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Multiple solutions for a class of Schrödinger–Bopp–Podolsky system with sublinear nonlinearity

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Abstract. In this paper, we investigate the existence and concentration of solutions for the following Schrödinger–Bopp–Podolsky system with sublinear nonlinearity:

$$\begin{cases} -\Delta u + \lambda V(x)u + G(x)\phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi G(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $V \in C(\mathbb{R}^3)$ is a potential well, $G \in L^2(\mathbb{R}^3)$ is nonnegative and $f \in C(\mathbb{R}^3 \times \mathbb{R})$ satisfies the sublinear conditions in $\mathbb{R}^3$. By imposing some suitable assumptions on $V(x)$, $G(x)$ and $f(x,u)$, we obtain the existence and multiplicity of negative energy solutions for the above system via variational methods. Moreover, the concentration of solutions is also explored on the set $V^{-}(0)$ as $\lambda \to \infty$.

Keywords: Schrödinger–Bopp–Podolsky system, sublinear nonlinearity, multiplicity, variational methods.

2020 Mathematics Subject Classification: 35J60, 35J20.

1 Introduction

In the past several decades, the following Schrödinger–Bopp–Podolsky system

$$\begin{cases} -\Delta u + \omega u + q^2\phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

has received more and more attention. Such a system appears when a Schrödinger field $\psi = \psi(t,x)$ couples with its electromagnetic field in the Bopp–Podolsky electromagnetic theory. The Bopp–Podolsky theory, developed by [6], and independently by Podolsky [31], is a second order gauge theory for the electromagnetic field. According to Mie theory [30], the Bopp–Podolsky theory was introduced to solve the so called infinity problem that appears in the

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classical Maxwell theory. Moreover, the Bopp–Podolsky theory may be interpreted as an effective theory for short distances (see [21]) and for large distances, which is experimentally indistinguishable from the Maxwell one. Thus, the Bopp–Podolsky parameter $a > 0$, which has the dimension of the inverse of mass, can be interpreted as a cut-off distance or can be linked to an effective radius for the electron. In fact, by the well-known Gauss law (or Poisson's equation), the electrostatic potential ϕ for a given charge distribution whose density is ρ satisfies equation

$$-\Delta\phi = \rho \quad \text{in } \mathbb{R}^3. \tag{1.2}$$

If $\rho = 4\phi\delta_{x_0}$ with $x_0 \in \mathbb{R}^3$, the fundamental solution of Eq. (1.2) is $\varsigma(x - x_0)$, where

$$\varsigma(x) = \frac{1}{|x|},$$

and the electrostatic energy is

$$\varepsilon_M(\varsigma) = \frac{1}{2}\int_{\mathbb{R}^3} |\Delta\varsigma|^2 \mathrm{d}x = \infty.$$

Thus, Eq. (1.2) is replaced by

$$\operatorname{div}\left(\frac{\Delta\phi}{\sqrt{1-|\Delta\phi|^2}}\right) = \rho \quad \text{in } \mathbb{R}^3$$

in the Bopp–Infeld theory and by

$$-\Delta\phi + a^2\Delta^2\phi = \rho \quad \text{in } \mathbb{R}^3$$

in the Bopp–Podolsky theory. In both cases, if $\rho = 4\phi\delta_{x_0}$, their solutions can be written explicitly and the corresponding energy is finite. In this paper, we focus on the Bopp–Podolsky theory $-\Delta + a^2\Delta^2$, the fundamental solution of equation

$$-\Delta\phi + a^2\Delta^2\phi = 4\pi\delta_{x_0}$$

is $\zeta(x - x_0)$, where

$$\zeta(x) := \frac{1 - e^{-\frac{|x|}{a}}}{|x|},$$

which presents no singularities at x_0, since

$$\lim_{x \to x_0} \zeta(x - x_0) = \frac{1}{a}.$$

Furthermore, its energy is

$$\varepsilon_{BP}(\zeta) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\zeta|^2 \mathrm{d}x + \frac{a^2}{2}\int_{\mathbb{R}^3} |\Delta\zeta|^2 \mathrm{d}x < \infty.$$

For more physical details, we refer the reader to [5,9,10,17] and the references therein.

In [19], d'Avenia and Siciliano firstly studied Schrödinger–Bopp–Podolsky system, and proved that the existence of nontrivial solutions for system (1.1) with $f(u) = |u|^{p-2}u$ $(2 < p < 6)$. Later, by using the fibering method, Siciliano and Silval [35] proved the multiplicity and nonexistence of solutions for system (1.1) with $f(u) = |u|^{p-2}u$ and $p \in (2, 3]$. Moreover, Wang et al. [38] established the existence and multiplicity of sign-changing solutions for system (1.1). In addition, the asymptotic behavior of sign-changing solutions was also established.

In [16], Chen and Tang considered the following Schrödinger–Bopp–Podolsky system with a subcritical perturbation:

$$\begin{cases} -\Delta u + V(x)u + \phi u = \mu g(u) + u^5 & \text{in } \mathbb{R}^3, \\ -\Delta \phi + a^2 \Delta^2 \phi = 4\pi u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.3}$$

By means of variational methods, the authors proved the existence of ground state solutions for system (1.3). In [26], by coupling the Pohožaev–Nehari manifold with the monotonicity trick, Li et al. obtained a ground state solution for system (1.3) with $g(u) = |u|^{p-1}u$ $(2 < p < 5)$. By virtue of the Nehari manifold technique and variational methods, Liu and Chen [28] studied the existence, nonexistence and asymptotic behavior of ground state solutions for system (1.3) with $\mu g(u) + u^5$ being replaced by $\lambda K(x) f(u) + |u|^4 u$.

In [40], Wang et al. studied obtained the following system:

$$\begin{cases} -\Delta u + (\lambda V(x) + V_0)u + K(x)\phi u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi + a^2 \Delta^2 \phi = K(x)\pi u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

where $4 < p < 6$ and the potential function $V(x)$ satisfies the following conditions:

(V_1) $V \in C(\mathbb{R}^3)$, $V(x) > 0$ on $\mathbb{R}^3$;

(V_2) there exists a constant $c > 0$ such that the set $V_c = \{x \in \mathbb{R}^3 : V(x) < c\}$ is nonempty and has finite Lebesgue measure;

(V_3) $\Omega = V^{-}(0) = \text{int}\{x \in \mathbb{R}^3, V(x) = 0\}$ is nonempty with locally Lipschitz boundary and $\bar{\Omega} = \{x \in \mathbb{R}^3, V(x) = 0\}$.

By using variational methods, the authors studied the existence of multi–bump solutions for system (1.4).

Remark 1.1. The conditions (V_1)–(V_3) imply that λV represents a potential well whose depth is controlled by λ. Thus, λV is called a steep potential well if λ is sufficiently large and one expects to find solutions which localize near its bottom Ω. This problem has found much interest after being first introduced by Bartsch and Wang [2] in the study of the existence of positive solutions for nonlinear Schrödinger equations and has been attracting much attention, see [24,25,34,41,42].

