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Study of singular elliptic equations with mixed boundary conditions and nonlocal source terms

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Abstract. In this article, we study the generalized solutions for singular elliptic equations with mixed boundary conditions, a nonlocal source term and a Hardy potential. More precisely, we use the variational methods to establish the existence of at least three generalized solutions for this problem.

Keywords: double-phase Laplacian operator, Hardy potential, mixed boundary condition, variable exponent space.

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1 Introduction

The study of the singular elliptic equations involving double-phase $(p_1(x), p_2(x))$-Laplacian operators with Hardy potential in variable exponent spaces has attracted the interest of many authors in different fields such as fluid dynamics, structural design and electrical systems [19]. In the recent years, several papers demonstrate the existence and multiplicity of solutions of this type of singular elliptic problems, for more details, see [2,4,5,7,14–18]. Recently Jian Liu et al. [16] established the existence of one weak solution, respectively two weak solutions to the following singular elliptic problem with mixed boundary conditions in variable exponent spaces

$$\begin{cases} -\Delta_{p(x)}u - \Delta_{q(x)}u + \frac{a(x)|u|^{h-2}u}{|x|^h} = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ \left(|\nabla u|^{p(x)-2}\nabla u + |\nabla u|^{q(x)-2}\nabla u\right)\cdot \nu + c(x)|u|^{t(x)-2}u = 0 & \text{on } \Gamma_2, \end{cases}$$

where Ω is a bounded subset in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega, \nu$ is the outward normal vector field on $\partial\Omega, \Gamma_1$ and Γ_2 are two smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$ such that $\Gamma_1 \cap \Gamma_2 = \emptyset, \overline{\Gamma_1} \cup \overline{\Gamma_2} = \partial\Omega, \overline{\Gamma_1} \cap \overline{\Gamma_2}$ is a $(N-2)$-dimensional submanifold

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of $\partial\Omega, \lambda > 0, 0 < a(x) \in L^{\infty}(\Omega), \Delta_{p(x)}u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$ is $p(x)$-Laplacian operator, $\Delta_{q(x)}u = \operatorname{div}\left(|\nabla u|^{q(x)-2}\nabla u\right)$ is $q(x)$-Laplacian operator, $p(x), q(x) \in C(\bar{\Omega})$ with $1 < q(x) < p(x) < p^*(x)$, where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & p(x) < N \\ +\infty, & p(x) \geq N \end{cases}$$

$1 < h < p(x)$, $t(x) \in C\left(\overline{\Gamma_2}\right)$ with $1 < t(x) < p_*(x)$, where

$$p_*(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & p(x) < N \\ +\infty, & p(x) \geq N \end{cases}$$

and $0 < c(x) \in L^{e(x)}(\Gamma_2)$, with $e(x) > \frac{p_*(x)t(x)}{p_*(x)-t(x)}$. The nonlinearity $f(x,u)$ is assumed to be a Carathéodory function such that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ which satisfy the following growth condition

$$|f(x,u)| \leq M_1(x) + M_2|u|^{s(x)-1}, \quad \text{a.e. } (x,u) \in \Omega \times \mathbb{R}$$

where $M_1(x) > 0$, $M_1(x) \in L^{\frac{s(x)}{s(x)-1}}(\Omega)$, $M_2 > 0$, $s(x) \in C_+(\bar{\Omega}) := \{\xi : \xi \in C(\overline{\Omega}), \xi(x) > 1, \text{ for all } x \in \overline{\Omega}\}$ with $s(x) < p^*(x)$. Moreover an Ambrosetti–Rabinowitz condition was imposed to guaranty the existence of two weak solutions when λ is in an appropriate range (See Theorem 3.1 in [16]).

In mathematical physics, the distinction between local and nonlocal source terms plays a key role in understanding the nature of interactions within a system and in developing appropriate models. A local source term depends only on the solution and its derivatives at a single point in the domain, capturing interactions that occur in a localized region. These terms are typically encountered in problems where effects are confined to immediate neighborhoods, such as local forces or reactions. However, in many physical systems, interactions can extend over longer distances or be influenced by global effects. In such cases, a nonlocal source term is more appropriate. This term involves the solution values from multiple points, often integrated over the domain, reflecting long-range interactions or global influences. Nonlocal source terms are commonly found in models of nonlocal elasticity, diffusion with memory, or biological systems with spatial coupling. This distinction motivates our study of a singular elliptic equation in variable exponent spaces, where we consider a more complex nonlocal source term to explore the existence of multiple weak solutions under mixed boundary conditions.

