Electronic Journal of Qualitative Theory of Differential Equations
2025, No. **44**, 1–18; https://doi.org/10.14232/ejqtde.2025.1.44 www.math.u-szeged.hu/ejqtde/

Global boundedness in a quasilinear predator–prey chemotaxis system with nonlinear indirect signal production

Zhangsheng Zhu ✉

School of Science, Jiujiang University, Jiujiang 332005, P.R. China

Received 2 January 2025, appeared 2 September 2025
Communicated by Dimitri Mugnai

Abstract. This paper deals with the quasilinear predator–prey chemotaxis system with nonlinear indirect signal production under homogeneous Neumann boundary conditions. Based on the maximal Sobolev regularity and a priori estimates, the global boundedness of solution is investigated. Meanwhile, this work partially improves or extends the results of Wang and Ke [*Acta Appl. Math.* **190**(2024), Paper No. 9, 14 pp.].

Keywords: chemotaxis, predator–prey, nonlinear production, boundedness.

2020 Mathematics Subject Classification: 35B35, 35A01, 35K55, 92C17.

1 Introduction

The mathematical modeling of predator–prey has a long and distinguished history. The research on the predator–prey model was pioneered by Lotka and Volterra [26]. Predators naturally move towards areas with many prey, while prey move towards areas with fewer predators, effectively avoiding theirs pursuers. Thus, their movements are not accidental but are directional, forming what we call a chemotaxis phenomenon. Chemotaxis is the widespread phenomena in nature, for instance, the directional movement of biological cells, bacteria or organisms in response to some chemical substance, including the positive (chemo-attraction) chemotaxis and negative (chemo-repulsion) chemotaxis. The research on the famous classical Keller–Segel model described the chemotaxis was initiated by Keller and Segel [18]. Chemotaxis is the ability of certain living organisms to orient their movement along a chemical concentration gradient of chemical signal substance in their environment, plays an important role in a wide range of biological processes. From then on, the studies on modeling and analysis of predator–prey chemotaxis system have received more attention from theoretical and mathematical ecologists due to it has various interesting issues, such as, global solvability, finite time blow-up, time asymptotic behavior.

✉ Email: zhuzhangsheng2010@163.com

In this paper, we study the following quasilinear predator–prey chemotaxis with nonlinear indirect signal production

$$\begin{cases} u_t = \nabla \cdot (D_1(u)\nabla u) + \chi \nabla \cdot (u(u+1)^{\alpha-1}\nabla \omega) + \lambda_1 u(1-u^{r_1-1}-\mu_1 v), & x \in \Omega,\, t>0, \\ v_t = \nabla \cdot (D_2(v)\nabla v) - \xi \nabla \cdot (v(v+1)^{\beta-1}\nabla \omega) + \lambda_2 v(1-v^{r_2-1}+\mu_2 u), & x \in \Omega,\, t>0, \\ \omega_t = \Delta \omega - \omega + z^{\gamma}, & x \in \Omega,\, t>0, \\ 0 = \Delta z - z + f(u) + g(v), & x \in \Omega,\, t>0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial \omega}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega,\, t>0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), \omega(x,0) = \omega_0(x) & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary, ν is the unit outer normal vector on the boundary, the parameters $\chi, \xi, \lambda_1, \lambda_2, \mu_1, \mu_2, \gamma > 0, r_1, r_2 > 1$ and $\alpha, \beta \in \mathbb{R}$. The initial date u_0, v_0 and ω_0 satisfy

$$u_0, v_0 \in C^0(\overline{\Omega}) \quad \text{and} \quad \omega_0 \in W^{1,\infty}(\Omega), \quad u_0, v_0, \omega_0 \geq 0 (\not\equiv 0) \text{ in } \overline{\Omega}. \tag{1.2}$$

Here $u(x,t)$ and $v(x,t)$ represent the population density of the prey and the predator, respectively. $\omega(x,t)$, $z(x,t)$ denote the concentration of chemical attractants, $z(x,t)$ is produced by $u(x,t)$ and $v(x,t)$, and $\omega(x,t)$ is secreted by $z(x,t)$. The nonlinear functions $D_1, D_2 \in C^2([0,\infty))$ satisfy

$$D_1(u) \geq (u+1)^{m_1-1}, \qquad D_2(v) \geq (v+1)^{m_2-1} \tag{1.3}$$

with $m_1, m_2 \in \mathbb{R}$. The functions $f(u), g(v)$ are smooth and satisfy

$$0 < f(u) \leq (u+1)^{\gamma_1}, \qquad 0 < g(v) \leq (v+1)^{\gamma_2} \tag{1.4}$$

with $\gamma_1, \gamma_2 > 0$.

Before entering our motivation and main results of this paper, we first retrace some important developments on the (1.1) and its variants. We begin with the following well-known chemotaxis system. The Keller–Segel system was first proposed in [17] to describe the chemotaxis of cellular slime system, which reads as

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), & x \in \Omega,\, t>0, \\ v_t = \Delta v - v + u, & x \in \Omega,\, t>0, \end{cases} \tag{1.5}$$

where $u(x,t)$ and $v(x,t)$ denote the cell density and chemical concentration, respectively. Osaki et al. [30] obtained that the system (1.5) possesses a unique global classical solution which is uniformly bounded if $n = 1$. The solution of the system (1.5) may blow up in finite or infinite time for $n = 2$ or $n \geq 3$ in [11,13,32,45,46].

The main focus of the above Keller–Segel system is chemoattraction, however, in practical application, chemorepulsion is also involved in many biological processes. In fact, there is a repulsive Keller–Segel system

$$\begin{cases} u_t = \Delta u + \xi \nabla \cdot (u \nabla v), & x \in \Omega,\, t>0, \\ v_t = \Delta v - v + u, & x \in \Omega,\, t>0. \end{cases} \tag{1.6}$$

Cieślak et al. [5] obtained the global existence of the smooth solutions to system (1.6) in the two dimensions and weak solutions in dimension $n = 3, 4$. for results on the repulsive Keller–Segel model were given in [35,57].

In order to better understand the system (1.1), we first mention the following chemotaxis system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v) + f(u), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + g(u), & x \in \Omega,\ t > 0, \end{cases} \tag{1.7}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$, $D(u)$ and $S(u)$ represent the self-diffusion of the cells and the density dependent cross-diffusion mechanism, respectively. Meanwhile, The function $f(u)$ and $g(u)$ denote the logistic source and the production of chemical substances, respectively.

