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On a special case of a difference equation with powers

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Abstract. Investigation of the long-term behaviour of solutions to the nonlinear difference equation

$$x_{n+1} = A + \frac{x_{n-m}^p}{x_{n-k}^r}, \quad n \in \mathbb{N}_0,$$

where $A, p, q \in \mathbb{R}$, $k, m \in \mathbb{N}_0$, $k \neq m$, was proposed by S. Stević about twenty years ago. A very special case of the equation ($p = 1$, $r = 2$, $m = 0$) has been recently considered in [*J. Appl. Math. Comput.* **67**(2021), 423–437]. We show that the main results therein are known or have some inaccuracies. Among other things, we show that the boundedness result therein is a consequence of some known results and using one of our previous methods we give a better upper bound for positive solutions to the equation, show that the proof of the global convergence result therein is not correct and provide a complete proof of a generalization, and also show that the results on semi-cycles of positive solutions are not correct and present some correct ones. Several comments are also given and some analyses are conducted.

Keywords: difference equation, positive solutions, boundedness, convergence, semi-cycles.

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1 Introduction

There has been a considerable recent interest in investigating concrete nonlinear difference equations and systems of difference equations (see, for instance, [1,3–19,29–61] and the related references therein).

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1.1 Notation

The set of positive natural numbers is denoted by $\mathbb{N}$, the set of whole numbers is denoted by $\mathbb{Z}$, the set of positive numbers $(0, +\infty)$ is denoted by $\mathbb{R}_+$, whereas the set of real numbers is denoted by $\mathbb{R}$. If $l \in \mathbb{Z}$ is fixed, then for the set $\{j \in \mathbb{Z} : j \geq l\}$ we use the notation $\mathbb{N}_l$. If $k, l \in \mathbb{Z}$ and $k \leq l$, then we use the notation $j = \overline{k, l}$ instead of writing the following expression: $k \leq j \leq l$, $j \in \mathbb{Z}$. By C_j^n where $n \in \mathbb{N}_0$, $0 \leq j \leq n$, we denote the binomial coefficients, that is, $C_j^n = \frac{n!}{j!(n-j)!}$, where $n \in \mathbb{N}_0$, $0 \leq j \leq n$, and where, as usual, we regard that $0! = 1$. For some basic facts on the binomial coefficients consult, for instance, [2,20,21,25,26]. Let us mention that in a part of the literature, such as [2,21], is used the notation C_n^j for the binomial coefficients. Another standard notation, that is, $\binom{n}{j}$ is also frequently used (see, e.g., [20,25,26]), but it is more robust, so, as usual, we use here the above notation.

1.2 Two important classes of difference equations with powers

Investigation of the long-term behaviour of positive solutions to the nonlinear difference equation

$$x_{n+1} = A + \frac{x_{n-m}^p}{x_{n-k}^r}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $A > 0$, $\min\{p, q\} \geq 0$, $k, m \in \mathbb{N}_0$, $k \neq m$, as well as of its max-type counterpart

$$x_{n+1} = \max\left\{A, \frac{x_{n-m}^p}{x_{n-k}^r}\right\}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

was suggested by S. Stević in January 2004, soon after acceptance of [42] where the boundedness, global attractivity, oscillations, and asymptotic periodicity of positive solutions to the equation (1.1) in the case $A \geq 0$, $p = r$, $m = 1$, $k = 0$ was investigated. This was probably the first paper with some new and non-trivial results on an equation of the form in (1.1) which is not rational (the case had been previously considered in [12], but the results therein were essentially known or incorrect). It should be mentioned that [42] also contains a result on convergence of solutions to equation (1.1) for the case of an arbitrary $m \in \mathbb{N}$.

Equations (1.1) and (1.2) are good prototypes for developing the theory of nonlinear difference equations, because of which S. Stević proposed their investigation. Since the time of his proposal for the investigation, the two classes of difference equations, as well as their various generalizations, including some systems of difference equations, have been studied a lot.

Remark 1.1. If $A = 0$ then equations (1.1) and (1.2) are reduced to the following product-type difference equation

$$x_{n+1} = \frac{x_{n-m}^p}{x_{n-k}^r}, \quad n \in \mathbb{N}_0. \tag{1.3}$$

If only positive solutions of equation (1.3) are considered, then employing the change of variables

$$y_n = \ln x_n$$

for $n \geq -\max\{k, m\}$, which can be found in old literature [22] (see, for instance, also [2,42]), the difference equation is transformed to the linear one with constant coefficients

$$z_{n+1} - p z_{n-m} + r z_{n-k} = 0, \quad n \in \mathbb{N}_0,$$

which is (theoretically) solvable in closed form ([22–24, 26–28]). Bearing in mind this fact and many known results on the linear difference equations with constant coefficients, the equations (1.1) and (1.2) with $A = 0$ are usually not of a special interest. Hence, although we have considered some special cases of the equation (1.3) with $A = 0$, as it was the case in [42], we frequently skipped considering the case in our papers.

Remark 1.2. It should be also mentioned that solvability of product-type systems of difference equations on the domain of complex numbers, as well as the difference equations and systems which are transformed to them by some changes of variables, has attracted some recent attention. See, for example, [59, 60] and the related references therein.