Motivated by the above advantages of a nonlocal source term and the previous work of Liu et al. [16], we shall study the singular elliptic equation with mixed boundary conditions in variable exponent spaces

$$\begin{cases} \displaystyle\sum_{i=1}^{2} -\Delta_{p_i(x)}u + \frac{a(x)|u|^{q-2}u}{|x|^q} = \lambda f(x,u)\Big(\int_\Omega F(x,u)\,dx\Big)^r & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ \left(\displaystyle\sum_{i=1}^{2} |\nabla u|^{p_i(x)-2}\nabla u\right)\nu + b(x)|u|^{h(x)-2}u = 0, & \text{on } \Gamma_2, \end{cases} \tag{P}$$

where $F(x,u) = \int_0^u f(x,t)\,dt$, for all $(x,u) \in \Omega \times \mathbb{R}$, r is a positive constant. In this given context, Ω is assumed to be an open bounded subset of $\mathbb{R}^N$, $(N > 2)$. The boundary of Ω

is denoted by $\partial\Omega$, is a smooth surface. The outward normal vectors on this boundary are represented by the vector field ν.

Additionally, there are two distinct smooth submanifolds, Γ_1 and Γ_2, which are parts of the boundary $\partial\Omega$. These submanifolds have a dimension of $N-1$ and do not intersect. The union of these submanifolds, denoted by $\overline{\Gamma_1} \cup \overline{\Gamma_2}$, covers the entire boundary, while their intersection, denoted by $\overline{\Gamma_1} \cap \overline{\Gamma_2}$, forms a submanifold of dimension $N-2$ within the boundary $\partial\Omega$. Moreover λ is a positive parameter and the function $a(x)$ is chosen to satisfy the assertion $0 < a(x) \in L^{\infty}(\Omega)$.

In the whole paper and for $i = 1, 2$, $-\Delta_{p_i(x)} u = \operatorname{div}(|\nabla u|^{p_i(x)-2} \nabla u)$ denotes the $p_i(x)$-Laplacian operator, where $p_i(x) \in C_+(\overline{\Omega})$. In the sequel, we denote by $\tilde{p}(x) = \max\{p_1(x), p_2(x)\}$ such that $1 < \tilde{p}(x) < \tilde{p}^*(x)$, where

$$\tilde{p}^*(x) = \begin{cases} \dfrac{N\tilde{p}(x)}{N-\tilde{p}(x)}, & \text{if } \tilde{p}(x) < N, \\ +\infty, & \text{if } \tilde{p}(x) \geq N, \end{cases}$$

$1 < q < \min\{N, \tilde{p}(x)\}$, $h(x) \in C(\overline{\Gamma_2})$ with $1 < h(x) < \tilde{p}_*(x)$, where

$$\tilde{p}_*(x) = \begin{cases} \dfrac{(N-1)\tilde{p}(x)}{N-\tilde{p}(x)}, & \text{if } \tilde{p}(x) < N, \\ +\infty, & \text{if } \tilde{p}(x) \geq N, \end{cases}$$

the function $b(x) \in L^{e(x)}(\Gamma_2)$, with $e(x) > \frac{\tilde{p}_*(x)h(x)}{\tilde{p}_*(x)-h(x)}$. We will assume in the next of this paper that $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function satisfying

(H) $$m_1|u|^{\alpha(x)-1} \leq f(x,u) \leq m_2|u|^{\beta(x)-1},$$

where, m_1, m_2 are positive constants. The functions $\alpha(x)$ and $\beta(x)$ belongs to the set $C_+(\overline{\Omega})$ and satisfy $\alpha(x) \leq \beta(x) < \tilde{p}(x)$, for all $x \in \Omega$.

The key differences between the problem of Liu et al. [16] and Problem (P) lie primarily in the structure of the source term, the conditions required for the existence of solutions, and the role of $b(x)$. In [16], the source term is a local nonlinearity $\lambda f(x,u)$, where $f(x,u)$ is a Carathéodory function satisfying specific growth conditions. The existence of two weak solutions relies on the Ambrosetti–Rabinowitz condition, which ensures the existence of solutions via variational methods. Additionally, the boundary condition on Γ_2 requires the positivity of $c(x)$.

In contrast, our work involves a more complex source term, which includes a nonlocal term $\left(\int_\Omega F(x,u)\,dx\right)^r$, where $F(x,u)$ is the primitive of $f(x,u)$. This nonlocal factor complicates the analysis but allows for the existence of three weak solutions without the need for the Ambrosetti-Rabinowitz condition. Moreover, the boundary condition on Γ_2 in our work involves $b(x)$, which is not required to be positive, providing more flexibility in the boundary conditions. These differences demonstrate how the second work handles a more intricate problem setup and proves the existence of multiple solutions with fewer restrictions compared to the first.

