and using (2.50), (2.51) and (2.52), we have

$$\lim_{n\to\infty} z_n = 1,$$

and so, from (2.49) ,

$$\lim_{n\to\infty} x_{2n} = 1.$$

From (2.46), we get

$$\lim_{n\to\infty} y_{2n+1} = 1.$$

Hence, (2.36) is true.

We mention that, difference equation (2.48) is with interlacing indices, and it was expected that its behavior is similar to (2.9) (for more details see, for example, [53]).

Arguing as in cases **II(a)** and **II(b)**, we can easily prove **III(a)** and **III(b)**. □

Remark 2.4. In statement **II** of Lemma 2.3, it is impossible to have

$$\frac{x_2^{\lambda}}{x_0^{q^2}} < 1. \tag{2.55}$$

Indeed, if, on the contrary, we assume that (2.55) holds, then, arguing as in **II(a)** of Lemma 2.3, for the solution w_n of (2.51) we get

$$\frac{w_0^{\lambda}}{w_{-1}^{q^2}} < 1,$$

and so, from statement **(iii)** of Lemma 2.2,

$$\lim_{n\to\infty} w_n = 0,$$

which means that

$$\lim_{n\to\infty} x_{2n} = 0,$$

which contradicts with the fact that $x_n \geq A$, for any $n \geq 1$. So, (2.55) cannot be satisfied.

Similarly, it is impossible to have

$$\frac{x_3^{\lambda}}{x_1^{q^2}} < 1,$$

in statement **III** of Lemma 2.3.

Proposition 2.5. *Consider the system of difference equations* (1.1)*, where relations* (2.1) *and* (2.2) *hold, and the initial values* x_{-1}, x_0, y_{-1}, y_0*, are positive real numbers. Let* (x_n, y_n) *be a solution of* (1.1)*. Then, the following statements are true:*

I. *Suppose that there exist integers* $l, r \geq 1$*, such that*

$$x_{2l+1} \leq B^{\frac{q+1}{p}}, \qquad x_{2r} \leq B^{\frac{q+1}{p}}. \tag{2.56}$$

Then, (x_n, y_n) *is eventually equal to the positive equilibrium* (A, B) *of* (1.1)*.*

II. *Suppose that there exists an integer* $l \geq 1$*, such that the first inequality of* (2.56) *and relations* (2.32)*,* (2.33) *hold. Then, relation* (2.34) *and*

$$x_{2n+1} = A, \quad y_{2n} = B, \quad n \geq l+1, \tag{2.57}$$

are satisfied.

III. *Suppose that there exists an integer* $l \geq 1$*, such that the first inequality of* (2.56) *and relations* (2.32)*,* (2.35) *hold. Then, relations* (2.36) *and* (2.57) *are satisfied.*

IV. *Suppose that there exists an integer* $r \geq 1$*, such that the second inequality of* (2.56) *and relations* (2.37)*,* (2.38) *hold. Then, relation* (2.39) *and*

$$x_{2n} = A, \quad y_{2n+1} = B, \quad n \geq r+1, \tag{2.58}$$

are satisfied.

V. *Suppose that there exists an integer* $r \geq 1$*, such that the second inequality of* (2.56) *and relations* (2.37)*,* (2.40) *hold. Then, relations* (2.41) *and* (2.58) *are satisfied.*

VI. *If relations* (2.32)*,* (2.33)*,* (2.37) *and* (2.38) *hold, then,*

$$\lim_{n\to\infty} x_n = \infty, \qquad \lim_{n\to\infty} y_n = \infty. \tag{2.59}$$

VII. *If relations* (2.32)*,* (2.33)*,* (2.37) *and* (2.40) *hold, then, relations* (2.34) *and* (2.41) *are satisfied.*

VIII. *If relations* (2.32)*,* (2.35)*,* (2.37) *and* (2.38) *hold, then, relations* (2.36) *and* (2.39) *are satisfied.*

IX. *If relations* (2.32)*,* (2.35)*,* (2.37) *and* (2.40) *hold, then,*

$$\lim_{n\to\infty} x_n = 1, \qquad \lim_{n\to\infty} y_n = 1. \tag{2.60}$$

Proof. **I.** From (2.56) and statement I of Lemma 2.3, we have

$$x_{2l+1+2k} = A, \quad y_{2l+2k} = B, \quad x_{2r+2k} = A, \quad y_{2r+2k-1} = B, \quad k = 1, 2, \ldots.$$

So, for $n \geq \max\{2l+3, 2r+2\}$, we get $x_n = A$, $y_n = B$. This implies that, (x_n, y_n) is eventually equal to (A, B).