Remark 1.3. Note that equations (1.1) and (1.2) are some kind of perturbations of equation (1.3), which is one of the reasons why they are good prototypes for some detailed investigations. Although, they have similar forms, the long-term behavior of their positive solutions can be quite different.

1.3 A bit on the history of investigations of equations (1.1) and (1.2)

In what follows we briefly mention some of the previous investigations of equations (1.1) and (1.2), and some of their generalizations. As we have already mentioned, in [42] was considered equation (1.1) in the case $A \geq 0$, $p = r > 0$, $m = 1$, $k = 0$. The case was also studied later in [5]. The boundedness, global attractivity, and periodicity of positive solutions to equations (1.1) and (1.2) in the case $A > 0$, $p = r > 0$, $m = 0$, $k = 1$ were investigated in [46] and [49], respectively. Equations (1.1) and (1.2) in the case $\min\{A, p, q\} > 0$, $m = 0$, $k = 1$ were considered in [47] (this was probably the first paper which also considered the case $p \neq r$). The boundedness character of positive solutions to equations (1.1) and (1.2) in the case $\min\{A, p, q\} > 0$, $m = 0$, $k = 2$ was studied in [52], where S. Stević used several new ideas and methods, among others, a new method that we called "oachkatzlschwoif/squirrel-tail" method. Equation (1.2) in the case $A > 0$, $p = r > 0$, $m = 0$, $k = 3$ was considered first in [51] and later in [16].

The boundedness character of positive solutions to the following extensions of the equations in [47]

$$x_{n+1} = A + \frac{x_n^p}{x_{n-1}^q x_{n-2}^r}, \quad n \in \mathbb{N}_0,$$

$$x_{n+1} = \max\left\{A, \frac{x_n^p}{x_{n-1}^q x_{n-2}^r}\right\}, \quad n \in \mathbb{N}_0,$$

was considered in [17] and [54], respectively.

The following generalization of equation (1.1) in the case $p = r$

$$x_n = A + \left(\frac{\sum_{i=1}^{k} \alpha_i x_{n-2p_i}}{\sum_{j=1}^{m} \beta_j x_{n-q_j}}\right)^p, \quad n \in \mathbb{N}_0,$$

where $\min\{p, A\} > 0$, $k, m \in \mathbb{N}$, $p_i \in \mathbb{N}$, $i = \overline{1,k}$, q_j, $j = \overline{1,m}$, are odd natural numbers such that $p_1 < p_2 < \cdots < p_k$, $q_1 < q_2 < \cdots < q_m$, $\min_{1\leq i\leq k, 1\leq j\leq m}\{\alpha_i, \beta_j\} > 0$ such that $\sum_{i=1}^{k} \alpha_i = \sum_{j=1}^{m} \beta_j$, was considered in [53].

The boundedness character of the equations (1.1) and (1.2) in the case $A > 0$, $p = r > 0$, $m = 0$, $k \in \mathbb{N}$ was completely characterized in [50]. Equation (1.2) in the case $p \neq q$ was considered in [55], whereas the corresponding results for equation (1.1) in the case were presented in [15]. The case $\min\{p,q\} > 0$, $m = 1$, $k = 0$ was considered in [35], whereas the case where the constant A is replaced by a sequence A_n was considered in [36]. For some special cases of equations (1.1) and (1.2) in rational form see [1,3,4,40] and the references therein, and with a variable coefficient A_n, see [31,32,41] and the related references therein. A cyclic system of difference equations corresponding to the equation (1.1) in the case $A > 0$, $m = 0$, $k = 1$, was considered in [57], whereas the corresponding cyclic system to the equation (1.2) in the case $A > 0$, $m = 0$, $k = 1$, was studied in [61]. A quite general boundedness result for a generalization of equation (1.1) was given in [58]. For some related difference equations and systems of difference equations, some of which also contain powers of dependent variables, see also [18,29,37,38,56].

1.4 A recent investigation

Recently, in [62], E. Tasdemir considered the following difference equation

$$x_{n+1} = A + B\frac{x_n}{x_{n-m}^2}, \quad n \in \mathbb{N}_0, \tag{1.4}$$

where $m \in \mathbb{N}$ and $\min\{A, B\} > 0$.

Note that by using the change of variables

$$x_n = \sqrt{B}\, y_n, \quad n \in \mathbb{N}_0, \tag{1.5}$$

equation (1.4) is transformed to the equation

$$y_{n+1} = \frac{A}{\sqrt{B}} + \frac{y_n}{y_{n-m}^2}, \quad n \in \mathbb{N}_0, \tag{1.6}$$

which is a very special case of equation (1.1). This immediately implies that equation (1.4) is equivalent with a special case of equation (1.1), and consequently it is not a new equation at all.

Instead of using the change of variables in (1.5) the author of [62] used the change of variables

$$y_n = \frac{x_n}{A}, \quad n \in \mathbb{N}_0, \tag{1.7}$$

and obtained the equation

$$y_{n+1} = 1 + p\frac{y_n}{y_{n-m}^2}, \quad n \in \mathbb{N}_0, \tag{1.8}$$

where $p = B/A^2$, which is, of course, equivalent with the special case of equation (1.1) mentioned above.

