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Optimal $C^{1,\alpha}$ regularity for quasilinear elliptic equations with Orlicz growth

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Abstract. In this paper we obtain the interior optimal $C^{1,\alpha}$ regularity of weak solutions for the following quasilinear elliptic equations with Orlicz growth in divergence form

$$-\operatorname{div} a(x, Du) = -\operatorname{div} \mathbf{F} + f \quad \text{in } \Omega \subset \mathbb{R}^n,$$

including the following two special models

$$-\operatorname{div}\left((ADu \cdot Du)^{\frac{p-2}{2}} ADu \right) = -\operatorname{div} \mathbf{F} + f$$

and

$$-\operatorname{div}\left((ADu \cdot Du)^{\frac{p-2}{2}} \ln^{\theta}\left(1 + (ADu \cdot Du)^{\frac{1}{2}}\right) ADu \right) = -\operatorname{div} \mathbf{F} + f$$

for $\theta > 0$, $n \geq 2$, $\mathbf{F}(x) \in C^{0,\sigma_0}(\Omega)$ for some $\sigma_0 \in (0,1)$ and $f \in L^q_{loc}(\Omega)$ with $n < q \leq \infty$, where the symmetric matrix $A(x)$ of coefficients is a Hölder continuous function satisfying the uniformly elliptic condition. Moreover, we would like to remark that this work can be viewed as a continuation and follow-up to the works [4, 33, 34].

Keywords: $C^{1,\alpha}$, regularity, divergence, weak solutions, Orlicz, quasilinear, elliptic.

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1 Introduction

In this paper we are mainly concerned with the interior optimal $C^{1,\alpha}$ regularity estimates for weak solutions of the following quasilinear elliptic equations with Orlicz growth in divergence form

$$-\operatorname{div} a(x, Du) = -\operatorname{div} \mathbf{F} + f \quad \text{in } \Omega \subset \mathbb{R}^n, \tag{1.1}$$

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where $n \geq 2$, $f \in L^q(\Omega)$ for $n < q \leq \infty$, $\mathbf{F}(x) \in C^{0,\sigma_0}(\Omega)$ for some $\sigma_0 \in (0,1)$ and the vector field $a(x,\xi) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1-regular in the variable ξ satisfying

$$|a(x,\xi)| + \left|D_\xi a(x,\xi)\right| |\xi| \leq C\varphi(|\xi|), \tag{1.2}$$

$$D_\xi a(x,\xi)\eta \cdot \eta \geq C\varphi'(|\xi|)|\eta|^2, \tag{1.3}$$

$$|a(x_1,\xi) - a(x_2,\xi)| \leq w\left(|x_1 - x_2|\right) \varphi(|\xi|). \tag{1.4}$$

Here $w : [0,+\infty) \to [0,+\infty)$ belongs to $C^{0,\sigma}\left([0,+\infty)\right)$ for some $\sigma \in (0,1)$, $w(0) = 0$ and the function $\varphi(t) : [0,+\infty) \to [0,+\infty)$ belongs to $C^1\left([0,+\infty)\right)$ satisfying

$$0 < i_a := \inf_{t>0} \frac{t\varphi'(t)}{\varphi(t)} \leq \sup_{t>0} \frac{t\varphi'(t)}{\varphi(t)} =: s_a < \infty. \tag{1.5}$$

Actually, the two special models of (1.1) are the nonhomogeneous p-Laplacian equation with varying coefficients in divergence form

$$-\operatorname{div}\left((ADu \cdot Du)^{\frac{p-2}{2}} ADu\right) = -\operatorname{div}\mathbf{F}+f$$

and the nonhomogeneous p-Laplacian equation with varying coefficients and the logarithmic growth in divergence form

$$-\operatorname{div}\left((ADu \cdot Du)^{\frac{p-2}{2}} \ln^\theta\left(1 + (ADu \cdot Du)^{\frac{1}{2}}\right) ADu\right) = -\operatorname{div}\mathbf{F}+f$$

for $\theta > 0$.

It is well-known that the classical elliptic p-Laplacian equation

$$-\operatorname{div}\left(|Du|^{p-2} Du\right) = f$$

can be derived from the variational problem

$$\mathcal{P}(u,\Omega) := \min_{v-g \in W_0^{1,p}(\Omega)} \mathcal{P}(v,\Omega),$$

where

$$\mathcal{P}(v,\Omega) := \int_\Omega \left(\frac{1}{p}|\nabla v|^p - fv\right) dx \quad \text{for any } v \in W^{1,p}(\Omega).$$

Then $\mathcal{P}(v,\Omega)$ attains its minimum at a unique function u, which implies u is the weak solution of elliptic p-Laplacian equation with the boundary condition $u = g$ on $\partial\Omega$, satisfying

$$\int_\Omega \left[|Du|^{p-2} Du \cdot D\phi - f\phi\right] dx = 0 \quad \text{for any } \phi \in W_0^{1,p}(\Omega).$$

In reality, the nonlinear elliptic and parabolic PDEs can be derived from many important practical problems among the natural sciences: nonlinear elasticity mechanics and dynamic glaciology, non-Newtonian fluid mechanics, turbulent flows of a gas in porous media, thermodynamics and so on. At the same time, they can also come from some financial and economic problems and simultaneously the solutions of the nonlinear PDEs and their properties illustrate the features of these problems. Since the structure models in some real financial products and the option price can be reduced to some nonlinear PDE boundary problems, it is useful to adopt the existing theory and methods of PDEs as a fundamental approach to the study of the financial and economic theory (see [22]). For these reasons it is very meaningful and useful for us to study various kinds of regularity estimates for the nonlinear elliptic and parabolic PDEs with different coefficients and domain conditions. There have been a wide research activities [5, 6, 8–10, 17, 23, 25] on L^p-type estimates for weak solutions of elliptic quasilinear equations of p-Laplacian type with different coefficient and domain assumptions. On the other hand, Duzaar, Kuusi and Mingione [19, 20, 24] also made a deep study of sharp local a priori estimates and regularity results for solutions to

$$-\operatorname{div} a(x, Du) = \mu,$$

whose prototype is the elliptic p-Laplacian equation with coefficients and the right-hand side measure μ

$$-\operatorname{div}\left(\gamma(x)\,|Du|^{p-2}\,Du\right) = \mu.$$

Moreover, Cianchi and Maz'ya [12–14] proved global Lipschitz regularity and sharp estimates for weak solutions of

$$\operatorname{div}\left(\varphi'\left(|\nabla u|\right)\nabla u\right) = f \quad \text{in } \Omega \tag{1.6}$$

with the condition (1.5). Meanwhile, Baroni [7] proved pointwise gradient estimates via linear Riesz potentials for solutions to the following nonlinear elliptic equations with the right-hand side measure

$$\operatorname{div}\left(\varphi'\left(|\nabla u|\right)\nabla u\right) = \mu.$$