• At first, we introduce the case where $f(u) \equiv 0, g(u) = u$. For the case $D(u)$ varies in algebraic form, the known results indicate that the asymptotic of $\frac{S(u)}{D(u)} \simeq u^{\frac{2}{n}} (u > 1)$ is the critical condition distinguishing between blow-up and global boundedness to (1.7). Winkler [43] proved that if $\frac{S(u)}{D(u)} \geq cu^{\frac{2}{n}+\varepsilon}$ for all $u > 1$, the model (1.7) with Ω being a ball admits smooth solutions that blow-up either in finite or infinite time, while Tao and Winkler [36] showed that if Ω is convex and $\frac{S(u)}{D(u)} \leq cu^{\frac{2}{n}-\varepsilon}$ for all $u > 1$, then the solutions are globally bounded. Later, if $D(u) = (u+1)^{\alpha-1}, S(u) = u(u+1)^{\beta-1}$, there existed a global bounded classical solution under $\alpha > \beta - \frac{2}{n} + 1$ [16]. and if $\alpha < \beta - \frac{2}{n} + 1$ and when $\Omega \subset \mathbb{R}^n$ is a ball, then for any $\alpha > 0$, the solutions for system (1.7) might blow up in finite or infinite time when $n \geq 2$ [6,48].

• Turn to the case that more general $g(u)$ satisfies $0 < g(u) \leq u^\gamma$. When $D(u) \equiv 1$ and $S(u) = u$, Liu [24] showed the solutions of (1.7) are bounded in time if $\gamma \in (0, \frac{2}{n})$ with $n \geq 2$; whereas when $\gamma > \frac{2}{n}$, Winkler obtained radial blow-up solutions to (1.7) with replacing the second equation by $0 = \Delta v + u^\gamma - \frac{1}{|\Omega|}\int_\Omega u^\gamma$ [47]. Moreover, Other more results on global boundedness or blow-up of solutions related to the system (1.7), we refer to [4,23,49] for more details.

• Now, we will introduce the situation with logistic source $f(u) = ru - \mu u^{1+\sigma}$. If $D(u) \equiv 1, S(u) = u, g(u) = u$, it has been proven that (1.7) admits a unique globally bounded solution with $\sigma = 1, n \geq 3$ and $\mu > \mu_0$, where $\mu_0 > 0$ [44]. Furthermore, Xiang [51] proved the boundedness and convergence of solutions to this model by giving the explicit formula of $\mu_0 > 0$. If there is no bigness restriction to $\mu > 0$, Lankeit [20] showed that the model has a global weak solution when $r > 0$ and $n \geq 3$, and then established the eventual smoothness of the weak solution under the condition that $n = 3$ and r is sufficiently small. If $D(u) \equiv 1, S(u) = u, 0 \leq g(u) \leq u(u+1)^{\gamma-1}$ with $\gamma > 0$, Zhuang et al. [59] established that (1.7) admits a globally bounded solution under $\beta + \gamma = \sigma + 1$ with sufficiently large μ, or under $\beta + \gamma < \sigma + 1$ for all $\mu > 0$; Next, we assume that $D(u) \simeq (u+1)^{-\alpha}$, $S(u) \simeq u(u+1)^{\beta-1}, g(u) = u$ with $\alpha, \beta \in \mathbb{R}, r, \mu, \sigma > 0$, it is shown that in [55] that the model admits globally bounded for $0 < \alpha + \beta < \max\{\sigma + \alpha, \frac{2}{n}\}$ or $\beta = \sigma$ with μ large enough. Whereas if $g(u) = u^\gamma$ with $\gamma > 0$, Ding et al. [9] proved that the model (1.7) possesses a globally bounded and classical solution if one of the following holds:

(i) $\alpha + \beta + \gamma < 1 + \frac{2}{n}$,

(ii) $\beta + \gamma < \sigma + 1$, or $\beta + \gamma = \sigma + 1$ with μ large enough.

One of the well-known variants form of (1.7) is the two-species and one-stimuli chemotaxis system. To describe the movement of two species, the following chemotaxis model

$$\begin{cases} u_t = \nabla\cdot(D_1(u)\nabla u - S_1(u)\nabla\omega) + f(u,v), & x\in\Omega,\ t>0,\\ v_t = \nabla\cdot(D_2(v)\nabla v - S_2(v)\nabla\omega) + g(u,v), & x\in\Omega,\ t>0,\\ \tau\omega_t = \Delta\omega - \omega + h(u,v), & x\in\Omega,\ t>0, \end{cases} \tag{1.8}$$

was firstly proposed by Tello and Winkler in [38], where $\tau\in\{0,1\}$. For system (1.8), $u(x,t)$ and $v(x,t)$ denote the density of different species respectively, and $\omega(x,t)$ represents the concentrations of chemical substances. The functions $f(u,v), g(u,v)$ are functional response functions which are used to describe the relationship between two species, for instance, the predator–prey effect or competition mechanisms, and $h(u,v)$ is the production of chemical substances.

Before stating our motivation and main results of this paper, we mention some previous contributions of model (1.8). On the one hand, if $D_1(u)\equiv 1, D_2(v)\equiv 1, S_1(u)=\chi u, S_2(v)=\xi v$ and $f(u,v)=\mu_1 u(1-u-a_1 v), g(u,v)=\mu_2 v(1-v-a_2 u), h(u,v)=u+v$, Tello and Winkler [38] showed that the system (1.8) with $\tau=0$ admits a global existence and asymptotic behavior of solutions when $0\le a_1,a_2<1$. For $a_1>1>a_2\ge 0$, Stinner et al. [34] proved that the global existence and convergence of classical solutions. Later on, for $\tau=1$, Bai and Winkler [2] showed that the global boundedness and asymptotical behavior of solution with sufficiently large $\mu_1,\mu_2>0$ for the two cases $a_1\ge 1, 0<a_2<1$ and $0<a_1,a_2<1$ when $n\le 2$. In corresponding of strong competition $a_1,a_2\ge 1$, Pan et al. [31] have developed a method to solve the asymptotic behavior. In addition, When $K_{1i}(s+1)^{\alpha_i-1}\le D_i(s)\le K_{2i}(s+1)^{\beta_i-1}, \frac{S_i(s)}{D_i(s)}\le K_i(s+1)^{\gamma_i}, (i=1,2)$, and $f(u,v),g(u,v)\equiv 0, h(u,v)=u+v$, it is proved that system (1.8) admits a global bounded solution for $0<\gamma_i<\frac{2}{n}$ in [39].