Inspired by the above works, more precisely by [40], in the present paper, we are interested in the existence and multiplicity of nontrivial solutions for system (1.1) with a steep potential well and sublinear nonlinearity, which has never been discussed in the available literature. Moreover, the concentration of solutions is also explored. Particularly, we consider the existence of multiple solutions for the following Schrödinger–Bopp–Podolsky system with sublinear nonlinearity:

$$\begin{cases} -\Delta u + \lambda V(x)u + G(x)\phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi + a^2 \Delta^2 \phi = 4\pi G(x) u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{P}$$

where $V(x)$ satisfies the conditions (V_1)–(V_3). In addition, we assume that $G(x)$ and $f(x,u)$ satisfy the following conditions:

(G) $G \in L^2(\mathbb{R}^3)$, $0 \leq G(x) \leq G_\infty$, where G_∞ is a positive constant, $G(x) \not\equiv 0$, and $\lim_{|x|\to\infty} G(x) = 0$;

(F_1) $f \in C(\mathbb{R}^3 \times \mathbb{R})$ and there exist $1 < \beta_1, \beta_2 < 2$ and positive functions $c_1 \in L^{\frac{2}{2-\beta_1}}(\mathbb{R}^3), c_2 \in L^{\frac{2}{2-\beta_2}}(\mathbb{R}^3)$ such that

$$|f(x,u)| \leq \beta_1 c_1(x)|u|^{\beta_1 - 1} + \beta_2 c_2(x)|u|^{\beta_2 - 1}, \qquad \forall\ (x,u) \in \mathbb{R}^3 \times \mathbb{R};$$

(F_2) there exists a bounded open set $\Lambda \subset \mathbb{R}^3$ and three constants $k_1, k_2 > 0$ and $k_3 \in (1,2)$ such that

$$F(x,u) \geq k_2|u|^{k_3}, \qquad \forall\ (x,u) \in \Lambda \times [-k_1, k_1],$$

where $F(x,u) = \int_0^u f(x,s)\mathrm{d}s$;

(F_3) $f(x,u) = -f(x,-u)$ for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$.

Now, we state our main results.

Theorem 1.2. *Assume that conditions (V_1)–(V_3), (G) and (F_1)–(F_2) hold, then problem (P) possesses at least one nontrivial solution.*

Theorem 1.3. *Assume that conditions (V_1)–(V_3), (G) and (F_1)–(F_3) hold, then problem (P) possesses infinitely many solutions $\{u_k\}$ such that*

$$\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_k|^2 + \lambda V(x)u_k^2)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} G(x)\phi_{u_k}{u_k}^2\mathrm{d}x - \int_{\mathbb{R}^3} F(x,u_k)\mathrm{d}x \to 0^- \quad \text{as } k \to \infty.$$

Evidently, the assumption (F_2) holds if the following condition holds:

(F_2') there exists a bounded open set $W \subset \mathbb{R}^3$ and three constants $k_1, k_2 > 0$ and $k_3 \in (1,2)$ such that

$$f(x,u)u \geq k_2 k_3 |u|^{k_3}, \qquad \forall\ (x,u) \in W \times [-k_1, k_1].$$

Therefore, by Theorems 1.2 and 1.3, we have the following corollary.

Corollary 1.4. *Assume that conditions (V_1)–(V_3), (G) and (F_1) and (F_2') hold, then problem (P) possesses at least one nontrivial solution. If additionally, (F_3) holds, then problem (P) possesses infinitely many solutions $\{u_k\}$ such that*

$$\frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u_k|^2 + \lambda V(x)u_k^2\right)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} G(x)\phi_{u_k}u_k^2\mathrm{d}x - \int_{\mathbb{R}^3} F(x,u_k)\mathrm{d}x \to 0^- \quad \text{as } k \to \infty.$$

Before giving the concentration of solutions, we first introduce the following space $\mathcal{D}$. Let $\mathcal{D}$ be the completion of $C_c^\infty(\mathbb{R}^3)$ with respect to the norm $\|\cdot\|_{\mathcal{D}}$ induced by the scalar product:

$$\langle \varphi, \psi \rangle_{\mathcal{D}} := \int_{\mathbb{R}^3} \nabla\varphi\nabla\psi\mathrm{d}x + a^2\int_{\mathbb{R}^3} \Delta\varphi\Delta\psi\mathrm{d}x.$$

Then $\mathcal{D}$ is a Hilbert space which is continuously embedded into $D^{1,2}(\mathbb{R}^3)$ and consequently into $L^6(\mathbb{R}^3)$.

Theorem 1.5. *Let $(u_{\lambda_n}, \phi_{\lambda_n})$ be a solution of problem (P) obtained in Theorem 1.2, then $u_{\lambda_n} \to \tilde{u}$ in $H^1(\mathbb{R}^3)$, $\phi_{\lambda_n} \to \tilde{\phi}$ in $\mathcal{D}$ as $\lambda_n \to \infty$, where $\tilde{u} \in H_0^1(\mathbb{R}^3)$ is a nontrivial solution of equation*

$$\begin{cases} -\Delta u + G(x)\phi u = f(x,u) & \text{in } \Omega, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi G(x)u^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is given by the condition (V_3).

Remark 1.6. If $a = 0$, then system (1.1) becomes the well–known Schrödinger–Poisson system as follows:

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.6}$$

which is also known as the nonlinear Schrödinger–Maxwell equations and have a strong physical meaning. Recently, with the development of critical point theory and variational methods, many researchers have studied the existence and multiplicity of solutions for the Schrödinger–Maxwell equations. Such as, by using the Mountain Pass theorem, D'Aprile and Mugnai [18] obtained the existence of radially symmetric solitary waves for system (1.6). In [33], when $V(x) \equiv 1$ and $f(u) = u^p$, $1 < p < 5$, Ruiz studied the existence and nonexistence of nontrivial solutions for the system (1.6). For more details about the Schrödinger–Poisson system, one can refer to [1, 11–15, 18, 22, 23, 29, 33, 39] and the references therein.

Remark 1.7. There are many functions $f(x,u)$ satisfying the conditions $(F_1) - (F_3)$. For example, let

$$f(x,u) = \frac{5\sin^2 x_1}{4(1+e^{|x|})}|u|^{-\frac{3}{4}}u + \frac{4\cos^2 x_1}{3(1+e^{|x|})}|u|^{-\frac{2}{3}}u,$$

where $x = \{x_1, x_2, x_3\}$. Then

$$|f(x,u)| \le \frac{5\sin^2 x_1}{4(1+e^{|x|})}|u|^{-\frac{3}{4}}u + \frac{4\cos^2 x_1}{3(1+e^{|x|})}|u|^{-\frac{2}{3}}u, \qquad \forall\, (x,u) \in (\mathbb{R}^3, \mathbb{R}),$$

and

$$\begin{aligned} F(x,u) &= \frac{\sin^2 x_1}{1+e^{|x|}}|u|^{\frac{5}{4}} + \frac{\cos^2 x_1}{1+e^{|x|}}|u|^{\frac{4}{3}} \\ &\ge \frac{\cos^2 1}{1+e}|u|^{\frac{4}{3}}, \qquad \forall\, (x,u) \in \Lambda \times [-1,1], \end{aligned}$$

where

$$\frac{5}{4} = \alpha_1 < \alpha_2 = \frac{4}{3}, \qquad c_1(x) = \frac{\sin^2 x_1}{1+e^{|x|}}, \qquad c_2(x) = \frac{\cos^2 x_1}{1+e^{|x|}},$$

and

$$a_1 = 1, \qquad a_2 = \frac{\cos^2 1}{1+e}, \qquad a_3 = \frac{4}{3}, \qquad \Lambda = B(0,1).$$

Notation: Throughout this paper, we shall denote by C various positive generic constants, which may vary from line to line. For $1 \le r \le \infty$, we shall also denote by $|\cdot|_r$ the L^r–norm. $B_r(y) := \{x \in \mathbb{R}^3 : |x-y| \le r\}$. If we take a subsequence of a sequence $\{u_k\}$, we shall denote it again by $\{u_k\}$. $D^{1,2}(\mathbb{R}^3)$ is the usual Sobolev space defined as the completion of $C_c^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|_{D^{1,2}} = \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{1/2}$.

The paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proofs of Theorems 1.2 and 1.3, respectively. In Section 4, we study the concentration of solutions and prove Theorem 1.5.

2 Preliminaries

Let

$$H^1(\mathbb{R}^3) := \{u \in L^2(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$$

with the inner product

$$\langle u, v\rangle_{H^1} = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv)\, \mathrm{d}x$$

and the norm

$$\|u\|_{H^1} = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right) \mathrm{d}x\right)^{\frac{1}{2}}.$$

Let

$$E = \left\{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 \mathrm{d}x < \infty\right\}.$$

In view of the potential $V(x)$, E is a Hilbert space with the following inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv)\, \mathrm{d}x$$

and the norm

$$\|u\| = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x\right)^{\frac{1}{2}}.$$

Throughout the paper, we will use the norm $\|\cdot\|$ in E. For $\lambda > 0$, we also need the following inner product

$$(u, v)_\lambda = \int_{\mathbb{R}^3} (\nabla u \nabla v + \lambda V(x)uv)\, \mathrm{d}x, \qquad \forall\, u, v \in E,$$

and the corresponding norm

$$\|u\|_\lambda = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + \lambda V(x)u^2\right) \mathrm{d}x\right)^{\frac{1}{2}}.$$

Obviously, for $\lambda \geq 1$, we have $\|u\| \leq \|u\|_\lambda$. Set $E_\lambda = (E, \|u\|_\lambda)$, it follows from the conditions (V_1) and (V_2) that

$$\begin{aligned}
\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right) \mathrm{d}x =& \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \int_{V(x)\leq b} u^2 \mathrm{d}x + \int_{V(x)>b} u^2 \mathrm{d}x \\
\leq& \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \left(\int_{V(x)\leq b} 1 \mathrm{d}x\right)^{\frac{2}{3}} \left(\int_{V(x)\leq b} |u|^6 \mathrm{d}x\right)^{\frac{1}{3}} \\
&+ \frac{1}{b} \int_{V(x)>b} V(x)u^2 \mathrm{d}x \\
\leq& \left(1 + \{V(x) \leq b\}S^{-2}\right) \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{1}{b} \int_{V(x)>b} V(x)u^2 \mathrm{d}x \\
\leq& \max\left\{1 + \{V(x) \leq b\}S^{-2}, \frac{1}{b}\right\} \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x,
\end{aligned}$$

where S is the best Sobolev constant of the embedding from $D^{1,2}(\mathbb{R}^3)$ into $L^6(\mathbb{R}^3)$. Then the embedding $E \hookrightarrow H^1(\mathbb{R}^3)$ is continuous, that is, there exists a constant $\eta > 0$ such that

$$\|u\|_{H^1} \leq \eta \|u\| \quad \text{for every } u \in E. \tag{2.1}$$

Furthermore, for every $r \in [2,6]$, there exist τ_r, $\lambda_0 > 0$ (independent of $\lambda \geq 1$) such that

$$|u|_r \leq \tau_r \|u\| \leq \tau_r \|u\|_\lambda \quad \text{for every } u \in E,\ \lambda \geq \lambda_0. \tag{2.2}$$

Note that problem (P) has a variational structure and its solutions can be regarded as critical points of the energy functional defined on the space E by

$$\begin{aligned} J_\lambda(u,\phi) = &\frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + \lambda V(x)u^2\right) \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} G(x)\phi u^2 \mathrm{d}x \\ &- \frac{1}{16\pi}|\nabla\phi|_2^2 - \frac{a^2}{16\pi}|\Delta\phi|_2^2 - \int_{\mathbb{R}^3} F(x,u)\mathrm{d}x. \end{aligned}$$

The functional J_λ exhibits strong indefiniteness, that is, it is unbounded both from below and from above on infinite–dimensional spaces. To avoid the indefiniteness, we can apply the reduction method described in [3, 4], which leads us to study a one–variable functional that does not present such a strong indefinite nature.

It is easy to see that the critical points of the C^1 functional $J_\lambda(u,\phi)$ on $H^1(\mathbb{R}^3) \times \mathcal{D}$ are weak solutions of problem (P). Thus, if $(u,\phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}$ is a critical point of $J_\lambda(u,\phi)$, then

$$\begin{aligned} 0 = \partial_u J_\lambda(u,\phi)[v] = &\int_{\mathbb{R}^3} (\nabla u \nabla v + \lambda V(x)uv)\,\mathrm{d}x + \int_{\mathbb{R}^3} G(x)\phi uv \mathrm{d}x \\ &- \int_{\mathbb{R}^3} f(x,u)v\mathrm{d}x \quad \text{for all } v \in H^1(\mathbb{R}^3) \end{aligned}$$

and

$$0 = \partial_\phi J_\lambda(u,\phi)[\xi] = \frac{1}{2}\int_{\mathbb{R}^3} u^2\xi \mathrm{d}x - \frac{1}{8\pi}\int_{\mathbb{R}^3} \nabla\phi\nabla\xi \mathrm{d}x - \frac{a^2}{8\pi}\int_{\mathbb{R}^3} \Delta\phi\Delta\xi \mathrm{d}x \quad \text{for all } \xi \in \mathcal{D}.$$

Define

$$K(x) := \frac{1 - \mathrm{e}^{-|x|/a}}{|x|}.$$

As stated in [19], for every fixed $u \in H^1(\mathbb{R}^3)$, the Riesz Theorem implies that equation

$$-\Delta\phi + a^2\Delta^2\phi = 4\pi G(x)u^2$$

admits a unique solution $\phi_u \in \mathcal{D}$, which can be presented as

$$\phi_u(x) := K * \left(Gu^2\right) = \int_{\mathbb{R}^3} \frac{1 - \mathrm{e}^{-|x-y|/a}}{|x-y|} G(y)u^2(y)\mathrm{d}y. \tag{2.3}$$

Then the following useful properties hold.