The equilibria of equation (1.8) satisfy the algebraic equation

$$\bar{y} = 1 + \frac{p}{\bar{y}}$$

and are equal to

$$\frac{1 + \sqrt{1 + 4p}}{2} \quad \text{and} \quad \frac{1 - \sqrt{1 + 4p}}{2}.$$

From this and since $p > 0$, we see that

$$\bar{y} = \frac{1 + \sqrt{1 + 4p}}{2} \tag{1.9}$$

is a unique positive equilibrium of equation (1.8).

A simple calculation shows that the linearized equation associated with (1.8) about the positive equilibrium $\bar{y}$ is given by

$$z_{n+1} - \frac{p}{\bar{y}^2} z_n + \frac{2p}{\bar{y}^2} z_{n-m} = 0, \quad n \in \mathbb{N}_0. \tag{1.10}$$

It is well-known that

$$\sum_{j=0}^{m} |a_j| < 1 \tag{1.11}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$z_{n+1} + a_0 z_n + a_1 z_{n-1} + \cdots + a_m z_{n-m} = 0, \tag{1.12}$$

from which it follows immediately that if $\frac{3p}{\bar{y}^2} < 1$ and $p > 0$, equation (1.10) is asymptotically stable. A simple calculation shows that this is satisfied for $p \in (0, 3/4)$. This implies that the equilibrium point $\bar{y}$ of equation (1.8) is locally asymptotically stable for $p \in (0, 3/4)$, which was noted in [62].

Remark 1.4. Condition (1.11) is a practical condition which guarantees asymptotic stability of equation (1.12), but his disadvantage is that it is not necessary. Moreover, there are better sufficient conditions guaranteeing asymptotic stability of the equation. Unfortunately, many authors use only this one and obtain pretty narrow sets of parameters which guarantee the stability.

The following claims should be the main results in [62].

Theorem 1.5. *Every solution $(y_n)_{n \in \mathbb{N}_{-m}}$ of equation (1.8) is bounded and persist and satisfies the following estimates*

$$1 < y_n < 1 + p(1+p)^m, \tag{1.13}$$

for $n \geq 2m + 2$.

Theorem 1.6. *Let $0 < p < 3/4$. Then, the positive equilibrium point $\bar{y}$ of equation (1.8) is globally asymptotically stable.*

Theorem 1.7. *Every solution of equation (1.8) satisfies the following statements:*

(a) Every solution of equation (1.8) has semi-cycles of length at most $2m + 1$.

(b) If every solution of equation (1.8) has a semi-cycle of length at least k, then, the following semi-cycle has at least $k + 1$ terms.

Beside these three claims, and the observation related to the local asymptotic stability of the equilibrium $\bar{y}$, [62] also gives a few other simple results, which are easily proved by well known methods and simple calculations such as the nonexistence of two periodic solutions and a growth rate result for solutions converging to the equilibrium $\bar{y}$.

Remark 1.8. At the end of [62] is proposed studying two special cases of equation (1.1) as 'Open Problems' despite the fact that these, as well as some general difference equations have been studied considerably so far, as we have mentioned in the previous subsection. The author of [62] might have not seen any of the papers: [3, 15–17, 31, 32, 35, 36, 42, 46, 47, 49–55, 57, 58, 61], which have been published in several popular journals.

1.5 Our aim

Here we show that Theorem 1.5 directly follows form some of our previous results, and that employing one of known methods can be obtained a better upper estimate than the one given in (1.13). Further, we show that the proof of Theorem 1.6 is not correct and provide a complete proof of a generalization. We also show that the proof of Theorem 1.7 is not correct, and present some correct ones related to it. We also conduct some analyses and present some additional comments.

2 Main results, counterexamples and comments

In this section we give several comments concerning the claims and results in [62], conduct some analyses, correct some of the proofs therein, and extend some of the claims and results therein.

2.1 On Theorem 1.5 and relating results

As far as concerning Theorem 1.5, the fact that all positive solutions to equation (1.8) are bounded follows from several results in the literature. For example, by using one of the methods presented in [50], which essentially originates from [47], in [15] was proved the following generalization of the boundedness results in [47] and [52]:

Theorem 2.1. *Assume $p, r > 0$ and $k \in \mathbb{N}_2$. Then every positive solution of the equation*

$$x_n = A + \frac{x_{n-1}^p}{x_{n-k}^r}, \quad n \in \mathbb{N}_0, \tag{2.1}$$

is bounded if

$$0 < p < \left(\frac{rk^k}{(k-1)^{k-1}}\right)^{1/k}. \tag{2.2}$$

Now note that in equation (1.4) or in the equivalent equation (1.8), we have $p = 1$ and $r = 2$ for which the condition in (2.2) is satisfied for any $k \in \mathbb{N}_2$, from which the boundedness part of Theorem 1.5 immediately follows.

