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A family of planar differential systems with hyperbolic algebraic limit cycles

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Abstract. In this paper, we characterize a family of planar polynomial differential systems of degree greater or equal than $n+1$, by presenting polynomial curves of degree n, which generally contain closed components. These systems admit precisely the bounded components of the curve as hyperbolic limit cycles.

Keywords: invariant curve, periodic solution, algebraic limit cycle, hyperbolic limit cycle.

2020 Mathematics Subject Classification: 34C25, 34C05, 34C07.

1 Introduction and the main result

The study of limit cycles in planar polynomial differential systems has been a central topic in the qualitative theory of ordinary differential equations for decades. Algebraic limit cycles have also received notable attention owing to their clear algebraic structure and geometric significance. The existence, classification, and stability of such cycles remain one of the most challenging problems, which is related to Hilbert's 16th problem and its generalizations. In this context, hyperbolic algebraic limit cycles are of particular interest, as their stability properties persist under small perturbations, making them a critical object of study in both theoretical and applied dynamical systems.

In this article, we consider the autonomous planar polynomial system of ordinary differential equations

$$\begin{aligned} \dot{x} &= \frac{dx}{dt} = f(x,y), \\ \dot{y} &= \frac{dy}{dt} = g(x,y), \end{aligned} \tag{1.1}$$

where f and g are two polynomials of $R[x,y]$ with no common factor, the derivatives are performed with respect to the time variable t. By definition, the degree of the system (1.1) is $n = \max(\deg(f), \deg(g))$. There is a huge literature about the analysis of the non-existence, existence, number, and stability of limit cycles of the system (1.1), see for instance [2, 7, 10, 6, 8] and the references therein.

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Starting with Colin Christopher [4] proved that if the line $D(x,y) = ux + vy + w = 0$ lies outside all non-singular compact components (ovals) of the algebraic curve $U = 0$ and the constants α and β are chosen such that $\alpha u + \beta v \neq 0$, then the system

$$\begin{aligned}\dot{x} &= \alpha U(x,y) + D(x,y)U_y(x,y),\\ \dot{y} &= \beta U(x,y) - D(x,y)U_x(x,y)\end{aligned}$$

admits all the bounded components of the curve $U = 0$ as hyperbolic limit cycles. Furthermore, the vector field (1.1) has no other limit cycles.

S. Benyoucef in his paper [3] analyses the existence of hyperbolic limit cycles of the differential system

$$\begin{aligned}\dot{x} &= \alpha U\left(x,y\right) + \left(ax + by + \Phi\left(\beta x - \alpha y\right)\right) U_y\left(x,y\right),\\ \dot{y} &= \beta U\left(x,y\right) - \left(ax + by + \Phi\left(\beta x - \alpha y\right)\right) U_x\left(x,y\right),\end{aligned}$$

where α and β are real numbers, Φ is a polynomial function of degree n and $U(x;y)$ is a polynomial of degree m.

As a continuation of [3], J. Llibre and C. Valls in [9] described a new class of polynomial differential systems that has the ovals of the algebraic curve as hyperbolic limit cycles

$$\begin{aligned}\dot{x} &= P(y)U\left(x,y\right) + \left(R(x,y) + \Phi(w(x,y))\right) U_y\left(x,y\right),\\ \dot{y} &= Q(x)U\left(x,y\right) - \left(R(x,y) + \Phi(w(x,y))\right) U_x\left(x,y\right),\end{aligned}$$

where $R(x,y) = \alpha x + \beta y + \int Q(x)dx - \int P(y)dy$, with $\alpha, \beta \in \mathbb{R}$ and the function $w = w(x,y) = \int Q(x)dx - \int P(y)dy$.

Our work builds on and extends earlier results by [3, 4, 9]. Specifically, we provide a new way to construct a new family of systems by presenting polynomial curves which generally contain closed components, and we investigate the conditions under which such systems admit bounded components of these algebraic curves as hyperbolic limit cycles.

