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On Shilnikov's scenario in 3D: Topological chaos for vectorfields of class C^1

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Abstract. Shilnikov's scenario in $\mathbb{R}^3$ means that the equation $x' = V(x) \in \mathbb{R}^3$ with $V(0) = 0$ has a homoclinic solution and the eigenvalues of $DV(0)$ are $u > 0$ and $\sigma \pm i\mu$ with $\sigma < 0 < \mu$ and $0 < u + \sigma$. For V once continuously differentiable we consider a flow which is equivalent to the flow of V and prove that topological chaos exists for a planar return map which describes flowline behaviour near the homoclinic orbit: For every sequence in 2 symbols $0, 1$ there are trajectories of the return map which take values in disjoint sets M_0, M_1 according to the symbol sequence. The proof is by the analysis of the action of the return map on curves and does not involve covering relations for 2-dimensional sets.

Keywords: Shilnikov's scenario, topological chaos, vectorfield, flow.

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1 Introduction

In his seminal paper [5] Shilnikov considered a differential equation

$$x'(t) = V(x(t)) \in \mathbb{R}^3 \tag{1.1}$$

with $V(0) = 0$ so that there is a homoclinic solution $h_V : \mathbb{R} \to \mathbb{R}^3$, $0 \neq h_V(t) \to 0$ for $|t| \to \infty$, and the derivative $DV(0)$ has eigenvalues $u > 0$ and $\sigma \pm i\,\mu \in \mathbb{C}$, $\sigma < 0 < \mu$, with

(H) $\qquad 0 < \sigma + u.$

Shilnikov's result in [5] is that for V analytic there exist countably many periodic orbits close to the homoclinic orbit $h_V(\mathbb{R})$. These periodic orbits arise from fixed points of a planar return map which is given by intersections of solutions with a transversal to the homoclinic orbit.

A related, stronger statement about complicated motion is conjugacy of the return map with the shift $(s_n)_{n=-\infty}^{\infty} \mapsto (s_{n+1})_{n=-\infty}^{\infty}$ in two symbols $s_n \in \{0, 1\}$. For work on the verification

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of this property see the monographs [1,2,9,10] and their references. A detailed presentation for V linear near the origin is contained in [2].

Shilnikov-type results on complicated motion close to a homoclinic orbit have also been obtained in infinite-dimensional spaces, for semiflows of solution operators which are linear close to equilibrium. See for example [4,7] on delay differential equations. These results are related to Shilnikov's scenario in $\mathbb{R}^4$ [6] with pairs of complex conjugate eigenvalues of $DV(0)$ in either halfplane.

The present paper deals with Shilnikov's scenario in $\mathbb{R}^3$ under the minimal smoothness assumption that V is once continuously differentiable – corresponding to the setting of a homoclinic solution combined with a spectral condition the statement of which does not require differentiability of higher order. Apart from this and with generalizations in mind, let us mention that for differential equations with state-dependent delay solution operators are in general not better than once continuously differentiable [3]. For a planar return map analogous to Shilnikov's in [5] we work out a proof that *topological chaos* exists, which means that for every given sequence $(s_n)_{n=-\infty}^{\infty}$, $s_n \in \{0,1\}$, there are trajectories $(x_n)_{n=-\infty}^{\infty}$ of the return map which take values in disjoint sets M_0, M_1 according to the rule $x_n \in M_{s_n}$. Notice that this is complementary to Shilnikov's result on periodic orbits [5] and weaker than conjugacy of the return map with a shift in two symbols.

The main results of the present paper are stated precisely in Section 8 below. They address return maps which are given by flowlines, as opposed to solutions of differential equations. Accordingly the subsequent sections 2–7 deal with flows and flowlines, and not with differential equations. Let us explain why. From the recent preprint [8] we know how to verify topological chaos for Shilnikov's scenario in $\mathbb{R}^3$ with the vectorfield V being twice continuously differentiable: The very first step is a transformation of V to a vectorfield whose local stable and unstable manifolds at the origin are flat, by a diffeomorphism which is twice continuously differentiable. The transformation reduces the order of smoothness for the vectorfields but preserves the smoothness of the flow F_V of Eq. (1.1). In the present situation, with V only continuously differentiable, the analogous transformation yields a vectorfield which is in general only continuous and thereby not good enough for arguments as used in [8]. The transformed flow F, however, is continuously differentiable with flat local invariant manifolds and further properties (F1)–(F5), from which we can proceed in Section 2 below. For more about the reduction by transformations see Section 9.

In Section 2 we immediately turn to scaled flows given by $F_\epsilon(t,x) = \frac{1}{\epsilon}F(t,\epsilon x)$, $\epsilon > 0$, all of which are equivalent to F. In the sequel we investigate the behaviour of these scaled flows inside and outside of a fixed neighbourhood of the origin, instead of studying F with respect to a family of shrinking neighbourhoods. As in [8] (and following Shilnikov [5]) we introduce a return map, now only for a sequence of small $\epsilon > 0$. The domain of the return map in a transversal to the homoclinic flowline is situated on one side of the flat local stable manifold, as shown at the top of Figure 1.1. Expressed in suitable coordinates the return map becomes a map from a rectangle into the plane. Section 7 shows how this map turns curves which connect certain horizontal levels in the rectangle into spirals around the origin. This suffices for the proof of Proposition 8.1 about one-directional topological chaos (with forward symbol sequences $(s_n)_{n=0}^{\infty}$). Theorem 8.2 extends the result of Proposition 8.1 to entire symbol sequences $(s_n)_{n=-\infty}^{\infty}$, by means of familiar compactness arguments.

The choice of Δ_2 in Section 7 shows that actually we obtain a countable family of sets of complicated trajectories of the return map. Another aspect which may be of interest is that the proofs of Proposition 8.1 and Theorem 8.2 do not involve covering relations for 2-dimensional

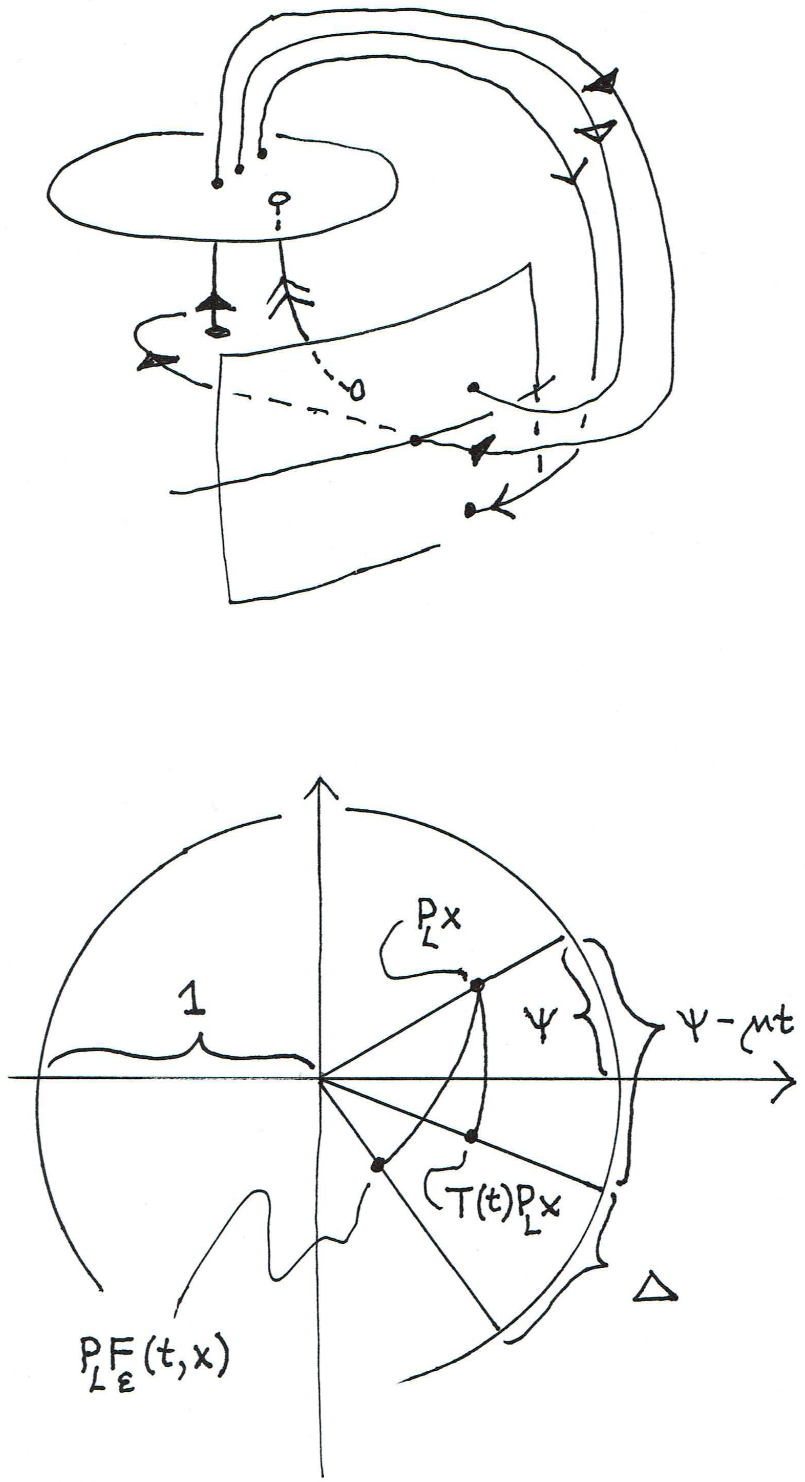

Figure 1.1: Top: The return map as a composition of the inner map with the exterior map. Bottom: Angles from Proposition 3.1.

sets.

Beyond results the emphasis in the present paper is on providing detailed proofs which may serve as a basis for future work. The content of Sections 2–6 parallels its counterpart in the preceding preprint [8], but working with the flows instead of differential equations necessitates modifications, among them in Section 3 another access to angles along projections of flowlines into the stable plane, and a rearrangement of arguments in Sections 4–6. Sections 7 and 8 are almost the same as their counterparts in the preprint [8]. We include all arguments in order to keep the paper self-contained.

What remains open, among others, is existence of periodic orbits corresponding to periodic symbol sequences. Also of interest might be a version of the present approach for Shilnikov's scenario in $\mathbb{R}^4$ [6].

Notation, preliminaries. A forward trajectory of a map $f : M \supset \text{dom} \to M$ is a sequence $(x_j)_{j=0}^{\infty}$ in dom with $x_{j+1} = f(x_j)$ for all integers $j \geq 0$. Entire trajectories are defined analogously, with all integers as indices.

For a vectorspace X, $x \in X$, and $M \subset X$, we set $x \pm M = \{y \in X : y \mp x \in M\}$. Similarly, for $A \subset \mathbb{R}$ and $x \in X$, $Ax = \{y \in X : \text{For some } a \in A, y = ax\}$.

The interior, the boundary, and the closure of a subset of a topological space are denoted by int $M, \partial M$, and cl M, respectively.

A curve is a continuous map from an interval $I \subset \mathbb{R}$ into a topological space.

Components of vectors in Euclidean spaces $\mathbb{R}^n$ are indicated by lower indices.The inner product on $\mathbb{R}^n$ is written as $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$, and we use the Euclidean norm given by $|x| = \sqrt{\langle x, x \rangle}$. The vectors of the canonical orthonormal basis on $\mathbb{R}^n$ are denoted by e_j, $j = 1, \ldots, n$, $e_{j,j} = 1$ and $e_{j,k} = 0$ for $j \neq k$. In $\mathbb{R}^3$ we write $L = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ and $U = \mathbb{R}e_3$. The associated projections $\mathbb{R}^3 \to \mathbb{R}^3$ onto L and onto U are denoted by P_L and P_U, respectively. For every $x \in \mathbb{R}^3$, $|P_U x|^2 + |P_L x|^2 = |x|^2$, and each of the projections has norm 1 in the space $L_c(\mathbb{R}^3, \mathbb{R}^3)$ of linear (continuous) maps $\mathbb{R}^3 \to \mathbb{R}^3$.

For a function $f : \mathbb{R}^n \supset \text{dom} \to \mathbb{R}^k$ on an open subset derivatives as linear maps $\mathbb{R}^n \to \mathbb{R}^k$ are denoted by $Df(x)$. For $n = k = 1$, $f'(x) = Df(x)1$. For partial derivatives in case $k = 1$, $\partial_j f(x) = Df(x)e_j$ for $j = 1, \ldots, n$.

Let $M \subset R^n$ be a continuously differentiable submanifold. For $x \in M$ the tangent space $T_x M$ is the set of tangent vectors $v = c'(0)$ of continuously differentiable curves $c : I \to \mathbb{R}^n$ with I an interval, not a singleton, $c(I) \subset M$, $0 \in I$, $c(0) = x$. A continuously differentiable map $f : M \supset \text{dom} \to N$, dom open in M and N a continuously differentiable submanifold of $\mathbb{R}^k$, is locally given by restrictions of continuously differentiable maps $g : \mathbb{R}^n \supset U \to \mathbb{R}^k$. For such U and g, and for $x \in \text{dom} \cap U$, the derivative of f at x is the linear map $T_x f : T_x M \to T_{f(x)} N$ given by $T_x f(v) = (g \circ c)'(0) = Dg(x)v$ for $v = c'(0)$ and c and g as above (with $c(I) \subset \text{dom} \cap U$).

The flow $\mathcal{F}$ generated by a vectorfield $\mathcal{V} : \mathbb{R}^n \supset \mathcal{U} \to \mathbb{R}^n$ which is locally Lipschitz continuous is the map $\mathbb{R} \times \mathbb{R}^n \supset \text{dom}_{\mathcal{F}} \to \mathbb{R}^n$ which is given by $(t, x) \in \text{dom}_{\mathcal{F}}$ if and only if t belongs to the domain of the maximal solution $y : I_x \to \mathbb{R}^n$ of the differential equation $x'(t) = \mathcal{V}(x(t))$ with initial value $y(0) = x$, and for such y, $\mathcal{F}(t, x) = y(t)$. $\mathcal{F}$ is of the same order of differentiability as $\mathcal{V}$. A subset $M \subset U$ is invariant under $\mathcal{F}$ if $x \in M$ implies $\mathcal{F}(t, x) \in M$ for all $t \in \mathbb{R}$ with $(t, x) \in \text{dom}_{\mathcal{F}}$. For a further subset $N \subset U$ the set M is called invariant under $\mathcal{F}$ in N if for every $x \in M \cap N$ and for every interval $I \ni 0$ with $\mathcal{F}(I \times \{x\}) \subset N$ we have $\mathcal{F}(I \times \{x\}) \subset M$. A flowline $\xi : \mathbb{R} \to \mathbb{R}^n$ of a flow $\mathcal{F}$ on $\text{dom}_{\mathcal{F}} = \mathbb{R} \times \mathbb{R}^n$ satisfies $\xi(t + s) = \mathcal{F}(t, \xi(s))$ for all reals s, t.

2 Transformation, scaling, projected flowlines

In the appendix Section 9 we describe how the flow F_V and the homoclinic flowline h_V can be transformed to a continuously differentiable flow $F : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ and a flowline $h : \mathbb{R} \to \mathbb{R}^3$ of F with the following properties.

(F1) For all $t \in \mathbb{R}$, $F(t, 0) = 0$.

(F2) $h(t) \neq 0 \neq h'(t)$ for all $t \in \mathbb{R}$, and $\lim_{|t|\to\infty} h(t) = 0$.

(F3) Every linear map $T(t) : \mathbb{R}^3 \ni x \mapsto D_2F(t, 0)x \in \mathbb{R}^3$, $t \in \mathbb{R}$, satisfies $T(t)L \subset L$ and $T(t)U \subset U$. For $t \in \mathbb{R}, x \in \mathbb{R}^3$, and $y = T(t)x$,

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = e^{\sigma t} \begin{pmatrix} \cos(\mu t) & \sin(\mu t) \\ -\sin(\mu t) & \cos(\mu t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad y_3 = e^{ut} x_3.$$

(F4) There exists $r_F > 0$ such that L and U are invariant under F in $\{x \in \mathbb{R}^3 : |x| < r_F\}$.

(F5) There exist reals $t_U < t_L$ with $h(t) \in U$ for all $t \leq t_U$ and $h(t) \in L$ for all $t \geq t_L$. Either $h(t) \in (0, \infty)e_3$ for all $t \leq t_U$, or $h(t) \in (-\infty, 0)e_3$ for all $t \leq t_U$.

In the sequel we focus on the case $h(t) \in (0, \infty)e_3$ for all $t \leq t_U$, the other case being analogous. We define

$$\begin{aligned} B_1 &= \{x \in \mathbb{R}^3 : |P_L x| \leq 1, |P_U x| \leq 1\}, \\ r_B &= 2 \left(\max_{0 \leq t \leq 1} |T(t)| + e^u + 2 \right), \quad \text{and} \\ B &= \{x \in \mathbb{R}^3 : |x| \leq r_B\}. \end{aligned}$$

Obviously, $B_1 \subset B$. For $\epsilon > 0$ we consider the scaled flows given by $F_\epsilon(t, x) = \frac{1}{\epsilon} F(t, \epsilon x)$. If x is a flowline of F_ϵ then ϵx is a flowline of F, and conversely, if y is a flowline of F, then $\frac{1}{\epsilon} y$ is a flowline of F_ϵ. Observe that for all $\epsilon > 0, x \in \mathbb{R}^3, y \in \mathbb{R}^3, t \in \mathbb{R}$,

$$D_2F_\epsilon(t, x)y = \frac{1}{\epsilon} D_2F(t, \epsilon x)\epsilon y = D_2F(t, \epsilon x)y.$$

Proposition 2.1.

(i) For every $\eta > 0$ there exists $\epsilon(\eta) > 0$ such that for all $\epsilon \in (0, \epsilon(\eta))$,

$$|D_2F_\epsilon(t, x) - T(t)| < \eta \quad \textit{for all } x \in B_1.$$

(ii) There exists $\epsilon_B > 0$ such that for $0 < \epsilon < \epsilon_B$,

$$F_\epsilon([0, 1] \times B_1) \subset B,$$

and L and U are invariant under F_ϵ in B.

Proof. 1. On (i). By compactness and continuity there exists $\epsilon(\eta) > 0$ such that for $0 < \epsilon < \epsilon(\eta)$ and $x \in B_1$ and for all $t \in [0, 1]$,

$$\eta \geq |D_2F(t, \epsilon x) - D_2F(t, 0)| = |D_2F_\epsilon(t, x) - T(t)|.$$

2. Let $0 \le t \le 1$. For $0 < \epsilon < \epsilon(1)$ assertion (i) yields $|D_2 F_\epsilon(t,y)| \le \max_{0\le s\le 1}|T(s)| + 1 \le r_B/2$ for all $y \in B_1$. For $x \in B_1$ we have $|x| \le 2$, and we infer

$$|F_\epsilon(t,x)| = |F_\epsilon(t,x) - 0| = |F_\epsilon(t,x) - F_\epsilon(t,0)| = \left|\int_0^1 D_2 F_\epsilon(t,sx)x\,ds\right| \le r_B|x|/2 \le r_B,$$

hence $F_\epsilon(t,x) \in B$.