On the other hand, if $f(u,v)=\mu_1 u(1-u-a_1 v)$ and $g(u,v)=\mu_2 v(1-v+a_2 u)$, the system (1.8) can be seen as a two-species predator–prey model with chemotaxis mechanism. When $D_1(u)\equiv 1, D_2(v)\equiv 1, S_1(u)=-\chi u, S_2(v)=\xi v$, in the case of $\tau=1$, Fu and Miao [10] proved that the system (1.8) admits a globally bounded classical solution under the parameters μ_1,μ_2,a_1,a_2 satisfy some suitable conditions and $n\le 2$. Furthermore, the unique positive equilibrium point and the semi-trivial equilibrium point were established, respectively. For $n=3$, Miao et al.[29] proved that the system (1.8) possesses a global and bounded classical solution. In the case $\tau=0$, Ma et al. [28] derived that the system (1.8) admits a boundedness and convergence of solutions for $n=2,3$. Recently, Yang-Xu-Pan [54] extended the work in [28] to arbitrary dimensional $n\ge 1$. When $D_1(u)\equiv 1, D_2(v)\equiv 1, S_1(u)=-\chi u^m, S_2(v)=\xi v^m$ and $f(u,v)=\mu_1 u(1-u-a_1 v), g(u,v)=\mu_2 v(1-v+a_2 u), h(u,v)\le \alpha u^l+\beta v^l$, it is shown that system (1.8) with $\tau=1$ possesses a global bounded solution if $\frac{\mu_2 a_2^{p+1}}{\mu_1}\le C$ with $p>\frac{n}{2}, n\ge 2$ and $l+m<1+\frac{2}{n}$ with $m\ge 1$ in [33]. Moreover, for more results on the predator–prey chemotaxis system with two chemicals, which can refer to the details in [22,25].

In the system mentioned above, the chemical stimulus are produced by population, directly. However, in some realistic biological processes, there are very few research results in multi-species and multi-stimuli chemotaxis systems, where the production mechanism of stimulus is indirect. This process can be described as

$$\begin{cases} u_t = d_1\Delta u - \chi_1\nabla\cdot(u\nabla\omega) + \mu_1 u(1-u-a_1 v), & x\in\Omega,\ t>0,\\ v_t = d_2\Delta v - \chi_2\nabla\cdot(v\nabla\omega) + \mu_2 v(1-v-a_2 u), & x\in\Omega,\ t>0,\\ \tau\omega_t = \Delta\omega - \omega + z, & x\in\Omega,\ t>0,\\ \tau z_t = \Delta z - z + u + v, & x\in\Omega,\ t>0, \end{cases} \tag{1.9}$$

where $\tau \in \{0,1\}$, $u(x,t)$ and $v(x,t)$ denote the density of two species, $\omega(x,t)$ and $z(x,t)$ represent the concentration of chemical substance, $\omega(x,t)$ is secreted by $z(x,t)$, $z(x,t)$ is secreted by $u(x,t), v(x,t)$. If $\tau = 1, n \le 2$, Xiang et al. [53] proved that the system (1.9) has a unique global classical solution, which is bounded in $\Omega \times (0,\infty)$. And if $\tau = 0$, it is shown that regardless of the size of u_0, v_0, the system admits a global bounded classical solution under two-dimension. Furthermore, asymptotic stabilization of global bounded solution to system (1.9) is researched under some suitable conditions of a_i, μ_i. Based on system (1.9), Xiang et al. [52] also discussed a two-competing-species chemotaxis model with indirect signal consumption, which admits a globally bounded and classical solution and obtains asymptotic stabilization under certain conditions. Besides, there are also other results of relevant systems in [41, 58]. More results involving the indirect signal production mechanism can refer to [3, 21, 27, 42, 50, 56].

Based on the above analysis, it is an interesting topic to study the impact of nonlinear indirect signal mechanisms on the dynamic behavior of predator–prey systems. Therefore, we will consider the following quasilinear predator–prey system with nonlinear indirect signal production

$$\begin{cases} u_t = \nabla\cdot(D_1(u)\nabla u) + \chi\nabla\cdot(u(u+1)^{\alpha-1}\nabla\omega) + \lambda_1 u(1-u^{r_1-1}-\mu_1 v), & x\in\Omega,\ t>0,\\ v_t = \nabla\cdot(D_2(v)\nabla v) - \xi\nabla\cdot(v(v+1)^{\beta-1}\nabla\omega) + \lambda_2 v(1-v^{r_2-1}+\mu_2 u), & x\in\Omega,\ t>0,\\ \omega_t = \Delta\omega - \omega + z^{\gamma}, & x\in\Omega,\ t>0,\\ 0 = \Delta z - z + f(u) + g(v), & x\in\Omega,\ t>0, \end{cases} \tag{1.10}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. When $D_1(u) = D_2(v) \equiv 1, \alpha = \beta = 1$ and $f(u) = u^{\gamma_1}, g(v) = v^{\gamma_2}$, Wang and Ke [40] proved that if $n \ge 1$ and $\gamma, \gamma_1, \gamma_2, n$ and r_1, r_2 satisfy suitable conditions, then the system (1.10) admitted a global bounded solution.