Lemma 2.1. *For every $u \in H^1(\mathbb{R}^3)$, we have:*

(i) $\phi_u \in \mathcal{D} \hookrightarrow L^\infty(\mathbb{R}^3)$;

(ii) $\phi_u \geq 0$;

(iii) *for every* $s \in (3,+\infty]$, $\phi_u \in L^s(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$;

(iv) *for every* $s \in (3,+\infty]$, $\nabla\phi_u = \nabla K * G(x)u^2 \in L^s(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$;

(v) $|\phi_u|_6 \leq C\|u\|_{H^1}^2$;

(*vi*) *for every* $y \in \mathbb{R}^3, \phi_{u(x+y)} = \phi_u(x+y)$*;*

(*vii*) *if* $u_n \rightharpoonup u \in H^1(\mathbb{R}^3)$*, then*

$$\phi_{u_n} \to \phi_u \quad \text{in } \mathcal{D}, \quad \int_{\mathbb{R}^3} G(x)\phi_{u_n} u_n^2 \mathrm{d}x \to \int_{\mathbb{R}^3} G(x)\phi_u u^2 \mathrm{d}x,$$

and

$$\int_{\mathbb{R}^3} G(x)\phi_{u_n} u_n \varphi \mathrm{d}x \to \int_{\mathbb{R}^3} G(x)\phi_u u \varphi \mathrm{d}x, \qquad \forall\, \varphi \in H^1(\mathbb{R}^3).$$

Proof. The proofs of (i)–(vi) can be found in [19, Lemma 3.4] and the proof of (vii) can be found in [37, Proposition 2.1]. We omit them here. □

If G_Φ is the graph of the map $\Phi\colon u \in H^1(\mathbb{R}^3) \mapsto \phi_u \in \mathcal{D}$, an application of the Implicit Function Theorem gives that

$$G_\Phi = \left\{(u,\phi) \in H^1(\mathbb{R}^3) \times \mathcal{D} : \partial_\phi J_\lambda(u,\phi) = 0\right\} \quad \text{and} \quad \Phi \in C^1(H^1(\mathbb{R}^3), \mathcal{D}).$$

Then we define the reduced functional

$$\begin{aligned} I_\lambda(u) = J_\lambda(u,\Phi(u)) &= \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + \lambda V(x)u^2\right) \mathrm{d}x \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3} G(x)\phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(x,u)\mathrm{d}x. \end{aligned} \tag{2.4}$$

Lemma 2.2. *Suppose that conditions* (V_1)–(V_3)*,* (G) *and* $(F_1)-(F_2)$ *hold, then the functional* I_λ *defined by* (2.4) *is well defined and of class* $C^1(E,\mathbb{R})$*.*

Proof. Set

$$T(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + \lambda V(x)u^2\right) \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} G(x)\phi_u u^2 \mathrm{d}x.$$

Following [19], $T(u)$ is well defined and of class C^1 functional. Thus, to prove $I_\lambda(u)$ is well defined and of class $C^1(E,\mathbb{R})$, we only need to prove that $Q(u) := \int_{\mathbb{R}^3} F(x,u)\mathrm{d}x$ is well defined and of class $C^1(E,\mathbb{R})$. It follows from the condition (F_1) that

$$|F(x,u)| \le \sum_{i=1}^{2} c_i(x)|u|^{\beta_i}, \qquad \forall\ (x,u) \in \mathbb{R}^3 \times \mathbb{R}. \tag{2.5}$$

Then for any $u \in E$, by (2.1), (2.2), (2.5) and the Hölder inequality, we derive

$$\begin{aligned} \int_{\mathbb{R}^3} |F(x,u)|\mathrm{d}x &\le \int_{\mathbb{R}^3} \sum_{i=1}^{2} c_i(x)|u|^{\beta_i}\mathrm{d}x \\ &\le \sum_{i=1}^{2} \left(\int_{\mathbb{R}^3} |c_i(x)|^{\frac{2}{2-\beta_i}}\mathrm{d}x\right)^{\frac{2-\beta_i}{2}} \left(\int_{\mathbb{R}^3} |u|^2\mathrm{d}x\right)^{\frac{\beta_i}{2}} \\ &\le \sum_{i=1}^{2} |c_i|_{\frac{2}{2-\beta_i}} \|u\|_{H^1}^{\beta_i}, \end{aligned} \tag{2.6}$$

which implies that $Q(u)$ is well defined.

Next, we show that $Q(u) \in C^1(E,\mathbb{R})$. For any function $\theta : \mathbb{R}^3 \to (0,1)$, by (F_1), (2.2) and the Hölder inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} \max_{t\in[0,1]} |f(x,u(x)+t\theta(x)\varphi(x))\varphi(x)|\mathrm{d}x \\
&= \int_{\mathbb{R}^3} \max_{t\in[0,1]} |f(x,u(x)+t\theta(x)\varphi(x))||\varphi(x)|\mathrm{d}x \\
&\le \sum_{i=1}^{2}\beta_i \int_{\mathbb{R}^3} c_i(x)|u(x)+t\theta(x)\varphi(x)|^{\beta_i-1}|\varphi(x)|\mathrm{d}x \\
&\le C\sum_{i=1}^{2}\int_{\mathbb{R}^3} c_i(x)\left(|u(x)|^{\beta_i-1}+|\varphi(x)|^{\beta_i-1}\right)|\varphi(x)|\mathrm{d}x \\
&\le C\sum_{i=1}^{2}\left(\int_{\mathbb{R}^3}|c_i(x)|^{\frac{2}{2-\beta_i}}\mathrm{d}x\right)^{\frac{2-\beta_i}{2}}\left(\int_{\mathbb{R}^3}|u(x)|^2\mathrm{d}x\right)^{\frac{\beta_i-1}{2}} \\
&\quad\times\left(\int_{\mathbb{R}^3}|\varphi(x)|^2\mathrm{d}x\right)^{\frac{1}{2}} \qquad (2.7)\\
&\quad+C\sum_{i=1}^{2}\left(\int_{\mathbb{R}^3}|c_i(x)|^{\frac{2}{2-\beta_i}}\mathrm{d}x\right)^{\frac{2-\beta_i}{2}}\left(\int_{\mathbb{R}^3}|\varphi(x)|^2\mathrm{d}x\right)^{\frac{\beta_i}{2}} \\
&\le C\sum_{i=1}^{2}|c_i|_{\frac{2}{2-\beta_i}}\left(\|u\|^{\beta_i-1}+\|\varphi\|^{\beta_i-1}\right)\|\varphi\| \\
&< +\infty, \qquad \forall\, u,\varphi\in E.
\end{aligned}$$

It follows from (2.7) and Lebesgue's Dominated Convergence Theorem that

$$\lim_{t\to 0}\frac{Q(u+tv)-Q(u)}{t} = \lim_{t\to 0}\int_{\mathbb{R}^3}\frac{F(u+tv)-F(u)}{t}\mathrm{d}x = \int_{\mathbb{R}^3} f(x,u)v\mathrm{d}x < +\infty, \qquad (2.8)$$

which implies that $Q(u)$ is Gâteaux differentiable in E. Let $u_n \to u$ in E, then $u_n \to u$ in $L^2(\mathbb{R}^3)$ and

$$\lim_{n\to\infty} u_n = u \quad \text{a.e. on } \mathbb{R}^3. \qquad (2.9)$$

Now, we claim that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|f(x,u_n)-f(x,u)|^2\mathrm{d}x = 0. \qquad (2.10)$$