3. On invariance. Let $0 < \epsilon < r_F/r_B$. Assume $x \in L$ and $F_\epsilon(s,x) \in B$ for s between 0 and $t \in \mathbb{R}$. Then we have $F(s,\epsilon x) \in \epsilon B$, or $|F(s,\epsilon x)| \le \epsilon r_B < r_F$. This yields $F(t,\epsilon x) \in L$. Consequently, $F_\epsilon(t,x) = \frac{1}{\epsilon}F(t,\epsilon x) \in \frac{1}{\epsilon}L = L$. Analogously for $x \in U$. □

The next proposition expresses closeness of the flow to its linearization at the origin in terms of components in U and L, for $0 \le t \le 1$ and $x \in B_1$.

Proposition 2.2. *For $\eta > 0$, $0 < \epsilon < \min\{\epsilon(\eta), \epsilon_B\}$, $x \in B_1$, and $0 \le t \le 1$,*

$$|P_U F_\epsilon(t,x) - T(t)P_U x)| \le \eta|P_U x| \quad \textit{and} \quad |P_L F_\epsilon(t,x) - T(t)P_L x)| \le \eta|P_L x|.$$

Moreover,

$$|P_U F_\epsilon(t,x)| \in (e^{ut} + [-\eta,\eta])|P_U x| \quad \textit{and} \quad |P_L F_\epsilon(t,x)| \in (e^{\sigma t} + [-\eta,\eta])|P_L x|.$$

Proof. Let $x \in B_1$ and $0 \le t \le 1$. We apply Proposition 2.1(ii) to $P_U x \in B_1$ and to $P_L x \in B_1$ and obtain $F_\epsilon(t,P_U x) \in U$ and $F_\epsilon(t,P_L x) \in L$, hence $P_L F_\epsilon(t,P_U x) = 0$ and $P_U F_\epsilon(t,P_L x) = 0$. It follows that

$$\begin{aligned}
|P_U F_\epsilon(t,x) - T(t)P_U x| &= |P_U F_\epsilon(t,x) - T(t)P_U x - (P_U F_\epsilon(t,P_L x) - P_U T(t)P_L x)| \\
&= \left|\int_0^1 (P_U D_2 F_\epsilon(t, P_L x + s(x - P_L x))[x - P_L x] - P_U T(t)[x - P_L x])ds\right| \\
&= \left|\int_0^1 P_U\{D_2 F_\epsilon(t, P_L x + s(x - P_L x))[P_U x] - P_U T(t)[P_U x]\})ds\right| \\
&\le |P_U| \max_{y\in B_1} |D_2 F_\epsilon(t,y) - T(t)||P_U x| \le \eta|P_U x| \\
&\quad \text{(with Proposition 2.1(i) and } |P_U| = 1).
\end{aligned}$$

We have $(P_U x)_3 = x_3$, and for $z = T(t)P_U x \in U$, $|z| = |z_3|$ with $z_3 = e^{ut}x_3$. Using the previous estimate we obtain

$$\begin{aligned}
(e^{ut} - \eta)|x_3| = |z| - \eta|x_3| &= |T(t)P_U x| - \eta|P_U x| \le |P_U F_\epsilon(t,x)| \\
&\le |T(t)P_U x| + \eta|P_U x| = |z| + \eta|x_3| = (e^{ut} + \eta)|x_3|
\end{aligned}$$

which yields

$$|P_U F_\epsilon(t,x)| \in (e^{ut} + [-\eta,\eta])|P_U x|.$$

The remaining assertions are shown analogously. □

Proposition 2.3. *Assume $0 < \eta < e^\sigma$, $e^\sigma + \eta < 1$, and $0 < \epsilon < \min\{\epsilon(\eta), \epsilon_B\}$. Let $n \in \mathbb{N}$ and $x \in B_1$ be given with $|P_U F_\epsilon(j+1,x)| \le 1$ for $j = 0, \ldots, n-1$. Then we have $F_\epsilon(j,x) \in B_1$ for $j = 0, \ldots, n$, and for $0 \le t \le 1$,*

$$\begin{aligned}
e^{\sigma(t+n)}(1 - \eta e^{-\sigma})^{n+1}|P_L x| &\le |P_L F_\epsilon(t+n,x)| \le e^{\sigma(t+n)}(1 + \eta e^{-\sigma})^{n+1}|P_L x|, \\
e^{u(t+n)}(1 - \eta)^{n+1}|P_U x| &\le |P_U F_\epsilon(t+n,x)| \le e^{u(t+n)}(1 + \eta)^{n+1}|P_U x|.
\end{aligned}$$

Proof. 1. From Proposition 2.2 for $t = 1$,

$$e^u(1-\eta)|P_U x| \le (e^u - \eta)|P_U x| \le |P_U F_\epsilon(1,x)| \le (e^u + \eta)|P_U x| \le e^u(1+\eta)|P_U x|$$

and

$$e^\sigma(1-\eta e^{-\sigma})|P_L x| = (e^\sigma - \eta)|P_L x| \le |P_L F_\epsilon(1,x)| \le (e^\sigma + \eta)|P_L x| = e^\sigma(1+\eta e^{-\sigma})|P_L x|.$$

Using $x \in B_1$ and $e^\sigma + \eta < 1$ we infer $|P_L F_\epsilon(1,x)| \le 1$. With $|P_U F_\epsilon(1,x)| \le 1$, we get $F_\epsilon(1,x) \in B_1$.

By induction we obtain $F_\epsilon(j,x) \in B_1$ for $j = 1,\dots,n$, with

$$e^{\sigma j}(1-\eta e^{-\sigma})^j|P_L x| \le |P_L F_\epsilon(j,x)| \le e^{\sigma j}(1+\eta e^{-\sigma})^j|P_L x| \quad \text{and}$$
$$e^{uj}(1-\eta)^j|P_U x| \le |P_U F_\epsilon(j,x)| \le e^{uj}(1+\eta)^j|P_U x|.$$

2. On the lower estimates of the assertion. For $j = n$ and $0 \le t \le 1$ Proposition 2.2 yields

$$(e^{ut} - \eta)|P_U F_\epsilon(n,x)| \le |P_U F_\epsilon(t+n,x)|.$$

Use $e^{ut}(1-\eta) \le e^{ut} - \eta$ and the lower estimate for $|P_U F_\epsilon(n,x)|$ in order to get

$$e^{u(t+n)}(1-\eta)^{n+1}|P_U x| \le |P_U F_\epsilon(t+n,x)|.$$

In the same way, now using $e^{\sigma t}(1-\eta e^{-\sigma}) \le e^{\sigma t} - \eta$, one finds

$$e^{\sigma(t+n)}(1-\eta e^{-\sigma})^{n+1}|P_L x| \le |P_L F_\epsilon(t+n,x)|.$$

3. The upper estimates of the assertion are shown analogously. □

For later use we turn to exponential estimates for flowlines which stay sufficiently long in B_1.

Proposition 2.4. *Let $\tilde\eta > 0$ be given. Assume $0 < \eta < e^\sigma$ and $e^\sigma + \eta < 1$ as in Proposition 2.3, and in addition*

$$\log(1+\eta e^{-\sigma}) < \tilde\eta \quad \text{and} \quad \log\left(\frac{1}{1-\eta e^{-\sigma}}\right) < \tilde\eta.$$

Let $n \in \mathbb{N}$ be given with

$$\frac{n+1}{n}\log(1+\eta e^{-\sigma}) < \tilde\eta \quad \text{and} \quad \frac{n+1}{n}\log\left(\frac{1}{1-\eta e^{-\sigma}}\right) < \tilde\eta.$$

Let $0 < \epsilon < \min\{\epsilon(\eta), \epsilon_B\}$ and consider $x \in B_1$ with $|P_U F_\epsilon(j+1,x)| \le 1$ for $j = 0,\dots,n-1$ as in Proposition 2.3. Then we have, for $0 \le t \le 1$,

$$e^{(\sigma-\tilde\eta)(t+n)}|P_L x| \le |P_L F_\epsilon(t+n,x)| \le e^{(\sigma+\tilde\eta)(t+n)}|P_L x|,$$
$$e^{(u-\tilde\eta)(t+n)}|P_U x| \le |P_U F_\epsilon(t+n,x)| \le e^{(u+\tilde\eta)(t+n)}|P_U x|.$$

Proof. In view of Proposition 2.3 the estimates of $|P_U F_\epsilon(t+n,x)|$ follow from the estimates

$$e^{(u-\tilde\eta)(t+n)} \le e^{u(t+n)}(1-\eta)^{n+1} \quad \text{and} \quad e^{u(t+n)}(1+\eta)^{n+1} \le e^{(u+\tilde\eta)(t+n)},$$

or equivalently,

$$-(t+n)\tilde{\eta} \le (n+1)\log(1-\eta) \quad \text{and} \quad (n+1)\log(1+\eta) \le (t+n)\tilde{\eta}.$$

Sufficient for the latter are

$$-n\tilde{\eta} \le (n+1)\log(1-\eta) \quad \text{and} \quad (n+1)\log(1+\eta) \le n\tilde{\eta},$$

which are a consequence of the hypotheses on η and n in combination with

$$\log\left(\frac{1}{1-\eta}\right) < \log\left(\frac{1}{1-\eta e^{-\sigma}}\right) \quad \text{and} \quad \log(1+\eta) < \log(1+\eta e^{-\sigma}).$$

Similarly the estimates of $|P_L F_\epsilon(t+n,x)|$ follow from the estimates

$$e^{(\sigma-\tilde{\eta})(t+n)} \le e^{\sigma(t+n)}(1-\eta e^{-\sigma})^{n+1} \quad \text{and} \quad e^{\sigma(t+n)}(1+\eta e^{-\sigma})^{n+1} \le e^{(\sigma+\tilde{\eta})(t+n)}$$

which are equivalent to

$$-\tilde{\eta}(t+n) \le (n+1)\log(1-\eta e^{-\sigma}) \quad \text{and} \quad (n+1)\log(1+\eta e^{-\sigma}) \le \tilde{\eta}(t+n).$$

Sufficient for the latter are

$$-\tilde{\eta}n \le (n+1)\log(1-\eta e^{-\sigma}) \quad \text{and} \quad (n+1)\log(1+\eta e^{-\sigma}) \le \tilde{\eta}n,$$

which are obvious from the hypotheses on η and n. □

3 Angles

This section deals with angles in the plane L, along projected flowlines $t \mapsto P_L F_\epsilon(t,x)$, for $x \in B_1 \setminus U$. The first step computes such angles for $0 \le t \le 1$.

Proposition 3.1. *Assume $0 < \eta < \frac{e^\sigma}{2}$, $0 < \epsilon < \min\{\epsilon(\eta), \epsilon_B\}$, $x \in B_1 \setminus U$, and*

$$\frac{1}{|P_L x|} P_L x = \begin{pmatrix} \cos(\psi) \\ \sin(\psi) \\ 0 \end{pmatrix} \quad \textit{for some } \psi \in \mathbb{R}.$$

Let $0 \le t \le 1$. For the unique $\Delta = \Delta(t,x,\psi) \in [-\pi,\pi)$ with

$$\frac{1}{|P_L F_\epsilon(t,x)|} P_L F_\epsilon(t,x) = \begin{pmatrix} \cos(\psi - \mu t + \Delta) \\ \sin(\psi - \mu t + \Delta) \\ 0 \end{pmatrix}$$

we have

$$\Delta(t,x,\psi) = \arcsin(\langle v, w^\perp \rangle) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

where

$$v = \frac{1}{|P_L F_\epsilon(t,x)|} P_L F_\epsilon(t,x), \quad w = \frac{1}{|T(t)P_L x|} T(t) P_L x, \quad \textit{and} \quad w^\perp = \begin{pmatrix} -w_2 \\ w_1 \\ 0 \end{pmatrix}.$$

In particular, $\Delta(0,x,\psi) = 0$. Moreover,

$$|\langle v, w^\perp \rangle| \le \frac{\eta}{e^\sigma - \eta} \quad \textit{and} \quad |\Delta| \le \arcsin\left(\frac{\eta}{e^\sigma - \eta}\right).$$

Compare Figure 1.1, bottom.

Proof. 1. Using $F_\epsilon(B_1 \times \{x\}) \subset B$ and invariance of U under F_ϵ in B from Proposition 2.1 (ii) we infer $F_\epsilon(t,x) \in B \setminus U$, hence $P_L F_\epsilon(t,x) \neq 0$. Also, $T(t)P_L x \neq 0$.

2. We have

$$\begin{aligned}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} &= \begin{pmatrix} \cos(\mu t) & \sin(\mu t) \\ -\sin(\mu t) & \cos(\mu t) \end{pmatrix} \cdot \begin{pmatrix} \cos(\psi) \\ \sin(\psi) \end{pmatrix} = \begin{pmatrix} \cos(\psi - \mu t) \\ \sin(\psi - \mu t) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\psi - \mu t) & -\sin(\psi - \mu t) \\ \sin(\psi - \mu t) & \cos(\psi - \mu t) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix},\end{aligned}$$

hence

$$\begin{pmatrix} -w_2 \\ w_1 \end{pmatrix} = \begin{pmatrix} \cos(\psi - \mu t) & -\sin(\psi - \mu t) \\ \sin(\psi - \mu t) & \cos(\psi - \mu t) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It follows that multiplication with the matrix

$$\begin{pmatrix} \cos(\psi - \mu t) & -\sin(\psi - \mu t) & 0 \\ \sin(\psi - \mu t) & \cos(\psi - \mu t) & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \cos(\psi - \mu t) & \sin(\psi - \mu t) & 0 \\ -\sin(\psi - \mu t) & \cos(\psi - \mu t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

defines a linear map $\rho : L \to L$ which satisfies $\rho w = e_1$ and $\rho w^\perp = e_2$. The map ρ preserves the inner product and the norm. We obtain

$$\begin{aligned}\begin{pmatrix} \cos \\ \sin \\ 0 \end{pmatrix}(\Delta) &= \begin{pmatrix} \cos(\psi - \mu t + \Delta - (\psi - \mu t)) \\ \sin(\psi - \mu t + \Delta - (\psi - \mu t)) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\psi - \mu t) & \sin(\psi - \mu t) & 0 \\ -\sin(\psi - \mu t) & \cos(\psi - \mu t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos(\psi - \mu t + \Delta) \\ \sin(\psi - \mu t + \Delta) \\ 0 \end{pmatrix} \\ &= \rho v = \langle \rho v, e_1 \rangle e_1 + \langle \rho v, e_2 \rangle e_2 = \langle \rho v, \rho w \rangle e_1 + \langle \rho v, \rho w^\perp \rangle e_2 \\ &= \langle v, w \rangle e_1 + \langle v, w^\perp \rangle e_2,\end{aligned}$$

hence $\cos(\Delta) = \langle v, w \rangle$ and $\sin(\Delta) = \langle v, w^\perp \rangle$. Let $\tilde{v} = P_L F_\epsilon(t,x) \neq 0$ and $\tilde{w} = T(t)P_L x \neq 0$. Then

$$\langle v, w \rangle = \frac{1}{|\tilde{v}||\tilde{w}|} \langle \tilde{v}, \tilde{w} \rangle.$$

Proposition 2.2 shows that $\tilde{z} = \tilde{v} - \tilde{w}$ satisfies $|\tilde{z}| \leq \eta |P_L x|$. Recall $|\tilde{w}| = e^{\sigma t}|P_L x|$. We infer

$$\langle \tilde{v}, \tilde{w} \rangle = \langle \tilde{w} + \tilde{z}, \tilde{w} \rangle \ \geq |\tilde{w}|^2 - |\tilde{z}||\tilde{w}| \geq |P_L x|^2 e^{\sigma t}(e^{\sigma t} - \eta) \geq |P_L x|^2 e^{\sigma}(e^{\sigma} - \eta) > 0,$$

which yields $\cos(\Delta) = \langle v, w \rangle > 0$. Recall $-\pi \leq \Delta < \pi$. It follows that $|\Delta| < \frac{\pi}{2}$. Consequently,

$$\Delta = \arcsin(\sin(\Delta)) = \arcsin(\langle v, w^\perp \rangle).$$

The equation $\Delta(0,x,\psi) = 0$ follows from $P_L F_\epsilon(0,x) = P_L x = T(0)P_L x$, which yields $v = w$, hence $\langle v, w^\perp \rangle = 0$.

3. Proof of the estimates of $\langle v, w^\perp \rangle$ and Δ. We have

$$|\langle v, w^\perp \rangle| = \frac{1}{|\tilde{v}||\tilde{w}|} |\langle \tilde{v}, \tilde{w}^\perp \rangle| = \frac{1}{|\tilde{v}||\tilde{w}|} |\langle \tilde{z}, \tilde{w}^\perp \rangle| \leq \frac{|\tilde{z}|}{|\tilde{v}|}.$$

Using $|\tilde{w}| = e^{\sigma t}|P_L x|$ and $|\tilde{z}| \le \eta |P_L x|$ we get

$$|\tilde{v}| \ge |\tilde{w}| - |\tilde{z}| = e^{\sigma t}|P_L x| - |\tilde{z}| \ge (e^{\sigma t} - \eta)|P_L x| \ge (e^{\sigma} - \eta)|P_L x|$$

and obtain

$$\frac{|\tilde{z}|}{|\tilde{v}|} \le \frac{\eta}{e^{\sigma} - \eta} < 1$$

where the last inequality holds by the hypothesis on η. Finally,

$$|\Delta| = |\arcsin(\langle v, w^{\perp}\rangle)| = \arcsin(|\langle v, w^{\perp}\rangle|) \le \arcsin\left(\frac{\eta}{e^{\sigma} - \eta}\right). \qquad \square$$

The next result describes the angles along projected flowlines $P_L F_\epsilon(\cdot, x)$, $x \in B_1 \setminus U$, by continuous functions, a little longer than the projected values $P_U F_\epsilon(\nu, x)$, $\nu \in \mathbb{N}$, remain in $B_1 \setminus U$. For later use we restrict attention to flowlines which start from the strip on the cylinder

$$M_I = \{x \in \mathbb{R}^3 : |P_L x| = 1\}$$

which is given by $0 < x_3 \le 1$,

$$x = \begin{pmatrix} \cos(\psi) \\ \sin(\psi) \\ \delta \end{pmatrix} \quad \text{with } \psi \in \mathbb{R} \text{ and } 0 < \delta \le 1. \tag{3.1}$$

See Figure 3.1, top.

For integers $n \ge 0$ we define

$$\mathrm{dom}_n = \big\{(t, \psi, \delta) \in [0, n+1] \times \mathbb{R} \times (0, 1] : \\ |P_U F_\epsilon(\nu, x)| \le 1 \text{ for } x \text{ given by (3.1) and } \nu = 0, \dots, n\big\}.$$

Proposition 3.2. *Assume* $0 < \eta < \frac{e^{\sigma}}{2}$, $0 < \epsilon < \min\{\epsilon(\eta), \epsilon_B\}$.