For the best of our knowledge, few rigorous result seem to be known about the quasilinear predator–prey system with nonlinear indirect signal production of (1.1). Thus, we will study the boundedness of the system (1.1), now we show the main result of the present paper.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n (n \ge 1)$ be a bounded domain with smooth boundary and the parameters fulfill $\chi, \xi, \lambda_1, \lambda_2, \mu_1, \mu_2, \gamma, \gamma_1, \gamma_2 > 0$ and $m_1, m_2, \alpha, \beta \in \mathbb{R}$. If $r_1 > 1, r_2 > 2$, $m_1 + \frac{2}{n} > \max\{\alpha + \gamma(\gamma_1+\gamma_2), \beta + \gamma(\gamma_1+\gamma_2), 1\}$ and $m_2 + \frac{2}{n} > \max\{\alpha + \gamma(\gamma_1+\gamma_2), \beta + \gamma(\gamma_1+\gamma_2), 1\}$. Assume that the initial date $u_0, v_0 \in C^0(\overline{\Omega})$ and $\omega_0 \in W^{1,\infty}(\Omega)$ are nonnegative, then the system (1.1) admits a nonnegative global classical solution*

$$(u,v,\omega,z) \in (C^0(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)))^3 \times (C^{2,0}(\overline{\Omega}\times(0,\infty))).$$

Moreover, the solution is bounded in $\Omega \times (0,\infty)$, in other words, there exists $C > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|\omega(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|z(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \tag{1.11}$$

for all $t > 0$.

2 Preliminaries

In this section, we first state the local existence of classical solutions to (1.1), which can be achieved by the fixed point theory and parabolic regularity. Refer to [8, 14, 19, 37] for more details.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary and the initial date $u_0, v_0 \in C^0(\overline{\Omega})$ and $\omega_0 \in W^{1,\infty}(\Omega)$ are nonnegative. Then there exists $T_{\max} \in (0, \infty]$ and a uniquely determined trip*

$$(u, v, \omega, z) \in \left(C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times [0, T_{\max}))\right)^3 \times (C^{2,0}(\overline{\Omega} \times (0, T_{\max})))$$

solving (1.1) in the classical sense in $\Omega \times (0, T_{\max})$.

With

$$u, v, \omega, z \geq 0 \quad in\ \overline{\Omega} \times (0, T_{\max}).$$

Furthermore, if $T_{\max} < \infty$, then

$$\lim_{t \to T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|\omega(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|z(\cdot, t)\|_{W^{1,\infty}(\Omega)} = \infty.$$

Lemma 2.2. *Assume that (u, v, ω, z) be a solution to (1.1). Then for any $\tilde{\eta} > 0$ and $\theta > 1$, there exists $\eta = \eta(\theta, \tilde{\eta}, |\Omega|)$ such that*

$$\int_\Omega z^\theta \leq \eta \int_\Omega (f(u)^\theta + g(v)^\theta) \ \text{for all } t \in (0, T_{\max}), \tag{2.1}$$

with $f(u), g(v)$ as given in system (1.1).

Proof. The proof can be obtained via the same method as in [12, Lemma 2.2], but we will provide a detailed proof process here. Integrating the fourth of (1.1) to obtain

$$\int_\Omega z \leq \int_\Omega (f(u) + g(v)). \tag{2.2}$$

Multiplying the fourth equation of (1.1) by $z^{\theta-1}$, and integrating on Ω, we get

$$\begin{aligned} \frac{4(\theta-1)}{\theta^2} \int_\Omega |\nabla z^{\frac{\theta}{2}}|^2 + \int_\Omega z^\theta &= \int_\Omega (f(u) + g(v)) z^{\theta-1} \\ &\leq \frac{\theta-1}{\theta} \int_\Omega z^\theta + \frac{1}{\theta} \int_\Omega \left(f(u) + f(v)\right)^\theta \end{aligned} \tag{2.3}$$

for all $t \in (0, T_{\max})$. By Young's inequality, and thus

$$\|z\|_{L^\theta(\Omega)} \leq \|f(u) + g(v)\|_{L^\theta(\Omega)}, \qquad t \in \ (0, T_{\max}), \tag{2.4}$$

$$\frac{4(\theta-1)}{\theta} \int_\Omega |\nabla z^{\frac{\theta}{2}}|^2 \leq \int_\Omega \left(f(u) + g(v)\right)^\theta, \qquad t \in \ (0, T_{\max}). \tag{2.5}$$

Using Ehrling's lemma, for any $\tilde{\eta} > 0$, $\theta > 1$, there exists $\tilde{C} = \tilde{C}(\tilde{\eta}, \theta) > 0$ such that

$$\|\phi\|^2_{L^2(\Omega)} \leq \tilde{\eta} \|\phi\|^2_{W^{1,2}(\Omega)} + \tilde{C} \|\phi\|^2_{L^{\frac{2}{\theta}}(\Omega)}, \ \phi \in W^{1,2}(\Omega). \tag{2.6}$$

Take $\phi = z^{\frac{\theta}{2}}$. we know from (2.2), (2.4) and (2.5) that

$$\int_\Omega z^\theta \leq \tilde{\eta} \int_\Omega \left(f(u) + g(v)\right)^\theta + c_0 \|f(u) + g(v)\|^\theta_{L^1} \tag{2.7}$$

with $c_0 = c_0(\tilde{\eta}, \theta) > 0$. We have by the Hölder inequality with (2.7) that

$$\begin{aligned} \int_\Omega z^\theta &\leq \tilde{\eta} \int_\Omega \left(f(u) + g(v)\right)^\theta + c_0 |\Omega|^{\theta-1} \int_\Omega \left(f(u) + g(v)\right)^\theta \\ &= (\tilde{\eta} + c_0 |\Omega|^{\theta-1}) \int_\Omega \left(f(u) + g(v)\right)^\theta \\ &\leq \eta \int_\Omega (f(u)^\theta + g(v)^\theta) \end{aligned} \tag{2.8}$$

with $\eta := 2^{\theta-1}(\tilde{\eta} + c_0 |\Omega|^{\theta-1})$. □

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary and the parameters satisfy $\chi, \xi, \lambda_1, \lambda_2, \mu_1, \mu_2, \gamma, \gamma_1, \gamma_2 > 0, r_1, r_2 > 1$ and $\alpha, \beta, m_1, m_2 \in \mathbb{R}$. Assume that (u, v, ω, z) be a solution to (1.1). Then there exists $C > 0$ such that*

$$\int_\Omega u + \int_\Omega v \leq C \tag{2.9}$$

for all $t \in (0, T_{\max})$.