In fact, since $u_n \to u$ in $L^2(\mathbb{R}^3)$, passing to a subsequence if necessary, it can be assumed that $\sum_{i=1}^{\infty}|u_{n_i}-u|_2^2 < +\infty$. Set $\omega(x) = \left(\sum_{i=1}^{\infty}|u_{n_i}-u|^2\right)^{\frac{1}{2}}$, then $\omega \in L^2(\mathbb{R}^3)$. Evidently,

$$\begin{aligned}
|f(x,u_{n_i})-f(x,u)|^2 &\le 2|f(x,u_{n_i})|^2+2|f(x,u)|^2 \\
&\le 4\sum_{j=1}^{2}\alpha_j^2|c_j(x)|^2\left(|u_{n_i}|^{2(\beta_j-1)}+|u|^{2(\beta_j-1)}\right) \\
&\le 4\sum_{j=1}^{2}(2^{\beta_j-1}+1)\beta_j^2|c_j(x)|^2\left(|u_{n_i}-u|^{2(\beta_j-1)}+|u|^{2(\beta_j-1)}\right) \qquad (2.11)\\
&\le 4\sum_{j=1}^{2}(2^{\beta_j-1}+1)\beta_j^2|c_j(x)|^2\left(|\omega(x)|^{2(\beta_j-1)}+|u|^{2(\beta_j-1)}\right) \\
&=: h(x), \qquad \forall\, i\in\mathbb{N},\ x\in\mathbb{R}^3,
\end{aligned}$$

and

$$\int_{\mathbb{R}^3} h(x)\mathrm{d}x = 4\sum_{j=1}^{2}\left(2^{\beta_j-1}+1\right)\beta_j^2\int_{\mathbb{R}^3}|c_j(x)|^2\left(|\omega(x)|^{2(\beta_j-1)}+|u|^{2(\beta_j-1)}\right)\mathrm{d}x$$
$$\leq 4\sum_{j=1}^{2}\left(2^{\beta_j-1}+1\right)\beta_j^2|c_j|^2_{\frac{2}{2-\beta_j}}\left(|\omega|_2^{2(\beta_j-1)}+|u|_2^{2(\beta_j-1)}\right) \tag{2.12}$$
$$< +\infty.$$

It follows from (2.11), (2.12) and Lebesgue's Dominated Convergence Theorem that (2.10) holds, then together with (2.8), we know that $Q(u) \in C^1(E,\mathbb{R})$. The proof is complete. □

Thus, for all $u, v \in H^1(\mathbb{R}^3)$,

$$I'_\lambda(u)[v] = \int_{\mathbb{R}^3}(\nabla u\nabla v + \lambda V(x)uv)\,\mathrm{d}x + \int_{\mathbb{R}^3} G(x)\phi_u uv\mathrm{d}x - \int_{\mathbb{R}^3} f(x,u)v\mathrm{d}x.$$

It can be proved that $(u,\phi) \in H^1(\mathbb{R}^3)\times\mathcal{D}$ is a solution of problem (P) if and only if $u \in E$ is a critical point of the functional I_λ.

Lemma 2.3. *Assume that there exists a sequence $\{u_n\} \subset E$ such that $u_n \rightharpoonup u$ in E as $n \to \infty$, then*

$$\left|\int_{\mathbb{R}^3} G(x)(\phi_{u_n}u_n - \phi_u u)(u_n - u)\mathrm{d}x\right| \to 0 \quad \text{as } n\to\infty. \tag{2.13}$$

Proof. By (2.1), we have $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then by (vii) of Lemma 2.1, as $n\to\infty$, there hold

$$\int_{\mathbb{R}^3} G(x)\phi_{u_n}u_n^2\mathrm{d}x \to \int_{\mathbb{R}^3} G(x)\phi_u u u_n\mathrm{d}x \tag{2.14}$$

and

$$\int_{\mathbb{R}^3} G(x)\phi_{u_n}u_n u\mathrm{d}x \to \int_{\mathbb{R}^3} G(x)\phi_u u^2\mathrm{d}x. \tag{2.15}$$

It follows from (2.14) and (2.15) that

$$\left|\int_{\mathbb{R}^3} G(x)(\phi_{u_n}u_n - \phi_u u)(u_n-u)\mathrm{d}x\right|$$
$$= \left|\int_{\mathbb{R}^3}\left(G(x)\phi_{u_n}u_n^2 - G(x)\phi_u u u_n - G(x)\phi_{u_n}u_n u + G(x)\phi_u u^2\right)\mathrm{d}x\right|$$
$$\leq \left|\int_{\mathbb{R}^3}\left(G(x)\phi_{u_n}u_n^2 - G(x)\phi_u u u_n\right)\mathrm{d}x\right| + \left|\int_{\mathbb{R}^3}\left(G(x)\phi_{u_n}u_n u - G(x)\phi_u u^2\right)\mathrm{d}x\right|$$
$$\to 0 \quad \text{as } n\to\infty.$$

The proof is complete. □

Lemma 2.4 ([20]). *Let E be a real Banach space and $I \in C^1(E,\mathbb{R})$ satisfies the (PS) condition. If I is bounded from below, then $c = \inf_E I$ is a critical value of I.*

Lemma 2.5. *Assume that conditions (V_1)–(V_3), (G) and (F_1)–(F_2) hold, then there exists $\Lambda_0 > 0$ such that I_λ is bounded from below whenever $\lambda > \Lambda_0$.*

Proof. It follows from (2.2), (2.4), (F_1) and the Hölder inequality that

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2+\lambda V(x)u^2\right)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} G(x)\phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(x,u)\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|_\lambda^2 - \int_{\mathbb{R}^3} F(x,u)\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^3} c_1(x)|u|^{\beta_1}\mathrm{d}x - \int_{\mathbb{R}^3} c_2(x)|u|^{\beta_2}\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|^2 - \sum_{i=1}^{2}\tau_2^{\beta_i}|c_i|_{\frac{2}{2-\beta_i}}\|u\|^{\beta_i}, \end{aligned}$$

which implies that $I_\lambda(u)\to\infty$ as $\|u\|\to\infty$, since $\beta_1,\beta_2\in(1,2)$. Consequently, there is $\Lambda_0 = \max\{1,\lambda_0\}$ such that for every $\lambda\geq\Lambda_0$, I_λ is bounded from below. The proof is complete. □

3 Proofs of main results

Lemma 3.1. *Assume that conditions (V_1)–(V_3), (G) and (F_1)–(F_2) hold, then I_λ satisfies the (PS) condition.*

Proof. Assume that $\{u_n\}$ is a (PS) sequence of I_λ such that $I_\lambda(u_n)$ is bounded and $I'_\lambda(u_n)\to 0$ as $n\to\infty$. It follows from Lemma 2.5 that $\{u_n\}$ is bounded in E_λ. Then there exists a constant $C>0$ such that

$$|u_n|_l \leq \tau_l\|u_n\|_\lambda \leq C, \qquad n\in\mathbb{N},\ \lambda\geq\Lambda_0,\ 2\leq l\leq 6. \tag{3.1}$$