Regarding the upper bound for positive solutions to equation (1.8) given in (1.13), we must say that there is a procedure, which nowadays can be regarded as standard one, for getting upper bounds for positive solutions of related difference equations. The procedure was essentially used, e.g., in [40] and it was a starting point for getting the squirrel-tail method. Namely, by iterating the numerators of the fractions in equation (1.8) m-times, and employing a simple inductive argument we have

$$\begin{aligned}
y_{n+1} &= 1 + p\frac{y_n}{y_{n-m}^2} = 1 + \frac{p}{y_{n-m}^2}\left(1 + p\frac{y_{n-1}}{y_{n-m-1}^2}\right) \\
&= 1 + \frac{p}{y_{n-m}^2} + p^2\frac{y_{n-1}}{y_{n-m}^2 y_{n-m-1}^2} = 1 + \frac{p}{y_{n-m}^2} + \frac{p^2}{y_{n-m}^2 y_{n-m-1}^2}\left(1 + p\frac{y_{n-2}}{y_{n-m-2}^2}\right) \\
&= 1 + \frac{p}{y_{n-m}^2} + \frac{p^2}{y_{n-m}^2 y_{n-m-1}^2} + \frac{p^3 y_{n-2}}{y_{n-m}^2 y_{n-m-1}^2 y_{n-m-2}^2}
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
&= 1 + \frac{p}{y_{n-m}^2} + \frac{p^2}{y_{n-m}^2 y_{n-m-1}^2} + \cdots + \frac{p^m y_{n-(m-1)}}{y_{n-m}^2 y_{n-m-1}^2 \cdots y_{n-2m+1}^2} \\
&= 1 + \frac{p}{y_{n-m}^2} + \frac{p^2}{y_{n-m}^2 y_{n-m-1}^2} + \cdots + \frac{p^m}{y_{n-m}^2 y_{n-m-1}^2 \cdots y_{n-2m+1}^2}\left(1 + p\frac{y_{n-m}}{y_{n-2m}^2}\right) \\
&= 1 + \frac{p}{y_{n-m}^2} + \cdots + \frac{p^m}{y_{n-m}^2 y_{n-m-1}^2 \cdots y_{n-2m+1}^2} + \frac{p^{m+1} y_{n-m}}{y_{n-m}^2 y_{n-m-1}^2 \cdots y_{n-2m}^2} \\
&= 1 + \frac{p}{y_{n-m}^2} + \cdots + \frac{p^m}{y_{n-m}^2 y_{n-m-1}^2 \cdots y_{n-2m+1}^2} + \frac{p^{m+1}}{y_{n-m} y_{n-m-1}^2 \cdots y_{n-2m}^2}.
\end{aligned} \tag{2.3}$$

From (2.3) and since

$$y_n > 1, \quad n \in \mathbb{N},$$

we have

$$y_{n+1} < 1 + p + \cdots + p^m + p^{m+1} = \frac{1 - p^{m+2}}{1 - p}, \tag{2.4}$$

for $n \geq 2m + 1$, when $p \neq 1$ and

$$y_{n+1} < m + 2, \tag{2.5}$$

for $n \geq 2m + 1$, when $p = 1$.

Let

$$S_m(p) := \sum_{j=0}^{m-1} p^j, \quad m \in \mathbb{N}.$$

Since

$$S_{m+2}(p) = 1 + p\sum_{j=0}^{m} p^j < 1 + p\sum_{j=0}^{m} C_j^m p^j = 1 + p(1+p)^m$$

we have that the upper bound given in (2.4) is better than the corresponding one in (1.13). Hence, the following result holds.

Theorem 2.2. *Let $p > 0$ and $m \in \mathbb{N}$. Then, every positive solution $(y_n)_{n \in \mathbb{N}_{-m}}$ to equation (1.8) is bounded and persistent and satisfies the following estimates*

$$1 < y_n < S_{m+2}(p),$$

for $n \geq 2m + 2$.

Remark 2.3. Some other methods which can be used in showing the boundedness of positive solutions to nonlinear difference equations, including applications of invariants, can be found, for instance, in [6, 15–17, 30, 33, 34, 39, 46, 47, 49–51, 53–55, 57, 58], as well as in the related references therein.

2.2 On Theorem 1.6 and some related results

The proof of Theorem 1.6 in [62] is not correct. Namely, to prove it, beside some nonessential conventional inaccuracies, the author of [62] only claims that from

$$m = 1 + p\frac{m}{M^2}, \quad M = 1 + p\frac{M}{m^2} \tag{2.6}$$

follows $m = M$ and apply a known result (i.e, [13, Theorem 1.15]) from which Theorem 1.6 immediately follows, but only if it is also employed Theorem 1.5 (this was not mentioned in [62], what can be regarded as an oversight). However, the above claim was not proved therein. It should be pointed out that its proof is neither obvious nor quite simple, although it uses only elementary mathematics (see the proof of Theorem 2.4 below). Moreover, it is possible to prove a better convergence result than Theorem 1.6. Namely, the following result holds.

Theorem 2.4. *Let $m \in \mathbb{N}$ and $p \in (0, 3/4] \cup [1, \infty)$. Then, the unique positive equilibrium point $\bar{y}$ of equation (1.8) is global attractor of all positive solutions.*

Proof. Since

$$I_p := [1, S_{m+2}(p)]$$

is an invariant interval for positive solutions to equation (1.8), as mentioned above, to apply [13, Theorem 1.15], it is enough to prove that the algebraic system (2.6) has only the solution $M = m = \bar{y}$ on the set I_p^2.