Proof. Integrating the first two equation in (1.1) with respect to $x \in \Omega$, we get

$$\frac{d}{dt}\int_\Omega u + \int_\Omega u = \int_\Omega \lambda_1 u(1 - u^{r_1 - 1} - \mu_1 v) + \int_\Omega u \tag{2.10}$$

and

$$\frac{d}{dt}\int_\Omega v + \int_\Omega v = \int_\Omega \lambda_2 v(1 - v^{r_2 - 1} + \mu_2 u) + \int_\Omega v \tag{2.11}$$

for all $t \in (0, T_{\max})$. Multiplying the equation (2.10) and the equation (2.11) with $\lambda_2\mu_2$ and $\lambda_1\mu_1$, respectively, then we add them up and invoke Young's inequality to derive that there exists $c_1 > 0$ such that

$$\begin{aligned}
&\lambda_2\mu_2\frac{d}{dt}\int_\Omega u + \lambda_2\mu_2\int_\Omega u + \lambda_1\mu_1\frac{d}{dt}\int_\Omega v + \lambda_1\mu_1\int_\Omega v \\
&= \lambda_2\mu_2(\lambda_1 + 1)\int_\Omega u - \lambda_1\lambda_2\mu_2\int_\Omega u^{r_1} + \lambda_1\mu_1(\lambda_2 + 1)\int_\Omega v - \lambda_1\lambda_2\mu_1\int_\Omega v^{r_2} \\
&\leq -\frac{\lambda_1\lambda_2\mu_2}{2}\int_\Omega u^{r_1} - \frac{\lambda_1\lambda_2\mu_1}{2}\int_\Omega v^{r_2} + c_1 \\
&\leq c_1
\end{aligned} \tag{2.12}$$

for all $t \in (0, T_{\max})$. Thus, applying the ODE comparison principle to (2.12), there exists $c_2 > 0$ such that

$$\int_\Omega u + \int_\Omega v \leq c_2 \tag{2.13}$$

for all $t \in (0, T_{\max})$. □

Lemma 2.4 ([15, Lemma 3.6] and [7, Lemma 3.2])**.** *Let $q \in (1, \infty)$ and $T \in (0, \infty]$, Then, for $g \in L^q([0, T); L^q(\Omega))$ and $z_0 \in W^{2,q}(\Omega)$, with $\frac{\partial z_0}{\partial \nu} = 0$ on $\partial\Omega$, there exists $C_q = C_q(\Omega, q, \|z_0\|_{W^{2,q}(\Omega)}) > 0$ such that every solution $z \in W^{1,q}_{loc}([0, T); L^q(\Omega)) \cap L^q_{loc}([0, T); W^{2,q}(\Omega))$ of*

$$\begin{cases}
z_t = \Delta z - z + g, & x \in \Omega,\ t > 0, \\
\frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\
z(x, 0) = z_0, & x \in \Omega,
\end{cases} \tag{2.14}$$

satisfies

$$\int_0^t e^s \int_\Omega |\Delta z(\cdot, t)|^q ds \leq C_q \left(1 + \int_0^t e^s \int_\Omega |g(\cdot, t)|^q ds\right) \quad \text{for all } t \in (0, T). \tag{2.15}$$

3 Global existence and boundedness

In this section, in order to prove the global existence and uniform boundedness of solutions to the system (1.1), we first establish the L^p-estimate for u, v.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary and the parameters satisfy $\chi, \xi, \lambda_1, \lambda_2, \mu_1, \mu_2, \gamma, \gamma_1, \gamma_2 > 0$ and $m_1, m_2, \alpha, \beta \in \mathbb{R}$. If $r_1 > 1, r_2 > 2$, $m_1 + \frac{2}{n} > \max\{\alpha + \gamma_(\gamma_1 + \gamma_2), \beta + \gamma(\gamma_1 + \gamma_2), 1\}$ and $m_2 + \frac{2}{n} > \max\{\alpha + \gamma(\gamma_1 + \gamma_2), \beta + \gamma(\gamma_1 + \gamma_2), 1\}$, then for any $p > \max\{1 - m_1, 1 - \alpha, 1 - \beta, 1 - m_2, 1 - \alpha + (1 - \gamma)(\gamma_1 + \gamma_2), 1 - \beta + (1 - \gamma)(\gamma_1 + \gamma_2), \frac{(2 - r_1)(r_2 - 1)}{r_2 - 2}, 1\}$, there exists $C > 0$ such that*

$$\int_\Omega u^p + \int_\Omega v^p \leq C \quad \textit{for all } t \in (0, T_{\max}). \tag{3.1}$$

Proof. For any $p > 1$, testing the first equation in system (1.1) by $(u+1)^{p-1}$, we have

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p =& -(p-1)\int_\Omega (u+1)^{p-2} D_1(u)|\nabla u|^2 - \chi(p-1)\int_\Omega u(u+1)^{p+\alpha-3}\nabla u \cdot \nabla \omega \\
&+ \lambda_1 \int_\Omega u(u+1)^{p-1} - \lambda_1 \int_\Omega u^{r_1}(u+1)^{p-1} - \lambda_1\mu_1 \int_\Omega u(u+1)^{p-1} v \\
\leq& -(p-1)\int_\Omega (u+1)^{p+m_1-3}|\nabla u|^2 - \chi(p-1)\int_\Omega u(u+1)^{p+\alpha-3}\nabla u \cdot \nabla \omega \\
&+ \lambda_1 \int_\Omega u(u+1)^{p-1} - \lambda_1 \int_\Omega u^{r_1}(u+1)^{p-1}
\end{aligned} \tag{3.2}$$

for all $t \in (0, T_{\max})$. Due to

$$(a+b)^s \leq 2^{s-1}(a^s + b^s)$$

for all $a, b > 0, s > 1$. We get

$$\frac{1}{2^{r_1-1}}\int_\Omega (u+1)^{p+r_1-1} \leq \int_\Omega (u+1)^{p-1}u^{r_1} + \int_\Omega (u+1)^{p-1} \tag{3.3}$$

with $t \in (0, T_{\max})$. Let $\varphi(t) := \frac{1}{p}\int_\Omega (u+1)^p$, $h_\alpha(u(\cdot, t)) := \int_0^u s(s+1)^{p+\alpha-3} ds$. Thus, by a simple calculation with (3.2) and (3.3), we obtain