Moreover, there exists $u\in E$ such that

$$u_n \rightharpoonup u \quad \text{in } E, \tag{3.2}$$

$$u_n \to u \quad \text{in } L^r_{\mathrm{loc}}(\mathbb{R}^3),\ r\in[2,6), \tag{3.3}$$

$$u_n \to u \quad \text{a.e on } \mathbb{R}^3. \tag{3.4}$$

By the condition (F_1), for any given $\epsilon>0$, there exists $R_\epsilon>0$ such that

$$\left(\int_{|x|>R_\epsilon}|c_i(x)|^{\frac{2}{2-\beta_i}}\mathrm{d}x\right)^{\frac{2-\beta_i}{2}} \leq \epsilon, \qquad i=1,2. \tag{3.5}$$

It follows from (3.3) that there exists $n_0>0$ such that

$$\int_{|x|\leq R_\epsilon}|u_n-u|^2\mathrm{d}x < \epsilon^2 \quad \text{for } n\geq n_0. \tag{3.6}$$

Then by (3.1), (3.6), (F_1) and the Hölder inequality, for any $n\geq n_0$, one has

$$\begin{aligned} &\int_{|x|\leq R_\epsilon}|f(x,u_n)-f(x,u)||u_n-u|\mathrm{d}x \\ &\leq \left(\int_{|x|\leq R_\epsilon}|f(x,u_n)-f(x,u)|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{|x|\leq R_\epsilon}|u_n-u|^2\mathrm{d}x\right)^{\frac{1}{2}} \\ &\leq \epsilon\left[\int_{|x|\leq R_\epsilon}2\left(|f(x,u_n)|^2+|f(x,u)|^2\right)\mathrm{d}x\right]^{\frac{1}{2}} \\ &\leq \epsilon\left[4\sum_{i=1}^{2}\beta_i^2\int_{|x|\leq R_\epsilon}|c_i(x)|^2\left(|u_n|^{2(\beta_i-1)}+|u|^{2(\beta_i-1)}\right)\mathrm{d}x\right]^{\frac{1}{2}} \end{aligned} \tag{3.7}$$

$$\leq C\epsilon\left[\sum_{i=1}^{2}\beta_i^2|c_i|^2_{\frac{2}{2-\beta_i}}\left(|u_n|_2^{2(\beta_i-1)}+|u|_2^{2(\beta_i-1)}\right)\right]^{\frac{1}{2}}$$
$$\leq C\epsilon\left[\sum_{i=1}^{2}\beta_i^2|c_i|^2_{\frac{2}{2-\beta_i}}\left(C^{2(\beta_i-1)}+|u|_2^{2(\beta_i-1)}\right)\right]^{\frac{1}{2}}.$$

On the other hand, for $n\in\mathbb{N}$, it follows from (F_1), (3.1), (3.5) and the Hölder inequality that

$$\begin{aligned}
&\int_{|x|>R_\epsilon}|f(x,u_n)-f(x,u)||u_n-u|\mathrm{d}x\\
&\leq\sum_{i=1}^{2}\beta_i^2\int_{|x|>R_\epsilon}|c_i(x)|\left(|u_n|^{\beta_i-1}+|u|^{\beta_i-1}\right)(|u_n|+|u|)\,\mathrm{d}x\\
&\leq 2\sum_{i=1}^{2}\beta_i^2\int_{|x|>R_\epsilon}|c_i(x)|\left(|u_n|^{\beta_i}+|u|^{\beta_i}\right)\mathrm{d}x\\
&\leq 2\sum_{i=1}^{2}\beta_i^2\left(\int_{|x|>R_\epsilon}|c_i(x)|^{\frac{2}{2-\beta_i}}\mathrm{d}x\right)^{\frac{2-\beta_i}{2}}\left(|u_n|_2^{\beta_i}+|u|_2^{\beta_i}\right)\\
&\leq 2\sum_{i=1}^{2}\beta_i^2\left(\int_{|x|>R_\epsilon}|c_i(x)|^{\frac{2}{2-\beta_i}}\mathrm{d}x\right)^{\frac{2-\beta_i}{2}}\left(C^{\beta_i}+|u|_2^{\beta_i}\right)\\
&\leq 2\epsilon\sum_{i=1}^{2}\beta_i^2\left(C^{\beta_i}+|u|_2^{\beta_i}\right).
\end{aligned}\tag{3.8}$$

Since ϵ is arbitrary, combining (3.7) and (3.8), we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\left(f(x,u_n)-f(x,u)\right)(u_n-u)\mathrm{d}x=0.\tag{3.9}$$

Thus, by (3.9), Lemma 2.3 and the weak convergence of $\{u_n\}$, one has

$$\begin{aligned}
o_n(1)&=\langle I'_\lambda(u_n)-I'_\lambda(u),u_n-u\rangle\\
&=\int_{\mathbb{R}^3}|\nabla(u_n-u)|^2\mathrm{d}x+\lambda\int_{\mathbb{R}^3}V(x)(u_n-u)^2\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}G(x)(\phi_{u_n}u_n-\phi_u u)(u_n-u)\mathrm{d}x\\
&\quad-\int_{\mathbb{R}^3}\left(f(x,u_n)-f(x,u)\right)(u_n-u)\mathrm{d}x\\
&=\|u_n-u\|_\lambda^2+o_n(1),
\end{aligned}$$

which implies that $u_n\to u$ in E_λ. The proof is complete. □

In order to find the multiplicity of nontrivial critical points of I_λ, we will use the genus properties, so we recall the following definitions and results (see[32]).

Let E be a Banach space, $c\in\mathbb{R}$ and $I\in C^1(E,\mathbb{R})$, set
$\Sigma=\{A\subset E\backslash\{0\}\colon A$ is closed in E and symmetric with respect to $0\}$,
$K_c=\{u\in E\colon I_\lambda(u)=c, I'_\lambda(u)=0\}$, $I_\lambda^c=\{u\in E\colon I_\lambda(u)\leq c\}$.

Definition 3.2. For $A\in\Sigma$, we say genus on A is n (denoted by $\gamma(A)=n$) if there is an odd map $\varphi\in C(A,\mathbb{R}^3\setminus\{0\})$ and n is the smallest integer with this property.

Lemma 3.3. *Let I be an even C^1 functional on E and satisfies the (PS) condition. For any $n \in \mathbb{N}$, set*

$$\Sigma_n = \{A \in \Sigma\colon \gamma(A) \geq n\}, \qquad c_n = \inf_{A \in \sum_n} \sup_{u \in A} I(u),$$

(i) if $\Sigma_n \neq \emptyset$, and $c_n \in \mathbb{R}$, then c_n is a critical value of I;

(ii) if there exists $r \in \mathbb{N}$ such that $c_n = c_{n+1} = \cdots = c_{n+r} = c \in \mathbb{R}$, and $c \neq I(0)$ then $\gamma(K_c) \geq r+1$.