The rest of this paper is structured as follows. In the next section we give some basic facts and variational structure concerning our problem. In Section 3, we state and prove the main result of this paper.

2 Mathematical backgrounds

In this section, we recall some definitions and basic properties of the generalized Lebesgue Sobolev spaces.

For any $p \in C_+(\overline{\Omega})$, let $p^+ = \max_{x\in\overline{\Omega}} p(x)$ and $p^- = \min_{x\in\overline{\Omega}} p(x)$. The generalized Lebesgue Sobolev space is defined as

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R}, u \text{ measurable} : \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}.$$

The so-called Luxemburg norm on this spaces defined by

$$\|u\|_{L^{p(x)}(\Omega)} = \|u\|_{p(x)} = \inf \left\{ \eta > 0 : \int_\Omega \left| \frac{u(x)}{\eta} \right|^{p(x)} dx \leq 1 \right\}.$$

The Lebesgue–Sobolev space $W^{1,p(x)}(\Omega)$ is defined as follows

$$W^{1,p(x)}(\Omega) := \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

and equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|\nabla u\|_{p(x)} + \|u\|_{p(x)},$$

where

$$|\nabla u| = \left(\sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^2 dx \right)^{\frac{1}{2}}.$$

Let $W_0^{1,p(x)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. In the following, let

$$E = W_0^{1,\tilde{p}(x)}(\Omega).$$

The set E is a closed subspace of $W^{1,\tilde{p}(x)}(\Omega)$, then E is a reflexive, separable and uniformly convex Banach space. So, we can get that $\|u\|_{W^{1,\tilde{p}(x)}(\Omega)}$ is equivalent to the norm

$$\|u\| = \|\nabla u\|_{\tilde{p}(x)} = \inf \left\{ \eta > 0 : \int_\Omega \left| \frac{\nabla u(x)}{\eta} \right|^{\tilde{p}(x)} dx \leq 1 \right\}.$$

Lemma 2.1 ([11]). *The continuous embedding $E \hookrightarrow L^{\alpha(x)}$ holds for any $\alpha \in C_+(\overline{\Omega})$ such that $\alpha(x) \leq \tilde{p}(x)$ a.e. on Ω. This leads to the inequality*

$$\|u\|_{\alpha(x)} \leq c_\alpha \|\nabla u\|_{\tilde{p}(x)}, \tag{2.1}$$

where c_α is a positive constant.

Lemma 2.2 ([12]). *If $\tilde{p} \in C_+(\overline{\Omega})$, then*

$$\min \left\{ \|\nabla u\|_{\tilde{p}(x)}^{\tilde{p}^-}, \|\nabla u\|_{\tilde{p}(x)}^{\tilde{p}^+} \right\} \leq \int_\Omega |\nabla u(x)|^{\tilde{p}(x)} dx \leq \max \left\{ \|\nabla u\|_{\tilde{p}(x)}^{\tilde{p}^-}, \|\nabla u\|_{\tilde{p}(x)}^{\tilde{p}^+} \right\},$$

for any $u \in L^{\tilde{p}(x)}(\Omega)$ and for a.e. $x \in \Omega$.

Lemma 2.3 ([10]). *If $s_1, s_2 \in C_+(\overline{\Omega})$ such that $s_1(x) \leq s_2(x)$ a.e. $x \in \Omega$, then exists the continuous embedding $W^{1,s_2(x)}(\Omega) \hookrightarrow W^{1,s_1(x)}(\Omega)$.*

Lemma 2.4 ([10]). *Let Ω having the cone property on its boundary, and let $\tilde{p} \in C_+(\overline{\Omega})$. Suppose that $s \in C_+(\overline{\Omega})$ and $s(x) < \tilde{p}^*(x)$ for all $x \in \overline{\Omega}$. Under these conditions, the embedding*

$$W^{1,\tilde{p}(x)}(\Omega) \hookrightarrow W^{1,s(x)}(\Omega)$$

is both continuous and compact.

Denote

$$C_+(\partial\Omega) := \{\xi : \xi \in C(\partial\Omega), \xi(x) > 1, \text{ on } \partial\Omega\},$$

and the surface measure on $\partial\Omega$ by $d\sigma$.

$$L^{\tilde{p}(x)}(\partial\Omega) := \left\{ u \mid u : \partial\Omega \to \mathbb{R} \text{ is measurable, and } \int_{\partial\Omega} |u(x)|^{\tilde{p}(x)}\, d\sigma < \infty \right\},$$

where $\tilde{p}(x)$ denotes a variable exponent that may vary with the point $x \in \partial\Omega$.