The theory of $C^{1,\alpha}$ regularity estimates for weak solutions of the elliptic p-Laplacian equation

$$-\operatorname{div}\left(|Du|^{p-2}\,Du\right) = 0 \tag{1.7}$$

is well-known. Evans, Lewis, Tolksdorf, Uhlenbeck and Ural'ceva (see [21, 26, 29–31] and references therein) have studied the theory of $C^{1,\alpha}$ regularity estimates for weak solutions of the elliptic p-Laplace equation (1.7) and the more general cases with variable coefficients. Moreover, Wang [32] used compactness methods to give a different proof of the interior $C^{1,\alpha}$ regularity for weak solutions of (1.7). In addition, Colombo and Mingione [15, 16] obtained the interior $C^{1,\alpha}$ regularity of weak solutions for a class of variational problems whose model is given by the functional

$$w \mapsto \int \left(|Dw|^p + a(x)\,|Dw|^q\right) dx.$$

A longstanding conjecture in elliptic regularity theory inquires whether a $W^{1,p}$ function whose p-Laplacian is bounded is locally of class $C^{p'} = C^{1,\frac{1}{p-1}}$. Recently, Araújo, Teixeira and Urbano [2] proved the planar counterpart of $C^{p'}$-regularity in the plane that weak solutions of the elliptic degenerate p-Laplacian equation

$$-\operatorname{div}\left(|Du|^{p-2}\,Du\right) = f(x) \quad \text{in } \Omega \subset \mathbb{R}^2$$

with a bounded source $f \in L^\infty$ are locally of class $C^{p'} = C^{1,\frac{1}{p-1}}$, in which they gave the precise control on a new oscillation of weak solutions in terms of the magnitude of its gradient and then improved $C^{1,\alpha}$ regularity estimates by geometric iteration. Moreover, we would like to remark that this regularity in [2] is optimal. Subsequently, Araújo, Teixeira and Urbano [3] solved the $C^{p'}$-regularity conjecture for weak solutions of the degenerate elliptic p-Laplacian equation in higher dimensions $n > 2$. Very recently, Araújo and Zhang [4] established the interior optimal sharp $C^{1,\alpha}$ estimates for weak solutions of quasilinear elliptic equations of p-Laplacian type with varying coefficients

$$-\operatorname{div} a(x, Du) = f \quad \text{in } \Omega \subset \mathbb{R}^n,$$

where $n \geq 2$, $f \in L^q(\Omega)$ for $n < q \leq \infty$ and the vector field $a(x,\xi) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1-regular in the variable ξ satisfying the following structural assumptions

$$\begin{aligned} |a(x,\xi)| + \left|D_\xi a(x,\xi)\right| |\xi| &\leq C|\xi|^{p-1}, \\ D_\xi a(x,\xi)\eta \cdot \eta &\geq C|\xi|^{p-2}|\eta|^2, \\ |a(x_1,\xi) - a(x_2,\xi)| &\leq w\left(|x_1 - x_2|\right)|\xi|^{p-1} \end{aligned}$$

with $w \in C^{0,\sigma}(\Omega)$ for some $\sigma \in (0,1)$, $w(0) = 0$ and $f \in L^q(\Omega)$. Additionally, Ding, Zhang and Zhou [18] then studied the optimal $C^{1,\alpha}$ estimates for the elliptic $p(x)$-Laplacian equation

$$\operatorname{div}\left(a(x)\,|Du|^{p(x)-2}\,Du\right) = \operatorname{div}\mathbf{h}(x) + f(x) \quad \text{in } \Omega.$$

Just recently Teixeira [28] undertook further research of the p-degenerate elliptic equations in a heterogeneous medium

$$-\operatorname{div} a(x, Du) = f(x, u, Du) \quad \text{in } \Omega,$$

which cover the following degenerate-elliptic nonhomogeneous PDE of the general form

$$-\operatorname{div}\left(|Du|^{p-2}\,Du\right) = f(x, u, Du) \quad \text{in } \Omega.$$

For the sake of convenience, we first elaborate on some definitions and fundamental results about the general Orlicz spaces, which have been widely used in the area of analysis as one of the most natural generalizations of Sobolev spaces (see [1,27]). A function $\Phi : [0,+\infty) \to [0,+\infty)$ is said to be a Young function if it is convex and $\Phi(0) = 0$. Moreover, a Young function Φ is called an N-function if $0 < \Phi(t) < \infty$ for $t > 0$ and

$$\lim_{t\to+\infty} \frac{\Phi(t)}{t} = \lim_{t\to 0+} \frac{t}{\Phi(t)} = +\infty.$$

Additionally, we call that a Young function Φ belongs to Δ_2 if there exists a positive constant K such that

$$\Phi(2t) \leq K\Phi(t) \quad \text{for any } t > 0.$$

Furthermore, we say that a Young function Φ belongs to ∇_2 if there exists a number $\theta > 1$ such that

$$\Phi(t) \leq \frac{\Phi(\theta t)}{2\theta} \quad \text{for any } t > 0.$$

Definition 1.1 (see [1]). *The Orlicz class $K^\Phi(\Omega)$ is the set of all measurable functions $g : \Omega \to \mathbb{R}$ satisfying*

$$\int_\Omega \Phi(|g|)dx < \infty.$$

The Orlicz space $L^\Phi(\Omega)$ is the linear hull of $K^\Phi(\Omega)$. Furthermore, we define $W^{1,\Phi}(\Omega)$ as

$$W^{1,\Phi}(\Omega) := \left\{u \in L^\Phi(\Omega) \mid Du \in L^\Phi(\Omega)\right\}.$$

The space $W_0^{1,\Phi}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,\Phi}(\Omega)$.