$$\begin{aligned}
\varphi'(t) \leq& -\frac{4(p-1)}{(p+m_1-1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 - \chi(p-1)\int_\Omega \nabla h_\alpha(u) \cdot \nabla \omega + \lambda_1 \int_\Omega u(u+1)^{p-1} \\
&- \lambda_1 \int_\Omega u^{r_1}(u+1)^{p-1} \\
\leq& -\frac{4(p-1)}{(p+m_1-1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 + \chi(p-1)\int_\Omega h_\alpha(u)\Delta \omega + \lambda_1 \int_\Omega (u+1)^p \\
&+ \lambda_1 \int_\Omega (u+1)^{p-1} - \frac{\lambda_1}{2^{r_1-1}}\int_\Omega (u+1)^{p+r_1-1}
\end{aligned} \tag{3.4}$$

for all $t \in (0, T_{\max})$. For the term $h_\alpha(u(\cdot t))$, we achieve

$$\begin{aligned}
h_\alpha(u(\cdot, t)) &= \int_0^{u(\cdot,t)} s(s+1)^{p+\alpha-3} ds \\
&\leq \frac{1}{p+\alpha-1}(u(\cdot, t)+1)^{p+\alpha-1}
\end{aligned} \tag{3.5}$$

for all $t \in (0, T_{\max})$. Thus, by the simple calculation with (3.5) and Young's inequality, we get

$$\begin{aligned}
\chi(p-1)\int_\Omega h_\alpha(u)\Delta \omega &\leq \frac{\chi(p-1)}{p+\alpha-1}\int_\Omega (u+1)^{p+\alpha-1}|\Delta \omega| \\
&\leq \int_\Omega (u+1)^{p+\alpha-1+\gamma(\gamma_1+\gamma_2)} + c_3 \int_\Omega |\Delta \omega|^{\frac{p+\alpha-1+\gamma(\gamma_1+\gamma_2)}{\gamma(\gamma_1+\gamma_2)}}
\end{aligned} \tag{3.6}$$

for all $t \in (0, T_{\max})$, where $c_3 = \left(\frac{\chi(p-1)}{p+\alpha-1}\right)^{\frac{p+\alpha-1+\gamma(\gamma_1+\gamma_2)}{\gamma(\gamma_1+\gamma_2)}}$. Using Young's inequality once more, for $r_1 > 1$ we have

$$\begin{aligned} \lambda_1 \int_\Omega (u+1)^p + \lambda_1 \int_\Omega (u+1)^{p-1} - \frac{\lambda_1}{2^{r_1-1}} \int_\Omega (u+1)^{p+r_1-1} \\ \le \frac{\lambda_1}{2^{r_1}} \int_\Omega (u+1)^{p+r_1-1} - \frac{\lambda_1}{2^{r_1-1}} \int_\Omega (u+1)^{p+r_1-1} + c_4 \\ = -\frac{\lambda_1}{2^{r_1}} \int_\Omega (u+1)^{p+r_1-1} + c_4 \end{aligned} \tag{3.7}$$

for all $t \in (0, T_{\max})$. Substituting (3.6)–(3.7) into (3.4), we obtain

$$\begin{aligned} \varphi'(t) \le & -\frac{4(p-1)}{(p+m_1-1)^2} \int_\Omega |\nabla (u+1)^{\frac{p+m_1-1}{2}}|^2 + \int_\Omega (u+1)^{p+\alpha-1+\gamma(\gamma_1+\gamma_2)} \\ & + c_3 \int_\Omega |\Delta \omega|^{\frac{p+\alpha-1+\gamma(\gamma_1+\gamma_2)}{\gamma(\gamma_1+\gamma_2)}} - \frac{\lambda_1}{2^{r_1}} \int_\Omega (u+1)^{p+r_1-1} + c_4 \end{aligned} \tag{3.8}$$

for all $t \in (0, T_{\max})$.

Now, we add to both sides of (3.8) the term $\frac{1}{p}\int_\Omega (u+1)^p$, and we multiply by e^t. Since $e^t\varphi'(t) + e^t\varphi(t) = \frac{d}{dt}(e^t\varphi(t))$, an integration over $(0,t)$ provides for all $t \in (0, T_{\max})$

$$\begin{aligned} e^t\varphi(t) \le \varphi(0) & - \frac{4(p-1)}{(p+m_1-1)^2} \int_0^t e^s \int_\Omega |\nabla (u+1)^{\frac{p+m_1-1}{2}}|^2 + \int_0^t e^s \int_\Omega (u+1)^{p+\alpha-1+\gamma(\gamma_1+\gamma_2)} \\ & + c_3 \int_0^t e^s \int_\Omega |\Delta\omega|^{\frac{p+\alpha-1+\gamma(\gamma_1+\gamma_2)}{\gamma(\gamma_1+\gamma_2)}} - \frac{\lambda_1}{2^{r_1}} \int_0^t e^s \int_\Omega (u+1)^{p+r_1-1} + \frac{1}{p} \int_0^t e^s \int_\Omega (u+1)^p \\ & + c_4 \int_0^t e^s ds. \end{aligned} \tag{3.9}$$

Taking $q = \frac{p+\alpha-1+\gamma(\gamma_1+\gamma_2)}{\gamma(\gamma_1+\gamma_2)} > 1$ in the Lemma 2.4, according to (2.15) and the third equation in system (1.1), we know

$$\int_0^t e^s \int_\Omega |\Delta\omega|^{\frac{p+\alpha-1+\gamma(\gamma_1+\gamma_2)}{\gamma(\gamma_1+\gamma_2)}} \le C_q\left(1 + \int_0^t e^s \int_\Omega z^{\frac{p+\alpha-1+\gamma(\gamma_1+\gamma_2)}{\gamma_1+\gamma_2}} ds\right) \tag{3.10}$$

for all $t \in (0, T_{\max})$. Applying Lemma 2.2 with $\theta = \frac{p+\alpha-1+\gamma(\gamma_1+\gamma_2)}{\gamma_1+\gamma_2} > 1$, and by (1.4), this implies

$$\int_\Omega z^{\frac{p+\alpha-1+\gamma(\gamma_1+\gamma_2)}{\gamma_1+\gamma_2}} \le \eta \int_\Omega \left((1+u)^{\frac{\gamma_1(p+\alpha-1+\gamma(\gamma_1+\gamma_2))}{\gamma_1+\gamma_2}} + (1+v)^{\frac{\gamma_2(p+\alpha-1+\gamma(\gamma_1+\gamma_2))}{\gamma_1+\gamma_2}} \right) \tag{3.11}$$