Consider $U(x,y) = 0$ be a curve of degree n with $n \geq 2$. We consider a polynomial differential system of degree m with $m \geq 3$:

$$\begin{aligned}\dot{x} &= \frac{\partial Q}{\partial y}(x,y)U(x,y) + \gamma\frac{\partial U}{\partial y}(x,y),\\ \dot{y} &= \frac{\partial Q}{\partial x}(x,y)U(x,y) - \gamma\frac{\partial U}{\partial x}(x,y),\end{aligned} \tag{1.2}$$

with $Q \in R[x,y]$, $\gamma \in \mathbb{R} - \{0\}$ and $(\cdot)_x, (\cdot)_y$ denoting the partial derivatives with respect to the x and y, respectively.

Our main result is the following.

Theorem 1.1. *Let $U = 0$ be a non-singular algebraic curve of degree n and Q a polynomial of degree $m \geq 2$, chosen so that the curve $Q_{xy}(x,y) = 0$ lies outside all oval components of $U = 0$ or $Q_{xy}(x,y) \neq 0$, then the polynomial differential equation (1.2) has all the oval components of $U = 0$ as hyperbolic limit cycles. Furthermore, the vector field has no other limit cycles.*

The paper is organized as follows. In Section 2, we introduce some preliminary results. Theorem 1.1 is proved in Section 3. The last part of Section 3 is devoted to providing some examples satisfying all the conditions of Theorem 1.1.

2 Preliminary results

In this section, we will present the basic concepts that we need to prove the main results of this paper.

As usual, the vector field

$$\mathcal{X} = f\left(x,y\right)\frac{\partial}{\partial x} + g\left(x,y\right)\frac{\partial}{\partial y},$$

is associated to the differential system (1.1).

Invariant algebraic curves

Let $U : \mathbb{R}^2 \to \mathbb{R}$ be a real polynomial. We say that U is an invariant for $\mathcal{X}$ if it satisfies

$$f(x,y)\frac{\partial U\left(x,y\right)}{\partial x} + g(x,y)\frac{\partial U\left(x,y\right)}{\partial y} = K\left(x,y\right)U\left(x,y\right), \tag{2.1}$$

for all $(x,y) \in \mathbb{R}^2$. Here, $K : \mathbb{R}^2 \to \mathbb{R}$, which is the cofactor of U. Simple analysis of equation (2.1) shows that the degree of the cofactor is at most $n-1$ where n represents the maximum of the degrees of f and g, and that the curve $U = 0$ is formed by trajectories of the system (1.1). Also, if the curve $U = 0$ is non-singular, the equilibrium points of the system are contained either in its non-bounded components or are located on the curve $K = 0$.

Note that if $K = 0$, then if $\mathcal{X}U = 0$, that is, $U(x,y) = 0$ is an invariant curve of (1.1) with zero cofactor. Equivalently, $U(x,y) = 0$ is constant if $(x(t), y(t))$ is a solution of (1.1).

Hyperbolic Limit Cycle

In the phase plane, a limit cycle $\Gamma = \{(x(t), y(t)), t \in [0, T]\}$ is a T-periodic solution isolated in the set of all periodic solutions of the system. If the limit cycle is contained in the zero of an invariant algebraic curve of the plane, then it is said to be algebraic; otherwise, it is said to be non-algebraic.

To see that all the oval components of Γ are hyperbolic limit cycles of system (1.1), we will use a classic result characterizing limit cycles among other periodic orbits, which states that: a T-periodic solution Γ is a hyperbolic limit cycle of system (1.1) if

$$\int_0^T \operatorname{Div}\left(\mathcal{X}\right)\left(\Gamma\left(t\right)\right)dt \neq 0,$$

where $\operatorname{Div}\left(\mathcal{X}\right) = \frac{\partial f(x,y)}{\partial x} + \frac{\partial g(x,y)}{\partial y}$, (see for instance [11] for more details).

The curve $U = 0$ is non-singular of system (1.1), i.e., it must not contain any singular point that satisfies

$$f\left(x,y\right) = 0, \qquad g\left(x,y\right) = 0.$$

Green's formula

In order to confirm the hyperbolicity of the limit cycles, we will make use of the following Green's theorem; see for instance [1].