(i) For every integer $n \ge 0$ *there exists a continuous function* $\phi^{(n)} : \mathrm{dom}_n \to \mathbb{R}$ *so that for each* $(t, \psi, \delta) \in \mathrm{dom}_n$*, with* x *given by (3.1), we have*

$$\frac{1}{|P_L F_\epsilon(t, x)|} P_L F_\epsilon(t, x) = \begin{pmatrix} \cos(\phi^{(n)}(t, \psi, \delta)) \\ \sin(\phi^{(n)}(t, \psi, \delta)) \\ 0 \end{pmatrix}, \tag{3.2}$$

and in case $n \le t \le n+1$,

$$\psi - t\mu - (n+1)\arcsin\left(\frac{\eta}{e^{\sigma} - \eta}\right) \le \phi^{(n)}(t, \psi, \delta) \le \psi - t\mu + (n+1)\arcsin\left(\frac{\eta}{e^{\sigma} - \eta}\right). \tag{3.3}$$

(ii) For every $n \in \mathbb{N}$, $([0, n] \times \mathbb{R} \times (0, 1]) \cap \mathrm{dom}_n \subset \mathrm{dom}_{n-1}$*, and on this set,* $\phi^{(n)}(t, \psi, \delta) = \phi^{(n-1)}(t, \psi, \delta)$.

Proof. 1. Obviously, $([0, n] \times \mathbb{R} \times (0, 1]) \cap \mathrm{dom}_n \subset \mathrm{dom}_{n-1}$ for all integers $n \ge 1$.

2. Proof of assertion (i). We construct the functions $\phi^{(n)}$ recursively.

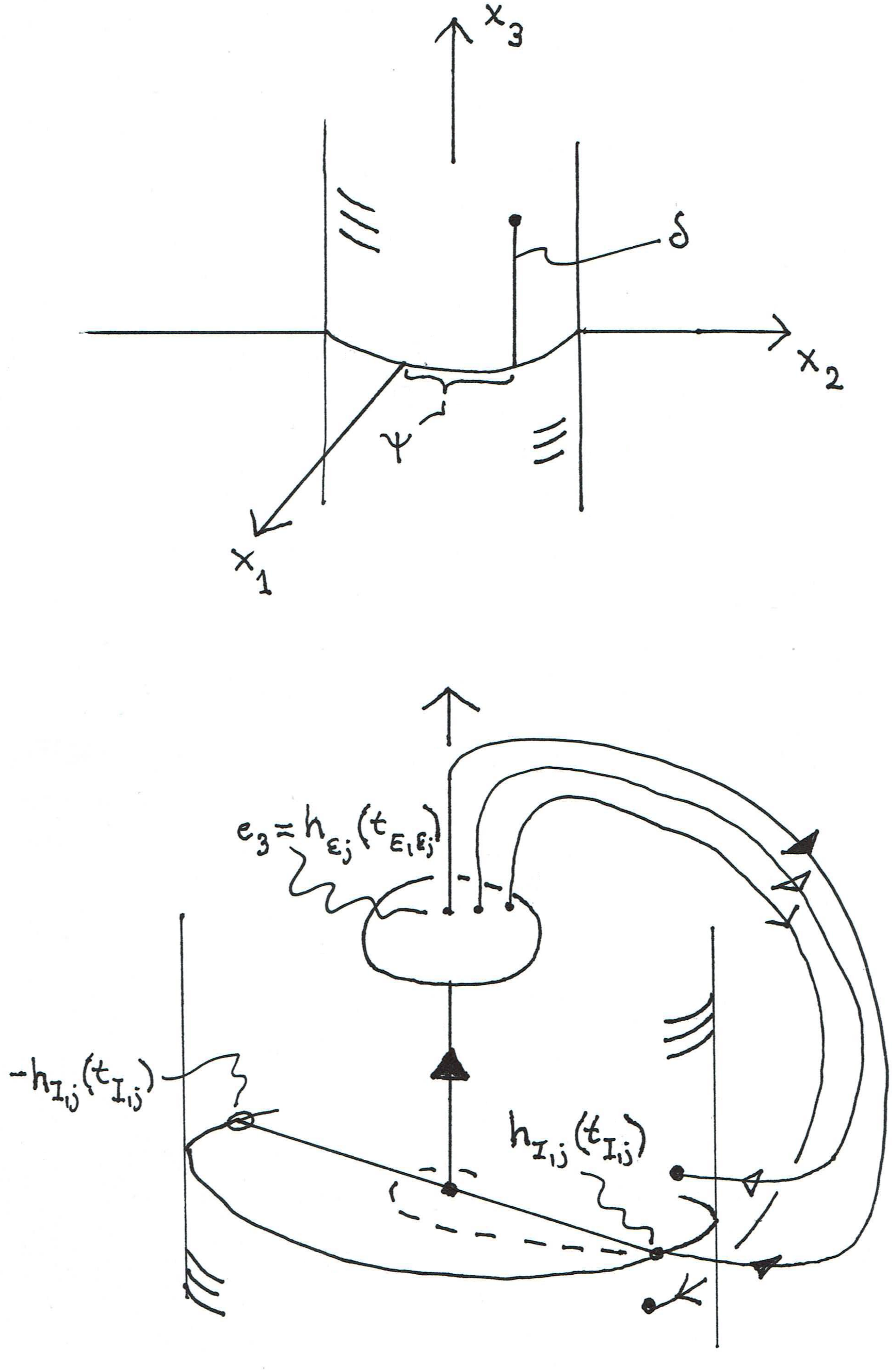

Figure 3.1: Top: The relations (3.1). Bottom: The exterior map.

2.1. For $n = 0$ Proposition 3.1 shows that the continuous function $\phi^{(0)} : \mathrm{dom}_0 \to \mathbb{R}$ defined by

$$\phi^{(0)}(t, \psi, \delta) = \psi - t\mu + \arcsin(\Delta) \quad \text{for } 0 \le t \le 1,$$

with $\Delta = \Delta(t, x, \psi)$ from Proposition 3.1, for x given by (3.1), satisfies (3.2) and (3.3).

2.2. Suppose now that for some integer $n \ge 0$ the continuous function $\phi^{(n)} : \mathrm{dom}_n \to \mathbb{R}$ satisfies (3.2) and (3.3). We define $\phi^{(n+1)} : \mathrm{dom}_{n+1} \to \mathbb{R}$ as follows.

For $(t, \psi, \delta) \in \mathrm{dom}_{n+1}$ with $0 \le t \le n+1$ we have $(t, \psi, \delta) \in \mathrm{dom}_n$ due to Part 1, and we set $\phi^{(n+1)}(t, \psi, \delta) = \phi^{(n)}(t, \psi, \delta)$.

For $(t, \psi, \delta) \in \mathrm{dom}_{n+1}$ with $n+1 \le t \le n+2$, observe first that for x given by (3.1) the property $|P_U F_\epsilon(\nu, x)| \le 1$ for $\nu = 0, \ldots, n+1$ yields $F_\epsilon(\nu, x) \in B_1$ for $\nu = 1, \ldots, n+1$, by means of Proposition 2.3. By Proposition 2.1 (ii), $F_\epsilon(s, x) \in B$ for $0 \le s \le n+2$. Using this in combination with $x \in B_1 \setminus U \subset B \setminus U$ we get $F_\epsilon(n+1, x) \in B \setminus U$, by invariance of U from Proposition 2.1 (ii). Altogether, $y = F_\epsilon(n+1, x)$ is contained in $B_1 \setminus U$, and

$$\frac{1}{|P_L y|} P_L y = \begin{pmatrix} \cos(\phi^{(n)}(n+1, \psi, \delta)) \\ \sin(\phi^{(n)}(n+1, \psi, \delta)) \\ 0 \end{pmatrix}.$$

An application of Proposition 3.1 to $y \in B_1 \setminus U$ shows that the continuous function

$$\phi^* : \mathrm{dom}_0 \to \mathbb{R}, \quad \phi^*(t, \psi, \delta) = \phi^{(n)}(n+1, \psi, \delta) - t\mu + \arcsin(\Delta) \quad \text{for } 0 \le t \le 1,$$

with $\Delta = \Delta(t, y, \phi^{(n)}(n+1, \psi, \delta))$ according to Proposition 3.1, satisfies (3.2) and (3.3) with y and $\phi^{(n)}(n+1, \psi, \delta)$ in place of x and ψ, respectively. Also,

$$\phi^{(n)}(n+1, \psi, \delta) - t\mu - \arcsin\left(\frac{\eta}{e^\sigma - \eta}\right) \le \phi^*(t, \psi, \delta) \le \phi^{(n)}(n+1, \psi, \delta) - t\mu + \arcsin\left(\frac{\eta}{e^\sigma - \eta}\right).$$

For $(t, \psi, \delta) \in \mathrm{dom}_{n+1}$ with $n+1 \le t \le n+2$ we have $(t - (n+1), \psi, \delta) \in \mathrm{dom}_0$. We complete the definition of $\phi^{(n+1)}$ by

$$\phi^{(n+1)}(t, \psi, \delta) = \phi^*(t - (n+1), \psi, \delta) \quad \text{for } (t, \psi, \delta) \in \mathrm{dom}_{n+1} \text{ with } n+1 < t \le n+2.$$

Proof that $\phi^{(n+1)}$ is continuous. Let $(t, \psi, \delta) \in \mathrm{dom}_{n+1}$ and a sequence $(t_m, \psi_m, \delta_m)_{m \in \mathbb{N}}$ in dom_{n+1} with $\lim_{m\to\infty}(t_m, \psi_m, \delta_m) = (t, \psi, \delta)$ be given. It is enough to show that for a subsequence we have $\phi^{(n+1)}(t_{m_\mu}, \psi_{m_\mu}, \delta_{m_\mu}) \to \phi^{(n+1)}(t, \psi, \delta)$ as $\mu \to \infty$. In case $t < n+1$, $(t, \psi, \delta) \in \mathrm{dom}_n$ and $\phi^{(n+1)}(t, \psi, \delta) = \phi^{(n)}(t, \psi, \delta)$, and for m sufficiently large, $t_m < n+1$, hence $(t_m, \psi_m, \delta_m) \in \mathrm{dom}_n$ and $\phi^{(n+1)}(t_m, \psi_m, \delta_m) = \phi^{(n)}(t_m, \psi_m, \delta_m)$ for such m, and continuity of $\phi^{(n)}$ yields $\phi^{(n+1)}(t_m, \psi_m, \delta_m) \to \phi^{(n+1)}(t, \psi, \delta)$ for $m \to \infty$. In case $n+1 < t \le n+2$ a similar argument with $\phi^*(\cdot - (n+1), \cdot, \cdot)$ in place of $\phi^{(n)}$ yields $\phi^{(n+1)}(t_m, \psi_m, \delta_m) \to \phi^{(n+1)}(t, \psi, \delta)$ for $m \to \infty$ as well. In case $t = n+1$ we distinguish the subcases that the set of indices m with $t_m \le n+1$ is bounded or unbounded. If it is unbounded then we can argue for a subsequence as in case $t < n+1$ above. If it is bounded we can argue as in case $n+1 < t \le n+2$ above and find

$$\begin{aligned}\lim_{m\to\infty} \phi^{(n+1)}(t_m, \psi_m, \delta_m) &= \lim_{m\to\infty} \phi^*(t_m - (n+1), \psi_m, \delta_m) = \phi^*(0, \psi, \delta) \\ &= \phi^{(n)}(n+1, \psi, \delta) = \phi^{(n+1)}(n+1, \psi, \delta).\end{aligned}$$

2.3. Proof of (3.2) for $\phi^{(n+1)}$. By the properties of $\phi^{(n)}$, for $(t, \psi, \delta) \in \mathrm{dom}_{n+1}$ with $0 \le t \le n+1$, obviously

$$\frac{1}{|P_L F_\epsilon(t,x)|} P_L F_\epsilon(t,x) = \begin{pmatrix} \cos(\phi^{(n)}(t,\psi,\delta)) \\ \sin(\phi^{(n)}(t,\psi,\delta)) \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\phi^{(n+1)}(t,\psi,\delta)) \\ \sin(\phi^{(n+1)}(t,\psi,\delta)) \\ 0 \end{pmatrix}.$$

For $(t, \psi, \delta) \in \mathrm{dom}_{n+1}$ with $n+1 < t \le n+2$ let $s = t-(n+1)$ and $y = F_\epsilon(n+1, x)$ with x given by (3.1). Using the version of (3.2) for ϕ^* we have

$$\begin{aligned} \frac{1}{|P_L F_\epsilon(t,x)|} P_L F_\epsilon(t,x) &= \frac{1}{|P_L F_\epsilon(s,y)|} P_L F_\epsilon(s,y) = \begin{pmatrix} \cos(\phi^*(s,\psi,\delta)) \\ \sin(\phi^*(s,\psi,\delta)) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\phi^*(t-(n+1),\psi,\delta)) \\ \sin(\phi^*(t-(n+1),\psi,\delta)) \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\phi^{(n+1)}(t,\psi,\delta)) \\ \sin(\phi^{(n+1)}(t,\psi,\delta)) \\ 0 \end{pmatrix}. \end{aligned}$$

2.4. Proof of (3.3) for $\phi^{(n+1)}$. For $(n+1+t, \psi, \delta) \in \mathrm{dom}_{n+1}$ with $0 < t \le 1$ we have

$$\begin{aligned} \phi^{(n+1)}(n+1+t,\psi,\delta) = \phi^*(t,\psi,\delta) &\le \phi^{(n)}(n+1,\psi,\delta) - t\mu + \arcsin\left(\frac{\eta}{e^\sigma - \eta}\right) \\ &\le \left(\psi - (n+1)\mu + (n+1)\arcsin\left(\frac{\eta}{e^\sigma-\eta}\right)\right) - t\mu + \arcsin\left(\frac{\eta}{e^\sigma-\eta}\right) \\ &\quad \text{(by (3.3) for } \phi^{(n)}) \\ &= \psi - (n+1+t)\mu + (n+2)\arcsin\left(\frac{\eta}{e^\sigma-\eta}\right). \end{aligned}$$

Analogously we get the lower estimate

$$\psi - (n+1+t)\mu - (n+2)\arcsin\left(\frac{\eta}{e^\sigma-\eta}\right) \le \phi^{(n+1)}(n+1+t,\psi,\delta)$$

for $0 < t \le 1$. By continuity both estimates hold also at $t = n+1$.

2.5. The proof of assertion (i) is complete. Assertion (ii) is obvious from the previous construction which includes the relation

$$\phi^{(n)}(t,\psi,\delta) = \phi^{(n-1)}(t,\psi,\delta)$$

for $(t,\psi,\delta) \in ([0,n] \times \mathbb{R} \times (0,1]) \cap \mathrm{dom}_n \subset \mathrm{dom}_{n-1}$. □

We proceed to estimates of $\phi^{(n)}$ from Proposition 3.2 on the interval $[n, n+1]$ in terms of affine linear maps with slopes close to $\pm\mu$.

Corollary 3.3. *Let $\tilde{\eta} > 0$ and $\eta > 0$ be given with $\eta < \frac{e^\sigma}{2}$ and*

$$\arcsin\left(\frac{\eta}{e^\sigma-\eta}\right) < \tilde{\eta}.$$

Let $0 < \epsilon < \min\{\epsilon(\eta), \epsilon_B\}$ and let $n \in \mathbb{N}$ be given with

$$\frac{n+1}{n}\arcsin\left(\frac{\eta}{e^\sigma-\eta}\right) < \tilde{\eta}.$$

Then the function $\phi^{(n)}$ obtained in Proposition 3.2 satisfies

$$\psi - (n+t)(\mu+\tilde{\eta}) \le \phi^{(n)}(n+t,\psi,\delta) \le \psi - (n+t)(\mu-\tilde{\eta})$$

for every $(n+t,\psi,\delta) \in \mathrm{dom}_n$ with $0 \le t \le 1$.

Proof. 1. The estimate (3.3) in Proposition 3.2 shows that the upper estimate of the assertion follows from

$$-(n+t)\mu + (n+1)\arcsin\left(\frac{\eta}{e^{\sigma}-\eta}\right) \le -(n+t)(\mu-\tilde{\eta})$$

which is equivalent to

$$(n+1)\arcsin\left(\frac{\eta}{e^{\sigma}-\eta}\right) \le (n+t)\tilde{\eta}.$$

The preceding estimate follows from

$$(n+1)\arcsin\left(\frac{\eta}{e^{\sigma}-\eta}\right) \le n\tilde{\eta}$$

which is obvious from the hypotheses on η and n.

2. Analogously the estimate (3.3) shows that the lower estimate of the assertion follows from

$$-(n+t)\mu - (n+1)\arcsin\left(\frac{\eta}{e^{\sigma}-\eta}\right) \ge -(n+t)(\mu+\tilde{\eta})$$

which is equivalent to

$$-(n+1)\arcsin\left(\frac{\eta}{e^{\sigma}-\eta}\right) \ge -(n+t)\tilde{\eta}.$$

The preceding estimate follows from

$$-(n+1)\arcsin\left(\frac{\eta}{e^{\sigma}-\eta}\right) \ge -n\tilde{\eta}$$

which is obvious from the hypotheses on η and n. □

4 Transversality, and exterior maps

For every $\epsilon > 0$ the flowline $h_\epsilon = \frac{1}{\epsilon}h$ of F_ϵ is homoclinic with $h_\epsilon(t) \ne 0 \ne h'_\epsilon(t)$ everywhere and $h_\epsilon(t) \to 0$ as $|t| \to \infty$.

Proposition 4.1.

(i) For $0 < \epsilon < |h(t_U)|$ there are reals $t_{E,\epsilon} \le t_U$ with

$$h_\epsilon(t_{E,\epsilon}) = e_3 \quad \text{and} \quad h_\epsilon(t) \in (0,\infty)e_3 \quad \text{for all } t \le t_{E,\epsilon}.$$

(ii) There are strictly montonic sequences $(t_{I,j})_{j\in\mathbb{N}}$ in $[t_L,\infty)$ and $(\epsilon_j)_{j\in\mathbb{N}}$ in $(0,\infty)$ with $t_{I,j} \to \infty$ and $\epsilon_j \to 0$ as $j \to \infty$ such that for every $j \in \mathbb{N}$,

$$\epsilon_j < |h(t_U)|, \quad |h_{\epsilon_j}(t_{I,j})| = 1, \quad (|h_{\epsilon_j}|^2)'(t_{I,j}) < 0, \quad h_{\epsilon_j}(t) \in L \quad \text{for } t \ge t_{I,j},\ h'_{\epsilon_j}(t_{I,j}) \in L.$$

Proof. 1. On (i). Recall that for all $t \le t_U$, $h(t) \in (0,\infty)e_3$. Let $0 < \epsilon < |h(t_U)|$. The relation $\lim_{t\to-\infty} h(t) = 0$ shows that for each $\epsilon \in (0, |h(t_U)|)$ there exists $t_{E,\epsilon} \le t_U$ with $|h(t_{E,\epsilon})| = \epsilon$. It follows that $h(t_{E,\epsilon}) = \epsilon e_3$, hence $h_\epsilon(t_{E,\epsilon}) = e_3$. For $0 < \epsilon < |h(t_U)|$ and for all $t \le t_{E,\epsilon} \le t_U$ we get $h_\epsilon(t) = \frac{1}{\epsilon}h(t) \in \frac{1}{\epsilon}(0,\infty)e_3 = (0,\infty)\epsilon$.