Proof of Theorem 1.2. By Lemmas 2.5 and 3.1, the conditions of Lemma 2.4 are satisfied. Thus, $c = \inf_{E_\lambda} I_\lambda(u)$ is a critical value of I_λ, that is, there exists a critical point u^* such that $I_\lambda(u^*) = c$. Now, we show that $u^* \neq 0$. Let $u \in (H_0^1(\Lambda) \cap E_\lambda) \backslash \{0\}$ and $|u|_\infty \leq 1$, it follows from (F_2) that

$$\begin{aligned} I_\lambda(tu) &= \frac{t^2}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + \lambda V(x)u^2)\mathrm{d}x + \frac{t^4}{4}\int_{\mathbb{R}^3} G(x)\phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(x,tu)\mathrm{d}x \\ &= \frac{t^2}{2}\|u\|_\lambda^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} G(x)\phi_u u^2\mathrm{d}x - \int_\Lambda F(x,tu)\mathrm{d}x \\ &\leq \frac{t^2}{2}\|u\|_\lambda^2 + \frac{t^4 G_\infty}{4}\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x - k_2 t^{k_3}\int_\Lambda |u|^{k_3}\mathrm{d}x, \end{aligned} \tag{3.10}$$

where $0 < t < k_1$ and k_1 is given in (F_2). Since $1 < k_3 < 2$, it follows from (3.10) that $I(tu) < 0$ for $t > 0$ small enough. Therefore, $I_\lambda(u^*) = c < 0$, that is, u^* is a nontrivial critical point of I_λ, and so u^* is a nontrivial solution of problem (P). The proof is complete. □

Proof of Theorem 1.3. By Lemmas 2.5 and 3.1, $I_\lambda \in C^1(E, \mathbb{R})$ is bounded from below and satisfies the (PS) condition. It follows from (2.4) and (F_3) that I_λ is even and $I_\lambda(0) = 0$. In order to apply Lemma 3.3, we now show that for any $n \in \mathbb{N}$, there exists $\epsilon > 0$ such that

$$\gamma(I_\lambda^{-\epsilon}) \geq n. \tag{3.11}$$

For any $n \in \mathbb{N}$, we take n disjoint open sets Λ_i such that

$$\bigcup_{i=1}^n \Lambda_i \subset \Lambda.$$

For $i = 1, 2, \ldots, n$, let $u_i \in (H_0^1(\Lambda_i) \cap E_\lambda) \backslash \{0\}$, $|u_i|_\infty \leq +\infty$, $\|u_i\|_\lambda = 1$, and

$$E_n = \operatorname{span}\{u_1, u_2, \ldots, u_n\}, \qquad S_n = \{u \in E_n : \|u\|_\lambda = 1\}.$$

Then for any $u \in E_n$, there exist $\mu_i \in \mathbb{R}, i = 1, 2, \ldots, n$ such that

$$u(x) = \sum_{i=1}^n \mu_i u_i(x), \qquad x \in \mathbb{R}^3. \tag{3.12}$$

Thus, we get

$$|u|_{k_3} = \left(\int_{\mathbb{R}^3} |u|^{k_3}\mathrm{d}x\right)^{\frac{1}{k_3}} = \left(\sum_{i=1}^n |\mu|^{k_3}\int_{\Lambda_i} |u|^{k_3}\mathrm{d}x\right)^{\frac{1}{k_3}}, \tag{3.13}$$

and

$$\begin{aligned}\|u\|_\lambda^2 &= \int_{\mathbb{R}^3}(|\nabla u|^2+\lambda V(x)|u|^2)\mathrm{d}x\\ &= \sum_{i=1}^n \mu_i^2 \int_{\Lambda_i}\left(|\nabla u_i|^2+\lambda V(x)|u_i|^2\right)\mathrm{d}x\\ &\le \sum_{i=1}^n \mu_i^2 \int_{\mathbb{R}^3}\left(|\nabla u_i|^2+\lambda V(x)|u_i|^2\right)\mathrm{d}x\\ &= \sum_{i=1}^n \mu_i^2\|u_i\|_\lambda^2 = \sum_{i=1}^n \mu_i^2.\end{aligned} \tag{3.14}$$

Since all norms are equivalent in a finite dimensional normed space, there exists $d_1 > 0$ such that

$$d_1\|u\|_\lambda \le |u|_{k_3} \quad \text{for all } u \in E_n. \tag{3.15}$$

Then, by (2.4), (F_2), (3.12)–(3.15), the Sobolev and Hölder inequalities, for $u \in S_n$, we have

$$\begin{aligned} I_\lambda(tu) &= \frac{t^2}{2}\int_{\mathbb{R}^3}(|\nabla u|^2+\lambda\int_{\mathbb{R}^3}V(x)u^2)\mathrm{d}x + \frac{t^4}{4}\int_{\mathbb{R}^3}G(x)\phi_u u^2\mathrm{d}x - \int_{\mathbb{R}^3}F(x,tu)\mathrm{d}x\\ &= \frac{t^2}{2}\|u\|_\lambda^2 + \frac{t^4}{4}\int_{\mathbb{R}^3}G(x)\phi_u u^2\mathrm{d}x - \sum_{i=1}^n\int_{\Lambda_i}F(x,t\mu_i u_i)\mathrm{d}x\\ &\le \frac{t^2}{2}\|u\|_\lambda^2 + \frac{t^4 G_\infty}{4}|\phi_u|_6|u|_3|u|_2 - k_2 t^{k_3}\sum_{i=1}^n|\mu_i|^{k_3}\int_{\Lambda_i}|u_i|^{k_3}\mathrm{d}x\\ &= \frac{t^2}{2}\|u\|_\lambda^2 + \frac{Ct^4}{4}\|u\|_\lambda^4 - k_2 t^{k_3}|u|_{k_3}^{k_3}\\ &\le \frac{t^2}{2}\|u\|_\lambda^2 + \frac{Ct^4}{4}\|u\|_\lambda^4 - k_2(d_1 t)^{k_3}\|u\|_\lambda^{k_3}\\ &= \frac{t^2}{2} + \frac{Ct^4}{4} - k_2(d_1 t)^{k_3}.\end{aligned} \tag{3.16}$$

Since $0 < t < k_1$ and $1 < k_3 < 2$, it follows from (3.16) that there exist $\epsilon > 0$ and $\delta > 0$ such that

$$I_\lambda(\delta u) < -\epsilon \quad \text{for all } u \in S_n. \tag{3.17}$$

Let

$$S_n^\delta = \{\delta u : u \in S_n\}, \qquad \Omega = \left\{(\mu_1,\mu_2,\ldots,\mu_n)\in\mathbb{R}^n : \sum_{i=1}^n \mu_i^2 < \delta^2\right\}.$$

It follows from (3.17) that

$$I_\lambda(u) < -\epsilon \quad \text{for all } u \in S_n^\delta,$$

which, together with the fact that $I_\lambda \in C^1(E,\mathbb{R})$ and is even, implies that

$$S_n^\delta \subset I_\lambda^{-\epsilon} \in \Sigma.$$

On the other hand, by (3.12) and (3.14), there exists an odd homeomorphism mapping $\phi \in C(S_n^\delta, \partial\Omega)$. By some properties of the genus, we have

$$\gamma(I_\lambda^{-\epsilon}) \ge \gamma(S_n^\delta) = n.$$

Thus, the proof of (3.11) holds. Set

$$c_n = \inf_{A\in\Sigma_n} \sup_{u\in A} I_\lambda(u). \tag{3.18}$$

It follows from (3.18) and the fact that I_λ is bounded from below on E_λ that $-\infty < c_n \le -\epsilon < 0$, that is to say, for any $n \in \mathbb{N}$, c_n is a real negative number. By Lemma 3.3, I_λ has infinitely many nontrivial critical points. Therefore, problem (P) possesses infinitely many nontrivial solutions. The proof is complete. □