Suppose Ω is a domain in $\mathbb{R}^2$ whose positively oriented boundary Γ is a finite collection of pairwise disjoint piecewise continuous simple closed curves. Suppose f and g are continuous functions defined on a larger open set, which contains both Ω and Γ, and suppose f and g have continuous first partial derivatives on this larger open set. Then

$$\oint_{\Gamma} f\left(x,y\right)dx + g\left(x,y\right)dy = \int\int_{\Omega}\left(\frac{\partial g\left(x,y\right)}{\partial x} - \frac{\partial f\left(x,y\right)}{\partial y}\right)dxdy.$$

3 Proof of Theorem 1.1

We assume that all bounded components of $U = 0$ are non-singular of system (1.2). Let $\Gamma = \{(x(t), y(t)), t \in [0, T]\}$ is an oval of the curve $U = 0$, corresponding to the periodic solution $(x(t), y(t))$ with period T. To see that all the oval components of $U = 0$ are hyperbolic limit cycles, we will prove that $U = 0$ is an invariant curve of the system (1.2), and $\int_0^T \mathrm{Div}\,(\mathcal{X})\,(\Gamma\,(t))\,dt \neq 0$.

(i) It is easy to see that $U = 0$ is an invariant algebraic curve with cofactor

$$K\,(x,y) = Q_x\,(x,y)\,U_y\,(x,y) + Q_y\,(x,y)\,U_x\,(x,y)\,,$$

because

$$\begin{aligned}\frac{dU\,(x,y)}{dt} &= \dot{x}\frac{\partial U\,(x,y)}{\partial x} + \dot{y}\frac{\partial U\,(x,y)}{\partial y}\\ &= \left(Q_x\,(x,y)\,U_y\,(x,y) + Q_y\,(x,y)\,U_x\,(x,y)\right)U\,(x,y)\,.\end{aligned}$$

(ii) The divergence of system (1.2) is

$$\mathrm{Div}(x,y) = Q_x\,(x,y)\,U_y\,(x,y) + Q_y\,(x,y)\,U_x\,(x,y) + 2Q_{xy}\,(x,y)\,U\,(x,y)$$

Therefore, on the curve $U = 0$, we have

$$\begin{aligned}\int_0^T \mathrm{Div}(x,y)\,dt &= \int_0^T Q_x(x,y)U_y(x,y) + Q_y(x,y)U_x(x,y)\,dt\\ &= \oint_\Gamma \frac{Q_x(x,y)U_y(x,y)}{\gamma U_y(x,y)}\,dx + \oint_\Gamma \frac{Q_y(x,y)U_x(x,y)}{-\gamma U_x(x,y)}\,dy\\ &= \frac{1}{\gamma}\left(\oint_\Gamma Q_x(x,y)\,dx - \oint_\Gamma Q_y(x,y)\,dy\right).\end{aligned}$$

Now, we apply the Green formula to get

$$\begin{aligned}\oint_\Gamma Q_x(x,y)\,dx - \oint_\Gamma Q_y(x,y)\,dy &= \iint_{\mathrm{int}(\Gamma)}\left(\frac{\partial(-Q_y(x,y))}{\partial x} - \frac{\partial Q_x(x,y)}{\partial y}\right)dx\,dy\\ &= -2\iint_{\mathrm{int}(\Gamma)} Q_{xy}(x,y)\,dx\,dy.\end{aligned}$$

where $\mathrm{int}(\Gamma)$ denotes the interior of the bounded region limited by Γ, thus

$$\int_0^T \mathrm{Div}\,(x,y)\,dt = -\frac{2}{\gamma}\iint_{\mathrm{int}(\Gamma)} Q_{xy}\,(x,y)\,dxdy.$$

Since by assumption $Q_{xy}\,(x,y) = 0$ lies outside all oval components of $U = 0$ or $Q_{xy}\,(x,y) \neq 0$, then $\int\int_{\mathrm{int}(\Gamma)} Q_{xy}\,(x,y)\,dxdy$ is non-zero, and the system (1.2) has all closed components of $U = 0$ as hyperbolic limit cycles.

Remark 3.1.

1. According to the hypotheses $Q_{xy}\,(x,y) \neq 0$, the polynomial $Q\,(x,y)$ must satisfy $\deg Q \geq 2$ and $Q\,(x,y) \neq h(x) + l(y)$, with $h \in R[x]$ and $l \in R[y]$.