II. From the first inequality of (2.56) and statement **I** of Lemma 2.3, we have

$$x_{2l+1+2k} = A, \qquad y_{2l+2k} = B, \qquad k = 1, 2, \ldots,$$

and so, (2.57) is true. Moreover, from (2.32), (2.33) and statement **II(a)** of Lemma 2.3, we have (2.34).

Arguing as above, using the corresponding relations and the appropriate statements of Lemma 2.3, we can easily prove statements **III–IX** of the proposition. This completes the proof of the proposition. □

Lemma 2.6. *Consider the system of difference equations* (1.1)*, where* (2.1) *and* (2.3) *hold, and the initial values* x_{-1}, x_0, y_{-1}, y_0*, are positive real numbers. Then, the following statements are true:*

I. *Suppose that there exists an integer* $m \geq 2$*, such that*

$$y_m = A^{\frac{p}{q+1}}, \tag{2.61}$$

then

$$x_{m+2k+1} = A, \quad y_{m+2k} = A^{\frac{p}{q+1}}, \quad k \geq 0. \tag{2.62}$$

II. *Suppose that*

$$y_{2n} > A^{\frac{p}{q+1}}, \quad \text{for any } n \geq 1. \tag{2.63}$$

(a). *If*

$$\frac{y_2^{\lambda}}{y_0^{q^2}} > 1, \tag{2.64}$$

where λ *was defined in* (2.33)*, then* (2.39) *holds.*

(b). *If*

$$\frac{y_2^{\lambda}}{y_0^{q^2}} = 1, \tag{2.65}$$

then (2.41) *holds.*

III. *Suppose that*

$$y_{2n+1} > A^{\frac{p}{q+1}}, \quad \text{for any } n \geq 1. \tag{2.66}$$

(a). *If*

$$\frac{y_3^{\lambda}}{y_1^{q^2}} > 1, \tag{2.67}$$

then (2.34) *holds.*

(b). *If*

$$\frac{y_3^{\lambda}}{y_1^{q^2}} = 1, \tag{2.68}$$

then (2.36) *holds.*

IV. *Suppose that there exists an integer $s \geq 2$, such that*

$$y_s < A^{\frac{p}{q+1}}. \tag{2.69}$$

Assume that

$$A^{\frac{p^2-q-1}{pq}} \leq B, \tag{2.70}$$

then

$$x_{s+2k-1} = A, \quad k \geq 1, \tag{2.71}$$

and, in addition,

(a). *If*

$$q < 1, \tag{2.72}$$

then

$$\lim_{k\to\infty} y_{s+2k} = A^{\frac{p}{q+1}}. \tag{2.73}$$

(b). *If*

$$q = 1, \tag{2.74}$$

then

$$y_{s+4k-2} = \frac{A^p}{y_s}, \quad y_{s+4k} = y_s, \quad k \geq 1. \tag{2.75}$$

(c). *If*

$$q > 1, \tag{2.76}$$

then there exists a positive integer k_0, such that

$$y_{s+4k-2} = \frac{A^p}{B^q}, \quad y_{s+4k-4} = B, \quad k \geq k_0 + 1. \tag{2.77}$$

Proof. From relations (2.1) and (2.3) we have $0 < A \leq 1$.

I. Since, from (2.1) and (2.3),

$$A^{\frac{p}{q+1}} \leq A^{\frac{q+1}{p}}, \tag{2.78}$$

from (1.1) and (2.61), we get

$$\frac{y_m^p}{x_{m-1}^q} \leq \frac{A^{q+1}}{A^q} = A,$$

and so, from (1.1),

$$x_{m+1} = A. \tag{2.79}$$

Then, from (2.3), (2.61) and (2.79), we get

$$\frac{x_{m+1}^p}{y_m^q} = \frac{A^p}{\left(A^{\frac{p}{q+1}}\right)^q} = A^{\frac{p}{q+1}} \geq B,$$

and so, from (1.1),

$$y_{m+2} = A^{\frac{p}{q+1}},$$

and working inductively, we get (2.62).