Assume that there is a solution to system (2.6) such that

$$M \neq m, \tag{2.7}$$

where $m, M \in I_p$.

From (2.6) we have

$$m - M = p\left(\frac{m}{M^2} - \frac{M}{m^2}\right) = \frac{p(m^3 - M^3)}{(mM)^2}$$

from which along with (2.7) it follows that

$$(Mm)^2 = p(m^2 + mM + M^2). \tag{2.8}$$

Further, from (2.6) we have

$$Mm = M + p\frac{m}{M} = m + p\frac{M}{m}$$

from which along with (2.7) it follows that

$$Mm = p(M + m). \tag{2.9}$$

From (2.6) we also have

$$M^2 m = M^2 + pm, \quad Mm^2 = m^2 + pM$$

from which along with (2.7) it follows that

$$Mm = M + m - p. \tag{2.10}$$

From (2.8) and (2.9) we have

$$p(M + m)^2 = m^2 + mM + M^2 = (M + m)^2 - Mm$$

and consequently

$$Mm = (1 - p)(M + m)^2. \tag{2.11}$$

If it were $p = 1$, then from (2.11) we would have $Mm = 0$, which contradicts to the assumption $m, M \in I_p$.

Now assume that $p \neq 1$. From (2.10) and (2.11) we have

$$(1-p)(M+m)^2 - (M+m) + p = (M+m-1)(M+m-p(M+m+1)) = 0. \tag{2.12}$$

Since $m, M \in I_p$ it is not possible to be $M + m = 1$. Hence, it must be $M + m = p(M + m + 1)$, that is

$$M + m = \frac{p}{1-p}. \tag{2.13}$$

From (2.13) and since $M + m > 0$ it follows that it must be $p \in (0, 1)$.

Using this fact, (2.10) and (2.13) we have

$$Mm = \frac{p}{1-p} - p = \frac{p^2}{1-p}. \tag{2.14}$$

From (2.13) and (2.14) we see that M and m are the two positive distinct roots of the quadratic polynomial

$$\widehat{P}_2(t) := t^2 - \frac{p}{1-p}t + \frac{p^2}{1-p}. \tag{2.15}$$

By direct calculation we see that the roots of polynomial (2.15) are

$$t_1 = \frac{p(1+\sqrt{4p-3})}{2(1-p)} \quad \text{and} \quad t_2 = \frac{p(1-\sqrt{4p-3})}{2(1-p)}. \tag{2.16}$$

From (2.16) we see that it is not possible to be $p \in (0, 3/4]$, since then $t_{1,2}$ will be complex numbers if $p \in (0, 3/4)$, or equal if $p = 3/4$.

Hence, the assumption $m \neq M$ is not possible, from which it follows that $M = m$, completing the proof of the theorem. □

Remark 2.5. From Theorem 2.4 and since the equilibrium $\bar{y}$ is locally asymptotically stable for $p \in (0, 3/4)$, the claim in Theorem 1.6 follows.

Remark 2.6. It is interesting to note that if $p \in (3/4, 1)$, then $t_2 > 1$. Indeed, we have

$$t_2 = \frac{2p}{1+\sqrt{4p-3}}$$

and consequently

$$t_2 - 1 = \frac{2p-1-\sqrt{4p-3}}{1+\sqrt{4p-3}}.$$

From this and since $2p - 1 - \sqrt{4p-3} > 0$ for $p \in (3/4, 1)$ is equivalent to $4(p-1)^2 > 0$, the claim follows.

To prove that Theorem 2.4 holds also for $p \in (3/4, 1)$ it will be enough to prove that

$$t_1 > \frac{1-p^{m+2}}{1-p}, \tag{2.17}$$

which is equivalent to

$$p(1+\sqrt{4p-3}) + 2p^{m+2} > 2.$$

Note that the function

$$f_m(t) := 2t^{m+2} + t(\sqrt{4t-3}+1) \tag{2.18}$$

is increasing for $t \geq 3/4$, and consequently on the interval $(3/4, 1)$.

On the other hand, we have

$$f_m\Big(\frac{3}{4}\Big) = 2\Big(\frac{3}{4}\Big)^{m+2} + \frac{3}{4} \leq \frac{51}{32} < 2$$

for any $m \in \mathbb{N}$, and

$$f_m(1) = 4 > 2.$$

Hence, for each $m \in \mathbb{N}$, there is a unique $\widehat{t}_m \in (3/4, 1)$, so that

$$f_m(\widehat{t}_m) = 2.$$

This with the monotonicity of the function f_m implies that inequality (2.17) holds when $p > \widehat{t}_m$.

This analysis shows that the following extension of Theorem 2.4 holds.

Theorem 2.7. *Let $m \in \mathbb{N}$, f_m be the function defined in (2.18), and $p \in (0, 3/4] \cup (\widehat{t}_m, +\infty)$, where $\widehat{t}_m$ is the unique solution to the equation $f_m(t) = 2$ on the interval $(3/4, 1)$. Then, the unique positive equilibrium point $\bar{y}$ of equation (1.8) is global attractor of all positive solutions.*

Remark 2.8. For each $m \in \mathbb{N}$ the value of $\widehat{t}_m$ can be approximated by a numeric calculation, but it seems cannot be expressed in closed form.