The norm associated with this space is given by the Luxemburg norm

$$\|u\|_{L^{\tilde{p}(x)}(\partial\Omega)} = \|u\|_{\tilde{p}(x)(\partial\Omega)} = \inf\left\{ \eta > 0 \mid \int_{\partial\Omega} \left| \frac{u(x)}{\eta} \right|^{\tilde{p}(x)} d\sigma \le 1 \right\}.$$

For a measurable function v on a subset $\Gamma_2 \subset \partial\Omega$, the space $L^{\tilde{p}(x)}(\Gamma_2)$ consists of functions that are measurable on Γ_2 and can be extended to some function $u \in L^{\tilde{p}(x)}(\partial\Omega)$ such that $u = v$ on Γ_2. Specifically, we define

$$L^{\tilde{p}(x)}(\Gamma_2) = \left\{ v \mid v \text{ is measurable on } \Gamma_2,\ \exists u \in L^{\tilde{p}(x)}(\partial\Omega) \text{ such that } u = v \text{ on } \Gamma_2 \right\}.$$

The corresponding norm on $L^{\tilde{p}(x)}(\Gamma_2)$ is

$$\|v\|_{L^{\tilde{p}(x)}(\Gamma_2)} = \|v\|_{\tilde{p}(x)(\Gamma_2)} = \inf\left\{ \|u\|_{\tilde{p}(x)(\partial\Omega)} \mid u = v \text{ on } \Gamma_2 \right\}.$$

Lemma 2.5 ([8]). *Assume that the boundary of Ω possesses the cone property and $\tilde{p} \in C_+(\overline{\Omega})$, if $h \in C_+(\partial\Omega)$ and $h(x) < \tilde{p}_*(x)$ for all $x \in \partial\Omega$ then the trace embedding*

$$W^{1,\tilde{p}(x)}(\Omega) \hookrightarrow L^{h(x)}(\partial\Omega)$$

is compact and continous.

Lemma 2.6 ([12]). *Let $G \subset \mathbb{R}^N$ be a measurable subset with $0 < meas(G) < +\infty$. Consider a Carathéodory function $f : G \times \mathbb{R} \to \mathbb{R}$ satisfying the estimate*

$$|f(x,u)| \le M_1(x) + M_2|u|^{\frac{p(x)}{q(x)}}, \quad \text{for almost every } (x,u) \in G \times \mathbb{R},$$

where $p(x), q(x) \in C_+(\overline{\Omega})$, $0 < M_1(x) \in L^{q(x)}(G)$, and $M_2 > 0$. In this case, the Nemytskii operator

$$N_f(u)(x) = f(x, u(x))$$

maps the space $L^{p(x)}(G)$ into $L^{q(x)}(G)$, and it is both continuous and bounded.

Lemma 2.7 (Hölder type inequality [1, 9]). *Let p, q, and s be measurable functions defined on Ω with the condition*

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}, \quad \textit{for almost every } x \in \Omega.$$

If $f \in L^{p(x)}(\Omega)$ and $g \in L^{q(x)}(\Omega)$, then the product fg belongs to $L^{s(x)}(\Omega)$, and the following inequality holds

$$\|fg\|_{s(x)} \leq 2\|f\|_{p(x)}\|g\|_{q(x)}.$$

Furthermore, if the condition

$$\frac{1}{s(x)} + \frac{1}{p(x)} + \frac{1}{q(x)} = 1, \quad \textit{for almost every } x \in \Omega,$$

is satisfied, then it follows that

$$\int_\Omega |f(x)g(x)h(x)|\, dx \leq 3\|f\|_{s(x)}\|g\|_{p(x)}\|h\|_{q(x)}.$$

Hereinafter, for $M > 0$ and $r(x) \in C_+(\overline{\Omega})$, we use the following notations

$$[M]^r = \max\{M^{r^-}, M^{r^+}\} \quad \text{and} \quad [M]_r = \min\{M^{r^-}, M^{r^+}\}.$$

In order to formulate the variational approach of the problem (P), we define the functional $J_\lambda : E \longrightarrow \mathbb{R}$ by

$$J_\lambda = J_1 - \lambda J_2,$$

where

$$J_1(u) = \sum_{i=1}^{2}\left(\int_\Omega \frac{1}{p_i(x)}|\nabla u|^{p_i(x)} dx\right) + \int_\Omega \frac{a(x)|u|^q}{q|x|^q}\, dx,$$
$$J_2(u) = \frac{1}{r+1}\left(\int_\Omega F(x,u)\, dx\right)^{r+1} - \frac{1}{\lambda}\int_{\Gamma_2} \frac{b(x)|u|^{h(x)}}{h(x)}\, d\sigma,$$

where $F(x,u) = \int_0^u f(x,t)\, dt$, $\forall\ (x,u) \in \Omega \times \mathbb{R}$.