Arguing as in statements **II** and **III** of Lemma 2.3, we can easily prove **II** and **III** of this Lemma.

IV. From (1.1), (2.69) and (2.78), we get

$$\frac{y_s^p}{x_{s-1}^q} \leq \frac{A^{q+1}}{A^q} = A,$$

and so, from (1.1),

$$x_{s+1} = A. \tag{2.80}$$

Then, from (2.3), (2.69) and (2.80), we get

$$\frac{x_{s+1}^p}{y_s^q} > \frac{A^p}{\left(A^{\frac{p}{q+1}}\right)^q} = A^{\frac{p}{q+1}} \geq B,$$

and so, from (1.1),

$$y_{s+2} = \frac{A^p}{y_s^q} > A^{\frac{p}{q+1}}. \tag{2.81}$$

From (1.1), (2.70), (2.80) and (2.81), we get

$$\frac{y_{s+2}^p}{x_{s+1}^q} = \frac{A^{p^2-q}}{y_s^{pq}} \leq \frac{A^{p^2-q}}{B^{pq}} \leq \frac{A^{p^2-q}}{A^{p^2-q-1}} = A,$$

and so, from (1.1),

$$x_{s+3} = A. \tag{2.82}$$

From (2.81) and (2.82), we have

$$\frac{x_{s+3}^p}{y_{s+2}^q} = A^{p-pq} y_s^{q^2},$$

and so, from (1.1),

$$y_{s+4} = \max\left\{B,\ A^{p-pq} y_s^{q^2}\right\}. \tag{2.83}$$

In addition, from (2.81) and (2.82), we have

$$\frac{x_{s+3}^p}{y_{s+2}^q} < \frac{A^p}{\left(A^{\frac{p}{q+1}}\right)^q} = A^{\frac{p}{q+1}},$$

and so, from (1.1) and (2.3), we get

$$y_{s+4} < A^{\frac{p}{q+1}}. \tag{2.84}$$

From (2.69), (2.80), (2.81), (2.82), (2.83), (2.84), and working inductively, we get (2.71), and

$$y_{s+4k-2} = \frac{A^p}{y_{s+4k-4}^q} > A^{\frac{p}{q+1}}, \qquad k \geq 1, \tag{2.85}$$

and

$$y_{s+4k} = \max\{B,\ A^{p-pq} y_{s+4k-4}^{q^2}\} < A^{\frac{p}{q+1}}, \qquad k \geq 1. \tag{2.86}$$

IV(a). First, suppose that (2.72) holds, then, from (1.1) and (2.3), we get

$$A^{p-pq} y_{s+4k-4}^{q^2} \geq A^{p-pq} B^{q^2} \geq B,$$

and so, from (2.86), we have

$$y_{s+4k} = A^{p-pq} y_{s+4k-4}^{q^2}, \qquad k \geq 1. \tag{2.87}$$

From (2.87), and by induction, (see also [55] and note that the equation is with interlacing indices [53]), it is easy to prove that

$$y_{s+4k} = A^{p\frac{1-(q^2)^k}{q+1}} y_s^{(q^2)^k}, \qquad k \geq 1. \tag{2.88}$$

From (2.88) and (2.72), we have

$$\lim_{k\to\infty} y_{s+4k} = A^{\frac{p}{q+1}}. \tag{2.89}$$

Using (2.85) and (2.89), we have

$$\lim_{k\to\infty} y_{s+4k-2} = A^{\frac{p}{q+1}}. \tag{2.90}$$

From (2.89) and (2.90), it is obvious that (2.73) is true.

IV(b). Now, suppose that (2.74) holds, then from (1.1) and (2.86), we have

$$y_{s+4k} = \max\{B, y_{s+4k-4}\} = y_{s+4k-4}, \qquad k \geq 1,$$

and so, the second relation of (2.75) is true. From the second relation of (2.75) and (2.85), we have that the first relation of (2.75) is also true.