2. On (ii). From $h(t) \ne 0$ everywhere and $h(t) \to 0$ as $t \to \infty$ we get a strictly increasing sequence $(t_{I,j})_{j\in\mathbb{N}}$ in $[t_L,\infty)$ with $t_{I,j} \to \infty$ for $j \to \infty$ so that $|h(t_{I,j})| < |h(t_U)|$ and $(|h|^2)'(t_{I,j}) < 0$ for all $j \in \mathbb{N}$, and the sequence given by $\epsilon_j = |h(t_{I,j})|$ is strictly decreasing. It follows that

$$|h_{\epsilon_j}(t_{I,j})| = \frac{1}{\epsilon_j}|h(t_{I,j})| = 1 \quad \text{and} \quad (|h_{\epsilon_j}|^2)'(t_{I,j}) = \frac{1}{\epsilon_j^2}|(|h|^2)'(t_{I,j}) < 0 \quad \text{for all } j \in \mathbb{N}.$$

Also, for $t \ge t_{I,j} \ge t_L$, $h_{\epsilon_j}(t) = \frac{1}{\epsilon_j}h(t) \in \frac{1}{\epsilon_j}L = L$, which yields $h'_{\epsilon_j}(t_{I,j}) \in L$. $\square$

We want to describe the behaviour of flowlines close to the homoclinic loop $h_\epsilon(\mathbb{R}) \cup \{0\}$ for small $\epsilon > 0$. This will be done in terms of a return map which is given by the return of flowlines $F_\epsilon(\cdot, x)$ from points x in the cylinder

$$M_I = \{z \in \mathbb{R}^3 : |P_L z| = 1\}$$

with $0 < x_3$ slightly above the plane L, to targets in M_I. The return map will be obtained as a composition of an *inner map*, which follows the flow until it reaches the plane

$$M_E = e_3 + L,$$

parallel to L , with an *exterior map* following the flow from a neighbourhood of e_3 in M_E until it reaches M_I.

The constructions and continuous differentiability of the inner and exterior maps requires that the homoclinic flowline intersects the smooth 2-dimensional submanifolds M_E and M_I of $\mathbb{R}^3$ transversally. Obviously, $T_x M_E = L$ for all $x \in M_E$, and at $x \in M_I \cap L$,

$$T_x M_I = \mathbb{R}x^\perp \oplus \mathbb{R}e_3, \quad \text{with } x^\perp = \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}.$$

Proposition 4.2.

(i) For $0 < \epsilon < |h(t_U)|$, $\partial_1 F_\epsilon(0, e_3) = D_1 F_\epsilon(0, e_3)1 = h'_\epsilon(t_{E,\epsilon}) \notin L = T_{e_3} M_E$.

(ii) Let $j \in \mathbb{N}$ be given and $x = h_{\epsilon_j}(t_{I,j})$. Then

$$\partial_1 F_{\epsilon_j}(0,x) = h'_{\epsilon_j}(t_{I,j}) = \partial_1 F_{\epsilon_j}(t_{I,j} - t_{E,\epsilon_j}, e_3), \quad \langle \partial_1 F_{\epsilon_j}(0,x), x\rangle < 0, \quad \textit{and} \quad \partial_1 F_{\epsilon_j}(0,x) \notin T_x M_I.$$

Proof. 1. On (i). From

$$F_\epsilon(t, e_3) - F_\epsilon(0, e_3) = F_\epsilon(t, h_\epsilon(t_{E,\epsilon})) - e_3 = h_\epsilon(t + t_{E,\epsilon}) - h_\epsilon(E, t_\epsilon)$$

we have $\partial_1 F_\epsilon(0, e_3) = h'_\epsilon(t_{E,\epsilon}) \ne 0$. Using $h_\epsilon(t) \in U$ for $t \le t_{E,\epsilon}$ we get $h'_\epsilon(t_{E,\epsilon}) \in U$. It follows that $\partial_1 F_\epsilon(0, e_3) \in U \setminus \{0\} \subset \mathbb{R}^3 \setminus L$.

2. On (ii). Let $j \in \mathbb{N}$ be given. Let $x = h_{\epsilon_j}(t_{I,j}) \in M_I \cap L$. Arguing as in Part 1 we get $\partial_1 F_{\epsilon_j}(0,x) = h'_{\epsilon_j}(t_{I,j}) = \partial_1 F_{\epsilon_j}(t_{I,j} - t_{E,\epsilon_j}, e_3)$. From Proposition 4.1 (ii),

$$0 > (|h_{\epsilon_j}|^2)'(t_{I,j}) = 2\langle h'_{\epsilon_j}(t_{I,j}), h_{\epsilon_j}(t_{I,j})\rangle = 2\langle \partial_1 F_{\epsilon_j}(0,x), x\rangle$$

and $h'_{\epsilon_j}(t_{I,j}) \in L$. From the preceding relations, $\partial_1 F_{\epsilon_j}(0,x) \notin \mathbb{R}x^\perp \oplus \mathbb{R}e_3 = T_x M_I$. $\square$

For $r > 0$ let
$$M_E(r) = \{y \in M_E : |y - e_3| < r\}.$$

Corollary 4.3. *There is a decreasing sequence $(r_j)_{j\in\mathbb{N}}$ with $r_j \to 0$ as $j \to \infty$ so that for every $j \in \mathbb{N}$ there exists a continuously differentiable map*
$$t_j : M_E(r_j) \to (0, \infty)$$
with $t_j(e_3) = t_{I,j} - t_{E,\epsilon_j}$, $F_{\epsilon_j}(t_j(y), y) \in \{z \in M_I : P_L z \neq -h_{\epsilon_j}(t_{I,j})\}$ for all $y \in M_E(r_j)$, and
$$\partial_1 F_{\epsilon_j}(0, y) \notin L \quad \textit{for all } y \in M_E(r_j).$$

Proof. Let $j \in \mathbb{N}$ be given. With the function $G : \mathbb{R}^3 \to \mathbb{R}$, $G(z) = z_1^2 + z_2^2 - 1$, the relation $F_{\epsilon_j}(t, y) \in M_I$ is equivalent to the equation $G(F_{\epsilon_j}(t, y)) = 0$. Using Proposition 4.2 (ii) with $x = F_{\epsilon_j}(t_{I,j} - t_{E,\epsilon_j}, e_3)$ we obtain
$$\partial_1(G \circ F_{\epsilon_j}(t_{I,j} - t_{E,\epsilon_j}, e_3) = 2\langle \partial_1 F_{\epsilon_j}(t_{I,j} - t_{E,\epsilon_j}, e_3), F_{\epsilon_j}(t_{I,j} - t_{E,\epsilon_j}, e_3)\rangle = 2\langle \partial_1 F_{\epsilon_j}(0, x), x)\rangle < 0.$$

Therefore the Implicit Function Theorem applies and yields a continuously differentiable positive function t_j^* on a neighbourhood N_j of e_3 in $\mathbb{R}^3$ which satisfies $t_j^*(e_3) = t_{I,j} - t_{E,\epsilon_j}$ and $F_{\epsilon_j}(t_j^*(y), y) \in M_I$ for all $y \in N_j$. Given any real $r > 0$ it follows from Proposition 4.2 (i) and continuity that there exists $r_j \in (0, r)$ with $M_E(r_j) \subset N_j$ and $\partial_1 F_{\epsilon_j}(0, y) \notin L$ for all $y \in M_E(r_j)$. Using $F_{\epsilon_j}(t_j^*(e_3), e_3) = h_{\epsilon_j}(t_{I,j}) \in L$ and continuity we also achieve $P_L F_{\epsilon_j}(t_j^*(y), y) \neq -h_{\epsilon_j}(t_{I,j}))$ for all $y \in M_E(r_j)$. Let t_j be the restriction of t_j^* to $M_E(r_j)$.

The desired decreasing sequence $(r_j)_{j\in\mathbb{N}}$ can be obtained recursively. □

For every $j \in \mathbb{N}$ the *exterior map*
$$E_j : M_E(r_j) \to \{z \in M_I : P_L z \neq -h_{\epsilon_j}(t_{I,j})\}, \quad E_j(y) = F_{\epsilon_j}(t_j(y), y),$$
into the open subset $\{z \in M_I : P_L z \neq -h_{\epsilon_j}(t_{I,j})\} = \{z \in M_I : P_L z \neq -E_j(e_3))\}$ of the manifold M_I is continuously differentiable. Compare Figure 3.1, bottom.

Corollary 4.4. *Let $j \in \mathbb{N}$ be given and $x = E_j(e_3)$. Then $T_{e_3} E_j(T_{e_3} M_E) = T_x M_I$.*

Proof. Let $v = \partial_1 F_{\epsilon_j}(0, e_3)$ and $w = h'_{\epsilon_j}(t_{I,j}) = \partial_1 F_{\epsilon_j}(t_j(e_3), e_3) = \partial F_{\epsilon_j}(0, x)$. Then $v \notin L = T_{e_3} M_E$ and $w \notin T_x M_I$. The map $T_{e_3} E_j : T_{e_3} M_E \to T_x M_I$ is given by
$$T_{e_3} E_j(z) = P_j D_2 F_{\epsilon_j}(t_j(e_3), e_3) z$$
with the projection $P_j : \mathbb{R}^3 \to \mathbb{R}^3$ along $\mathbb{R}w$ onto $T_x M_I$. The isomorphism $D_2 F_{\epsilon_j}(t_j(e_3), e_3)$ sends v to w and maps $T_{e_3} M_E$ onto a 2-dimensional space Q which is complementary to $\mathbb{R}w$. The projection P_j maps the complementary space Q onto the complementary space $T_x M_I$. □

5 Inner maps

We begin with the travel time from the strip $\{x \in M_I : 0 < x_3 < 1\}$ to the plane M_E.

Proposition 5.1. *Assume $0 < \eta < e^\sigma$, $e^\sigma + \eta < 1$, and $0 < \epsilon < \min\{\epsilon(\eta), \epsilon_B\}$. Let $x \in M_I$ with $0 < x_3 < 1$ be given. Then there exists $t = \tau_\epsilon(x) > 0$ with $0 < F_{\epsilon,3}(s, x) < 1$ for $0 \leq s < t$ and $F_{\epsilon,3}(t, x) = 1$.*

Proof. 1. We show $|F_{\epsilon,3}(s,x)| > 1$ for some $s > 0$. Suppose this is false. Then Proposition 2.3 yields $F_\epsilon(n,x) \in B_1$ for all integers $n \geq 0$. Using an estimate from Proposition 2.3 and $1-\eta > e^\sigma$ we get

$$|F_{\epsilon,3}(n,x)| = |P_U F_\epsilon(n,x)| \geq e^{un}(1-\eta)^{n+1}|P_U x| \geq e^{(u+\sigma)n}e^\sigma|x_3| > 0$$

for all integers $n \geq 0$, and the hypothesis $u+\sigma > 0$ yields a contradiction to the assumption that $|F_{\epsilon,3}(s,x)|$, $s \geq 0$, is bounded.

2. It follows that

$$0 < \inf\{s \geq 0 : |F_{\epsilon,3}(s,x)| \geq 1\} < \infty.$$

Set $t = \tau_\epsilon(x) = \inf\{s \geq 0 : |F_{\epsilon,3}(s,x)| \geq 1\}$. Then $|F_{\epsilon,3}(s,x)| < 1$ for $0 \leq s < t$, and by continuity, $|F_{\epsilon,3}(t,x)| = 1$. Let $n = n_\epsilon(x)$ denote the largest integer in $[0,t]$. Proposition 2.3 yields $F_\epsilon(j,x) \in B_1$ for $j = 0,\ldots,n$. By $0 < x_3$, $x \in B_1 \setminus L$. Using Proposition 2.1 (ii) we infer $F_\epsilon(s,x) \in B \setminus L$ on $[0,t]$, hence $|F_{\epsilon,3}(s,x)| > 0$ on $[0,t]$. Finally, $0 < x_3$ and continuity combined yield $0 < F_{\epsilon,3}(s,x)$ on $[0,t]$. □

In order to use the travel time $\tau_\epsilon(x)$ of Proposition 5.1 for an *inner map* with values $F_\epsilon(\tau_\epsilon(x),x)$ in the domain $M_E(r_j)$ of an exterior map E_j we observe that due to hyperbolic behavior of flowlines close to the stationary point 0 the relation $F_\epsilon(\tau_\epsilon(x),x) \in M_E(r_j)$ should hold for $x \in M_I$ with $0 < x_3$ sufficiently small. Recall ϵ_j from Proposition 4.1 and r_j from Corollary 4.3.

Proposition 5.2. *Assume $0 < \eta < e^\sigma$ and $1+\eta e^{-\sigma} < e^{-\sigma/2}$. Consider an integer j so large that*

$$\epsilon_j < \min\{\epsilon(\eta), \epsilon_B\}.$$

For every $x \in M_I$ with $0 < x_3 < 1$ the largest integer $n = n_j(x)$ in $[0, \tau_{\epsilon_j}(x))$ satisfies

$$n > \frac{1}{u+\log(2)}\log\left(\frac{1}{x_3}\right) - 1. \tag{5.1}$$

For $\delta_j^* \in (0,1)$ so small that

$$\left(1 + \frac{2}{\sigma}\log(r_j)\right) < \frac{1}{u+\log(2)}\log\left(\frac{1}{\delta_j^*}\right) - 1. \tag{5.2}$$

we have

$$|F_{\epsilon_j}(\tau_{\epsilon_j}(x),x) - e_3| = |P_L F_{\epsilon_j}(\tau_{\epsilon_j}(x),x)| < r_j \quad \textit{for all } x \in M_I \textit{ with } 0 < x_3 < \delta_j^*.$$

Proof. 1. On (5.1). Notice that we have $e^\sigma + \eta < 1$, so that the hypothesis concerning $\eta > 0$ in Proposition 5.1 is satisfied. Let $x \in M_I$ with $0 < x_3 < 1$ be given and let $t = \tau_{\epsilon_j}(x)$. We derive an estimate of the largest integer $n = n_j(x)$ in $[0,t)$. Proposition 5.1 in combination with Proposition 2.3, both for $\epsilon = \epsilon_j$, yield

$$\begin{aligned} 1 = |P_U F_{\epsilon_j}(t,x)| &\leq e^{ut}(1+\eta)^{n+1}|P_U x| = e^{ut}(1+\eta)^{n+1}x_3 \\ &< (2e^u)^{n+1}x_3 \quad (\text{with } t \leq n+1 \text{ and } \eta < 1), \end{aligned}$$

hence

$$n > \frac{1}{u+\log(2)}\log\left(\frac{1}{x_3}\right) - 1.$$

2. Now consider $\delta_j^* \in (0,1)$ which satisfies the inequality (5.1), and let $x \in M_I$ with $0 < x_3 < \delta_j^*$ be given. Using $|P_L x| = 1$ from $x \in M_I$ and an upper estimate from Proposition 2.3 we have, with $n = n_j(x)$,

$$|P_L F_{\epsilon_j}(t,x)| \le e^{\sigma t}(1+\eta e^{-\sigma})^{n+1}|P_L x| \le e^{\sigma \cdot n} e^{(-\sigma/2)(n+1)} = e^{(\sigma/2)(n-1)}.$$

Thereby the desired inequality $|P_L F_{\epsilon_j}(t,x)| < r_j$ follows from $e^{(\sigma/2)(n-1)} < r_j$ which is equivalent to

$$n > 1 + \frac{2}{\sigma}\log(r_j).$$

The preceding equation follows from the lower estimate (5.1) of $n = n_j(x)$ in Part 1 in combination with $x_3 < \delta_j^*$ and with the smallness assumption (5.2) on δ_j^*. □

Next we use transversality of the flow at points of $M_E(r_j)$ as prepared in Corollary 4.3 in order to obtain smoothness of the travel time.

Corollary 5.3. *Let η, j, and δ_j^* satisfy the hypotheses of Proposition 5.2. Then the map*

$$\tau_j^* : \{x \in M_I : 0 < x_3 < \delta_j^*\} \to (0,\infty), \quad \tau_j^*(x) = \tau_{\epsilon_j}(x),$$

is continuously differentiable.

Proof. 1. Continuity. Let $x \in M_I$ with $0 < x_3 < \delta_j^*$ be given, and let $t = \tau_j^*(x) = \tau_{\epsilon_j}(x)$. Let $y = F_{\epsilon_j}(t,x)$ and observe that because of $F_{\epsilon_j,3}(s,x) < 1 = F_{\epsilon_j,3}(t,x)$ for $0 \le s < t$ we have $\partial_1 F_{\epsilon_j,3}(0,y) = \partial_1 F_{\epsilon_j,3}(t,x) \ge 0$. Recall $y \in M_E(r_j)$ from Proposition 5.2. The relation $\partial_1 F_{\epsilon_j}(0,y) \notin L$ from Corollary 4.3 yields $\partial_1 F_{\epsilon_j,3}(0,y) \ne 0$. We conclude that $\partial_1 F_{\epsilon_j,3}(t,x) = \partial_1 F_{\epsilon_j,3}(0,y) > 0$. Now let $\rho \in (0,t)$ be given. Then for some $s \in (t, t+\rho)$, $F_{\epsilon_j,3}(s,x) > 1$. By continuity there is a neighbourhood N_1 of x in $\mathbb{R}^3$ with $F_{\epsilon_j,3}(s,z) > 1$ for all $z \in N_1$. By continuity and compactness we also find a neighbourhood $N \subset N_1$ of x in $\mathbb{R}^3$ so that for all $z \in N$ and for all $s \in [0, t-\rho]$ we have $0 < F_{\epsilon_j,3}(s,z) < 1$. It follows that for all $z \in N \cap \{\xi \in M_I : 0 < \xi_3 < \delta_j^*\}$ we have $t - \rho < \tau_{\epsilon_j}(z) < s < t + \rho$, which yields continuity of the map τ_j^* at x.