for all $t \in (0, T_{\max})$. By Young's inequality, we have

$$\int_\Omega z^{\frac{p+\alpha-1+\gamma(\gamma_1+\gamma_2)}{\gamma_1+\gamma_2}} \le \int_\Omega \left((1+u)^{p+\alpha-1+\gamma(\gamma_1+\gamma_2)} + (1+v)^{p+\alpha-1+\gamma(\gamma_1+\gamma_2)}\right) + c_5 \tag{3.12}$$

for all $t \in (0, T_{\max})$. Collecting (3.10)–(3.12), we know

$$\begin{aligned} e^t\varphi(t) \le \varphi(0) & - \frac{4(p-1)}{(p+m_1-1)^2} \int_0^t e^s \int_\Omega |\nabla (u+1)^{\frac{p+m_1-1}{2}}|^2 + c_6 \int_0^t e^s \int_\Omega (u+1)^{p+\alpha-1+\gamma(\gamma_1+\gamma_2)} \\ & + c_7 \int_0^t e^s \int_\Omega (v+1)^{p+\alpha-1+\gamma(\gamma_1+\gamma_2)} + \frac{1}{p} \int_0^t e^s \int_\Omega (u+1)^p - \frac{\lambda_1}{2^{r_1}} \int_0^t e^s \int_\Omega (u+1)^{p+r_1-1} \\ & + c_8 \int_0^t e^s ds \end{aligned} \tag{3.13}$$

for all $t \in (0, T_{\max})$. We get from (2.9) with the Gagliardo–Nirenberg inequality that

$$
\begin{aligned}
&\hat{c}\int_\Omega (u+1)^{p+\alpha+\gamma(\gamma_1+\gamma_2)-1} \\
&= \hat{c}\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^{\frac{2(p+\alpha+\gamma(\gamma_1+\gamma_2)-1)}{p+m_1-1}}(\Omega)}^{\frac{2(p+\alpha+\gamma(\gamma_1+\gamma_2)-1)}{p+m_1-1}} \\
&\leq c_9\left\|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^2(\Omega)}^{\theta_1\frac{2(p+\alpha+\gamma(\gamma_1+\gamma_2)-1)}{p+m_1-1}}\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^{\frac{2}{p+m_1-1}}(\Omega)}^{(1-\theta_1)\frac{2(p+\alpha+\gamma(\gamma_1+\gamma_2)-1)}{p+m_1-1}} \\
&\quad + c_9\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^{\frac{2}{p+m_1-1}}(\Omega)}^{\frac{2(p+\alpha+\gamma(\gamma_1+\gamma_2)-1)}{p+m_1-1}} \\
&\leq c_{10}\left(\int_\Omega\left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2\right)^{\theta_1\frac{p+\alpha+\gamma(\gamma_1+\gamma_2)-1}{p+m_1-1}} + c_{11}
\end{aligned}
\tag{3.14}
$$

for all $t \in (0, T_{\max})$ and $\theta_1 = \frac{\frac{p+m_1-1}{2} - \frac{p+m_1-1}{2(p+\alpha+\gamma(\gamma_1+\gamma_2)-1)}}{\frac{p+m_1-1}{2} - \frac{1}{2} + \frac{1}{n}} \in (0,1)$.

Since $m_1 + \frac{2}{n} > \alpha + \gamma(\gamma_1+\gamma_2)$ implies $\theta_1\frac{p+\alpha+\gamma(\gamma_1+\gamma_2)-1}{p+m_1-1} < 1$, we have by Young's inequality that

$$
\hat{c}\int_\Omega (u+1)^{p+\alpha+\gamma(\gamma_1+\gamma_2)-1} \leq \frac{(p-1)}{(p+m_1-1)^2}\int_\Omega\left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2 + c_{12} \tag{3.15}
$$

for all $t \in (0, T_{\max})$. By (2.9) with the Gagliardo–Nirenberg inequality once more, we get

$$
\begin{aligned}
\hat{c}\int_\Omega (u+1)^p &= \hat{c}\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^{\frac{2p}{p+m_1-1}}(\Omega)}^{\frac{2p}{p+m_1-1}} \\
&\leq c_{13}\left\|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^2(\Omega)}^{\theta_2\frac{2p}{p+m_1-1}}\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^{\frac{2}{p+m_1-1}}(\Omega)}^{(1-\theta_2)\frac{2p}{p+m_1-1}} \\
&\quad + c_{13}\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^{\frac{2}{p+m_1-1}}(\Omega)}^{\frac{2p}{p+m_1-1}} \\
&\leq c_{14}\left(\int_\Omega\left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2\right)^{\theta_2\frac{p}{p+m_1-1}} + c_{15}
\end{aligned}
\tag{3.16}
$$

for all $t \in (0, T_{\max})$ and $\theta_2 = \frac{\frac{p+m_1-1}{2} - \frac{p+m_1-1}{2p}}{\frac{p+m_1-1}{2} - \frac{1}{2} + \frac{1}{n}} \in (0,1)$. Due to $m_1 + \frac{2}{n} > 1$ implies $\theta_2\frac{p}{p+m_1-1} < 1$, we get by Young's inequality that

$$
\hat{c}\int_\Omega (u+1)^p \leq \frac{(p-1)}{(p+m_1-1)^2}\int_\Omega\left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2 + c_{15} \tag{3.17}
$$

for all $t \in (0, T_{\max})$. Combining (3.15) and (3.17), we obtain

$$
\begin{aligned}
e^t\varphi(t) \leq \varphi(0) &- \frac{2(p-1)}{(p+m_1-1)^2}\int_0^t e^s\int_\Omega\left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2 + c_7\int_0^t e^s\int_\Omega (v+1)^{p+\alpha-1+\gamma(\gamma_1+\gamma_2)} \\
&- \frac{\lambda_1}{2^{r_1}}\int_0^t e^s\int_\Omega (u+1)^{p+r_1-1} + c_{16}\int_0^t e^s ds
\end{aligned}
\tag{3.18}
$$

for all $t \in (0, T_{\max})$. Multiplying the second equation in system (1.1) with $(1+v)^{p-1}$, for any