In this paper we define

$$\Phi(t) := \int_0^t \varphi(\tau)d\tau \quad \text{for } t \geq 0. \tag{1.8}$$

Then from (1.5) it is easy to check that

$$\varphi(t) \text{ is strictly increasing and continuous over } [0, +\infty], \tag{1.9}$$

and

$$\Phi(t) \text{ is increasing over } [0, +\infty]. \tag{1.10}$$

As usual, the solutions of (1.1) are taken in a weak sense. We now state the definition of weak solutions.

Definition 1.2. A function $u \in W^{1,\Phi}_{loc}(\Omega)$ is a local weak solution of of (1.1) in Ω if

$$\int_\Omega \left[a(x, Du) \cdot D\phi - \mathbf{F} \cdot D\phi - f\phi\right] dx = 0$$

holds for any $\phi \in C_0^\infty(\Omega)$.

A direct example

$$v(x) = \frac{p-1}{p}|x|^{\frac{p}{p-1}} = \frac{p-1}{p}|x|^{1+\frac{1}{p-1}} \quad \text{for } p > 1,$$

which satisfies the elliptic p-Laplacian equation

$$\operatorname{div}\left(|Dv|^{p-2}Dv\right) = n,$$

shows that if the optimal regularity for the elliptic p-Laplacian equations is $C^{1,\alpha}$, then we can get to know that α is surely smaller than $\frac{1}{p-1}$. In particular, in [34] the authors proved that weak solutions for a class of the homogeneous quasilinear elliptic equations

$$\operatorname{div}\left(\varphi'\left(|\nabla u|\right)\nabla u\right) = 0 \quad \text{in } \Omega \tag{1.11}$$

with $f, w \equiv 0$ are locally $C^{1,\alpha}$ for some $\alpha \in (0,1)$ depending on n, i_a, s_a. Following the same approach as Definition 3 of the article [28], we define the maximal C^{1,α_m} regularity for weak solutions of (1.11) as corresponding to the maximal $\alpha_m \in (0,1]$, where α_m is given by

$$\alpha_m := \sup\left\{\begin{array}{r} \alpha \in (0,1) \mid h \text{ belongs to } C^{1,\alpha}_{loc}(\Omega) \text{ for every local weak solution} \\ h \in W^{1,\Phi}_{loc}(\Omega) \text{ of (1.11) with the condition (1.5)} \end{array}\right\}.$$

Furthermore, in [33] authors have also extended the corresponding results in [34] to the non-uniformly nonlinear elliptic equations.

Now let us state the main result of this work.

Theorem 1.3. *Assume that the conditions (1.2)–(1.5) hold and let $u \in W^{1,\Phi}_{loc}(\Omega)$ be a local weak solution of (1.1). Then we have $u \in C^{1,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0,1)$, where*

$$\alpha := \min\left\{\alpha_m - \epsilon, \alpha_{\sigma, i_a, s_a, q}\right\} \tag{1.12}$$

for any positive number ϵ less than α_m, and

$$\alpha_{\sigma,i_a,s_a,q} := \begin{cases} \min\left\{\sigma,\sigma_0,1-\dfrac{n}{q}\right\}\cdot\min\left\{1,\dfrac{1}{s_a}\right\} & \text{if } n<q<\infty, \\ \min\{\sigma,\sigma_0\}\cdot\min\left\{1,\dfrac{1}{s_a}\right\} & \text{if } q=\infty. \end{cases} \tag{1.13}$$

Moreover, for every $\Omega' \subset\subset \Omega$, the following estimates

$$\sup_{\substack{x,y\in\Omega' \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}} \le C \tag{1.14}$$

and

$$\sup_{\substack{x,y\in\Omega' \\ x\neq y}} \frac{|Du(x)-Du(y)|}{|x-y|^{\alpha}} \le C \tag{1.15}$$

hold, where C is a positive constant depending on $n, i_a, s_a, \alpha, \sigma, \sigma_0$, $\|u\|_{W^{1,\Phi}_{loc}(\Omega)}$, $\|w\|_{C^{0,\sigma}_{loc}(\Omega)}$, $[\boldsymbol{F}]_{C^{0,\sigma_0}_{loc}(\Omega)}$, $\|f\|_{L^q_{loc}(\Omega)}$ and dist $(\Omega',\partial\Omega)$.

2 Auxiliary lemmas

In the following, we will introduce several auxiliary lemmas, which will be employed in the proof of the conclusions stated in Theorem 1.3. Moreover, we stress the point that the proof is much influenced by the papers [2–4, 11]. Let us first state a crucial lemma under present assumptions on the function $\varphi(t)$.

Lemma 2.1. *If $\varphi(t)$ satisfies (1.5), then we find that*

$$\theta^{i_a}\varphi(t) \le \varphi(\theta t) \le \theta^{s_a}\varphi(t) \quad \text{for } \theta \ge 1 \tag{2.1}$$

and

$$C\theta^{i_a-1}\varphi'(t) \le \varphi'(\theta t) \le C\theta^{s_a-1}\varphi'(t) \quad \text{for } \theta \ge 1. \tag{2.2}$$

Proof. Firstly, (1.5) implies that

$$\frac{i_a}{\theta t} \le \frac{\varphi'(\theta t)}{\varphi(\theta t)} \le \frac{s_a}{\theta t} \quad \text{for any } t>0,$$

which implies that

$$\int_1^{\theta} \frac{i_a}{\theta t} d\theta \le \frac{1}{t}\ln\left[\frac{\varphi(\theta t)}{\varphi(t)}\right] = \int_1^{\theta}\frac{\varphi'(\theta t)}{\varphi(\theta t)}d\theta \le \int_1^{\theta}\frac{s_a}{\theta t}d\theta.$$

So, we can get the desired estimate (2.1)

$$\theta^{i_a}\varphi(t) \le \varphi(\theta t) \le \theta^{s_a}\varphi(t) \quad \text{for } \theta \ge 1.$$