2. We show that locally the map τ_j^* is given by continuously differentiable maps. Let $x \in M_I$ with $0 < x_3 < \delta_j^*$ be given and let $t = \tau_j^*(x)$. Then $F_{\epsilon_j,3}(t,x) = 1$, and $\partial_1 F_{\epsilon_j,3}(t,x) > 0$, see Part 1. The Implicit Function Theorem yields an open neighbourhood N of x in $\mathbb{R}^3$ and $\rho > 0$ and a continuously differentiable map $\tau : N \to (t-\rho, t+\rho)$ with $F_{\epsilon_j,3}(\tau(z),z) = 1$ for all $z \in N$, and on $(t-\rho, t+\rho) \times N$,

$$F_{\epsilon_j,3}(s,z) = 1 \quad \text{if and only if} \quad s = \tau(z).$$

By continuity according to Part 1, there is an open neighbourhood $N_1 \subset N$ of x in $\mathbb{R}^3$ so that for all $z \in N_1 \cap \{\xi \in M_I : 0 < \xi_3 < \delta_j^*\}$ we have $t - \rho < \tau_j^*(z) < t + \rho$. Recall $F_{\epsilon_j,3}(\tau_j^*(z),z) = F_{\epsilon_j,3}(\tau_{\epsilon_j}(z),z) = 1$ for all $z \in M_I$ with $0 < z_3 < \delta_j^*$. It follows that on $N_1 \cap \{\xi \in M_I : 0 < \xi_3 < \delta_j^*\}$ we have $\tau_j^*(z) = \tau(z)$. The restriction of τ to $N_1 \cap \{\xi \in M_I : 0 < \xi_3 < \delta_j^*\}$ is a continuously differentiable function on the open subset $N_1 \cap \{\xi \in M_I : 0 < \xi_3 < \delta_j^*\}$ of the submanifold M_I. □

In the sequel we arrange for an inner map on a subset of $\{x \in M_I : 0 < x_3 < \delta_j^*\}$ which can be estimated in the same way as its counterpart in [8]. Upon that we will be able to follow [8] in proving existence of chaotic motion.

For $0 < \eta < \frac{e^\sigma}{2}$ we abbreviate

$$m(\eta,\sigma) = \max\left\{\log(1+\eta e^{-\sigma}), \log\left(\frac{1}{1-\eta e^{-\sigma}}\right), \arcsin\left(\frac{\eta}{e^\sigma - \eta}\right)\right\}$$

and consider the following hypotheses.

For $\tilde{\eta} > 0$ given,

$$0 < \eta < \frac{e^\sigma}{2} \quad \text{and} \quad 1 + \eta e^{-\sigma} < e^{-\sigma/2} \tag{5.3}$$

(observe that this yields $\eta < e^\sigma$ and $e^\sigma + \eta < e^{\sigma/2} < 1$) and

$$m(\eta,\sigma) < \tilde{\eta}. \tag{5.4}$$

$j \in \mathbb{N}$ is so large that

$$\epsilon_j < \min\{\epsilon(\eta), \epsilon_B\}. \tag{5.5}$$

$\delta_j^* \in (0,1)$ satisfies (5.2) and $\delta_j \in (0, \delta_j^*)$ is so small that

$$\frac{1}{u + \log(2)} \log\left(\frac{1}{\delta_j}\right) - 1 > \frac{m(\eta,\sigma)}{\tilde{\eta} - m(\eta,\sigma)}. \tag{5.6}$$

Proposition 5.4. *Let $\tilde{\eta} > 0$ be given. Assume that the relations (5.3)–(5.6) hold for $\eta > 0$, $j \in \mathbb{N}$, and δ_j. For every $x \in M_I$ with $0 < x_3 < \delta_j$ the largest integer $n = n_j(x)$ in $[0, \tau_j^*(x))$ satisfies*

$$\frac{n+1}{n} m(\eta,\sigma) < \tilde{\eta}.$$

Proof. Using the relations (5.1) and (5.6) we get

$$n > \frac{1}{u+\log(2)} \log\left(\frac{1}{x_3}\right) - 1 > \frac{1}{u+\log(2)} \log\left(\frac{1}{\delta_j}\right) - 1 > \frac{m(\eta,\sigma)}{\tilde{\eta} - m(\eta,\sigma)}$$

which yields $\frac{n+1}{n} m(\eta,\sigma) < \tilde{\eta}$. □

We are ready for the definition of the inner map I_j, given $\tilde{\eta} > 0$ and $\eta > 0$, $j \in \mathbb{N}$, and δ_j which satisfy the relations (5.3)–(5.6). As a domain for I_j we take

$$M_{I,j} = \{x \in M_I : 0 < x_3 < \delta_j \quad \text{and} \quad P_L x \neq -E_j(e_3)\}.$$

Here the line given by $P_L x = -E_j(e_3) = -h_{\epsilon_j}(t_{I,j})$ is excluded for later use, in order to have a global parametrization of $M_{I,j}$ available. With $\tau_j(x) = \tau_j^*(x)$ on $M_{I,j}$ we set

$$I_j(x) = F_{\epsilon_j}(\tau_j(x), x)$$

and obtain a continuously differentiable map $I_j : M_{I,j} \to M_E$, with values in $M_E(r_j)$ according to Proposition 5.1. Compare Figure 5.1, top.

Corollary 5.5. *Let $\tilde{\eta} > 0$ and assume that $\eta > 0$, $j \in \mathbb{N}$, and δ_j satisfy the relations (5.3)–(5.6). Let $x \in M_{I,j}$. Then we have*

$$e^{(\sigma - \tilde{\eta})\tau_j(x)} \le |P_L I_j(x)| \le e^{(\sigma + \tilde{\eta})\tau_j(x)},$$
$$e^{(u - \tilde{\eta})\tau_j(x)} x_3 \le 1 \le e^{(u + \tilde{\eta})\tau_j(x)} x_3.$$

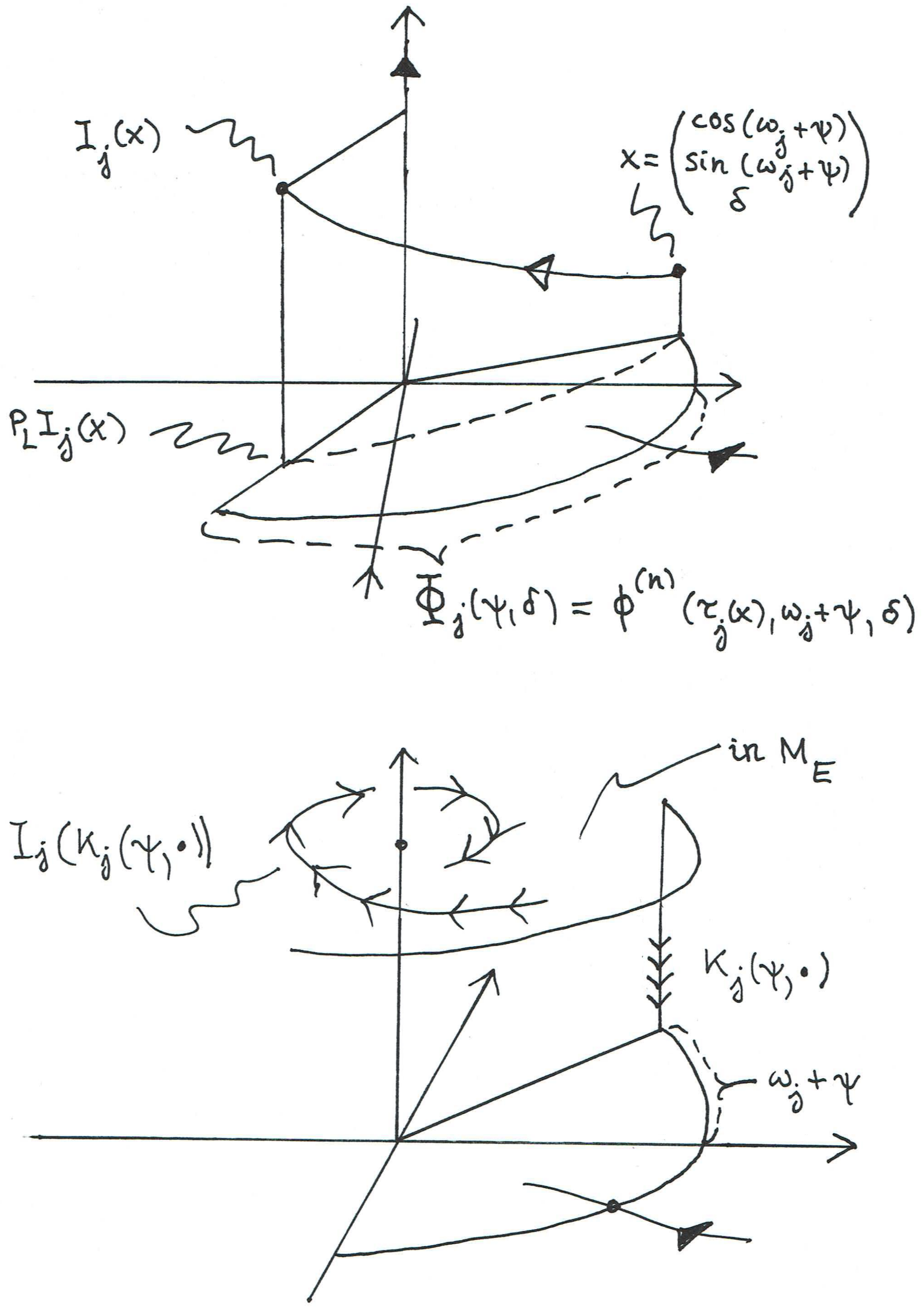

Figure 5.1: Top: The inner map and related angles. Bottom: The inner map along vertical line segments.

For $\psi \in \mathbb{R}$ with

$$x = \begin{pmatrix} \cos(\psi) \\ \sin(\psi) \\ x_3 \end{pmatrix}$$

and $\delta = x_3$ the function $\phi^{(n)}$ obtained in Proposition 3.2, with $n = n_j(x)$, satisfies

$$\psi - \tau_j(x)(\mu + \tilde{\eta}) \le \phi^{(n)}(\tau_j(x), \psi, \delta) \le \psi - \tau_j(x)(\mu - \tilde{\eta}).$$

Proof. Proposition 5.4 shows that the hypotheses on the integer n in Proposition 2.4 and Corollary 3.3 are satisfied for $n = n_j(x)$. Apply Proposition 2.4 to $t = \tau_j(x) - n_j(x) \in [0,1]$, with $|P_L x| = 1$ and $|P_U x| = x_3$ and $P_U I_j(x) = e_3$. This yields the estimates of $P_L I_j(x)$ and $P_U I_j(x)$.

From $|P_L x| = 1$, $x \in B_1 \setminus U$. An application of Corollary 3.3 to $t = \tau_j(x) - n_j(x) \in [0,1]$ yields the estimate of the angle function $\phi^{(n)}$ with $n = n_j(x)$. □

Corollary 5.6. *Let $\tilde{\eta} > 0$ be given with $\tilde{\eta} < u$ and assume that $\eta > 0$, $j \in \mathbb{N}$, and δ_j satisfy the relations (5.3)–(5.6). For every $x \in M_{I,j}$,*

$$\frac{1}{u + \tilde{\eta}} \log\left(\frac{1}{x_3}\right) \le \tau_j(x) \le \frac{1}{u - \tilde{\eta}} \log\left(\frac{1}{x_3}\right).$$

6 The return map in the plane

We begin with parametrizations of the open subsets $\{x \in M_I : P_L x \neq -E_j(e_3)\}$ of the submanifold M_I. Let $j \in \mathbb{N}$ be given. Consider $\omega_j \in [-\pi, \pi)$ determined by

$$\begin{pmatrix} \cos(\omega_j) \\ \sin(\omega_j) \\ 0 \end{pmatrix} = E_j(e_3) \quad (= h_{\epsilon_j}(t_{I,j})).$$

The map

$$K_j : (-\pi, \pi) \times \mathbb{R} \to \{x \in M_I : P_L x \neq -E_j(e_3)\}, \quad K_j(\psi, \delta) = \begin{pmatrix} \cos(\omega_j + \psi) \\ \sin(\omega_j + \psi) \\ \delta \end{pmatrix},$$

is a continuously differentiable diffeomorphism with $K_j(0,0) = E_j(e_3)$. The return map $R_j = E_j \circ I_j$ sends its domain $M_{I,j} = \{x \in M_I : 0 < x_3 < \delta_j, P_L x \neq -E_j(e_3)\}$ into the set $\{z \in M_I : P_L z \neq -E_j(e_3)\}$ which equals the image $K_j((-\pi, \pi) \times \mathbb{R})$. By the *return map in the plane* we mean the continuously differentiable map

$$Q_j : (-\pi, \pi) \times (0, \delta_j) \to (-\pi, \pi) \times \mathbb{R}$$

given by $Q_j(\psi, \delta) = K_j^{-1}(R_j(K_j(\psi, \delta)))$.

We also need information about a coordinate representation of the exterior map E_j alone. Corollary 4.4 yields that the derivative $T_{e_3}(K_j^{-1} \circ E_j)$ is an isomorphism from $L = T_{e_3} M_E$ onto the plane $\mathbb{R}^2$. So it sends basis vectors v_j and w_j of L to the vectors $\binom{1}{0}$ and $\binom{0}{1}$, respectively.

Let κ_j denote the isomorphism $L \to \mathbb{R}^2$ given by

$$\kappa_j(\xi v_j + \eta w_j) = \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

The restriction P_j of $\kappa_j \circ P_L$ to the open subset $M_E(r_j)$ of the submanifold M_E defines a continuously differentiable diffeomorphism onto an open neighbourhood of $0 = P_j(e_3)$ in $\mathbb{R}^2$. Obviously,

$$P_j(e_3 + \xi v_j + \eta w_j) = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad \text{for all reals } \xi, \eta.$$

As $T_{e_3} P_j$ and $T_{e_3}(K_j^{-1} \circ E_j)$ act the same on the basis v_j, w_j of $T_{e_3} M_E = L$ the *exterior map in coordinates*

$$K_j^{-1} \circ E_j \circ P_j^{-1} : \mathbb{R}^2 \supset P_j(M_E(r_j)) \to \mathbb{R}^2$$

satisfies

$$D(K_j^{-1} \circ E_j \circ P_j^{-1})(0) = \mathrm{id}_{\mathbb{R}^2}. \tag{6.1}$$

Corollary 6.1. *Let $j \in \mathbb{N}$ and $\beta > 0$ be given. There exists $\alpha_j = \alpha_j(\beta) \in (0, \pi)$ so that for all $(\psi, \delta) \in [-\alpha_j, \alpha_j] \times [-\alpha_j, \alpha_j]$ we have $(\psi, \delta) \in P_j(M_E(r_j))$, and*

$$|(K_j^{-1} \circ E_j \circ P_j^{-1})(\psi, \delta) - (\psi, \delta)| \le \beta |(\psi, \delta)|, \tag{6.2}$$

$$|D(K_j^{-1} \circ E_j \circ P_j^{-1})(\psi, \delta) - id_{\mathbb{R}^2}| \le \beta. \tag{6.3}$$

We proceed to estimates of the range of the *inner map in coordinates*

$$(-\pi, \pi) \times (0, \delta_j) \ni (\psi, \delta) \mapsto P_j(I_j(K_j(\psi, \delta))) \in \mathbb{R}^2$$

and of the return map in the plane Q_j.

Proposition 6.2. *Assume $0 < \tilde{\eta} < -\sigma/2$ and consider $\eta > 0, j \in \mathbb{N}, \delta_j$ so that the relations (5.3)–(5.6) hold. Let $\beta \in (0, \frac{1}{2}]$ be given. Consider $\alpha_j = \alpha_j(\beta) > 0$ according to Corollary 6.1, with $\alpha_j < \delta_j$. Then*

$$\delta_{\beta,j} = \left(\frac{2}{3(|\kappa_j| + 1)} \alpha_j \right)^{\frac{3u}{-\sigma}}$$

satisfies $\delta_{\beta,j} \le \frac{2}{3}\alpha_j$, and for all $(\psi, \delta) \in [-\alpha_j, \alpha_j] \times (0, \delta_{\beta,j}]$ we have

$$|P_j(I_j(K_j(\psi, \delta)))| \le \frac{2}{3}\alpha_j, \tag{6.4}$$

$$Q_j(\psi, \delta) \in [-\alpha_j, \alpha_j] \times [-\alpha_j, \alpha_j]. \tag{6.5}$$

Proof. 1. In order to show $\delta_{\beta,j} \le \frac{2}{3}\alpha_j$ notice first that

$$\delta_{\beta,j} < \alpha_j < \delta_j < 1.$$

The hypothesis on $\tilde{\eta}$ yields

$$\frac{u + \tilde{\eta}}{-\sigma - \tilde{\eta}} < \frac{\frac{3u}{2}}{\frac{-\sigma}{2}} = \frac{3u}{-\sigma}.$$

It follows that

$$\delta_{\beta,j} = \left(\frac{2}{3(|\kappa_j| + 1)} \alpha_j \right)^{\frac{3u}{-\sigma}} \le \left(\frac{2}{3(|\kappa_j| + 1)} \alpha_j \right)^{\frac{u+\tilde{\eta}}{-\sigma-\tilde{\eta}}},$$

hence

$$\delta_{\beta,j}^{\frac{-\sigma-\tilde{\eta}}{u+\tilde{\eta}}} \le \frac{2}{3(|\kappa_j| + 1)} \alpha_j.$$

Consequently, with $0 < -\sigma - \tilde{\eta} < u + \tilde{\eta}$ and $\delta_{\beta,j} < 1$,

$$\delta_{\beta,j} \leq \delta_{\beta,j}^{\frac{-\sigma-\tilde{\eta}}{u+\tilde{\eta}}} \leq \frac{2}{3(|\kappa_j|+1)}\alpha_j \leq \frac{2}{3}\alpha_j. \tag{6.6}$$

2. From Corollary 6.1 with $0 < \beta \leq 1/2$ we get $\alpha_j < \delta_j$ so that for all $(\tilde{\psi}, \tilde{\delta}) \in [-\alpha_j, \alpha_j] \times [-\alpha_j, \alpha_j]$, we have

$$|(K_j^{-1} \circ E_j \circ P_j^{-1})(\tilde{\psi}, \tilde{\delta})| \leq (1+\beta)|(\tilde{\psi}, \tilde{\delta})| \leq \frac{3}{2}|(\tilde{\psi}, \tilde{\delta})|.$$

3. Proof of (6.4). For $(\psi, \delta) \in [-\alpha_j, \alpha_j] \times (0, \delta_{\beta,j}]$ we have $K_{j,3}(\psi, \delta) = \delta \in (0, \delta_{\beta,j}] \subset (0, \delta_j)$. With $x = K_j(\psi, \delta)$, Corollary 5.5 yields

$$|P_j(I_j(K_j(\psi, \delta)))| = |\kappa_j P_L I_j(x)| \leq |\kappa_j| e^{(\sigma+\tilde{\eta})\tau_j(x)}.$$

Notice that $\tilde{\eta} < u$. Using the lower estimate of $\tau_j(x)$ from Corollary 5.6 we infer

$$\begin{aligned} |P_j(I_j(K_j(\psi, \delta)))| &\leq |\kappa_j| \left(\frac{1}{x_3}\right)^{\frac{\sigma+\tilde{\eta}}{u+\tilde{\eta}}} \\ &\leq |\kappa_j| \delta_{\beta,j}^{\frac{-\sigma-\tilde{\eta}}{u+\tilde{\eta}}} \leq \frac{2}{3}\alpha_j \quad \text{(with (6.6))}. \end{aligned}$$

4. Proof of (6.5). For (ψ, δ) as in Part 3 let $(\tilde{\psi}, \tilde{\delta}) = (P_j(I_j(K_j)))(\psi, \delta)$. Then

$$|(\tilde{\psi}, \tilde{\delta})| \leq \frac{2}{3}\alpha_j,$$

hence $(\tilde{\psi}, \tilde{\delta}) \in [-\alpha_j, \alpha_j] \times [-\alpha_j, \alpha_j]$, which according to Part 2 yields

$$|(K_j^{-1} \circ E_j \circ P_j^{-1})(\tilde{\psi}, \tilde{\delta})| \leq \frac{3}{2}|(\tilde{\psi}, \tilde{\delta})|.$$

It follows that

$$\begin{aligned} |Q_j(\psi, \delta)| &= |K_j^{-1}(R_j(K_j(\psi, \delta)))| = |(K_j^{-1} \circ E_j \circ P_j^{-1} \circ P_j \circ I_j)(K_j(\psi, \delta))| \\ &= |(K_j^{-1} \circ E_j \circ P_j^{-1})(\tilde{\psi}, \tilde{\delta})| \\ &\leq \frac{3}{2}|(\tilde{\psi}, \tilde{\delta})| \leq \alpha_j. \end{aligned}$$

Finally, use that the disk of radius α_j and center $0 \in \mathbb{R}^2$ is contained in the square $[-\alpha_j, \alpha_j] \times [-\alpha_j, \alpha_j]$. □

The last result of this section concerns continuity of the angle corresponding to $P_L I_j(K_j(\psi, \delta))$, as a function of $(\psi, \delta) \in (-\pi, \pi) \times (0, \delta_j)$.