2. In the case when $Q(x,y) = h(x) + l(y) + \alpha xy$, with $\alpha \neq 0$, the integral

$$\int_0^T \mathrm{Div}(x,y)\, dt = -\frac{2a}{\gamma} \iint_{\mathrm{int}(\Gamma)} dx\, dy.$$

So, we have the particular case; if $\alpha \neq 0$, the system (1.2) admits all bounded components of Γ as hyperbolic algebraic limit cycles.

Remark 3.2. If we take $Q(x,y) = yP(x)$, with $P \in \mathbb{R}[x]$, system (1.2) becomes

$$\begin{aligned} \dot{x} &= P(x)U(x,y) + \gamma \frac{\partial U}{\partial y}(x,y), \\ \dot{y} &= yP'(x)U(x,y) - \gamma \frac{\partial U}{\partial x}(x,y), \end{aligned} \tag{3.1}$$

then, the integral

$$\int_0^T \mathrm{Div}\,(x,y)\, dt = -\frac{2}{\gamma} \iint_{\mathrm{int}(\Gamma)} P'\,(x)\, dxdy.$$

So, if we choose that the curve $P'\,(x) = 0$ lies outside all oval components of $U = 0$, then the polynomial differential system (3.1) has all the oval components of $U = 0$ as hyperbolic limit cycles. Furthermore, the vector field has no other limit cycle.

By the change of variables $(x,y) \to (u,v)$ with $x = u$, $y = v + h(u)$, the system (3.1) is transformed into

$$\begin{aligned} \dot{u} &= -\,(h(u)+v)^2 \frac{\partial}{\partial v}\left(\frac{P(u)}{h(u)+v}\right) \tilde{U}(u,v) + \gamma \frac{\partial \tilde{U}}{\partial v}(u,v), \\ \dot{v} &= (h(u)+v)^2 \frac{\partial}{\partial u}\left(\frac{P(u)}{h(u)+v}\right) \tilde{U}(u,v) - \gamma \frac{\partial \tilde{U}}{\partial u}(u,v), \end{aligned} \tag{3.2}$$

where $h, P \in R[x]$, with $R(0) = 0, \tilde{U}\,(u,v) = \tilde{U}\,(u, v + h(u))$ and $\gamma \in \mathbb{R} - \{0\}$. For this last system we can choose $P'(u) = 0$ lies outside all oval components of $\tilde{U} = 0$, then in this case all the oval components of $\tilde{U} = 0$ are hyperbolic limit cycles.

Now we provide some examples of polynomial differential systems of the form (1.2) satisfying all the assumptions of Theorem 1.1 and admitting all oval components of $U = 0$ as hyperbolic limit cycles.

Example 3.3. Let $\gamma = 1$ and $Q\,(x,y) = y\,(x^2 - 4x)$. The system

$$\begin{aligned} \dot{x} &= x(x-4)(3x^2 + 5y^2 - 7) + 10y, \\ \dot{y} &= 2y(x-2)(3x^2 + 5y^2 - 7) - 6x, \end{aligned} \tag{3.3}$$

has an invariant algebraic curve given by $3x^2 + 5y^2 - 7 = 0$. Since $2\,(x-2) = 0$ does not intersect the curve $3x^2 + 5y^2 - 7 = 0$, this oval is an algebraic limit cycle hyperbolic of system (3.3) (see Figure 3.1).

Example 3.4. Let $\gamma = 1$ and $Q\,(x,y) = yx^3 + y^3 x$. The system

$$\begin{aligned} \dot{x} &= x\,(x^2 + 3y^2)\left(x^4 - 2x^2 + y^2 + \frac{1}{4}\right) + 2y^2, \\ \dot{y} &= y\,(3x^2 + y^2)\left(x^4 - 2x^2 + y^2 + \frac{1}{4}\right) - 4x\,(x^2 - 1) \end{aligned} \tag{3.4}$$

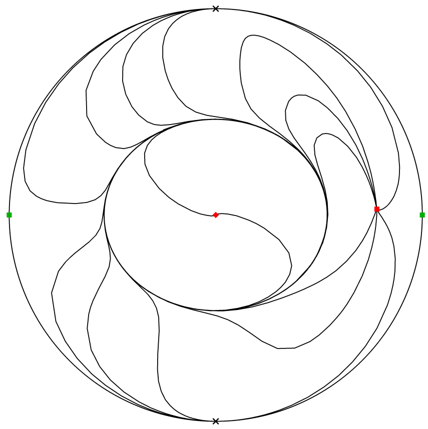


Figure 3.1: The limit cycle of (3.3).