On the other hand, by using (1.5) again we deduce that

$$i_a\frac{\varphi(\theta t)}{\theta t} \le \varphi'(\theta t) \le s_a\frac{\varphi(\theta t)}{\theta t} \quad \text{for any } t>0$$

and

$$i_a \frac{\varphi(t)}{t} \leq \varphi'(t) \leq s_a \frac{\varphi(t)}{t} \quad \text{for any } t > 0,$$

which imply, thanks to (2.1), that

$$\frac{i_a}{s_a} \theta^{i_a - 1} \varphi'(t) \leq i_a \frac{\theta^{i_a} \varphi(t)}{\theta t} \leq \varphi'(\theta t) \leq s_a \frac{\theta^{s_a} \varphi(t)}{\theta t} \leq \frac{s_a}{i_a} \theta^{s_a - 1} \varphi'(t).$$

Thus, we finish the proof of this lemma. □

In the subsequent analysis, we proceed to verify that the weak solution u of equation (1.1) can be approximated by that of a reference equation with constant coefficients, which exhibits favorable analytical properties. Before proceeding, we first define the ball $B_r(x_0)$ as the open ball with center x_0 and radius r, and specify B_r as $B_r := B_r(0)$.

Lemma 2.2. *Assume that the conditions (1.2)–(1.5) hold and let $u \in W^{1,\Phi}_{loc}(\Omega)$ be a local weak solution of (1.1) in $\Omega \supset B_1$ with $\|u\|_{L^\infty(B_1)} \leq 1$ and $u(0) = 0$. For any $\epsilon > 0$, there exists a positive constant δ, depending on $n, q, i_a, s_a, \sigma, \sigma_0$ and ϵ, such that if*

$$\|f\|_{L^q(B_1)} \leq \delta, \qquad [\mathbf{F}]_{C^{0,\sigma_0}(B_1)} \leq \delta$$

and

$$\sup_{B_1} |a(x,\xi) - a(0,\xi)| \leq \varphi(|\xi|)\delta,$$

then there exists a function h weak solution of

$$-\operatorname{div} \bar{a}(Dh) = 0 \quad \text{in } B_{3/4} \text{ with } h(0) = 0 \tag{2.3}$$

for a constant coefficient field $\bar{a}$ satisfying (1.2)–(1.5) and $w \equiv 0$, such that

$$\sup_{B_{1/2}} |u - h| + |Du(0) - Dh(0)| \leq \epsilon.$$

Proof. We prove this by contradiction. Suppose, for the sake of contradiction, that the result is false. Then there exist $\epsilon_0 > 0$, $\{f_j\}_{j=1}^\infty$, $\{\mathbf{F}_j\}_{j=1}^\infty$, $\{a_j(x,\xi)\}_{j=1}^\infty$ and $\{u_j\}_{j=1}^\infty$ satisfying

$$\int_\Omega \left[a_j(x, Du_j) \cdot D\phi - \mathbf{F}_j \cdot D\phi - f_j \phi\right] dx = 0 \quad \text{for any } \phi \in C_0^\infty(\Omega), \tag{2.4}$$

$$\|u_j\|_{L^\infty(B_1)} \leq 1,$$
$$u_j(0) = 0,$$
$$\|f_j\|_{L^q(B_1)} \leq \frac{1}{j},$$
$$[\mathbf{F}_j]_{C^{0,\sigma_0}(B_1)} \leq \frac{1}{j},$$
$$|a_j(x,\xi) - a_j(0,\xi)| \leq \frac{1}{j} \varphi(|\xi|),$$

so that for any weak solution h of the homogeneous constant-coefficient equation (2.3) in $B_{3/4}$ satisfying $h(0) = 0$, we have

$$\sup_{B_{1/2}} |u_j - h| + |Du_j(0) - Dh(0)| \geq \epsilon_0. \tag{2.5}$$

In light of this, a standard regularity estimate for weak solutions of (2.4) guarantees that the sequence $\{u_j\}_{j=1}^{\infty}$ satisfies $|Du_j| \le L/2$ for some $L > 0$ and all $x \in B_{3/4}$, and forms a pre-compact sequence in the C^1-topology (see [7, Corollary 1.3]). This further implies the existence of a function $u_\infty \in C^1$ with $u_\infty(0) = 0$ and

$$\sup_{B_{1/2}} |u_j - u_\infty| + |Du_j(0) - Du_\infty(0)| \to 0 \quad \text{as } j \to +\infty. \tag{2.6}$$

Define

$$b_j(x,\xi) := a_j(x,\xi)\chi_{\{|\xi|\le L\}} + L\,\chi_{\{|\xi|\ge L\}}.$$

Since $\{b_j(0,\cdot)\}$ is equicontinuous and uniformly bounded, by the Ascoli–Arzelà theorem, we can find $b_\infty(0,\cdot)$ such that

$$b_j(0,\cdot) \to b_\infty(0,\cdot) \quad \text{uniformly in } B_L := \{|\xi| \le L\} \text{ as } j \to \infty,$$

which implies that

$$\left|a_j(x,\xi) - b_\infty(0,\xi)\right| \le \varphi(|\xi|)\frac{1}{j} + \left|a_j(0,\xi) - b_\infty(0,\xi)\right| \to 0 \quad \text{as } j \to +\infty \tag{2.7}$$

uniformly in $B_{3/4} \times B_L$. Now, applying (2.4), (2.6)–(2.7) and standard arguments, we find that u_∞ is a weak solution of the constant-coefficient equation

$$\int_{B_{3/4}} b_\infty(0, Du_\infty)\cdot D\phi\,dx = 0 \quad \text{for any } \phi \in C_0^\infty(B_{3/4}) \tag{2.8}$$

with $u_\infty(0) = 0$. However, this contradicts (2.5) by (2.6) because u_∞ can serve as the function h described in (2.5). Thus, the proof is complete. □

In addition, by making full use of the above lemma we are able to prove the following crucial results which play an essential role in the subsequent discussion.