Proposition 6.3. *Assume $0 < \tilde{\eta} < -\sigma/2$ and consider $\eta > 0, j \in \mathbb{N}, \delta_j$ so that the relations (5.3)–(5.6) hold. Then the function $\Phi_j : (-\pi, \pi) \times (0, \delta_j) \to \mathbb{R}$ given by*

$$\Phi_j(\psi, \delta) = \phi^{(n)}(\tau_j(x), \omega_j + \psi, \delta)$$

with $\phi^{(n)}$ according to Proposition 3.2, $n = n_j(x)$ the largest integer in $[0, \tau_j(x))$ and $x = K_j(\psi, \delta)$, satisfies

$$\frac{1}{|P_L I_j(K_j(\psi,\delta))|} P_L I_j(K_j(\psi,\delta)) = \begin{pmatrix} \cos(\Phi_j(\psi,\delta)) \\ \sin(\Phi_j(\psi,\delta)) \\ 0 \end{pmatrix} \quad \text{for all} \quad (\psi,\delta) \in (-\pi,\pi) \times (0,\delta_j) \tag{6.7}$$

and is continuous.

Proof. 1. The definition of Φ_j makes sense because for $(\psi, \delta) \in (-\pi, \pi) \times (0, \delta_j)$ and $x = K_j(\psi, \delta)$ we have $|P_U F_{\epsilon_j}(t, x)| < 1$ on $[0, \tau_j(x))$, hence $|P_U F_{\epsilon_j}(\nu, x)| < 1$ for $\nu = 0, \ldots, n_j(x)$, which in combination with $n_j(x) < \tau_j(x) \le n_j(x) + 1$ yields $(\tau_j(x), \omega_j + \psi, \delta) \in \mathrm{dom}_{n_j(x)}$.

Eq. (6.7) holds because for $\phi = \phi^{n_j(x)}(\tau_j(x), \omega_j + \psi, \delta) = \Phi_j(\psi, \delta)$ we have

$$\frac{1}{|P_L I_j(K_j(\psi,\delta))|} P_L I_j(x) = \frac{1}{|P_L F_{\epsilon_j}(\tau_j(x), x)|} P_L F_{\epsilon_j}(\tau_j(x), x) = \begin{pmatrix} \cos(\phi) \\ \sin(\phi) \\ 0 \end{pmatrix}$$

due to Proposition 3.2 (i).

2. Proof that Φ_j is continuous at $(\psi, \delta) \in (-\pi, \pi) \times (0, \delta_j)$. Let $x = K_j(\psi, \delta)$. We have $n_j(x) < \tau_j(x) \le n_j(x) + 1 < n_j(x) + 2$. Let a sequence $(\psi_m, \delta_{(m)})_{m \in \mathbb{N}}$ in $(-\pi, \pi) \times (0, \delta_j)$ be given which converges to (ψ, δ). It is enough to find a subsequence so that $\Phi_j(\psi_{m_\mu}, \delta_{(m_\mu)}) \to \Phi_j(\psi, \delta)$ as $\mu \to \infty$.

2.1. In case $\tau_j(x) < n + 1$ the continuity of τ_j yields an integer $m_x \ge 0$ so that for all indices $m \ge m_x$ we have $n_j(x) < \tau_j(x_m) < n_j(x) + 1$ for $x_m = K_j(\psi_m, \delta_{(m)})$. Hence $n_j(x) = n_j(x_m)$ for $m \ge m_x$. Consequently, $(\tau_j(x_m), \omega_j + \psi_m, \delta_{(m)}) \in \mathrm{dom}_{n_j(x_m)} = \mathrm{dom}_{n_j(x)}$ and

$$\Phi_j(\psi_m, \delta_{(m)}) = \phi^{(n_j(x_m))}(\tau_j(x_m), \omega_j + \psi_m, \delta_{(m)}) = \phi^{(n_j(x))}(\tau_j(x_m), \omega_j + \psi_m, \delta_{(m)}) \quad \text{for} \quad m \ge m_x.$$

As $\phi^{(n_j(x))}$ is continuous according to Proposition 3.2 we arrive at

$$\lim_{m\to\infty} \Phi_j(\psi_m, \delta_{(m)}) = \lim_{m\to\infty} \phi^{(n_j(x))}(\tau_j(x_m), \omega_j + \psi_m, \delta_{(m)}) = \phi^{(n_j(x))}(\tau_j(x), \omega_j + \psi, \delta) = \Phi_j(\psi, \delta).$$

2.2. In case $\tau_j(x) = n_j(x) + 1$ we have $|P_U F_{\epsilon_j}(n_j(x) + 1, x)| = |P_U F_{\epsilon_j}(\tau_j(x), x)| = 1$ in addition to $|P_U F_{\epsilon_j}(\nu, x)| < 1$ for $\nu = 0, \ldots, n_j(x)$ and conclude that $(\tau_j(x), \omega_j + \psi, \delta) \in \mathrm{dom}_{n_j(x)+1}$.

By Proposition 3.2 (ii), $\phi^{(n_j(x)+1)}(n_j(x) + 1, \omega_j + \psi, \delta) = \phi^{(n_j(x))}(n_j(x) + 1, \omega_j + \psi, \delta)$.

We distinguish the subcases that the indices m with $\tau_j(x_m) \le n_j(x) + 1$ are bounded or not.

2.2.1. If the indices with $\tau_j(x_m) \le n_j(x) + 1$ are unbounded then there is a strictly increasing sequence $(m_\mu)_{\mu \in \mathbb{N}}$ of positive integers with $\tau_j(x_{m_\mu}) \le n_j(x) + 1$ for all $\mu \in \mathbb{N}$. As in Part 2.1 we find $\Phi_j(\psi_{m_\mu}, \delta_{(m_\mu)}) \to \Phi_j(\psi, \delta)$ for $\mu \to \infty$.

2.2.2. If there is an upper bound $m_x \in \mathbb{N}$ for the indices m with $\tau_j(x_m) \le n_j(x) + 1$ then $n_j(x) + 1 < \tau_j(x_m)$ for all indices $m > m_x$. In addition we may assume $\tau_j(x_m) < n_j(x) + 2$ for all $m > m_x$. It follows that $n_j(x) + 1 = n_j(x_m)$ and

$$\Phi_j(\psi_m, \delta_{(m)}) = \phi^{(n_j(x_m))}(\tau_j(x_m), \omega_j + \psi_m, \delta_{(m)}) = \phi^{(n_j(x)+1)}(\tau_j(x_m), \omega_j + \psi_m, \delta_{(m)})$$

for $m > m_x$. Using continuity of $\phi^{(n_j(x)+1)}$ from Proposition 3.2 we find

$$\begin{aligned} \lim_{m\to\infty} \Phi_j(\psi_m, \delta_{(m)}) &= \lim_{m\to\infty} \phi^{(n_j(x)+1)}(\tau_j(x_m), \omega_j + \psi_m, \delta_{(m)}) \\ &= \phi^{(n_j(x)+1)}(\tau_j(x), \omega_j + \psi, \delta) = \phi^{(n_j(x)+1)}(n_j(x) + 1, \omega_j + \psi, \delta) \\ &= \phi^{(n_j(x))}(n_j(x) + 1, \omega_j + \psi, \delta) = \phi^{(n_j(x))}(\tau_j(x), \omega_j + \psi, \delta) = \Phi_j(\psi, \delta). \quad \square \end{aligned}$$

7 Curves expanded by the return map in the plane

In this section we consider curves $g : [a,b] \to (-\pi,\pi) \times (0,\delta_j)$ which connect level sets $(-\pi,\pi) \times \{\Delta_1\}$ and $(-\pi,\pi) \times \{\Delta_2\}$ with $0 < \Delta_1 < \Delta_2 < \delta_j$. We find subintervals $[a_0, b_0]$ and $[a_1, b_1]$ so that the angle function Φ_j sends $g((a_0, b_0))$ and $g((a_1, b_1))$ into disjoint sets M_0 and M_1, and transport by the return map in the plane Q_j yields two curves which again connect the said level sets. Be aware that the angle function Φ_j is not directly related to the return map in the plane but only to the inner map in coordinates (depicted in Figure 5.1, bottom), which is the first composite of the return map in the plane.

Throughout this section we assume $0 < \tilde{\eta} < \min\{\mu, -\sigma/2\}$, and that $\eta > 0$, $j \in \mathbb{N}$, and $\delta_j \in (0,1)$ satisfy the relations (5.3)–(5.6). We begin with a comparison of angles $\Phi_j(\psi,\delta)$ for arguments in level sets as above. Let

$$\begin{aligned} c = c_{\tilde{\eta}} &= \frac{(u+\tilde{\eta})(\mu+\tilde{\eta})}{(u-\tilde{\eta})(\mu-\tilde{\eta})} \\ &> 1, \\ k = k_{\tilde{\eta}} &= e^{-6\pi\frac{u+\tilde{\eta}}{\mu-\tilde{\eta}}} \\ &< 1. \end{aligned}$$

For $\Delta_2 \in (0,\delta_j)$ we set

$$\begin{aligned} \Delta_1 = \Delta_1(\Delta_2) &= k\Delta_2^c \\ &< \Delta_2. \end{aligned}$$

Proposition 7.1. *Let $\Delta_2 \in (0,\delta_j)$ be given and consider $\Delta_1 = \Delta_1(\Delta_2) = k\Delta_2^c$. Then*

$$4\pi \le \Phi_j(\psi,\Delta_2) - \Phi_j(\gamma,\Delta_1) \quad \textit{for all } \psi,\gamma \textit{ in } (-\pi,\pi).$$

Proof. Assume $-\pi < \psi < \pi$, $-\pi < \gamma < \pi$. Recall the definition of Φ_j in Proposition 6.3 and apply the last estimate in Corollary 5.5, and the estimate of the travel time in Corollary 5.6. This yields the inequalities

$$\Phi_j(\psi,\Delta_2) - (\omega_j + \psi) \ge (-\mu - \tilde{\eta}) \cdot \frac{1}{u-\tilde{\eta}} \log\left(\frac{1}{\Delta_2}\right)$$

and

$$\Phi_j(\gamma,\Delta_1) - (\omega_j + \gamma) \le (-\mu + \tilde{\eta}) \cdot \frac{1}{u+\tilde{\eta}} \log\left(\frac{1}{\Delta_1}\right)$$

from which we obtain

$$\begin{aligned} \Phi_j(\psi,\Delta_2) - \Phi_j(\gamma,\Delta_1) &\ge -2\pi + \frac{\mu+\tilde{\eta}}{u-\tilde{\eta}} \log(\Delta_2) - \frac{\mu-\tilde{\eta}}{u+\tilde{\eta}} \log(\Delta_1) \\ &\ge -2\pi + \frac{\mu+\tilde{\eta}}{u-\tilde{\eta}} \log(\Delta_2) - \frac{\mu-\tilde{\eta}}{u+\tilde{\eta}} [\log(k) + c\log(\Delta_2)] \\ &= -2\pi + 6\pi + \left(\frac{\mu+\tilde{\eta}}{u-\tilde{\eta}} - c\frac{\mu-\tilde{\eta}}{u+\tilde{\eta}}\right) \log(\Delta_2) = 4\pi. \end{aligned}$$

□

From here on let also $\beta \in (0,1/2]$ be given and consider $\alpha_j = \alpha_j(\beta) \in (0,\delta_j)$ according to Proposition 6.2. Proposition 7.1 shows that the quantities

$$m_1 = m_1(j,\Delta_1) = \max_{|\gamma|\le\alpha_j} \Phi_j(\gamma,\Delta_1) \quad \text{and} \quad m_2 = m_2(j,\Delta_2) = \min_{|\psi|\le\alpha_j} \Phi_j(\psi,\Delta_2)$$

satisfy $m_1 + 4\pi \le m_2$. Also, there exists $\psi_j \in [m_1+\pi, m_2-\pi]$ with

$$\begin{pmatrix} \cos(\psi_j) \\ \sin(\psi_j) \\ 0 \end{pmatrix} = \frac{1}{|w_j|} w_j.$$

Proposition 7.2 (Angles along curves connecting vertical levels)**.** *Consider $\Delta_2 \in (0,\delta_j)$ and $\Delta_1 = \Delta_1(\Delta_2)$ as above. Let a curve $g : [a,b] \to (-\pi,\pi)\times(0,\delta_j)$ be given with $g_2(b) = \Delta_2$ and $g_2(a) = \Delta_1$. Then there exist $a_0' < b_0' \le a_1' < b_1'$ in $[a,b]$ such that*

$$\Phi_j(g(t)) \in (\psi_j - \pi, \psi_j) \quad \text{on} \quad (a_0', b_0'), \quad \Phi_j(g(a_0')) = \psi_j - \pi, \quad \Phi_j(g(b_0')) = \psi_j,$$

$$\Phi_j(g(t)) \in (\psi_j, \psi_j + \pi) \quad \text{on} \quad (a_1', b_1'), \quad \Phi_j(g(a_1')) = \psi_j, \quad \Phi_j(g(b_1')) = \psi_j + \pi.$$

Compare Figure 7.1.

Proof. 1. We construct a_1' and b_1'. From

$$\Phi_j(g(a)) \le m_1 < m_1 + \pi \le \psi_j \le m_2 - \pi < m_2 \le \Phi_j(g(b))$$

we have

$$\Phi_j(g(a)) \le \psi_j - \pi < \psi_j < \psi_j + \pi \le \Phi_j(g(b)).$$

By continuity, $\psi_j = \Phi_j(g(t))$ for some $t \in (a,b)$. Again by continuity there exists $b_1' \in (t,b]$ with $\Phi_j(g(s)) < \psi_j + \pi$ on $[t, b_1')$ and $\Phi_j(g(b_1')) = \psi_j + \pi$. Upon that, there exists $a_1' \in [t, b_1')$ with $\psi_j < \Phi_j(g(s))$ on $(a_1', b_1']$ and $\Phi_j(g(a_1')) = \psi_j$.

2. The construction of a_0' and b_0' with $b_0' \le a_1'$ is analogous. □

We turn to the position of $Q_j(\psi,\delta)$ for arguments $(\psi,\delta) \in (-\pi,\pi)\times[\Delta_1,\Delta_2]$. A look at Eq. (6.7) in Proposition 6.3 confirms that in the cases

$$\Phi_j(\psi,\delta) = \psi_j - \pi, \quad \Phi_j(\psi,\delta) = \psi_j, \quad \Phi_j(\psi,\delta) = \psi_j + \pi$$

the value $P_L I_j(K_j(\psi,\delta))$ belongs to the rays

$$(0,\infty)(-w_j), \quad (0,\infty)w_j, \quad (0,\infty)(-w_j), \quad \text{respectively,}$$

hence $P_j(I_j(K_j(\psi,\delta)))$ is on the vertical axis in $\mathbb{R}^2$.

Proposition 7.3 (From angles to vertical levels)**.** *Assume in addition to the hypotheses made in the present section up to here that $\tilde\eta$ satisfies*

$$c_{\tilde\eta} \frac{-\sigma + \tilde\eta}{u - \tilde\eta} < 1. \tag{7.1}$$

Let $\beta \in (0,1/2]$ be given and choose reals $\alpha_j = \alpha_j(\beta) \in (0,\delta_j)$ and $\delta_{\beta,j} \in (0, 2\alpha_j/3]$ according to Proposition 6.2. Consider $\Delta_2 \in (0,\delta_{\beta,j})$ so small that

$$2\sqrt{2}\Delta_2 < \frac{1}{|\kappa_j^{-1}|} k^{\frac{-\sigma+\tilde\eta}{u-\tilde\eta}} \Delta_2^{c\frac{-\sigma+\tilde\eta}{u-\tilde\eta}}, \quad \textit{with} \quad c = c_{\tilde\eta} \quad \textit{and} \quad k = k_{\tilde\eta}. \tag{7.2}$$

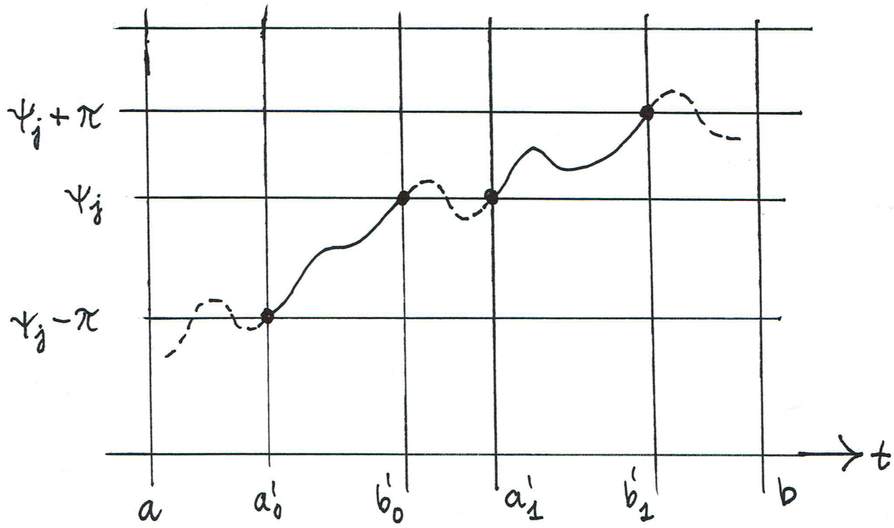

Figure 7.1: The map $\Phi_j \circ g$.

Let $\Delta_1 = \Delta_1(\Delta_2)$, $(\psi,\delta) \in [-\alpha_j,\alpha_j] \times [\Delta_1,\Delta_2]$, and $z = Q_j(\psi,\delta)$. Then

(i) $|z| > \sqrt{2}\Delta_2$.