2.3 On Theorem 1.7 and relating results

The proof of the claim *(a)* in Theorem 1.7 is not correct. Before we present the arguments for the claim, let us mention that the constant/equilibrium solution

$$y_n = \bar{y}, \quad n \in \mathbb{N}_{-m},$$

should have been excluded from the consideration therein, since for the solution the claim is obviously not true. Namely, the solution has only one semi-cycle, which can be regarded as positive, but also it can be regarded as negative, depending on a definition of semi-cycles.

To prove the claim (for the nonconstant solutions), the author of [62] unsuccessfully copied a known method, which can be found, for example, in [1]. Namely, he chooses a positive solution $(y_n)_{n \geq \mathbb{N}_{-m}}$ to equation (1.8) and considers one of its negative semi-cycles (it is claimed that a similar argument proves the claim also for positive semi-cycles), say

$$y_N, y_{N+1}, \dots, y_{N+2m} < \bar{y}$$

where y_N is the first term in the negative semi-cycle.

It is claimed that

$$\begin{aligned} y_{N+m+i} &= 1 + p\frac{y_{N+m+i-1}}{y^2_{N+i-1}} > 1 + p\frac{y_{N+m+i-1}}{\bar{y}^2} \\ &> \Big(1 + \frac{p}{\bar{y}^2}\Big) y_{N+m+i-1} > y_{N+m+i-1} \end{aligned} \tag{2.19}$$

for $i = \overline{0, m-1}$, from which it is concluded that

$$y_{N+m} < y_{N+m+1} < \cdots < y_{N+2m-1} < y_{N+2m},$$

and consequently

$$y_{N+2m+1} = 1 + p\frac{y_{N+2m}}{y_{N+m}^2} > \bar{y}.$$

However, the second inequality in (2.19) is not correct, since if it was true, then it would be $y_{N+m+i-1} < 1$, which is not possible for any N satisfying the condition $N + m + i - 1 \geq 1$.

The above proof can be applied to positive solutions to the difference equation

$$y_{n+1} = 1 + p\frac{y_n}{y_{n-m}}, \tag{2.20}$$

however, as we have just noticed, it does not hold for the case of equation (1.8).

One of the interesting problems is the existence of solutions of difference equations which do not oscillate around an equilibrium. The characteristic polynomials associated to their linearizations can be of some help in the study. Namely, if a characteristic polynomial has a zero in the interval $(0, 1)$, then it suggests the existence of solutions of the original difference equation which do not oscillate around an equilibrium. Some methods for showing the existence of such solutions, which use the idea, have been developed in the last few decades (see, e.g., [6–11, 14, 43–45, 48] and the related references therein; for another method see [19]).

These facts suggest to consider the problem of the existence of zeros of the associated characteristic polynomial

$$P_{m+1}(t) = t^{m+1} - \frac{p}{\bar{y}^2}t^m + \frac{2p}{\bar{y}^2} = 0, \quad m \in \mathbb{N}_0. \tag{2.21}$$

to equation (1.10), in the interval $(0, 1)$.

First note that

$$P_{m+1}(0) = \frac{2p}{\bar{y}^2} = \frac{4p}{1 + 2p + \sqrt{1 + 4p}} \in (0, 2) \tag{2.22}$$

and

$$P_{m+1}(1) = 1 + \frac{p}{\bar{y}^2} = 1 + \frac{2p}{1 + 2p + \sqrt{1 + 4p}} \in (1, 2) \tag{2.23}$$

when $p > 0$.

We have

$$P'_{m+1}(t) = (m+1)t^{m-1}\left(t - \frac{2mp}{(m+1)(1 + 2p + \sqrt{1 + 4p})}\right), \quad m \in \mathbb{N}_0. \tag{2.24}$$

Hence, the polynomial $P'_{m+1}(t)$ has two zeros $\widetilde{t}_1 = 0$ and

$$\widetilde{t}_2 = \frac{2mp}{(m+1)(1 + 2p + \sqrt{1 + 4p})} \in (0, 1).$$

From (2.24) we have that the polynomial $P_{m+1}(t)$ has a local minimum at $\widetilde{t}_2$, which is equal to

$$P_{m+1}(\widetilde{t}_2) = \frac{p}{\bar{y}^2}\left(2 - \frac{m^m}{(m+1)^{m+1}}\left(\frac{2p}{1 + 2p + \sqrt{1 + 4p}}\right)^m\right). \tag{2.25}$$

From (2.25) it is easy to see that

$$P_{m+1}(\widetilde{t}_2) > \frac{p}{\bar{y}^2} > 0.$$

Thus, for any $m \in \mathbb{N}$, the polynomial $P_{m+1}(t)$ does not have a zero on the interval $[0,1]$.

If m is odd, i.e., $m = 2l - 1$ for some $l \in \mathbb{N}$, than t_2 is a unique local and global extremum (minimum).