Lemma 2.3. *Assume that the conditions (1.2)–(1.5) hold and let $u \in W^{1,\Phi}_{loc}(\Omega)$ be a local weak solution of (1.1) in $\Omega \supset B_1$ with $\|u\|_{L^\infty(B_1)} \le 1$ and $u(0) = 0$.*

1. *There exist two positive constants δ_0 and $\rho_0 \in (0,1)$, depending on $n, q, i_a, s_a, \sigma, \sigma_0$ and α, such that if*

$$\|f\|_{L^q(B_1)} \le \delta_0, \quad [\mathbf{F}]_{C^{0,\sigma_0}(B_1)} \le \delta_0, \quad \sup_{B_1}\left|a(x,\xi) - a(0,\xi)\right| \le \varphi(|\xi|)\delta_0 \tag{2.9}$$

and

$$|Du(0)| \le \frac{1}{4}\rho_0^\alpha,$$

then we have

$$\sup_{B_{\rho_0}} |u(x)| \le \rho_0^{1+\alpha}.$$

2. *If*

$$|Du(0)| \le \frac{1}{4}\rho_0^{i\alpha}$$

for some positive integer $i \in \mathbb{N}$ and (2.9) hold, then we have

$$\sup_{B_{\rho_0^i}} |u(x)| \le C\rho_0^{i(1+\alpha)},$$

where C depends on $n, q, i_a, s_a, \sigma, \sigma_0$ and α.

3. *If*

$$|Du(0)| \le \frac{1}{4}\rho^{\alpha}$$

for any $0 < \rho \le \rho_0 < 1$ *and* (2.9) *hold, then we have*

$$\sup_{B_\rho} |u(x)| \le C\rho^{1+\alpha},$$

where C *depends on* $n, q, i_a, s_a, \sigma, \sigma_0$ *and* α.

Proof. (1) In accordance with Lemma 2.2, for any $\epsilon > 0$, there exists a weak solution h to the constant-coefficient equation (2.3) satisfying

$$|Du(0) - Dh(0)| \le \epsilon$$

and

$$\sup_{B_\rho} |u| \le \epsilon + \sup_{B_\rho} |h|$$

for any $0 < \rho \le 1/2$. Meanwhile, from the interior C^{1,α_m}-regularity estimates for the constant coefficient equation (2.3) together with the fact that $h(0) = 0$, we conclude that

$$\sup_{B_\rho} |h| \le C\rho^{1+\alpha_m} + |Dh(0)|\rho,$$

where the positive constant $C > 1$ depends on n, i_a and s_a, which implies that

$$\begin{aligned}
\sup_{B_{\rho_0}} |u| &\le \epsilon + C\rho_0^{1+\alpha_m} + (\epsilon + |Du(0)|)\,\rho_0 \\
&\le \epsilon + C\rho_0^{1+\alpha_m} + \left(\epsilon + \frac{1}{4}\rho_0^{\alpha}\right)\rho_0 \\
&\le \rho_0^{1+\alpha}
\end{aligned}$$

by choosing $\rho_0 = (4C)^{\frac{1}{\alpha - \alpha_m}}$ and $\epsilon = \frac{1}{4}\rho_0^{1+\alpha}$.

(2) We prove it by induction. From (1) we know that the result is true for $i = 1$. Let us assume the conclusion is true for $1, 2, \ldots i$. Suppose u satisfies

$$|Du(0)| \le \frac{1}{4}\rho_0^{(i+1)\alpha}. \tag{2.10}$$

We denote $u_i(x)$ by

$$u_i(x) := \frac{u\left(\rho_0^i x\right)}{\rho_0^{i(\alpha+1)}}.$$

Then $u_i(x)$ solves

$$-\operatorname{div} a_i(x, Du_i) = f_i \quad \text{in } B_1,$$

where

$$\begin{aligned}
a_i(x,\xi) &:= \frac{a(\rho_0^i x, \rho_0^{i\alpha}\xi)}{\varphi(\rho_0^{i\alpha})}, \\
f_i &:= \frac{\rho_0^i f(\rho_0^i x)}{\varphi(\rho_0^{i\alpha})}, \\
\mathbf{F}_i &:= \frac{\mathbf{F}(\rho_0^i x)}{\varphi(\rho_0^{i\alpha})}.
\end{aligned}$$

We set

$$\varphi_i(t) := \frac{\varphi(\rho_0^{i\alpha} t)}{\varphi(\rho_0^{i\alpha})}.$$

It is easy to check that $a_i(x,\xi)$ and φ_i still satisfy the assumptions (1.2)–(1.5), and $f_i(x)$ and $\mathbf{F}_i(x)$ satisfy

$$\|f_i\|_{L^q(B_1)} \leq \frac{\rho_0^{i(1-\frac{n}{q})}}{\varphi(\rho_0^{i\alpha})}\|f\|_{L^q(B_1)} \leq \rho_0^{i(1-\frac{n}{q}-\alpha s_a)}\|f\|_{L^q(B_1)} \leq \delta_0$$

and

$$\begin{aligned}
[\mathbf{F}_i]_{C^{0,\sigma_0}(B_1)} &:= \sup_{x,y\in B_1} \frac{|\mathbf{F}_i(x)-\mathbf{F}_i(y)|}{|x-y|^{\sigma_0}} \\
&= \frac{1}{\varphi(\rho_0^{i\alpha})} \sup_{x,y\in B_1} \frac{|\mathbf{F}(\rho_0^i x)-\mathbf{F}(\rho_0^i y)|}{|x-y|^{\sigma_0}} \\
&\leq \frac{\rho_0^{i\sigma_0}}{\varphi(\rho_0^{i\alpha})}[\mathbf{F}]_{C^{0,\sigma_0}(B_1)} \\
&\leq \rho_0^{i(\sigma_0-\alpha s_a)}[\mathbf{F}]_{C^{0,\sigma_0}(B_1)} \\
&\leq \delta_0
\end{aligned}$$

in view of (1.12), (1.13), (2.1) and the facts that

$$1-\frac{n}{q}-\alpha s_a \geq 0 \quad \text{and} \quad \sigma_0 - \alpha s_a \geq 0.$$

Moreover, using the inductive result for the case i, one can deduce that

$$|Du_i(0)| \leq \frac{1}{4}\rho_0^{\alpha}$$

by the assumed condition (2.10). Then it follows from (1) that

$$\sup_{B_{\rho_0}} |u_i(x)| \leq \rho_0^{1+\alpha}.$$

Finally, the definition of u_i implies that

$$\sup_{B_{\rho_0^{i+1}}} |u(x)| \leq \rho_0^{(i+1)(1+\alpha)},$$

which completes the proof of (2).