(ii) In the cases $\Phi_j(\psi,\delta) = \psi_j - \pi$, $\quad \Phi_j(\psi,\delta) = \psi_j$, $\quad \Phi_j(\psi,\delta) = \psi_j + \pi$, we have

$$z_2 < -\Delta_2, \quad z_2 > \Delta_2, \quad z_2 < -\Delta_2, \quad \textit{respectively}.$$

Compare Figure 7.2.

Proof. 1. On assertion (i). Observe first that due to Proposition 6.2 the rectangle $[-\alpha_j,\alpha_j] \times [\Delta_1,\Delta_2]$ is contained in the domain of definition of the maps $P_j(I_j(K_j(\cdot,\cdot)))$ and Q_j. Let $(\psi,\delta) \in [-\alpha_j,\alpha_j] \times [\Delta_1,\Delta_2]$ be given, and let

$$x = P_j(I_j(K_j(\psi,\delta))) \quad (= \kappa_j P_L I_j(K_j(\psi,\delta))).$$

Observe $K_{j,3}(\psi,\delta) = \delta$ and apply Corollaries 5.5 and 5.6. This yields

$$|x| \geq \frac{1}{|\kappa_j^{-1}|}\delta^{\frac{-\sigma+\tilde{\eta}}{u-\tilde{\eta}}} \geq \frac{1}{|\kappa_j^{-1}|}\Delta_1^{\frac{-\sigma+\tilde{\eta}}{u-\tilde{\eta}}} = \frac{1}{|\kappa_j^{-1}|}k^{\frac{-\rho+\tilde{\eta}}{u-\tilde{\eta}}}\Delta_2^{c\frac{-\sigma+\tilde{\eta}}{u-\tilde{\eta}}} > 2\sqrt{2}\Delta_2.$$

We have

$$z = Q_j(\psi,\delta) = (K_j^{-1} \circ E_j \circ I_j)(K_j(\psi,\delta)) = (K_j^{-1} \circ E_j \circ P_j^{-1} \circ P_j \circ I_j)(K_j(\psi,\delta)) = (K_j^{-1} \circ E_j \circ P_j^{-1})(x).$$

The inequality (6.4) in Proposition 6.2 gives us $|x| \leq \frac{2}{3}\alpha_j$, hence $x \in [-\alpha_j,\alpha_j] \times [-\alpha_j,\alpha_j]$. Using the relation (6.2) in Corollary 6.1 we infer $|z - x| \leq \beta|x|$. It follows that

$$|z| \geq |x| - \beta|x| \geq \frac{1}{2}|x| > \sqrt{2}\Delta_2.$$

2. On assertion (ii) for the case $(\psi,\delta) \in [-\alpha_j,\alpha_j] \times [\Delta_1,\Delta_2]$ with $\Phi_j(\psi,\delta) = \psi_j$. Let $z = Q_j(\psi,\delta) \in \mathbb{R}^2$.

2.1. From (6.7) in Proposition 6.3 we have that $P_L I_j(K_j(\psi,\delta))$ is a positive multiple of

$$\begin{pmatrix} \cos(\Phi_j(\psi,\delta)) \\ \sin(\Phi_j(\psi,\delta)) \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\psi_j) \\ \sin(\psi_j) \\ 0 \end{pmatrix} \in (0,\infty) w_j,$$

hence

$$P_j(I_j(K_j(\psi,\delta))) = \kappa_j P_L I_j(K_j(\psi,\delta)) \in (0,\infty) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

2.2. From Part 1, $|z| > \sqrt{2}\Delta_2$. For $x = P_j(I_j(K_j(\psi,\delta))) = \kappa_j P_L I_j(K_j(\psi,\delta))$ Part 2.1 yields $x = \binom{0}{x_2}$ with $x_2 > 0$.

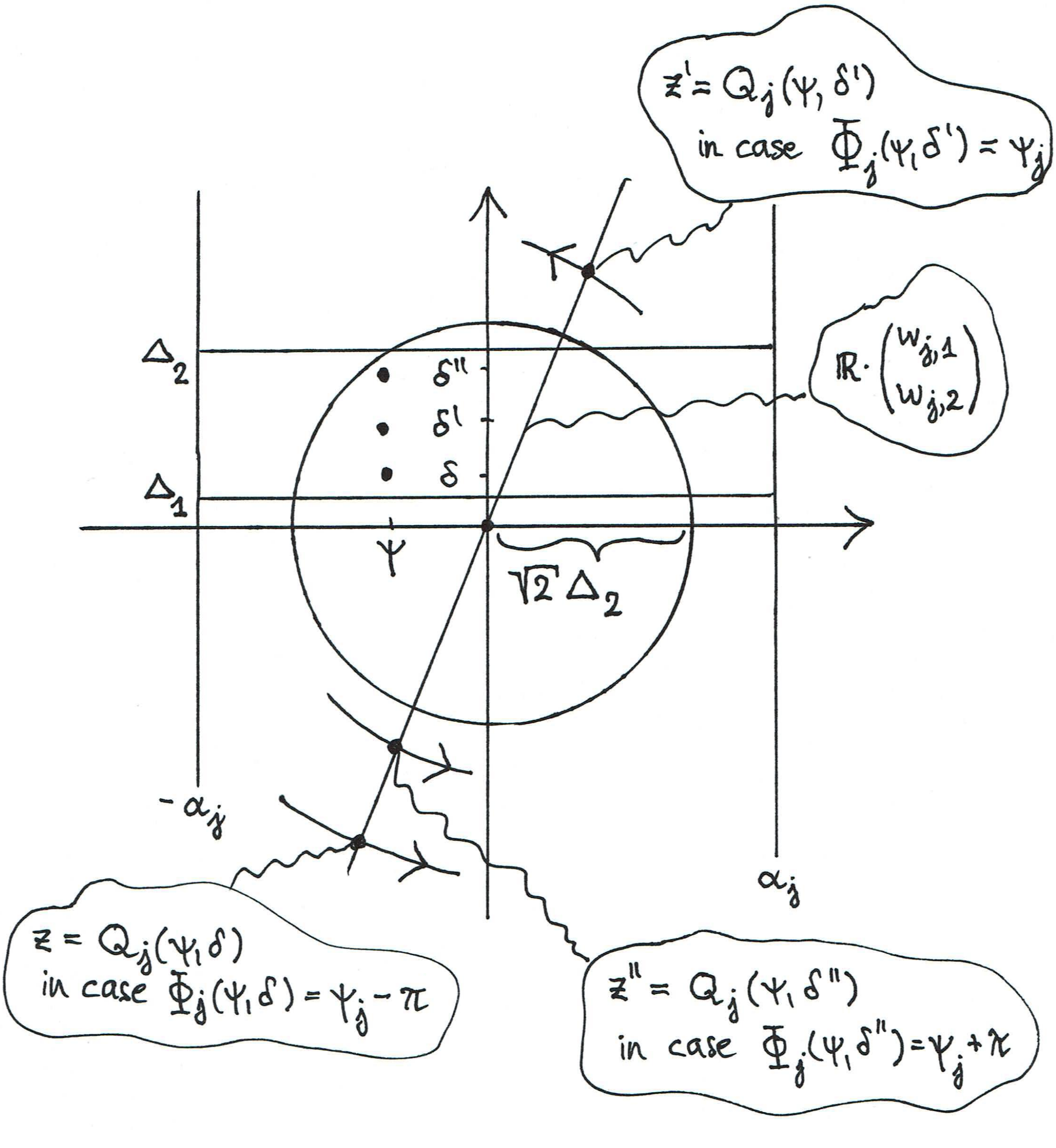

Figure 7.2: Positions of $Q_j(\psi,\delta)$ depending on the angle $\Phi_j(\psi,\delta)$.

2.3. Proof of $|z| \le \sqrt{2} z_2$: We have $|x| = x_2$. From $x_2 - z_2 \le |x_2 - z_2| \le |z - x| \le \beta|x| = \beta x_2$, $z_2 \ge (1-\beta)x_2 > 0$. Also, from $x_1 = 0$, $|z_1| \le |x_1| + \beta|x| = \beta x_2$. It follows that

$$|z|^2 = z_1^2 + z_2^2 \le \beta^2 x_2^2 + z_2^2 \le \frac{\beta^2}{(1-\beta)^2} z_2^2 + z_2^2 \le 2z_2^2.$$

2.4. Consequently, $z_2 = |z_2| \ge \frac{1}{\sqrt{2}}|z| > \Delta_2$.

3. The proofs of assertion (ii) in the two remaining cases are analogous, making use of the fact that in both cases we have that $P_j(I_j(K_j(\psi, \delta))) = \kappa_j P_L I_j(K_j(\psi, \delta))$ is a positive multiple of $\binom{0}{-1}$. □

The next result makes precise what was briefly announced at the begin of the section. The disjoint sets mentioned there will be given in terms of the angle $\Phi_j(\psi, \delta)$ corresponding to the value $P_j(I_j(K_j(\psi, \delta)))$ of the inner map in coordinates, and not by the position of the value $Q_j(\psi, \delta)$ of the return map in the plane to the left or right of the vertical axis. Our choice of disjoint sets circumvents a discussion how the latter are related to the more accessible angles $\Phi_j(\psi, \delta)$.

Proposition 7.4. *Assume the hypotheses of Proposition 7.3 are satisfied and let $\Delta_1 = \Delta_1(\Delta_2)$. Consider the disjoint sets*

$$M_0 = \{(\psi, \delta) \in [-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2] : \psi_j - \pi < \Phi_j(\psi, \delta) < \psi_j\}$$

and

$$M_1 = \{(\psi, \delta) \in [-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2] : \psi_j < \Phi_j(\psi, \delta) < \psi_j + \pi\}.$$

For every curve $g : [a, b] \to [-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$ with $g_2(a) = \Delta_1$ and $g_2(b) = \Delta_2$ there exist $a_0 < b_0 < a_1 < b_1$ in $[a, b]$ such that

$$\begin{aligned} \textit{on} \quad (a_0, b_0), \quad & g(t) \in M_0 \quad \textit{and} \quad Q_{j,2}(g(t)) \in (\Delta_1, \Delta_2), \\ \textit{with} \quad & Q_{j,2}(g(a_0)) = \Delta_1 \quad \textit{and} \quad Q_{j,2}(g(b_0)) = \Delta_2, \\ \textit{while on} \quad (a_1, b_1), \quad & g(t) \in M_1 \quad \textit{and} \quad Q_{j,2}(g(t)) \in (\Delta_1, \Delta_2), \\ \textit{with} \quad & Q_{j,2}(g(a_1)) = \Delta_2 \quad \textit{and} \quad Q_{j,2}(g(b_1)) = \Delta_1. \end{aligned}$$

Proof. Proposition 7.2 yields $a_0' < b_0' \le a_1' < b_1'$ in $[a, b]$ such that

$$\begin{aligned} \text{on} \quad & (a_0', b_0'), \quad \Phi_j(g(t)) \in (\psi_j - \pi, \psi_j), \\ & \text{with} \quad \Phi_j(g(a_0')) = \psi_j - \pi \quad \text{and} \quad \Phi_j(g(b_0')) = \psi_j, \\ \text{and on} \quad & (a_1', b_1'), \quad \Phi_j(g(t)) \in (\psi_j, \psi_j + \pi), \\ & \text{with} \quad \Phi_j(g(a_1')) = \psi_j \quad \text{and} \quad \Phi_j(g(b_1'))_2 = \psi_j + \pi. \end{aligned}$$

From Proposition 7.3 (ii),

$$Q_{j,2}(g(a_0')) < -\Delta_2, \quad Q_{j,2}(g(b_0')) > \Delta_2, \quad Q_{j,2}(g(a_1')) > \Delta_2, \quad Q_{j,2}(g(b_1')) < -\Delta_2.$$

As in the proof of Proposition 7.2 one finds $a_0 < b_0$ in (a_0', b_0') and $a_1 < b_1$ in (a_1', b_1') with

$$\begin{aligned} Q_{j,2}(g(a_0)) = \Delta_1 \quad \text{and} \quad Q_{j,2}(g(b_0)) = \Delta_2, \quad \text{and} \quad Q_{j,2}(g(t)) \in (\Delta_1, \Delta_2) \quad \text{on} \quad (a_0, b_0), \\ Q_{j,2}(g(a_1)) = \Delta_2 \quad \text{and} \quad Q_{j,2}(g(b_1)) = \Delta_1, \quad \text{and} \quad Q_{j,2}(g(t)) \in (\Delta_1, \Delta_2) \quad \text{on} \quad (a_1, b_1). \end{aligned}$$

Observe that on $(a_0, b_0) \subset (a_0', b_0')$ we have $g(t) \in M_0$ while on $(a_1, b_1) \subset (a_1', b_1')$ we have $g(t) \in M_1$. □

See Figure 7.3.

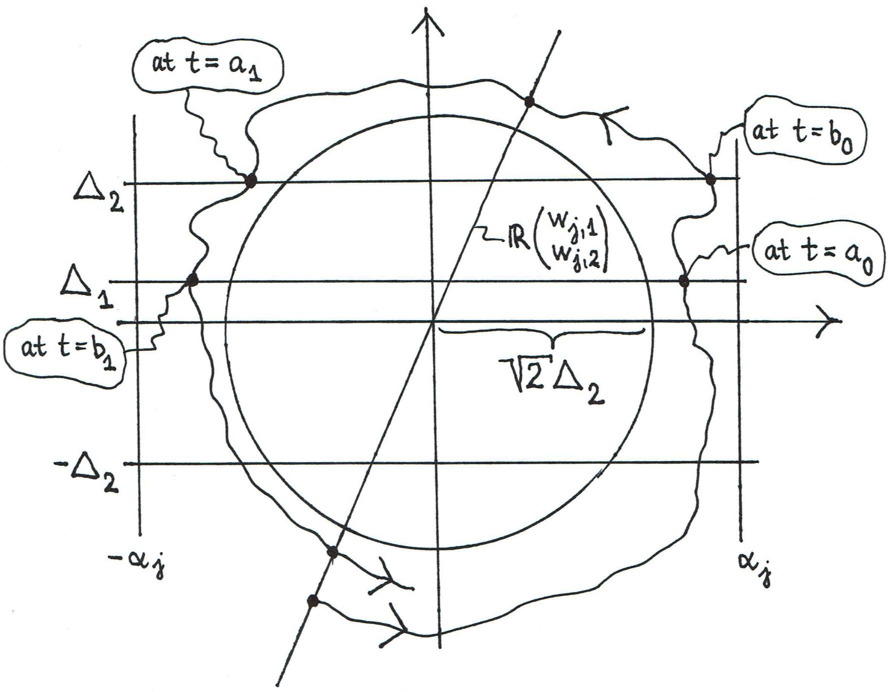

Figure 7.3: The values $Q_j(g(t))$ for $a \le t \le b$.

8 Complicated dynamics

For the results of this section we assume that the hypotheses of Proposition 7.4 are satisfied. It may be convenient to repeat all of these assumption here, beginning with the choice of a real number $\tilde{\eta} > 0$ with

$$\tilde{\eta} < \min\{\mu, -\sigma/2\}$$

which satisfies the inequality (7.1).

The numbers $\eta > 0$, $j \in \mathbb{N}$, and $\delta_j \in (0,1)$ are chosen so that the relations (5.3)–(5.6) hold.

For given $\beta \in (0, 1/2]$, the reals $\alpha_j = \alpha_j(\beta) \in (0, \delta_j)$ and $\delta_{\beta,j} \in (0, 2\alpha_j/3]$ are chosen according to Proposition 6.2.

$\Delta_2 \in (0, \delta_{\beta,j})$ is chosen so that the inequality (7.2) holds, and $\Delta_1 = \Delta_1(\Delta_2)$.

Recall the disjoint sets M_0 and M_1 from Proposition 7.4.

Proposition 8.1. *For every sequence $(s_n)_{n=0}^{\infty}$ in $\{0,1\}$ there are forward trajectories $(x_n)_{n=0}^{\infty}$ of Q_j in $[-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$ with $x_n \in M_{s_n}$ and $\Delta_1 \le Q_{j,2}(x_n) \le \Delta_2$ for all integers $n \ge 0$.*

Proof. 1. Let a sequence $(s_n)_{n=0}^{\infty}$ in $\{0,1\}$ be given. Choose a curve $g : [a,b] \to [-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$ such that $g_2(t) \in (\Delta_1, \Delta_2)$ for $a < t < b$ and $g_2(a) = \Delta_1$, $g_2(b) = \Delta_2$, for example, $g(t) = (0, t)$ for $a = \Delta_1 \le t \le \Delta_2 = b$.

For integers $n \ge 0$ we construct recursively curves $g_n : [A_n, B_n] \to [-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$ with decreasing domains in $[a,b]$ as follows.

1.1. In order to find g_0 we apply Proposition 7.4 to the curve g and obtain $a_0 < b_0 < a_1 < b_1$ in $[a,b]$ with the properties stated in Proposition 7.4. In case $s_0 = 0$ we define g_0 by $A_0 =$

a_0, $B_0 = b_0$, $g_0(t) = g(t)$ for $A_0 \le t \le B_0$. Notice that $g_0(t) \in M_{s_0}$ for all $t \in (A_0, B_0)$, $Q_{j,2}(g_0(t)) \in (\Delta_1, \Delta_2)$ on (A_0, B_0), $Q_{j,2}(g_0(A_0)) = \Delta_1$, and $Q_{j,2}(g_0(B_0)) = \Delta_2$. In case $s_0 = 1$ we define g_0 by $A_0 = a_1$, $B_0 = b_1$, $g_0(t) = g(a_1 + b_1 - t)$ for $A_0 \le t \le B_0$. Notice that also in this case $g_0(t) \in M_{s_0}$ for all $t \in (A_0, B_0)$, $Q_{j,2}(g_0(t)) \in (\Delta_1, \Delta_2)$ on (A_0, B_0), and $Q_{j,2}(g_0(A_0)) = Q_{j,2}(g(a_1 + b_1 - a_1)) = \Delta_1$, $Q_{j,2}(g_0(B_0)) = Q_{j,2}(g(a_1 + b_1 - b_1)) = \Delta_2$.

1.2. For an integer $n \ge 0$ let a curve $g_n : [A_n, B_n] \to [-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$ be given with $g_n(t) \in M_{s_n}$ for all $t \in (A_n, B_n)$ and $Q_{j,2}(g_n(t)) \in (\Delta_1, \Delta_2)$ on (A_n, B_n), $Q_{j,2}(g_n(A_n)) = \Delta_1$, $Q_{j,2}(g_n(B_n)) = \Delta_2$. Proceeding as in Part 1.1, with the curve $[A_n, B_n] \ni t \mapsto Q_j(g_n(t)) \in [-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$ in place of the former curve g, we obtain $A_{n+1} < B_{n+1}$ in $[A_n, B_n]$ and a curve $g_{n+1} : [A_{n+1}, B_{n+1}] \to [-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$ with $g_{n+1}(t) \in M_{s_{n+1}}$ for all $t \in (A_{n+1}, B_{n+1})$ and $Q_{j,2}(g_{n+1}(t)) \in (\Delta_1, \Delta_2)$ on (A_{n+1}, B_{n+1}), $Q_{j,2}(g_{n+1}(A_{n+1})) = \Delta_1$, $Q_{j,2}(g_{n+1}(B_{n+1})) = \Delta_2$.