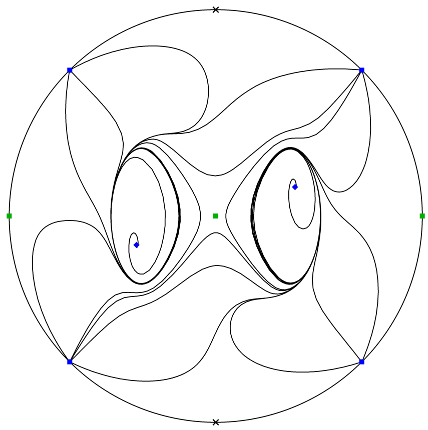


Figure 3.2: The two limit cycles of (3.4).

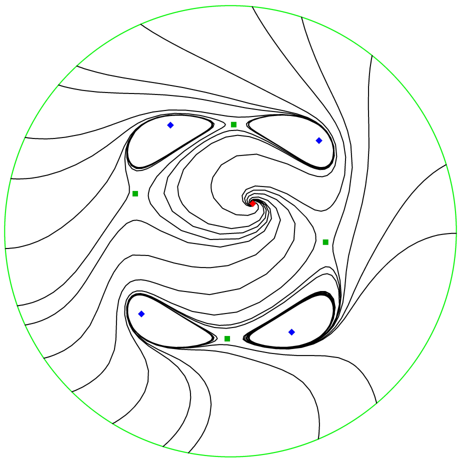
has an invariant algebraic curve given by $x^4 - 2x^2 + y^2 + \frac{1}{4} = 0$. Since $Q_{xy}(x, y) = 3\left(x^2 + y^2\right) \neq 0$. So the two ovals of $x^4 - 2x^2 + y^2 + \frac{1}{4} = 0$ are hyperbolic algebraic limit cycles of system (3.4) (see Figure 3.2).

Example 3.5. Let $\gamma = -2$ and $Q(x) = \alpha xy - \beta y + \gamma, \alpha \neq 0$. The family

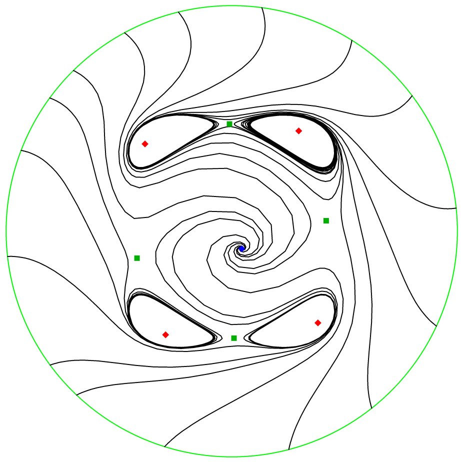
$$\begin{aligned} \dot{x} &= (\alpha x - \beta)\left(y^4 + x^4 - 3y^2 - 2x^2 + \frac{7}{3}\right) - 4y\left(2y^2 - 3\right), \\ \dot{y} &= \alpha y\left(y^4 + x^4 - 3y^2 - 2x^2 + \frac{7}{3}\right) + 8x\left(x^2 - 1\right) \end{aligned} \tag{3.5}$$

has an invariant algebraic curve given by $y^4 + x^4 - 3y^2 - 2x^2 + \frac{7}{3} = 0$. Since $Q_{x,y}(x, y) = \alpha \neq 0$, the four ovals of $y^4 + x^4 - 3y^2 - 2x^2 + \frac{7}{3} = 0$ are hyperbolic algebraic limit cycles of system (3.5) (see Figure 3.3).

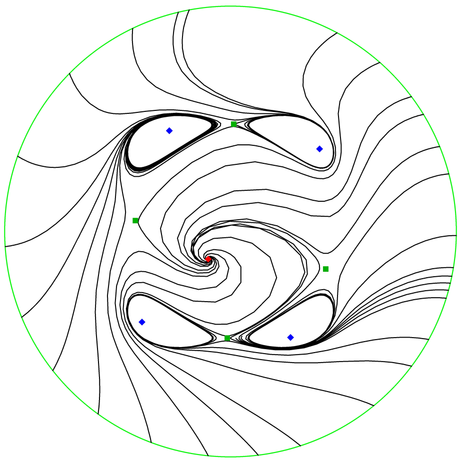
Note: All figures are plotted on the Poincaré disk by using a program for polynomial planar phase portraits, such as the one described in [5, pp. 233–257].