It is clear that $u \in E$ is a weak solution of the problem (P) if

$$J'_\lambda(u)(v) = J'_1(u)(v) - \lambda J'_2(u)(v) = 0, \qquad \forall\ v\ \in E,$$

where J_1, J_2 are continuously Gâteaux differentiable and we have

$$J'_1(u)(v) = \sum_{i=1}^{2}\left(\int_\Omega |\nabla u|^{p_i(x)-2}\nabla u \nabla v\, dx\right) + \int_\Omega \frac{a(x)|u|^{q-2}uv}{|x|^q}\, dx,$$

and

$$J'_2(u)(v) = \left(\int_\Omega F(x,u)\, dx\right)^r \int_\Omega f(x,u)v\, dx - \frac{1}{\lambda}\int_{\Gamma_2} b(x)|u|^{h(x)-2}uv\, d\sigma,$$

Lemma 2.8 ([16]). *The operator J'_1 verifies the following properties*

- ***(i)*** *J'_1 is coercive and strictly monotone in E.*
- ***(ii)*** *J'_1 is an homeomorphism.*

Lemma 2.9. *The operator $J'_2 : E \longrightarrow E^*$ is compact.*

Proof. Let $J_2' = \varphi_1'(u) - \frac{1}{\lambda}\varphi_2'(u)$, where

$$\varphi_1'(u)(v) = \left(\int_\Omega F(x,u)\,dx\right)^r \int_\Omega f(x,u)v\,dx, \qquad \varphi_2'(u)(v) = \int_{\Gamma_2} b(x)|u|^{h(x)-2}uv\,d\sigma, \quad \forall v \in E.$$

From condition (**H**), the compact embeddings $E \hookrightarrow L^{\beta(x)}(\Omega)$, where $1 < \beta(x) < \tilde{p}^*(x)$, and $E \hookrightarrow L^{h(x)}(\partial\Omega)$, where $1 < h(x) < \tilde{p}_*(x)$, imply the compactness of $\varphi_1'(u)$ and $\varphi_2'(u)$. Consequently, the functional $J_2'(u) : E \to E^*$ is compact.

To demonstrate this, let $(u_k)_k \subset E$ be a sequence such that $u_k \rightharpoonup u$ weakly in E. Due to the compactness of the embeddings, there exists a subsequence, still denoted by $(u_k)_k$, such that $u_k \to u$ strongly in $L^{\beta(x)}(\Omega)$, and $u_k(x) \to u(x)$ for almost every $x \in \Omega$. The continuity of $F(x,u)$ with respect to u ensures that

$$F(x,u_k) \to F(x,u) \quad \text{for almost every } x.$$

Also, there exists $C > 0$ such that

$$|F(x,u_k)| < C|u_k|^{\beta(x)}.$$

By using the dominated convergence theorem, we can write

$$\int_\Omega F(x,u_k)\,dx \to \int_\Omega F(x,u_k)\,dx \quad \text{as } k \to +\infty. \tag{2.2}$$

In the subsequent part of the proof, we employ the approach introduced by Liu et al. [16]. Specifically, we assert that the Nemytskii operator $N_f(u)(x) = f(x,u(x))$ is continuous, as $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that satisfies condition (**H**). Consequently, it follows that $N_f(u_k) \to N_f(u)$ in $L^{\frac{\beta(x)}{\beta(x)-1}}(\Omega)$.

Next, utilizing Lemma 2.1 and the Hölder inequality, for any $v \in E$, we have the estimate

$$\begin{aligned}
\left|\int_\Omega f(x,u_k)v\,dx - \int_\Omega f(x,u)v\,dx\right| &\le \int_\Omega |(f(x,u_k) - f(x,u))v|\,dx \\
&\le 2\|N_f(u_k) - N_f(u)\|_{\frac{\beta(x)}{\beta(x)-1}} \|v\|_{\beta(x)}, \\
&\le 2c_\beta \|N_f(u_k) - N_f(u)\|_{\frac{\beta(x)}{\beta(x)-1}} \|\nabla v\|_{\tilde{p}(x)},
\end{aligned} \tag{2.3}$$

where c_β is the constant associated with the embedding $E \hookrightarrow L^{\beta(x)}(\Omega)$ with $1 < \beta(x) < \tilde{p}^*(x)$.

By combining the results from equations (2.2) and (2.3), we deduce that $\varphi_1'(u_k) \to \varphi_1'(u)$ in E, which implies that φ_1' is completely continuous. Therefore, φ_1' is a compact operator.