2. From $A_n \le A_{n+1} < B_{n+1} \le B_n$ for all integers $n \ge 0$ we get $\cap_{n\ge 0}[A_n, B_n] = [A, B]$ with $A = \lim_{n\to\infty} A_n \le \lim_{n\to\infty} B_n = B$. Choose $t \in [A, B]$ and define

$$x_n = g_n(t) \in [-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$$

for every integer $n \ge 0$. This yields a forward trajectory of Q_j with $\Delta_1 \le Q_{j,2}(x_n) \le \Delta_2$ for all integers $n \ge 0$.

Let an integer $n \ge 0$ be given. Proof of $x_n \in M_{s_n}$. We have $x_n = g_n(t)$ with $A_n \le A \le t \le B \le B_n$, and for all $v \in (A_n, B_n)$, $g_n(v) \in M_{s_n}$. By continuity, $x_n \in \text{cl}\, M_{s_n}$. In case $s_n = 0$ this yields

$$\psi_j - \pi \le \Phi_j(x_n) \le \psi.$$

Assume $\Phi_j(x_n) \in \{\psi - \pi, \psi\}$, Then Proposition 7.3 (ii) gives $|Q_{j,2}(x_n)| > \Delta_2$, which contradicts the previous estimate of $Q_{j,2}(x_n)$. Consequently,

$$\psi_j - \pi < \Phi_j(x_n) < \psi, \quad \text{or,} \quad x_n \in M_0 = M_{s_n}.$$

In case $s_n = 1$ the proof is analogous. □

The final result extends Proposition 8.1 to entire trajectories.

Theorem 8.2. *For every sequence $(s_n)_{n=-\infty}^{\infty}$ in $\{0, 1\}$ there exist entire trajectories $(y_n)_{n=-\infty}^{\infty}$ of Q_j with $y_n \in M_{s_n}$ for all integers n.*

Proof. 1. Let $(s_n)_{n=-\infty}^{\infty}$ in $\{0, 1\}$ be given. Proposition 8.1 guarantees that for every integer k there is a forward trajectory $(y_{k,n})_{n=0}^{\infty}$ of Q_j in $[-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$ so that for each integer $n \ge 0$,

$$y_{k,n} \in M_{s_{n-k}} \quad \text{and} \quad \Delta_1 \le Q_{j,2}(y_{k,n}) \le \Delta_2.$$

For integers k, n with $k \ge -n$ we define

$$z_{k,n} = y_{k,n+k},$$

so that

$$\begin{aligned} z_{k,n} &= y_{k,n+k} \in M_{s_{n+k-k}} = M_{s_n}, \\ z_{k,n+1} &= y_{k,n+1+k} = Q_j(y_{k,n+k}) = Q_j(z_{k,n}), \\ Q_{j,2}(z_{k,n}) &= Q_{j,2}(y_{k,n+k}) \in [\Delta_1, \Delta_2]. \end{aligned}$$

1.1. Choice of subsequences for integers $N \ge 0$.

1.1.1. The case $N = 0$: For every integer $k \geq 0$, $z_{k,0} \in M_{s_0}$. The sequence $(z_{k,0})_{k=0}^{\infty}$ in the compact set $[-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$ has a convergent subsequence $(z_{\kappa_0(k),0})_{k=0}^{\infty}$ given by a strictly increasing map $\kappa_0 : \mathbb{N}_0 \to \mathbb{N}_0$. Let $y_0 = \lim_{k\to\infty} z_{\kappa_0(k),0} \in \text{cl } M_{s_0} \subset [-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$.

1.1.2. The case $N = 1$: We choose a convergent subsequence of $(z_{\kappa_0(k),-1})_{k=0}^{\infty}$ given by a strictly increasing map $\mu_{-1} : \mathbb{N}_0 \to \mathbb{N}_0$, and upon this a convergent subsequence of $(z_{\kappa_0 \circ \mu_{-1}(k),1})_{k=0}^{\infty}$ given by a strictly increasing map $\mu_1 : \mathbb{N}_0 \to \mathbb{N}_0$, and define $\kappa_1 = \mu_{-1} \circ \mu_1$. For $k \to \infty$,

$$z_{\kappa_0 \circ \kappa_1(k),-0}, \quad z_{\kappa_0 \circ \kappa_1(k),-1} \quad \text{and} \quad z_{\kappa_0 \circ \kappa_1(k),1}$$

converge to elements $y_0 \in \text{cl } M_{s_0}$, $y_{-1} \in \text{cl } M_{s_{-1}}$, and $y_1 \in \text{cl } M_{s_1}$, respectively.

1.1.3. The general case $N \in \mathbb{N}$: Consecutively choosing further convergent subsequences analogously to Part 1.1.2 we obtain strictly increasing maps $\kappa_n : \mathbb{N}_0 \to \mathbb{N}_0$, $n = 0, \ldots, N$, so that for each $n \in \{-N, \ldots, N\}$ the sequence

$$(z_{\kappa_0 \circ \cdots \circ \kappa_N(k),n})_{k=0}^{\infty}$$

converges for $k \to \infty$ to some $y_n \in \text{cl } M_{s_n}$.

2. The *diagonal sequence* $K : \mathbb{N}_0 \to \mathbb{N}_0$ defined by $K(N) = (\kappa_0 \circ \cdots \circ \kappa_N)(N)$ is strictly increasing since for every $N \in \mathbb{N}_0$ we have

$$K(N+1) = (\kappa_0 \circ \cdots \circ \kappa_N)(\kappa_{N+1}(N+1)) > (\kappa_0 \circ \cdots \circ \kappa_N)(\kappa_{N+1}(N)) \geq (\kappa_0 \circ \cdots \circ \kappa_N)(N) = K(N)$$

due to strict monotonicity of all maps involved.

3. Let an integer n be given and set $N = |n|$. Proof that

$$(z_{K(k),n})_{k=N+1}^{\infty} \quad \text{is a subsequence of} \quad (z_{(\kappa_0 \circ \cdots \circ \kappa_N)(k),n})_{k=N+1}^{\infty}.$$

Consider the map $\lambda : \{k \in \mathbb{N}_0 : k > N\} \to \mathbb{N}_0$ given by $\lambda(k) = (\kappa_{N+1} \circ \cdots \circ \kappa_k)(k)$. For every integer $k \geq N+1$,

$$K(k) = (\kappa_0 \circ \cdots \circ \kappa_N)(\lambda(k)),$$

and λ is strictly increasing because analogously to Part 2 we have

$$\begin{aligned}\lambda(k+1) &= (\kappa_{N+1} \circ \cdots \circ \kappa_{k+1})(k+1) = (\kappa_{N+1} \circ \cdots \circ \kappa_k)(\kappa_{k+1}(k+1)) \\ &\geq (\kappa_{N+1} \circ \cdots \circ \kappa_k)(k+1) > (\kappa_{N+1} \circ \cdots \circ \kappa_k)(k) = \lambda(k)\end{aligned}$$

for every integer $k \geq N+1$.

Being a subsequence of $(z_{(\kappa_1 \circ \cdots \circ \kappa_{|n|})(k),n})_{k=N+1}^{\infty}$ the sequence $(z_{K(k),n})_{k=N+1}^{\infty}$ converges for $k \to \infty$ to $y_n \in \text{cl } M_{s_n}$.

4. We show that $(y_n)_{n=-\infty}^{\infty}$ is an entire trajectory of Q_j. Let an integer n be given and set $N = |n|$. From Part 3 in combination with Part 1.1.3 we get that $(z_{K(k),n})_{k=N+1}^{\infty}$ converges to $y_n \in [-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$ and that $(z_{K(k),n+1})_{k=N+2}^{\infty}$ converges to y_{n+1}. Recall $z_{k,n+1} = Q_j(z_{k,n})$ for all integers k, n with $k \geq -n$. For integers $k > N = |n|$ we have $K(k) \geq k \geq -n$, and the preceding statement yields

$$z_{K(k),n+1} = Q_j(z_{K(k),n}).$$

It follows that

$$y_{n+1} = \lim_{N+2 \leq k \to \infty} z_{K(k),n+1} = \lim_{N+1 \leq k \to \infty} Q_j(z_{K(k),n}) = Q_j(y_n).$$

5. Proof of $y_n \in M_{s_n}$ for all integers n. Let an integer n be given. We have $y_n \in \text{cl } M_{s_n}$. In case $s_n = 0$ this yields $\psi_j - \pi \leq \Phi_j(y_n) \leq \psi_j$. Therefore the assumption $y_n \notin M_{s_n}$ results in $\Phi_j(y_n) \in \{\psi_j - \pi, \psi_j\}$, which according to Proposition 7.3 (ii) means $|Q_{j,2}(y_n)| > \Delta_2$, in contradiction to $\Delta_1 \leq Q_{j,2}(y_n) \leq \Delta_2$. The proof in case $s_n = 1$ is analogous. $\square$

9 Appendix: From the vectorfield V to the flow F

Consider Shilnikov's scenario according to Section 1, with a continuously differentiable vectorfield $V : \mathbb{R}^3 \supset \text{dom}_V \to \mathbb{R}^3$, dom_V open and $V(0) = 0$, so that there is a homoclinic solution h_V of Eq. (1.1), and the eigenvalues $u > 0$ and $\sigma \pm i\mu$, $\sigma < 0 < \mu$, of $DV(0)$ satisfy

(H) $\quad 0 < \sigma + u.$

For simplicity assume in addition $\text{dom}_V = \mathbb{R}^3$ and that V is bounded. This is not a severe restriction since we are only interested in solutions close to the compact homoclinic loop $\text{cl}\, h_V(\mathbb{R}) = h_V(\mathbb{R}) \cup \{0\}$, and one can achieve the desired properties by a modification of the vectorfield outside a neighbourhood of $\text{cl}\, h_V(\mathbb{R})$. — Then the solutions of Eq. (1.1) constitute a continuously differentiable flow $F_V : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$. In the sequel we describe how to transform F_V into a flow F with the properties (F1)–(F5) stated in Section 2.

Choose eigenvectors $w \in \mathbb{R}^3$ for the eigenvalue $u > 0$ of $DV(0)$, and $z \in \mathbb{C}^3$ for the eigenvalue $\sigma + i\,\mu$ of the complexification of $DV(0)$. Set $w_1 = \Re z \in \mathbb{R}^3$ and $w_2 = \Im z \in \mathbb{R}^3$. Then $E_s = \mathbb{R}w_1 \oplus \mathbb{R}w_2$ and $E_u = \mathbb{R}w$ are invariant under $DV(0)$. For the isomorphism $B : \mathbb{R}^3 \to \mathbb{R}^3$ given by $Be_1 = w_1, Be_2 = w_2, Be_3 = w$ we have that the linear map $B^{-1}DV(0)B$ is multiplication with the matrix

$$A = \begin{pmatrix} \sigma & \mu & 0 \\ -\mu & \sigma & 0 \\ 0 & 0 & u \end{pmatrix},$$

which leaves $L = B^{-1}E_s$ and $U = B^{-1}E_u$ invariant.

Recall the local stable and unstable manifolds W_s and W_u of Eq. (1.1) at the origin. Given any $\lambda > 0$ we may assume that with some $r = r(\lambda) > 0$ they have the form

$$W_s = \{x + w_s(x) : x \in E_s, |x| < r\} \quad \text{and} \quad W_u = \{y + w_u(y) : y \in E_u, |y| < r\}$$

for continuously differentiable maps

$$w_s : \{x \in E_s : |x| < r\} \to E_u \quad \text{and} \quad w_u : \{y \in E_u : |y| < r\} \to E_s$$

which satisfy $w_s(0) = 0, Dw_s(0) = 0, w_u(0) = 0, Dw_u(0) = 0$, and

$$|Dw_s(x)| < \lambda \quad for \quad x \in E_s \quad \text{with} \quad |x| < r, \quad |Dw_u(y)| < \lambda \quad \text{for} \quad y \in E_u \quad \text{with} \quad |y| < r.$$

For sufficiently small neigbourhoods N of the origin in $\mathbb{R}^3$ the submanifolds W_s and W_u are invariant under F_V in N, and $F_V(t,x) \in N$ for all $t \geq 0$ implies $x \in W_s$ while $F_V(t,x) \in N$ for all $t \leq 0$ implies $x \in W_u$.

There exist $\hat{r} \in (0,r)$ and a continuously differentiable extension $\hat{w}_s : E_s \to E_u$ of the restriction of w_s to $\{x \in E_s : |x| \leq \hat{r}\}$ with $|D\hat{w}_s(x)| < \lambda$ on E_s, and analogously there is a continuously differentiable extension $\hat{w}_u : E_u \to E_s$ of the restriction of w_u to $\{y \in E_u : |y| \leq \hat{r}\}$ with $|D\hat{w}_u(y)| < \lambda$ on E_u. For $\lambda > 0$ sufficiently small the continuously differentiable map

$$S : \mathbb{R}^3 \to \mathbb{R}^3, \quad S(x+y) = x + y - \hat{w}_s(x) - \hat{w}_u(y) \quad \text{for } x \in E_s, y \in E_u,$$

with $S(0) = 0$ and $DS(0) = id_{\mathbb{R}^3}$ satisfies

$$|DS(z) - id_{\mathbb{R}^3}| \leq \frac{1}{2} \quad \text{for all } z \in \mathbb{R}^3.$$

It follows that all derivatives of S are isomorphisms. Using integration along line segments and the previous estimate one shows that S is one-to-one. In order to see that S is onto notice that for every $\zeta \in \mathbb{R}^3$ the map $z \mapsto \zeta - (S(z) - z)$ is a strict contraction, whose fixed point is a preimage of ζ under S. As all derivatives of S are isomorphisms applications of the Inverse Mapping Theorem yield that S^{-1} is continuously differentiable. Altogether, S is a diffeomorphism which maps $\hat{W}_s = \{x + \hat{w}_s(x) : x \in E_s\}$ onto E_s and $\hat{W}_u = \{y + \hat{w}_u(y) : y \in E_u\}$ onto E_u .

Define the continuously differentiable flow $F : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ by

$$F(t, x) = B^{-1}S(F_V(t, S^{-1}(Bx))).$$

For all $t \in \mathbb{R}$, $F(t, 0) = 0$, which is property (F1).

On (F2): Necessarily, $h_V'(t) \neq 0$ for all $t \in \mathbb{R}$. From the properties $h_V(t) \neq 0 \neq h_V'(t)$ for all $t \in \mathbb{R}$ and $\lim_{|t|\to\infty} h_V(t) = 0$ in combination with the fact that $B^{-1} \circ S$ is a diffeomorphism with fixed point 0 we infer that the flowline $h = B^{-1} \circ S \circ h_V$ of F satisfies $h(t) \neq 0 \neq h'(t)$ for all $t \in \mathbb{R}$ and $\lim_{|t|\to\infty} h(t) = 0$.

On (F3): For $t \in \mathbb{R}$ let $T(t) = D_2F(t, 0)$. Then

$$T(t) = B^{-1}DS(0)D_2F_V(t, S^{-1}(B0))DS^{-1}(0)B = B^{-1}D_2F_V(t, 0)B = B^{-1}e^{t\,DV(0)}B = e^{t\,B^{-1}DV(0)B}.$$

As the linear map $B^{-1}DV(0)B$ is multiplication by the matrix A we get $T(t)x = e^{t\,B^{-1}DV(0)B}x = e^{t\,A} \cdot x$ for all $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$. It follows that $T(t)L \subset L$ and $T(t)U \subset U$ for every $t \in \mathbb{R}$. The representation of $y = T(t)x = e^{t\,A} \cdot x$ described in property (F3) is confirmed by solving the initial value problems

$$x' = Ax, \quad x(0) = e_\nu, \quad \nu = 1, 2, 3,$$

whose solutions coincide at $t \in \mathbb{R}$ with the columns of the matrix $e^{t\,A}$.

On (F4). We show that for a neighbourhood N of the origin in $\mathbb{R}^3$ so small that W_s is invariant under F_V in N and

$$|x| < \hat{r} \quad \text{for all} \quad x \in E_s \quad \text{and} \quad y \in E_u \quad \text{with} \quad x + y \in N$$

the space L is invariant under F in $\tilde{N} = B^{-1}S(N)$.

Proof. Let $z \in L \cap \tilde{N}$ and $t \in \mathbb{R}$ be given with $F(\tau, z) \in \tilde{N}$ for all $\tau \in t \cdot [0, 1]$. As $B^{-1} \circ S$ maps $\hat{W}_s$ onto L we get $x = S^{-1}(Bz) \in \hat{W}_s \cap N$, and for every $\tau \in t \cdot [0, 1]$,

$$F_V(\tau, x) = S^{-1}(BF(\tau, B^{-1}S(x))) = S^{-1}(BF(\tau, z)) \in S^{-1}(B\tilde{N}) = N.$$

By the choice of N, $|\xi| < \hat{r}$ for $\xi \in E_s$ with $x = \xi + \hat{w}_s(\xi)$. Hence $\hat{w}_s(\xi) = w_s(\xi)$, and thereby $x \in W_s \cap N$. The invariance of W_s under F_V in N yields $F_V(t, x) \in W_s \cap N$. As above, $|\xi_t| < \hat{r}$ for $\xi_t \in E_s$ with $F_V(t, x) = \xi_t + w_s(\xi_t)$. It follows that $w_s(\xi_t) = \hat{w}_s(\xi_t)$, and thereby

$$F(t, z) = B^{-1}S(F_V(t, S^{-1}(Bz))) = B^{-1}S(F_V(t, x)) = B^{-1}S(\xi_t + \hat{w}_s(\xi_t)) \in L,$$

which shows the desired invariance of L under F in $\tilde{N}$.

It follows easily that for some $r_L > 0$ the space L is invariant under F in $\{z \in \mathbb{R}^3 : |z| < r_L\}$. Analogously U is invariant under F in $\{z \in \mathbb{R}^3 : |z| < r_U\}$, for some $r_U > 0$. Let

$r_F = \min\{r_L, r_U\}$. Then both L and U are invariant under F in $\{z \in \mathbb{R}^3 : |z| < r_F\}$, which is property (F4).

On (F5). From $\lim_{t\to\infty} h_V(t) = 0$, $h_V(t) \in W_s \cap \{x + y : x \in E_s, y \in E_u, |x| < \hat{r}\}$ for $t > 0$ sufficiently large. By $\hat{w}_s(x) = w_s(x)$ for $x \in E_s$ with $|x| < \hat{r}$,

$$B^{-1}S(\{x + w_s(x) : x \in E_s, |x| < \hat{r}\}) \subset B^{-1}S(\hat{W}_s) = L.$$

Consequently, there is $t_L \in \mathbb{R}$ with $h(t) = B^{-1}S(h_V(t)) \in L$ for all $t \geq t_L$. Analogously, $h(t) \in U$ for all $t \leq t_U$, with some $t_u < t_L$. By continuity and $h(t) \neq 0$ everywhere, either $h(t) \in (0,\infty)e_3$ for all $t \leq t_U$, or $h(t) \in (-\infty, 0)e_3$ for all $t \leq t_U$.

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