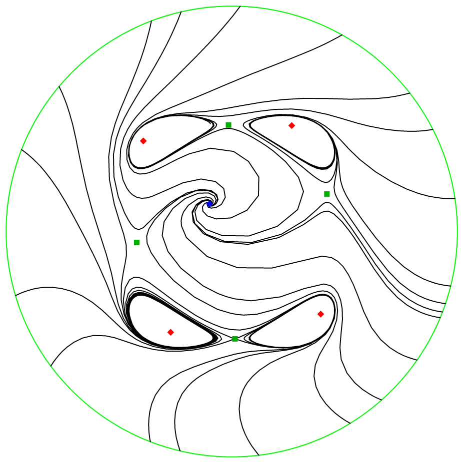
(a) $\alpha = 3$ and $\beta = 2$



(b) $\alpha = -2$ and $\beta = -1$



(c) $\alpha = 3$ and $\beta = -2$



(d) $\alpha = -3$ and $\beta = 2$

Figure 3.3: The four limit cycles of (3.5) with different values of α and β.

References

[1] T. M. Apostol, *Calculus. Vol. II: Multi-variable calculus and linear algebra, with applications to differential equations and probability*, Wiley, 1969. MR248290

[2] A. Bendjeddou, R. Cheurfa, Cubic and quartic planar differential systems with exact algebraic limit cycles, *Electron. J. Differential Equations* **2011**, No. 15, 1–12. MR2781050

[3] S. Benyoucef, Polynomial differential systems with hyperbolic algebraic limit cycles, *Electron. J. Qual. Theory Differ. Equ.* **2020**, No. 34, 1–7. https://doi.org/10.14232/ejqtde.2020.1.34; MR4108771; Zbl 1463.34132

[4] C. Christopher, Polynomial vector fields with prescribed algebraic limit cycles, *Geom. Dedicata* **88**(2001), 255–258. https://doi.org/10.1023/A:1013171019668; MR1877219; Zbl 1072.14534

[5] F. Dumortier, J. Llibre, J. Artés, *Qualitative theory of planar differential systems*, Universitext, Springer-Verlag, Berlin, 2006. https://doi.org/10.1007/978-3-540-32902-2; MR2256001; Zbl 1110.34002

[6] J. Llibre, R. Ramírez, N. Sadovskaia, On the 16th Hilbert problem for algebraic limit cycles, *J. Differential Equations* **248**(2010), 1401–1409. https://doi.org/10.1016/j.jde.2009.11.02; MR2593047; Zbl 1269.34038

[7] J. Llibre, R. Ramírez, N. Sadovskaia, On the 16th Hilbert problem for limit cycles on nonsingular algebraic curves, *J. Differential Equations* **250**(2011), 983–999. https://doi.org/10.1016/j.jde.2010.06.009; MR2737821; Zbl 1269.34038

[8] J. Llibre, C. Valls, Normal forms and hyperbolic algebraic limit cycles for a class of polynomial differential systems, *Electron. J. Differential Equations* **2018**, No. 83, 1–7. MR3831829; Zbl 1391.34065

[9] J. Llibre, C. Valls, Polynomial differential systems with hyperbolic limit cycles, *J. Geom. Phys.* **194**(2023), 104983. https://doi.org/10.1016/j.geomphys.2023.104983; MR4642970; Zbl 1535.34031

[10] J. Llibre, Y. Zhao, Algebraic limit cycles in polynomial systems of differential equations, *J. Phys. A* **40**(2007), 14207–14222. https://doi.org/10.1088/1751-8113/40/47/012; MR2438121; Zbl 1135.34025

[11] L. Perko, *Differential equations and dynamical systems*, Texts in Applied Mathematics, Vol. 7, Springer-Verlag, New York, 2001. https://doi.org/10.1007/978-1-4613-0003-8; MR1801796