By similar argument for φ_2' on Γ_2, The Nemytskii operator $N_h(u) = |u|^{h(x)-2}u$ is continuous and $N_h(u_k) \longrightarrow N_h(u)$ in $L^{\frac{h(x)}{h(x)-1}}(\Gamma_2)$. According to the Hölder type inequality in Lemma 2.7 and the compact embedding $E \hookrightarrow L^{h(x)}(\partial\Omega)$, $1 < h(x) < \tilde{p}_*(x)$, thus for all $v \in E$, we have

$$\begin{aligned}
\left|\varphi_2'(u_k)(v) - \varphi_2'(u)(v)\right| &= \left|\int_{\Gamma_2} b(x)|u_k|^{h(x)-2}u_k v\,d\sigma - \int_{\Gamma_1} b(x)|u|^{h(x)-2}uv\,d\sigma\right| \\
&\le 3\|b\|_{e(x)}\|N_h(u_k) - N_h(u)\|_{\frac{h(x)}{h(x)-1}(\Gamma_2)}\|v\|_{\frac{e(x)h(x)}{e(x)-h(x)}(\Gamma_2)} \\
&\le 3\|b\|_{e(x)}\|N_h(u_k) - N_h(u)\|_{\frac{h(x)}{h(x)-1}(\Gamma_2)}\|v\|_{\frac{e(x)h(x)}{e(x)-h(x)}(\partial\Omega)} \\
&\le 3b_h\|b\|_{e(x)}\|N_h(u_k) - N_h(u)\|_{\frac{h(x)}{h(x)-1}(\Gamma_2)}\|v\|,
\end{aligned}$$

where b_h is the embedding constant of the trace embedding $W^{1,\tilde{p}(x)} \hookrightarrow L^{\frac{e(x)h(x)}{e(x)-h(x)}}(\partial\Omega)$. Thus $\varphi_2'(u_k) \longrightarrow \varphi_2'(u)$ in E^* as $n \longrightarrow +\infty$, i.e. φ_2' is completely continuous, so φ_2' is compact. Therefore J_2' is compact. □

To obtain our result, we use the following critical points theorem.

Theorem 2.10 ([6, Theoem 3.6])**.** *Consider a reflexive real Banach space E, and $J_1 : E \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional. Assume that the Gâteaux derivative of J_1 has a continuous inverse on E^*. Furthermore, let $J_2 : E \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, assume that*

$$\inf_E J_1 = J_1(0) = J_2(0) = 0.$$

Suppose that there exist $d > 0$ and $\overline{x} \in E$, with $d < J_1(\overline{x})$, such that

(i) $\frac{\sup_{J_1(x)<d} J_2(x)}{d} < \frac{J_2(\overline{x})}{J_1(\overline{x})}$,

(ii) for every $\lambda \in \Psi_d := \left]\frac{J_1(\overline{x})}{J_2(\overline{x})}, \frac{d}{\sup_{J_1(x)\le d} J_2(x)}\right[$, $J_\lambda := J_1 - \lambda J_2$ *is coercive.*

Then, for every $\lambda \in \Psi_d$, $J_1 - \lambda J_2$ has at least three distinct critical points in E.

3 Main results

Before giving our result, we recall the Hardy inequality (for more details, see [13]), if $1 < q < N$, then

$$\int_\Omega \frac{|u(x)|^q}{|x|^q}\,dx \le \frac{1}{H}\int_\Omega |\nabla u|^q\,dx, \qquad \forall u \in W_0^{1,q}(\Omega) = \{u \in W^{1,q}(\Omega) : u|_{\Gamma_1} = 0\},$$

where $H = \left(\frac{N-q}{q}\right)^q$.

Using (2.2) and due to $q < \tilde{p}(x)$ for any $x \in \Omega$, we conclude that $E \hookrightarrow W_0^{1,q}(\Omega)$, thus there is a positive constant k such that

$$\int_\Omega \frac{|u(x)|^q}{|x|^q}\,dx \le \frac{k^q}{H}\|u\|^q, \qquad \forall\ u \in E. \tag{3.1}$$

To present our main result, we define

$$D(x) = \sup\{D > 0 \mid B(x, D) \subseteq \Omega\},$$

for all $x \in \Omega$, here $B(x, D)$ represents a ball centered at x with radius D. It is simple to see that there exists a point $x_0 \in \Omega$ such that $B(x_0, R) \subseteq \Omega$, where $R = \sup_{x\in\Omega} D(x)$.