IV(c). Finally, suppose that (2.76) holds. We prove that, there exists a positive integer k_0, such that,

$$y_{s+4k_0} = B. \tag{2.91}$$

On the contrary, we assume that

$$y_{s+4k} > B, \qquad k \geq 0,$$

then, from (2.86), we get (2.87), and so, (2.88) is true. Then, from (2.88), we get

$$y_{s+4k} = A^{\frac{p}{q+1}} \left(\frac{y_s}{A^{\frac{p}{q+1}}} \right)^{(q^2)^k},$$

and so, from (2.69) and (2.76),

$$\lim_{k\to\infty} y_{s+4k} = 0,$$

which is a contradiction, since from (1.1),

$$y_{s+4k} \geq B > 0, \qquad k \geq 0.$$

So, there exists a positive integer k_0, such that (2.91) holds.
From (2.1), (2.3), (2.76), (2.85), (2.86) and (2.91), we have

$$y_{s+4k_0+2} = \frac{A^p}{B^q}, \qquad y_{s+4k_0+4} = \max\{B, A^{p-pq}B^{q^2}\} = B,$$

and working inductively, we get (2.77). This completes the proof of the lemma. □

Using Lemma 2.6 and arguing as in Proposition 2.5, we can easily prove the following proposition.

Proposition 2.7. *Consider the system of difference equations (1.1), where relations (2.1) and (2.3) hold, and the initial values x_{-1}, x_0, y_{-1}, y_0, are positive real numbers. Let (x_n, y_n) be a solution of (1.1). Then, the following statements are true:*

I. *Suppose that there exist integers $l, r \geq 1$, such that*

$$y_{2l+1} = A^{\frac{p}{q+1}}, \qquad y_{2r} = A^{\frac{p}{q+1}}. \tag{2.92}$$

Then, (x_n, y_n) is eventually equal to the positive equilibrium $(A, A^{\frac{p}{q+1}})$ of (1.1).

II. *Suppose that there exists an integer $l \geq 1$, such that the first equality of (2.92) and relations (2.63), (2.64) hold. Then, relation (2.39) and*

$$x_{2n} = A, \quad y_{2n-1} = A^{\frac{p}{q+1}}, \quad n \geq l+1, \tag{2.93}$$

are satisfied.

III. *Suppose that there exists an integer $l \geq 1$, such that the first equality of (2.92) and relations (2.63), (2.65) hold. Then, relations (2.41) and (2.93) are satisfied.*

IV. *Suppose that there exists an integer $r \geq 1$, such that the second equality of (2.92) and relations (2.66), (2.67) hold. Then, relation (2.34) and*

$$x_{2n+1} = A, \quad y_{2n} = A^{\frac{p}{q+1}}, \quad n \geq r, \tag{2.94}$$

are satisfied.

V. *Suppose that there exists an integer $r \geq 1$, such that the second equality of (2.92) and relations (2.66), (2.68) hold. Then, relations (2.36) and (2.94) are satisfied.*

VI. *If relations (2.63), (2.64), (2.66) and (2.67) hold, then (2.59) is satisfied.*

VII. *If relations (2.63), (2.64), (2.66) and (2.68) hold, then, relations (2.36) and (2.39) are satisfied.*

VIII. *If relations (2.63), (2.65), (2.66) and (2.67) hold, then, relations (2.34) and (2.41) are satisfied.*

IX. *If relations (2.63), (2.65), (2.66) and (2.68) hold, then (2.60) is satisfied.*

If, in addition, relation (2.70) holds, then the following statements are true:

X. *Suppose that there exists an integer $l \geq 1$, such that the first equality of (2.92) holds and an integer $t \geq 1$, such that*

$$y_{2t} < A^{\frac{p}{q+1}}. \tag{2.95}$$

Then

(a). *if $q < 1$, then,*

$$x_n = A, \quad n \geq \max\{2l+2, 2t+1\},$$
$$y_{2n+1} = A^{\frac{p}{q+1}}, \quad n \geq l, \quad \lim_{n\to\infty} y_{2n} = A^{\frac{p}{q+1}},$$