If m is even, i.e., $m = 2l$ for some $l \in \mathbb{N}$, then $P_{2l+1}(t)$ has a local maximum at $\widetilde{t}_1$, and increases on the interval $(-\infty, 0]$. Hence, it has a unique negative zero. From this, (2.22) and since

$$P_{2l+1}(-1) = -1 + \frac{p}{\bar{y}^2} = -\frac{1+\sqrt{1+4p}}{1+2p+\sqrt{1+4p}} < 0 \tag{2.26}$$

when $p > 0$, we have that the polynomial $P_{2l+1}(t)$ has a unique zero t_0 in the interval $(-1, 0)$. This fact suggests the existence of oscillatory solutions to equation (1.8) in this case, which is not so surprising. It also suggests that the asymptotics of some oscillatory solutions can be find by using some of the methods presented in [6–11, 14, 43–45, 48]. We will not pursue the investigation in the direction here, since the above conducted analysis also suggests that all non-equilibrium solutions are oscillatory. Indeed, the following result on the oscillatory character of positive solutions to equation (1.8) holds.

Theorem 2.9. *Let $m \in \mathbb{N}$ and $p > 0$. Then, equation (1.8) does not have positive non-equilibrium solutions which are eventually bigger or smaller than the positive equilibrium $\bar{y}$.*

Proof. Assume that $(y_n)_{n \ge \mathbb{N}_{-m}}$ is a positive non-equilibrium solution to equation (1.8) such that

$$y_n \ge \bar{y}, \tag{2.27}$$

for $n \ge n_0$, for some $n_0 \in \mathbb{N}_{-m}$. Since equation (1.8) is autonomous we may assume that $n_0 = -m$.

From (1.8) and (2.27) we have

$$y_{n+1} - \bar{y} = p\frac{y_n}{y_{n-m}^2} - \frac{p}{\bar{y}} = p\frac{y_n\bar{y} - y_{n-m}^2}{y_{n-m}^2\bar{y}} \ge 0,$$

for $n \in \mathbb{N}_0$, from which along with (2.27) it follows that

$$y_n \ge \frac{y_{n-m}^2}{\bar{y}} \ge y_{n-m}, \tag{2.28}$$

for $n \in \mathbb{N}_0$.

From (2.28) we see that the subsequences y_{mk+i}, $k \in \mathbb{N}_{-1}$, $i = \overline{0, m-1}$, are nondecreasing. From this and Theorem 1.5 it follows that there are finite limits

$$\lim_{k\to+\infty} y_{mk+i} = \bar{y}_i, \quad i = \overline{0, m-1}. \tag{2.29}$$

Note also that it must be

$$\bar{y}_i \ge \bar{y}, \quad i = \overline{0, m-1}. \tag{2.30}$$

Letting $k \to +\infty$ in the relations

$$y_{mk+i+1} = 1 + p\frac{y_{mk+i}}{y_{m(k-1)+i}^2}, \quad i = \overline{0, m-1}$$

we get that $(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m)$ is a solution to the algebraic system

$$\bar{y}_{i+1} = 1 + \frac{p}{\bar{y}_i}, \quad i = \overline{0, m-1}, \tag{2.31}$$

where we use the standard convention $\bar{y}_i = \bar{y}_j$ when $i \equiv j\,(\mod m)$.

If $\bar{y}_{i_0+1} = \bar{y}_{i_0}$, for some $i_0 \in \{0, 1, \ldots, m-1\}$, then from (2.31) we see that it must be $\bar{y}_{i_0+1} = \bar{y}_{i_0} = \bar{y}$, from which together with (2.31) we easily get $\bar{y}_i = \bar{y}$, for $i = \overline{0, m-1}$, which contradicts to (2.30), the monotonicity of the subsequences $(y_{mk+i})_{k\in\mathbb{N}_{-1}}$, $i = \overline{0, m-1}$, and the assumption that the solution is non-equilibrium.

Now we assume that

$$\bar{y}_i \neq \bar{y}_{i+1}, \quad i = \overline{0, m-1}. \tag{2.32}$$

From (2.31) we have

$$\bar{y}_{i+1} - \bar{y}_i = \frac{p}{\bar{y}_i} - \frac{p}{\bar{y}_{i-1}} = \frac{p(\bar{y}_{i-1} - \bar{y}_i)}{\bar{y}_{i-1}\bar{y}_i}, \quad i = \overline{0, m-1}, \tag{2.33}$$

and

$$\bar{y}_i \bar{y}_{i+1} = \bar{y}_i + p, \quad i = \overline{0, m-1}. \tag{2.34}$$

From (2.33) it follows that

$$\prod_{i=0}^{m-1} (\bar{y}_{i+1} - \bar{y}_i) \left(\prod_{i=0}^{m-1} \bar{y}_i \right)^2 = (-1)^m p^m \prod_{i=0}^{m-1} (\bar{y}_{i+1} - \bar{y}_i), \tag{2.35}$$

from which along with (2.32) we immediately get a contradiction when m is an odd number. If m is an even number we get

$$\left(\prod_{i=0}^{m-1} \bar{y}_i \right)^2 = p^m. \tag{2.36}$$

On the other hand, from (2.34) we have

$$\left(\prod_{i=0}^{m-1} \bar{y}_i \right)^2 = \prod_{i=0}^{m-1} (\bar{y}_i + p). \tag{2.37}$$

From (2.36) and (2.37) and since

$$\prod_{i=0}^{m-1} (\bar{y}_i + p) \geq \prod_{i=0}^{m-1} (\bar{y} + p) > p^m \tag{2.38}$$

we get a contradiction.