Theorem 3.1. *Assume that $\tilde{p}^- > \beta^+(r+1)$, and that there exists $d, \delta > 0$ such that*

$$\left(\frac{1}{p_1^-}\left[\frac{2\delta}{R}\right]^{p_1} + \frac{1}{p_2^-}\left[\frac{2\delta}{R}\right]^{p_2}\right)\left|B(x_0, R) \setminus \overline{B}\left(x_0, \frac{R}{2}\right)\right| = d,$$

then for every $\lambda \in]A_\delta, B_d[$, where

$$A_\delta = \frac{\left(\frac{1}{p_1^-}\left[\frac{2\delta}{R}\right]^{p_1} + \frac{1}{p_2^-}\left[\frac{2\delta}{R}\right]^{p_2}\right)\left|B(x_0, R) \setminus \overline{B}(x_0, \frac{R}{2})\right| + \frac{k^q\|a\|_\infty}{qH}\tilde{M}^q}{\frac{m_1^{r+1}\left([\delta]_\alpha\right)^{r+1}}{(r+1)(\alpha^+)^{r+1}}\left|B(x_0, \frac{R}{2})\right|^{r+1}},$$

and

$$B_d = \frac{d}{\frac{m_2^{r+1}([c_\beta]^\beta)^{r+1}(\tilde{p}^+)^{\frac{\beta^+(r+1)}{\tilde{p}^-}}}{(r+1)(\beta^-)^{r+1}}\left(\left[[d]^{\frac{1}{\tilde{p}}}\right]^\beta\right)^{r+1}},$$

problem (P) *admits at least three weak solutions.*

Proof. It is worth noting that J_1 and J_2 satisfy the regularity assumptions outlined in Theorem 2.10. We will now establish the fulfillment of conditions (i) and (ii). To this end, let's consider

$$\left(\frac{1}{p_1^-}\left[\frac{2\delta}{R}\right]^{p_1} + \frac{1}{p_2^-}\left[\frac{2\delta}{R}\right]^{p_2}\right)\left|B(x_0,R)\setminus \overline{B}\left(x_0,\frac{R}{2}\right)\right| = d,$$

and consider $v_\delta \in E$ such that

$$v_\delta \begin{cases} 0 & x \in \Omega\setminus B(x_0,R), \\ \frac{2\delta}{R}(R-|x-x_0|) & \in B(x_0,R)\setminus B\left(x_0,\frac{R}{2}\right), \\ \delta & x \in B\left(x_0,\frac{R}{2}\right). \end{cases}$$

Then, from Lemma 2.2 and by a direct calculation, shows that $\nabla v_\delta(x) = \frac{2\delta}{R}$, thus

$$\begin{aligned} &\|\nabla v_\delta(x)\|_{\tilde{p}(x)} \\ &\quad \le \max\{\|\nabla v_\delta\|_{\tilde{p}(x)}^{\tilde{p}^-}, \|\nabla v_\delta\|_{\tilde{p}(x)}^{\tilde{p}^+}\}, \\ &\quad \le \max\left\{\left(\left[\frac{2\delta}{R}\right]^{\tilde{p}}\right)^{\tilde{p}^-}\left|B(x_0,R)\setminus\overline{B}\left(x_0,\frac{R}{2}\right)\right|^{\tilde{p}^-}, \left(\left[\frac{2\delta}{R}\right]^{\tilde{p}}\right)^{\tilde{p}^+}\left|B(x_0,R)\setminus\overline{B}\left(x_0,\frac{R}{2}\right)\right|^{\tilde{p}^+}\right\}, \\ &\quad = \tilde{M}. \end{aligned}$$

By equation (3.1), we have

$$\begin{aligned} J_1(v_\delta) &= \sum_{i=1}^{2}\left(\int_\Omega \frac{1}{p_i(x)}|\nabla v_\delta|^{p_i(x)}\,dx\right) + \int_\Omega \frac{a(x)|v_\delta|^q}{q|x|^q}\,dx, \\ &\le \left(\frac{1}{p_1^-}\left[\frac{2\delta}{R}\right]^{p_1} + \frac{1}{p_2^-}\left[\frac{2\delta}{R}\right]^{p_2}\right)\left|B(x_0,R)\setminus\overline{B}\left(x_0,\frac{R}{2}\right)\right| + \frac{k^q\|a\|_\infty}{qH}\|v_\delta\|^q, \\ &= \left(\frac{1}{p_1^-}\left[\frac{2\delta}{R}\right]^{p_1} + \frac{1}{p_2^-}\left[\frac{2\delta}{R}\right]^{p_2}\right)\left|B(x_0,R)\setminus\overline{B}\left(x_0,\frac{R}{2}\right)\right| + \frac{k^q\|a\|_\infty}{qH}\|\nabla v_\delta(x)\|_{\tilde{p}(x)}^q, \\ &\le \left(\frac{1}{p_1^-}\left[\frac{2\delta}{R}\right]^{p_1} + \frac{1}{p_2^-}\left[\frac{2\delta}{R}\right]^{p_2}\right)\left|B(x_0,R)\setminus\overline{B}\left(x_0,\frac{R}{2}\right)\right| + \frac{k^q\|a\|_\infty}{qH}\tilde{M}^q. \end{aligned}$$