(3) For $0 < \rho \leq \rho_0 < 1$, there exists a positive integer $k \in \mathbb{N}$ such that $\rho_0^{k+1} < \rho \leq \rho_0^k$. By the given conditions, we derive

$$|Du(0)| \leq \frac{1}{4}\rho^{\alpha} \leq \frac{1}{4}\rho_0^{k\alpha}$$

and then by (2),

$$\sup_{B_\rho} |u(x)| \leq \sup_{B_{\rho_0^k}} |u(x)| \leq \rho_0^{k(1+\alpha)} = \rho_0^{-(1+\alpha)}\rho_0^{(k+1)(1+\alpha)} \leq \rho_0^{-(1+\alpha)}\rho^{1+\alpha}.$$

This completes our proof. $\square$

Our approach is based on the following two oscillation estimates, which are similarly to the classical Taylor expansion, themselves interesting and can reveal some essential qualities or characteristics of the solutions to the problem (1.1). More precisely, they can give the precise control on the oscillation of weak solutions to the problem (1.1) in term of the magnitude of the gradients

$$\sup_{B_r} |u(x) - u(0)| \lesssim r^{1+\alpha} + |Du(0)|\, r.$$

It is noteworthy that we shall need to solve some difficulties cased by the differences of balls B_r with large radii: $|Du(0)| \lesssim r$ and with small radii: $r \lesssim |Du(0)|$. Next, we shall first derive the following sharp regularity estimates for the large radii. Moreover, we would like to remark that the positive constants C are called universal if they only depend on $n, i_a, s_a, \sigma, \sigma_0, \alpha$, $\|u\|_{W^{1,\Phi}_{loc}(\Omega)}$, $\|w\|_{C^{0,\sigma}_{loc}(\Omega)}$, $[\mathbf{F}]_{C^{0,\sigma_0}_{loc}(\Omega)}$, $\|f\|_{L^q_{loc}(\Omega)}$ and $\operatorname{dist}(\Omega', \partial\Omega)$.

Proposition 2.4. *Assume that the conditions (1.2)–(1.5) hold and let $u \in W^{1,\Phi}_{loc}(\Omega)$ be a local weak solution of (1.1). Let $\Omega' \subset\subset \Omega$ and let $x_0 \in \Omega'$, there exist universal positive constants κ, C and $\bar\rho$ such that if*

$$|Du(x_0)| \le \kappa \rho^{\alpha} \tag{2.11}$$

for some $0 < \rho \le \rho_0$ with $B_{\rho_0}(x_0) \subset \Omega' \subset\subset \Omega$, then we have

$$\sup_{B_\rho(x_0)} \left|u(x) - u(x_0)\right| \le C\rho^{1+\alpha} \tag{2.12}$$

and

$$\sup_{B_\rho(x_0)} \left|u(x) - u(x_0) - Du(x_0)\cdot(x - x_0)\right| \le C\rho^{1+\alpha} \tag{2.13}$$

for any $0 < \rho \le \bar\rho < \rho_0 < 1$.

Proof. We denote $v(x)$ by

$$v(x) := \frac{u(x_0 + A_0 x) - u(x_0)}{B_0}$$

and then $v(x)$ solves

$$-\operatorname{div} a_0(x, Dv) = -\operatorname{div} \mathbf{F}_0 + f_0 \quad \text{in } B_1,$$

where

$$\begin{aligned}
A_0 &:= \min\left\{1, \frac{d_0}{2}, \frac{w^{-1}(\delta_0)}{d_0}\right\} \quad \text{with } d_0 = \operatorname{dist}\left(\Omega', \partial\Omega\right),\\
B_0 &:= \max\left\{1, 2\|u\|_{L^\infty(\Omega')}, \left(\|f\|_{L^q(\Omega')}\delta_0^{-1}\right)^{\frac{1}{i_a}}, \left([\mathbf{F}]_{C^{0,\sigma_0}(\Omega')}\delta_0^{-1}\right)^{\frac{1}{i_a}}\right\},\\
a_0(x,\xi) &:= \frac{a(x_0 + A_0 x, \frac{B_0}{A_0}\xi)}{\varphi\left(\frac{B_0}{A_0}\right)},\\
f_0 &:= \frac{A_0 f(x_0 + A_0 x)}{\varphi\left(\frac{B_0}{A_0}\right)},\\
\mathbf{F}_0 &:= \frac{\mathbf{F}_0(x_0 + A_0 x)}{\varphi\left(\frac{B_0}{A_0}\right)}.
\end{aligned}$$

We set

$$\varphi_0(t) := \frac{\varphi(\frac{B_0}{A_0}t)}{\varphi\left(\frac{B_0}{A_0}\right)}.$$

It is easy to check that $a_0(x,\xi)$ and φ_0 still satisfy the assumptions (1.2)–(1.5), $v(0) = 0$, $\|v\|_{L^\infty(B_1)} \leq 1$,

$$\begin{aligned}
|a_0(x,\xi) - a_0(0,\xi)| &= \left| \frac{a(x_0 + A_0 x, \frac{B_0}{A_0}\xi)}{\varphi\left(\frac{B_0}{A_0}\right)} - \frac{a(x_0, \frac{B_0}{A_0}\xi)}{\varphi\left(\frac{B_0}{A_0}\right)} \right| \\
&\leq w\left(A_0|x|\right)\varphi_0(|\xi|) \\
&\leq \delta_0 \varphi_0(|\xi|),
\end{aligned}$$

$$\begin{aligned}
\|f_0\|_{L^q(B_1)} &\leq \frac{A_0^{1-\frac{n}{q}}}{\varphi\left(\frac{B_0}{A_0}\right)} \|f\|_{L^q(\Omega')} \\
&\leq A_0^{1-\frac{n}{q}+i_a} B_0^{-i_a} \|f\|_{L^q(\Omega')} \\
&\leq B_0^{-i_a} \|f\|_{L^q(\Omega')} \leq \delta_0,
\end{aligned}$$