(b). *if $q = 1$, then,*

$$x_n = A, \quad n \geq \max\{2l+2, 2t+1\}, \quad y_{2n+1} = A^{\frac{p}{q+1}}, \quad n \geq l,$$
$$y_{2t+4n-2} = \frac{A^p}{y_{2t}}, \quad y_{2t+4n} = y_{2t}, \quad n \geq 1,$$

(c). *if $q > 1$, then, there exists a positive integer n_0, such that*

$$x_n = A, \quad n \geq \max\{2l+2, 2t+1\}, \quad y_{2n+1} = A^{\frac{p}{q+1}}, \quad n \geq l,$$
$$y_{2t+4n-2} = \frac{A^p}{B^q}, \quad y_{2t+4n-4} = B, \quad n \geq n_0 + 1,$$

XI. *Suppose that there exists an integer $r \geq 1$, such that the second equality of (2.92) holds and an integer $v \geq 1$, such that*

$$y_{2v+1} < A^{\frac{p}{q+1}}. \tag{2.96}$$

Then

(a). *if $q < 1$, then,*

$$x_n = A, \quad n \geq \max\{2r+1, 2v+2\},$$
$$y_{2n} = A^{\frac{p}{q+1}}, \quad n \geq r, \quad \lim_{n\to\infty} y_{2n+1} = A^{\frac{p}{q+1}},$$

(b). *if $q = 1$, then,*

$$x_n = A, \quad n \geq \max\{2r+1, 2v+2\}, \quad y_{2n} = A^{\frac{p}{q+1}}, \quad n \geq r,$$
$$y_{2v+4n-1} = \frac{A^p}{y_{2v+1}}, \quad y_{2v+4n+1} = y_{2v+1}, \quad n \geq 1,$$

(c). *if $q > 1$, then, there exists a positive integer n_1, such that*

$$x_n = A, \quad n \geq \max\{2r+1, 2v+2\}, \quad y_{2n} = A^{\frac{p}{q+1}}, \quad n \geq r,$$
$$y_{2v+4n-1} = \frac{A^p}{B^q}, \quad y_{2v+4n-3} = B, \quad n \geq n_1 + 1.$$

XII. *If relations (2.63), (2.64) and (2.96) hold, then*

(a). *if $q < 1$, then,*

$$x_{2n} = A, \quad n \geq v+1, \quad \lim_{n\to\infty} x_{2n+1} = \infty,$$
$$\lim_{n\to\infty} y_{2n} = \infty, \quad \lim_{n\to\infty} y_{2n+1} = A^{\frac{p}{q+1}},$$

(b). *if $q = 1$, then,*

$$x_{2n} = A, \quad n \geq v+1, \quad \lim_{n\to\infty} x_{2n+1} = \infty,$$
$$\lim_{n\to\infty} y_{2n} = \infty, \quad y_{2v+4n-1} = \frac{A^p}{y_{2v+1}}, \quad y_{2v+4n+1} = y_{2v+1}, \quad n \geq 1,$$

(c). *if $q > 1$, then, there exists a positive integer n_1, such that*

$$x_{2n} = A, \quad n \geq v+1, \quad \lim_{n\to\infty} x_{2n+1} = \infty,$$
$$\lim_{n\to\infty} y_{2n} = \infty, \quad y_{2v+4n-1} = \frac{A^p}{B^q}, \quad y_{2v+4n-3} = B, \quad n \geq n_1 + 1.$$

XIII. *If relations (2.63), (2.65) and (2.96) hold, then*

(a). *if* $q < 1$, *then,*

$$x_{2n} = A, \quad n \geq v+1, \quad \lim_{n\to\infty} x_{2n+1} = 1,$$
$$\lim_{n\to\infty} y_{2n} = 1, \quad \lim_{n\to\infty} y_{2n+1} = A^{\frac{p}{q+1}},$$

(b). *if* $q = 1$, *then,*

$$x_{2n} = A, \quad n \geq v+1, \quad \lim_{n\to\infty} x_{2n+1} = 1,$$
$$\lim_{n\to\infty} y_{2n} = 1, \quad y_{2v+4n-1} = \frac{A^p}{y_{2v+1}}, \quad y_{2v+4n+1} = y_{2v+1}, \quad n \geq 1,$$