Therefore, $J_1(v_\delta) > d$. however, it is important to consider the following

$$\begin{aligned} J_2(v_\delta) \ge \frac{1}{r+1}\left(\int_\Omega F(x,v_\delta)dx\right)^{r+1} &\ge \frac{m_1^{r+1}}{(r+1)(\alpha^+)^{r+1}}\left(\int_{B(x_0,\frac{R}{2})}|\delta|^{\alpha(x)}\,dx\right)^{r+1}, \\ &\ge \frac{m_1^{r+1}([\delta]_\alpha)^{r+1}}{(r+1)(\alpha^+)^{r+1}}\left|B(x_0,\frac{R}{2})\right|^{r+1}. \end{aligned} \tag{3.2}$$

In addition, for every $u \in J_1^{-1}(]-\infty, d[)$, one has the following

$$\frac{1}{\tilde{p}^+}\left[\|\nabla u\|_{\tilde{p}(x)}\right]_{\tilde{p}} \leq d, \tag{3.3}$$

therefore,

$$\|\nabla u\|_{\tilde{p}(x)} \leq \left[\tilde{p}^+ J_1(u)\right]^{\frac{1}{\tilde{p}}} < \left[\tilde{p}^+ d\right]^{\frac{1}{\tilde{p}}}.$$

Under the assumption (**H**), we conclude the following

$$\begin{aligned}
J_2(u) &\leq \frac{1}{r+1}\left(\int_\Omega F(x,u)\,dx\right)^{r+1}, \\
&\leq \frac{m_2^{r+1}}{(r+1)(\beta^-)^{r+1}}\left(\int_\Omega |u|^{\beta(x)}\,dx\right)^{r+1}, \\
&\leq \frac{m_2^{r+1}}{(r+1)(\beta^-)^{r+1}}\left(\left[\|u\|_{\beta(x)}\right]^{\beta}\right)^{r+1}, \\
&\leq \frac{m_2^{r+1}}{(r+1)(\beta^-)^{r+1}}\left(\left[c_\beta\|\nabla u\|_{\tilde{p}(x)}\right]^{\beta}\right)^{r+1}.
\end{aligned} \tag{3.4}$$

This leads to the following result

$$\sup_{J_1(u)<d} J_2(u) \leq \frac{m_2^{r+1}([c_\beta]^\beta)^{r+1}(\tilde{p}^+)^{\frac{\beta^+(r+1)}{\tilde{p}^-}}}{(r+1)(\beta^-)^{r+1}}\left(\left[[d]^{\frac{1}{\tilde{p}}}\right]^{\beta}\right)^{r+1},$$

and

$$\frac{1}{d}\sup_{J_1(u)<d} J_2(u) < \frac{1}{\lambda}.$$

Furthermore, we can establish the coerciveness of J_λ for any positive value of λ by employing inequality (3.4) once more. This yields the following result

$$J_2(u) \leq \frac{m_2^{r+1}([c_\beta]^\beta)^{r+1}}{(r+1)(\beta^-)^{r+1}}\left(\left[\|\nabla u\|_{\tilde{p}(x)}\right]^{\beta}\right)^{r+1}.$$

When $\|u\| > 1$, the following can be inferred

$$J_1(u) - \lambda J_2(u) \geq \frac{1}{\tilde{p}^+}\|\nabla u\|_{\tilde{p}(x)}^{\tilde{p}^-} - \lambda\frac{m_2^{r+1}([c_\beta]^\beta)^{r+1}}{(r+1)(\beta^-)^{r+1}}\left(\left[\|\nabla u\|_{\tilde{p}(x)}\right]^{\beta}\right)^{r+1}.$$

By considering the fact that $\tilde{p}^- > \beta^+(r+1)$, we can reach the desired conclusion. In conclusion, considering the aforementioned fact that

$$\Psi_d^- = (A_\delta, B_d) \subseteq \left(\frac{J_1(v)}{J_2(v)}, \frac{d}{\sup_{J_1(u)<d} J_2(u)}\right).$$

Based on Theoem 2.10, it can be deduced that for $\lambda \in \Psi_d^-$, the function $J_1 - \lambda J_2$ possesses at least three critical points in E. These critical points correspond to weak solutions of problem (P). $\square$

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