$$\begin{aligned}
[\mathbf{F}_0]_{C^{0,\sigma_0}(B_1)} &\leq \frac{A_0^{\sigma_0}}{\varphi\left(\frac{B_0}{A_0}\right)} [\mathbf{F}]_{C^{0,\sigma_0}(\Omega')} \\
&\leq A_0^{\sigma_0+i_a} B_0^{-i_a} [\mathbf{F}]_{C^{0,\sigma_0}(\Omega')} \\
&\leq \delta_0
\end{aligned}$$

in view of (1.5) and (2.1), and

$$|Dv(0)| \leq C\rho^\alpha$$

for $\rho \leq \rho_0$, by the assumed condition (2.11). Then by Lemma 2.3, we obtain

$$\sup_{B_\rho} |v(x)| \leq C\rho^{1+\alpha}$$

for some $0 < \rho \leq \rho_0$, which implies that

$$\sup_{B_{A_0\rho}(x_0)} |u(x) - u(x_0)| \leq C\rho^{1+\alpha} \leq C\left(A_0\rho\right)^{1+\alpha}.$$

Thus, we can finish the proof of (2.12)–(2.13) by choosing $\bar{\rho} = A_0\rho_0 \leq \rho_0$. □

Here we are going to prove another regularity estimates in the balls B_r with small radii $r \lesssim |Du|$ which enable the vector field $a(x,\xi)$ can be viewed as essentially a function with linear growth in the variable ξ.

Proposition 2.5. *Assume that the conditions (1.2)–(1.5) hold and let $u \in W^{1,\Phi}_{loc}(\Omega)$ be a local weak solution of (1.1). Let $\Omega' \subset\subset \Omega$ and let $x_0 \in \Omega'$, there exist universal positive constants κ, C and $\tilde{\rho}$ such that if*

$$|Du(x_0)| > \kappa\rho^\alpha \tag{2.14}$$

for $0<\rho\le\tilde{\rho}$ *with* $B_{\tilde{\rho}}(x_0)\subset\Omega'\subset\subset\Omega$, *then for* $0<\rho\le\tilde{\rho}$ *we have*

$$\sup_{B_\rho(x_0)}\left|u(x)-u(x_0)\right|\le C\rho^\alpha \tag{2.15}$$

and

$$\sup_{B_\rho(x_0)}\left|u(x)-u(x_0)-Du(x_0)\cdot(x-x_0)\right|\le C\rho^{1+\alpha}. \tag{2.16}$$

Proof. We define $v(x)$ by

$$v(x):=\frac{u(x_0+\rho_* x)-u(x_0)}{\rho_*^{1+\alpha}}$$

and then $v(x)$ solves

$$-\operatorname{div} a_*(x,Dv)=-\operatorname{div}\mathbf{F}_*(x)+f_*(x)\quad\text{in } B_1, \tag{2.17}$$

where

$$\begin{aligned}
\rho_* &:= \left[\kappa^{-1}|Du(x_0)|\right]^{\frac{1}{\alpha}},\\
a_*(x,\xi) &:= \frac{a(x_0+\rho_* x,\rho_*^\alpha\xi)}{\varphi\left(\rho_*^\alpha\right)},\\
f_* &:= \frac{\rho_* f(x_0+\rho_* x)}{\varphi\left(\rho_*^\alpha\right)},\\
\mathbf{F}_* &:= \frac{\mathbf{F}(x_0+\rho_* x)}{\varphi\left(\rho_*^\alpha\right)}.
\end{aligned}$$

We set

$$\varphi_*(t):=\frac{\varphi(\rho_*^\alpha t)}{\varphi(\rho_*^\alpha)}.$$

It is easy to check that $a_*(x,\xi)$ and φ_* still satisfy the assumptions (1.2)–(1.5). Now we divide into two cases.

Case 1: $\rho_*\le\bar{\rho}<1$, where $\bar{\rho}$ is defined in Proposition 2.4. Then we find that

$$|Du(x_0)|=\kappa\rho_*^\alpha\le\kappa\bar{\rho}^\alpha.$$

From Proposition 2.4, we know that

$$\sup_{B_{\rho_*}(x_0)}\left|u(x)-u(x_0)\right|\le C\rho_*^{1+\alpha} \tag{2.18}$$

and

$$\sup_{B_{\rho_*}(x_0)}\left|u(x)-u(x_0)-Du(x_0)\cdot(x-x_0)\right|\le C\rho_*^{1+\alpha}, \tag{2.19}$$

which implies that

$$\sup_{B_1}|v(x)|\le C\sup_{B_{\rho_*}(x_0)}\frac{\left|u(x)-u(x_0)\right|}{\rho_*^{1+\alpha}}\le C.$$

Consequently, by applying the C^0-estimates to Dv, there exists $\tau_*>0$ such that

$$\operatorname{osc}_{B_{\tau_*}}|Dv|<\frac{\kappa}{2}.$$

Since $Dv(0) = \kappa$, it follows that $|Dv(x)| > \kappa/2$ in B_{τ_*}. Thus, there is a universal constant $c_0 > 1$ satisfying

$$c_0^{-1} \le |Dv(x)| \le c_0 \quad \text{in } B_{\tau_*}.$$

At this stage, the nonlinear PDE (2.17) can be treated as essentially linear because

$$\begin{gathered}
|a_*(x,\xi)| + |D_\xi a_*(x,\xi)||\xi| \le C\varphi'(c_0)|\xi|, \\
D_\xi a_*(x,\xi)\eta \cdot \eta \ge C\varphi'\left(\frac{1}{c_0}\right)|\eta|^2, \\
|a_*(x_1,\xi) - a_*(x_2,\xi)| \le Cw(|x_1 - x_2|)\,\varphi'(c_0)|\xi|
\end{gathered}$$

in view of (1.5) and Lemma 2.1. Hence, by the $C^{1,\beta}$ estimates for linear equations (see Theorem 3.1 in [4]), we obtain

$$\sup_{B_r} |v(x) - v(0)| \le Cr^\beta$$

and

$$\sup_{B_r} |v(x) - v(0) - Dv(0)\cdot x| \le Cr^{1+\beta}$$

for any $0 < r \le \tau_*/2$ and

$$\beta := \begin{cases} \min\left\{\sigma, \sigma_0, 1 - \frac{n}{q}\right\} & \text{if } n < q < \infty, \\ \min\{\sigma, \sigma_0\} & \text{if } q = \infty. \end{cases}$$