(c). *if* $q > 1$, *then, there exists a positive integer* n_1, *such that*

$$x_{2n} = A, \quad n \geq v+1, \quad \lim_{n\to\infty} x_{2n+1} = 1,$$
$$\lim_{n\to\infty} y_{2n} = 1, \quad y_{2v+4n-1} = \frac{A^p}{B^q}, \quad y_{2v+4n-3} = B, \quad n \geq n_1 + 1.$$

XIV. *If relations* (2.66), (2.67) *and* (2.95) *hold, then*

(a). *if* $q < 1$, *then,*

$$x_{2n+1} = A, \quad n \geq t, \quad \lim_{n\to\infty} x_{2n} = \infty,$$
$$\lim_{n\to\infty} y_{2n+1} = \infty, \quad \lim_{n\to\infty} y_{2n} = A^{\frac{p}{q+1}},$$

(b). *if* $q = 1$, *then,*

$$x_{2n+1} = A, \quad n \geq t, \quad \lim_{n\to\infty} x_{2n} = \infty,$$
$$\lim_{n\to\infty} y_{2n+1} = \infty, \quad y_{2t+4n-2} = \frac{A^p}{y_{2t}}, \quad y_{2t+4n} = y_{2t}, \quad n \geq 1,$$

(c). *if* $q > 1$, *then, there exists a positive integer* n_0, *such that*

$$x_{2n+1} = A, \quad n \geq t, \quad \lim_{n\to\infty} x_{2n} = \infty,$$
$$\lim_{n\to\infty} y_{2n+1} = \infty, \quad y_{2t+4n-2} = \frac{A^p}{B^q}, \quad y_{2t+4n-4} = B, \quad n \geq n_0 + 1.$$

XV. *If relations* (2.66), (2.68) *and* (2.95) *hold, then*

(a). *if* $q < 1$, *then,*

$$x_{2n+1} = A, \quad n \geq t, \quad \lim_{n\to\infty} x_{2n} = 1,$$
$$\lim_{n\to\infty} y_{2n+1} = 1, \quad \lim_{n\to\infty} y_{2n} = A^{\frac{p}{q+1}},$$

(b). *if* $q = 1$, *then,*

$$x_{2n+1} = A, \quad n \geq t, \quad \lim_{n\to\infty} x_{2n} = 1,$$
$$\lim_{n\to\infty} y_{2n+1} = 1, \quad y_{2t+4n-2} = \frac{A^p}{y_{2t}}, \quad y_{2t+4n} = y_{2t}, \quad n \geq 1,$$

(c). *if $q > 1$, then, there exists a positive integer n_0, such that*

$$x_{2n+1} = A, \quad n \geq t, \quad \lim_{n\to\infty} x_{2n} = 1,$$
$$\lim_{n\to\infty} y_{2n+1} = 1, \quad y_{2t+4n-2} = \frac{A^p}{B^q}, \quad y_{2t+4n-4} = B, \quad n \geq n_0 + 1.$$

XVI. *Suppose that there exist integers $t, v \geq 1$, such that (2.95) and (2.96) hold, then*

(a). *if $q < 1$, then,*

$$x_n = A, \quad n \geq \max\{2t+1, 2v+2\}, \quad \lim_{n\to\infty} y_n = A^{\frac{p}{q+1}},$$

(b). *if $q = 1$, then,*

$$x_n = A, \quad n \geq \max\{2t+1, 2v+2\},$$
$$y_{2t+4n-2} = \frac{A^p}{y_{2t}}, \quad y_{2t+4n} = y_{2t}, \quad n \geq 1,$$
$$y_{2v+4n-1} = \frac{A^p}{y_{2v+1}}, \quad y_{2v+4n+1} = y_{2v+1}, \quad n \geq 1,$$