If we assume that the inequality

$$y_n \leq \bar{y},$$

holds eventually, then as above is obtained that the subsequences y_{mk+i}, $k \in \mathbb{N}_{-1}$, $i = \overline{0, m-1}$, are nonincreasing and convergent, whereas the proof that the algebraic system (2.31) in this case, does not have a nontrivial solution follows by using the same argument and the inequalities

$$\prod_{i=0}^{m-1} (\bar{y}_i + p) \geq \prod_{i=0}^{m-1} (p+1) > p^m$$

first of which follows from the first inequality in (1.13). □

Concerning the proof of the claim *(b)* in Theorem 1.7, it is quite confusing. Namely, the number k is not specified at all. It does not have any connection with the claim of the theorem and the proof given therein is incorrect.

Here we formulate a claim which should generalize what should have been the claim of Theorem 1.7 *(b)* and give a detailed and complete proof.

Before it, we quote the following convention: a positive semicycle of a solution $(y_n)_{n\in\mathbb{N}_{-m}}$ to a difference equation is a finite or infinite subsequence/string $y_l, y_{l+1}, \dots, y_{N-1}$ satisfying the condition

$$y_j \geq \bar{y}, \quad j = \overline{l, N-1},$$

with $l \geq -m$ and $N \leq +\infty$, such that either $l = -m$, or $l > -m$ and $y_{l-1} < \bar{y}$, and either $N = +\infty$, or $N < +\infty$ and $y_N < \bar{y}$; a negative semicycle of a solution $(y_n)_{n\in\mathbb{N}_{-m}}$ to the difference equation is a finite or infinite subsequence/string $y_l, y_{l+1}, \dots, y_{N-1}$ satisfying the condition

$$y_j < \bar{y}, \quad j = \overline{l, N-1},$$

with $l \geq -m$ and $N \leq +\infty$, such that either $l = -m$, or $l > -m$ and $y_{l-1} \geq \bar{y}$, and either $N = +\infty$, or $N < +\infty$ and $y_N \geq \bar{y}$.

Theorem 2.10. *Let $A > 0$, $m \in \mathbb{N}$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing continuous function. Consider the difference equation*

$$y_{n+1} = A + \frac{y_n}{y_{n-m}g(y_{n-m})}, \quad n \in \mathbb{N}_0. \tag{2.39}$$

If a solution to equation (2.39) has a semi-cycle of length at least m, then every semi-cycle after that has at least $m+1$ terms.

Proof. First note that equation (2.39) has a unique positive equilibrium $\bar{y}$ which satisfies the relation

$$\bar{y} = A + \frac{1}{g(\bar{y})}. \tag{2.40}$$

Indeed the function

$$f(t) := A + \frac{1}{g(t)}$$

is a positive nonincreasing continuous function, from which the claim follows.

Let $y_{N-m}, y_{N-m+1}, \dots, y_{N-1}$ be a semi-cycle of length at least m. We may assume that it is a negative one and that $y_N \geq \bar{y}$. So, we have

$$\max\{y_{N-m}, y_{N-m+1}, \dots, y_{N-1}\} < \bar{y} \leq y_N. \tag{2.41}$$

We may also assume that $N - m \geq 1$.

From (2.40), (2.41), the monotonicity of the function g, and since

$$y_n > A, \quad n \in \mathbb{N},$$

we have

$$y_{N+1} = A + \frac{y_N}{y_{N-m}g(y_{N-m})} \geq A + \frac{y_N}{y_{N-m}g(\bar{y})} > A + \frac{1}{g(\bar{y})} = \bar{y}. \tag{2.42}$$

If we have proved that

$$y_{N+i} \geq \bar{y}$$

for some $0 \le i < m$, then similar to (2.42) we have

$$y_{N+i+1} = A + \frac{y_{N+i}}{y_{N+i-m}g(y_{N+i-m})} \ge A + \frac{y_{N+i}}{y_{N+i-m}g(\bar{y})} > A + \frac{1}{g(\bar{y})} = \bar{y}. \tag{2.43}$$

From the relations in (2.42) and (2.43), and employing the method of mathematical induction, we obtain that

$$y_{N+i} \ge \bar{y},$$

for $i = \overline{0, m}$, proving the claim for the case of the first semi-cycle after the chosen one.

If $y_{N-m}, y_{N-m+1}, \dots, y_{N-1}$ is a positive semi-cycle of length at least m and $y_N < \bar{y}$, that is

$$\min\{y_{N-m}, y_{N-m+1}, \dots, y_{N-1}\} \ge \bar{y} > y_N,$$

the proof is similar/dual, so is omitted.

Repeating the same procedure/arguments for the each next semi-cycle the theorem follows. □

A historical remark. A slightly different version of this paper was submitted to the Journal of Applied Mathematics and Computing on February 19, 2023. However, the paper has not been accepted for publication in the journal, and we have not received a real scientific explanation for the unexpected (at least for us) decision.

Author contributions. S. Stević initiated the investigation, proposed many ideas and methods, and conducted detailed investigations. B. Iričanin and W. Kosmala analyzed the proposed ideas, made some calculations, and gave some comments. All authors read and approved the final manuscript.

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