Consequently, we deduce that

$$\sup_{B_{\rho_* r}(x_0)} |u(x) - u(x_0)| \le C\rho_*^{1+\alpha} r^\beta \le C(\rho_* r)^\alpha$$

and

$$\sup_{B_{\rho_* r}(x_0)} |u(x) - u(x_0) - Du(x_0)\cdot(x - x_0)| \le C\rho_*^{1+\alpha} r^{1+\beta} \le C(\rho_* r)^{1+\alpha}$$

for any $0 < r \le \tau_*/2$, which implies that

$$\sup_{B_\rho(x_0)} |u(x) - u(x_0)| \le C\rho^\alpha$$

and

$$\sup_{B_\rho(x_0)} |u(x) - u(x_0) - Du(x_0)\cdot(x - x_0)| \le C\rho^{1+\alpha}$$

for any $0 < \rho \le \tau_*\rho_*/2$. On the one hand, if $\tau_*\rho_*/2 < \rho \le \rho_* \le \bar{\rho} < 1$, (2.18) implies that

$$\sup_{B_\rho(x_0)} |u(x) - u(x_0)| \le C\rho_*^{1+\alpha} \le C\rho_*^\alpha \le C\left(\frac{2}{\tau_*}\right)^\alpha \rho^\alpha \le C\rho^\alpha$$

and then

$$\sup_{B_\rho(x_0)} |u(x) - u(x_0) - Du(x_0)\cdot(x - x_0)| \le C\rho_*^{1+\alpha} \le C\left(\frac{2}{\tau_*}\right)^{1+\alpha} \rho^{1+\alpha} \le C\rho^{1+\alpha},$$

which completes the proof of Case 1.

Case 2: $\rho_* > \bar{\rho}$. Then the definition of ρ_* implies that

$$|Du(x_0)| = \kappa\rho_*^\alpha > \kappa\bar{\rho}^\alpha.$$

Since the argument applied previously in Case 1 for the function v can also be directly used to the function u for the universal constant $\kappa\bar{\rho}^\alpha > 0$, we can directly get the desired results (2.15)–(2.16) for any $0 < \rho < \rho'$ with some positive constant ρ' and then finish the final proof of this proposition by choosing $\tilde{\rho} := \min\{\rho', \bar{\rho}\}$. □

3 Final proof

Finally, we shall combine the previous Proposition 2.4 and Proposition 2.5 for the large radii and small radii cases respectively to finish the proof of the main result in this paper.

Proof of Theorem 1.3. From Proposition 2.4 and Proposition 2.5 we can find that for any compact subset Ω' of Ω there exist positive constants κ, C and $\tilde{\rho}$, depending on $n, i_a, s_a, \alpha, \sigma, \sigma_0$, $\|u\|_{W^{1,\Phi}_{loc}(\Omega)}$, $\|w\|_{C^{0,\sigma}_{loc}(\Omega)}$, $[\mathbf{F}]_{C^{0,\sigma_0}_{loc}(\Omega)}$, $\|f\|_{L^q_{loc}(\Omega)}$ and $\operatorname{dist}(\Omega', \partial\Omega)$, such that for $0 < \rho \le \tilde{\rho} < 1$ with $B_{\tilde{\rho}}(x_0) \subset \Omega' \subset\subset \Omega$ we have

$$\sup_{B_\rho(x_0)} \left|u(x) - u(x_0)\right| \le C\rho^\alpha \tag{3.1}$$

and

$$\sup_{B_\rho(x_0)} \left|u(x) - u(x_0) - Du(x_0)\cdot(x - x_0)\right| \le C\rho^{1+\alpha}, \tag{3.2}$$

where α is defined in (1.13). Let $x, y \in \Omega'$ and $R = |x - y| \le \tilde{\rho} < 1$. Then from (3.1) we deduce that

$$\left|u(x) - u(y)\right| \le CR^\alpha.$$

And then we can conclude that

$$\sup_{\substack{x,y\in\Omega' \\ x\ne y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \le C$$

holds true by a standard finite covering argument, which implies that (1.14) is true. Furthermore, without loss of generality we may as well assume that $x = (x_1, x') \in \Omega'$ and $y = (x_2, x') \in \Omega'$ with $x' \in \mathbb{R}^{n-1}$. At this time we can obtain that $R = |x - y| = |x_1 - x_2| \le \tilde{\rho} < 1$,

$$u(y) = u(x) + \frac{\partial u(x)}{\partial x_1}(x_2 - x_1) + O\left(|x_1 - x_2|^{\alpha+1}\right)$$

and

$$u(x) = u(y) - \frac{\partial u(y)}{\partial x_1}(x_2 - x_1) + O\left(|x_1 - x_2|^{\alpha+1}\right),$$

which implies that

$$\left|\frac{\partial u(x)}{\partial x_1} - \frac{\partial u(y)}{\partial x_1}\right| |x_2 - x_1| = O\left(|x_1 - x_2|^{\alpha+1}\right).$$

Therefore, we can reach the conclusion that

$$\sup_{\substack{x,y\in\Omega' \\ x\ne y}} \frac{\left|\frac{\partial u(x)}{\partial x_1} - \frac{\partial u(y)}{\partial x_1}\right|}{|x - y|^\alpha} \le C$$

by a standard finite covering argument. Similarly, we also can prove that

$$\sup_{\substack{x,y\in\Omega' \\ x\neq y}} \frac{\left|\frac{\partial u(x)}{\partial x_i} - \frac{\partial u(y)}{\partial x_i}\right|}{|x-y|^{\alpha}} \leq C \quad \text{for } 2 \leq i \leq n.$$

Thus, we can finish the final proof of the conclusion (1.15). □

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