(c). *if $q > 1$, then, there exist positive integers n_0, n_1, such that*

$$x_n = A, \; n \geq \max\{2t+1, 2v+2\},$$
$$y_{2t+4n-2} = \frac{A^p}{B^q}, \quad y_{2t+4n-4} = B, \quad n \geq n_0 + 1,$$
$$y_{2v+4n-1} = \frac{A^p}{B^q}, \quad y_{2v+4n-3} = B, \quad n \geq n_1 + 1.$$

Remark 2.8. If for the system of difference equations (1.1) relations (2.1), (2.3) and (2.70) hold and the initial values x_{-1}, x_0, y_{-1}, y_0, are positive real numbers, then it is impossible to exist integers $l, v \geq 1$, such that the first equality of (2.92) and (2.96) to be valid simultaneously.

Indeed, if the first equality of (2.92) holds then, from statement I of Lemma 2.6, we have

$$y_{2n+1} = A^{\frac{p}{q+1}}, \quad n \geq l,$$

and so, (2.96) can not be true for $v \geq l$.

Now, suppose that $v < l$. From statement IV of Lemma 2.6 and relations (2.69), (2.85), (2.86) and (2.96), we get

$$y_{2v+4k-1} > A^{\frac{p}{q+1}}, \quad k \geq 1,$$

and

$$y_{2v+4k+1} < A^{\frac{p}{q+1}}, \quad k \geq 1,$$

which contradicts with the first equality of (2.92), since we assume that $v < l$.

Similarly, it is impossible to exist integers $r, t \geq 1$, such that the second equality of (2.92) and (2.95) to be valid simultaneously.

Remark 2.9. If for the system of difference equations (1.1) relations (2.1) and (2.4) hold and the initial values x_{-1}, x_0, y_{-1}, y_0, are positive real numbers, then the behavior of its solutions is similar to the behavior we have in Proposition 2.7. For this reason, we omit the reference.

Proposition 2.10. *Consider the system of difference equations* (1.1)*, where relations* (2.1) *and* (2.5) *hold, and the initial values* x_{-1}, x_0, y_{-1}, y_0*, are positive real numbers. Then, all the solutions of* (1.1) *are unbounded from above.*

Proof. Suppose that $A > 1$ and (x_n, y_n) is a solution of (1.1). We prove that x_n is unbounded from above. On the contrary, we assume that there exists a positive real number M, such that

$$x_n \leq M, \quad \text{for any } n \geq 1. \tag{2.97}$$

From (1.1) and (2.97), we get

$$\frac{y_n^p}{x_{n-1}^q} \leq M, \quad \text{for any } n \geq 0,$$

and so, from (2.97), and since $q > 0$, we have

$$y_n^p \leq M x_{n-1}^q \leq M M^q = M^{q+1}, \quad \text{for any } n \geq 2,$$

which means that

$$y_n \leq M^{\frac{q+1}{p}}, \quad \text{for any } n \geq 2. \tag{2.98}$$

From (1.1) and (2.98), we get

$$\frac{x_n^p}{y_{n-1}^q} \leq M^{\frac{q+1}{p}}, \quad \text{for any } n \geq 2,$$

and so, from (2.1) and (2.98),

$$x_n^p \leq M^{\frac{q+1}{p}} y_{n-1}^q \leq M^{\frac{q+1}{p}} M^{q\frac{q+1}{p}} = M^{\frac{(q+1)^2}{p}}, \quad \text{for any } n \geq 3,$$

which means that

$$x_n \leq M^{(\frac{q+1}{p})^2}, \quad \text{for any } n \geq 3.$$

Working inductively, we get

$$\begin{aligned} x_n &\leq M^{(\frac{q+1}{p})^{2k}}, \quad \text{for any} \quad n \geq 2k+1, \quad k = 1, 2, \ldots, \\ y_n &\leq M^{(\frac{q+1}{p})^{2k-1}}, \quad \text{for any} \quad n \geq 2k, \quad k = 1, 2, \ldots \end{aligned} \tag{2.99}$$

From (2.1), it is obvious that

$$\lim_{k\to\infty} M^{(\frac{q+1}{p})^{2k}} = 1, \qquad \lim_{k\to\infty} M^{(\frac{q+1}{p})^{2k-1}} = 1. \tag{2.100}$$

Since $A > 1$, from (1.1) we have

$$x_n \geq A > 1, \quad \text{for any } n \geq 1,$$

and so, there exists a positive real number ϵ, such that

$$x_n \geq A > 1 + \epsilon, \quad \text{for any } n \geq 1. \tag{2.101}$$

From the first inequality of (2.99) and the first limit of (2.100), we get that there exists a positive integer n_0 such that

$$x_n < 1 + \epsilon, \quad \text{for any } n \geq n_0,$$

which contradicts with (2.101). So, x_n is unbounded from above, if $A > 1$.