4 Concentration of solutions

In the following, we study the concentration of solutions for problem (P) as $\lambda \to \infty$. Define

$$\tilde{c} = \inf_{u\in H_0^1(\Omega)} I_\lambda(u)|_{H_0^1(\Omega)},$$

where $I_\lambda(u)|_{H_0^1(\Omega)}$ is a restricted of $I_\lambda(u)$ on $H_0^1(\Omega)$, that is,

$$I_\lambda(u)|_{H_0^1(\Omega)} = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \lambda V(x)u^2\right) \mathrm{d}x + \frac{1}{4}\int_\Omega G(x)\phi_u u^2 \mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x.$$

Similar to the proof of Theorem 1.2, it is easy to prove that $\tilde{c} < 0$ can be achieved. Since $H_0^1(\Omega) \subset E_\lambda$ for all $\lambda > 0$, we derive

$$c \le \tilde{c} < 0 \quad \text{for all } \lambda > \Lambda_0.$$

Proof of Theorem 1.5. Inspired by the ideas of Sun and Wu [36], we give the proof as follows. For any $\lambda_n \to \infty$, let $u_n := u_{\lambda_n}$ be the critical point of I_{λ_n} obtained in Theorem 1.2, then

$$I_{\lambda_n}(u_n) \le \tilde{c} < 0, \tag{4.1}$$

and

$$\begin{aligned} I_{\lambda_n}(u_n) &= \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + \lambda V(x)u_n^2\right) \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} G(x)\phi_{u_n} u_n^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(x,u_n)\mathrm{d}x \\ &\ge \frac{1}{2}\|u_n\|_{\lambda_n}^2 - \sum_{i=1}^{2} \tau_2^{\beta_i} |c_i|_{\frac{2}{2-\beta_i}} \|u_n\|_{\lambda_n}^{\beta_i}, \end{aligned}$$

showing that

$$\|u_n\|_{\lambda_n} \le C, \tag{4.2}$$

where C is independent of λ_n. Thus, we may assume that $u_n \rightharpoonup \tilde{u}$ in E_λ and $u_n \to \tilde{u}$ in $L_{\mathrm{loc}}^r(\mathbb{R}^3)$ for $r \in [2,6)$. It follows from Fatou's Lemma that

$$\int_{\mathbb{R}^3} V(x)|\tilde{u}|^2 \mathrm{d}x \le \liminf_{n\to\infty} \int_{\mathbb{R}^3} V(x)|u_n|^2 \mathrm{d}x \le \liminf_{n\to\infty} \frac{\|u_n\|_{\lambda_n}^2}{\lambda_n} = 0,$$

which implies that $\tilde{u} = 0$ a.e. in $\mathbb{R}^3 \backslash V^-(0)$ and $\tilde{u} \in H_0^1(\Omega)$ by (V_3). Then for any $\varphi \in C_0^\infty(\Omega)$, since $\langle I'_{\lambda_n}(u_n), \varphi\rangle = 0$, it is easy to verify that

$$\int_\Omega \nabla\tilde{u}\nabla\varphi \mathrm{d}x + \int_\Omega G(x)\phi_{\tilde{u}}\tilde{u}\varphi \mathrm{d}x - \int_\Omega f(x,\tilde{u})\varphi \mathrm{d}x = 0,$$

which implies that $\tilde{u}$ is a weak solution of Eq.(1.5) by the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$.

Next, we show that $u_n \to \tilde{u}$ in $L^r(\mathbb{R}^3)$ for $r \in [2,6)$. Otherwise, by Lions' vanishing lemma [27], there exist $\delta > 0, \rho > 0$ and $x_n \in \mathbb{R}^3$ such that

$$\int_{B_\rho(x_n)} |u_n - \tilde{u}|^2 \mathrm{d}x \geq \delta.$$

Since $u_n \to \tilde{u}$ in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$ as $|x_n| \to \infty$ and $n \to \infty$. Therefore, $\mathrm{meas}\{B_\rho(x_n) \cap V_b\} \to 0$ as $n \to \infty$. It follows from the Hölder inequality that

$$\int_{B_\rho(x_n)\cap V_b} |u_n - \tilde{u}|^2 \mathrm{d}x \leq \left(\mathrm{meas}\{B_\rho(x_n) \cap V_b\}\right)^{\frac{2^*-2}{2^*}} \left(\int_{\mathbb{R}^3} |u_n - \tilde{u}|^{2^*} \mathrm{d}x\right)^{\frac{2}{2^*}} \to 0 \quad \text{as } n \to \infty.$$

Then for n sufficiently large,

$$\begin{aligned}
\|u_n\|_{\lambda_n}^2 &\geq \lambda_n b \int_{B_\rho(x_n)\cap\{x\in\mathbb{R}^3 : V(x)\geq b\}} |u_n|^2 \mathrm{d}x \\
&= \lambda_n b \int_{B_\rho(x_n)\cap\{x\in\mathbb{R}^3 : V(x)\geq b\}} |u_n - \tilde{u}|^2 \mathrm{d}x \\
&= \lambda_n b \left(\int_{B_\rho(x_n)} |u_n - \tilde{u}|^2 \mathrm{d}x - \int_{B_\rho(x_n)\cap V_b} |u_n - \tilde{u}|^2 \mathrm{d}x\right) \\
&\to \infty \quad \text{as } n \to \infty,
\end{aligned}$$

which is a contradiction with (4.2). Next, we prove that $u_n \to \tilde{u}$ in $H^1(\mathbb{R}^3)$. By virtue of $\langle I'_{\lambda_n}(u_n), u_n\rangle = \langle I'_{\lambda_n}(u_n), \tilde{u}\rangle = 0$ and the fact that $u_n \to \tilde{u}$ in $L^r(\mathbb{R}^3)$ for $r \in [2,6)$, we derive

$$\lim_{n\to\infty} \|u_n\|_{\lambda_n}^2 = \lim_{n\to\infty} (u_n, \tilde{u})_{\lambda_n} = \lim_{n\to\infty} \langle u_n, \tilde{u}\rangle = \|\tilde{u}\|^2.$$

Observe that $\|u_n\| \leq \|u_n\|_{\lambda_n}$, therefore,

$$\limsup_{n\to\infty} \|u_n\|^2 \leq \|\tilde{u}\|^2.$$

On the other hand, from the weak semi–continuity of norm, we have

$$\|\tilde{u}\|^2 \leq \limsup_{n\to\infty} \|u_n\|^2.$$

Therefore,

$$u_n \to \tilde{u} \quad \text{in } H^1(\mathbb{R}^3).$$

It follows from (4.1) that

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla \tilde{u}|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} G(x)\phi_{\tilde{u}}\tilde{u}^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(x,\tilde{u})\mathrm{d}x \leq \tilde{c} < 0,$$

which implies that $\tilde{u} \neq 0$. The proof is complete. □

Conflict of interest statement

Authors state no conflict of interest.

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