Similarly, we can prove that y_n is unbounded from above, if $B > 1$.

We mention that, under some initial values, we can find solutions such that

$$\lim_{n\to\infty} x_n = \infty \quad \text{or} \quad \lim_{n\to\infty} y_n = \infty.$$

Indeed, we consider the difference equation (2.9) of Lemma 2.2, where $a = p$ and $b = q$. Since, from statement (i) of Lemma 2.2, if

$$\frac{z_0^{\lambda_1}}{z_{-1}^q} > 1, \qquad \lambda_1 = \frac{p + \sqrt{p^2 - 4q}}{2},$$

then

$$\lim_{n\to\infty} z_n = \infty, \tag{2.102}$$

there exists a positive integer n_1, such that

$$z_{n+1} > \max\{A, B\}, \quad \text{for any } n \geq n_1. \tag{2.103}$$

If we set

$$x_{-1} = z_{n_1+j-1}, \qquad y_0 = z_{n_1+j}, \tag{2.104}$$

$$y_{-1} = z_{n_1+w-1}, \qquad x_0 = z_{n_1+w}, \tag{2.105}$$

where j, w are arbitrary integers in $\{0, 1, 2, \dots\}$, then, from (1.1), (2.103) and (2.104),

$$x_1 = \max\left\{A, \frac{z_{n_1+j}^p}{z_{n_1+j-1}^q}\right\} = \max\{A, z_{n_1+j+1}\} = z_{n_1+j+1},$$

$$y_2 = \max\left\{B, \frac{z_{n_1+j+1}^p}{z_{n_1+j}^q}\right\} = \max\{B, z_{n_1+j+2}\} = z_{n_1+j+2},$$

and from (1.1), (2.103) and (2.105),

$$y_1 = \max\left\{B, \frac{z_{n_1+w}^p}{z_{n_1+w-1}^q}\right\} = \max\{B, z_{n_1+w+1}\} = z_{n_1+w+1},$$

$$x_2 = \max\left\{A, \frac{z_{n_1+w+1}^p}{z_{n_1+w}^q}\right\} = \max\{A, z_{n_1+w+2}\} = z_{n_1+w+2},$$

and working inductively,

$$x_{2n+1} = z_{n_1+j+2n+1}, \quad y_{2n+2} = z_{n_1+j+2n+2}, \quad \text{for any } n \geq 0, \tag{2.106}$$

and

$$y_{2n+1} = z_{n_1+w+2n+1}, \quad x_{2n+2} = z_{n_1+w+2n+2}, \quad \text{for any } n \geq 0. \tag{2.107}$$

So, if (2.104) holds, from (2.102) and (2.106) we get

$$\lim_{n\to\infty} x_{2n+1} = \infty, \qquad \lim_{n\to\infty} y_{2n} = \infty,$$

and if (2.105) holds, from (2.102) and (2.107) we get

$$\lim_{n\to\infty} x_{2n} = \infty, \qquad \lim_{n\to\infty} y_{2n+1} = \infty.$$

This completes the proof of the proposition. □

Open Problem 2.1. *Consider the system of difference equations* (1.1), *where* (2.1) *and* (2.3) *hold, and the initial values* x_{-1}, x_0, y_{-1}, y_0, *are positive real numbers. Suppose that there exists an integer* $l \geq 2$, *such that*

$$y_l < A^{\frac{p}{q+1}}.$$

Assume that

$$A^{\frac{p^2-q-1}{pq}} > B.$$

Study the behavior of the solution (x_n, y_n) *of this system.*

Acknowledgements

The authors would like to thank the referees for their helpful suggestions.

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