$p > 1$, we derive from integration by parts that

$$\begin{aligned} \tilde{\varphi}'(t) \le & -\frac{4(p-1)}{(p+m_2-1)^2}\int_\Omega \left|\nabla (v+1)^{\frac{p+m_2-1}{2}}\right|^2 - \frac{\xi(p-1)}{p+\beta-1}\int_\Omega (v+1)^{p+\beta-1}|\Delta\omega| \\ & + \lambda_2\left(\int_\Omega (v+1)^p + \int_\Omega (v+1)^{p-1}\right) \\ & - \frac{\lambda_2}{2^{r_2-1}}\int_\Omega (v+1)^{p+r_2-1} + \lambda_2\mu_2\int_\Omega (v+1)^p u \end{aligned} \tag{3.19}$$

for all $t \in (0, T_{\max})$, where $\tilde{\varphi}(t) = \frac{1}{p}\int_\Omega (v+1)^p$. Due to $r_2 > 1$, applications of Young's inequality imply that

$$\lambda_2\int_\Omega (v+1)^p + \lambda_2\int_\Omega (v+1)^{p-1} \le \frac{\lambda_2}{2^{r_2-2}}\int_\Omega (v+1)^{p+r_2-1} + c_{17} \tag{3.20}$$

and

$$\begin{aligned} \lambda_2\mu_2\int_\Omega (1+v)^p u & \le \frac{\lambda_2}{2^{r_2-2}}\int_\Omega (v+1)^{p+r_2-1} + c_{18}\int_\Omega u^{\frac{p+r_2-1}{r_2-1}} \\ & \le \frac{\lambda_2}{2^{r_2-2}}\int_\Omega (v+1)^{p+r_2-1} + c_{18}\int_\Omega (u+1)^{\frac{p+r_2-1}{r_2-1}} \end{aligned} \tag{3.21}$$

for all $t \in (0, T_{\max})$. Substituting (3.20)–(3.21) into (3.19), we conclude that

$$\begin{aligned} \tilde{\varphi}'(t) \le & -\frac{4(p-1)}{(p+m_2-1)^2}\int_\Omega \left|\nabla (v+1)^{\frac{p+m_2-1}{2}}\right|^2 - \frac{\xi(p-1)}{p+\beta-1}\int_\Omega (v+1)^{p+\beta-1}|\Delta\omega| \\ & + c_{18}\int_\Omega (u+1)^{\frac{p+r_2-1}{r_2-1}} + c_{19} \end{aligned} \tag{3.22}$$

for all $t \in (0, T_{\max})$. Invoking the same procedure as dealing with u, we have

$$\begin{aligned} e^t\tilde{\varphi}(t) \le \tilde{\varphi}(0) & - \frac{2(p-1)}{(p+m_2-1)^2}\int_0^t e^s\int_\Omega \left|\nabla (v+1)^{\frac{p+m_2-1}{2}}\right|^2 + c_{20}\int_0^t e^s\int_\Omega (u+1)^{p+\beta-1+\gamma(\gamma_1+\gamma_2)} \\ & + c_{18}\int_0^t e^s\int_\Omega (u+1)^{\frac{p+r_2-1}{r_2-1}} + c_{21}\int_0^t e^s ds \end{aligned} \tag{3.23}$$

for all $t \in (0, T_{\max})$. By (2.9) with the Gagliardo–Nirenberg inequality once more, we obtain

$$\begin{aligned} & \hat{c}\int_\Omega (u+1)^{p+\beta+\gamma(\gamma_1+\gamma_2)-1} \\ & = \hat{c}\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^{\frac{2(p+\beta+\gamma(\gamma_1+\gamma_2)-1)}{p+m_1-1}}(\Omega)}^{\frac{2(p+\beta+\gamma(\gamma_1+\gamma_2)-1)}{p+m_1-1}} \\ & \le c_{22}\left\|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^2(\Omega)}^{\theta_3\frac{2(p+\beta+\gamma(\gamma_1+\gamma_2)-1)}{p+m_1-1}}\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^{\frac{2}{p+m_1-1}}(\Omega)}^{(1-\theta_1)\frac{2(p+\beta+\gamma(\gamma_1+\gamma_2)-1)}{p+m_1-1}} \\ & \quad + c_{22}\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^{\frac{2}{p+m_1-1}}(\Omega)}^{\frac{2(p+\beta+\gamma(\gamma_1+\gamma_2)-1)}{p+m_1-1}} \\ & \le c_{23}\left(\int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2\right)^{\theta_3\frac{p+\beta+\gamma(\gamma_1+\gamma_2)-1}{p+m_1-1}} + c_{24} \end{aligned} \tag{3.24}$$

for all $t \in (0, T_{\max})$ and $\theta_3 = \frac{\frac{p+m_1-1}{2} - \frac{p+m_1-1}{2(p+\beta+\gamma(\gamma_1+\gamma_2)-1)}}{\frac{p+m_1-1}{2} - \frac{1}{2} + \frac{1}{n}} \in (0,1)$.

Since $m_1 + \frac{2}{n} > \beta + \gamma(\gamma_1+\gamma_2)$ implies $\theta_3\frac{p+\beta+\gamma(\gamma_1+\gamma_2)-1}{p+m_1-1} < 1$, which in conjunction with

Young's inequality deduces that

$$\hat{c}\int_\Omega (u+1)^{p+\beta+\gamma(\gamma_1+\gamma_2)-1} \le \frac{2(p-1)}{(p+m_1-1)^2}\int_\Omega |\nabla(u+1)^{\frac{p+m_1-1}{2}}|^2 + c_{25} \tag{3.25}$$

for all $t \in (0, T_{\max})$. In view of the Gagliardo–Nirenberg inequality, there exists $c_{26} > 0$ such that

$$\begin{aligned}
&\hat{c}\int_\Omega (v+1)^{p+\alpha+\gamma(\gamma_1+\gamma_2)-1} \\
&\quad = \hat{c}\left\|(v+1)^{\frac{p+m_2-1}{2}}\right\|_{L^{\frac{2(p+\alpha+\gamma(\gamma_1+\gamma_2)-1)}{p+m_2-1}}(\Omega)}^{\frac{2(p+\alpha+\gamma(\gamma_2+\gamma_3)-1)}{p+m-1}} \\
&\quad \le c_{26}\left\|\nabla(v+1)^{\frac{p+m_2-1}{2}}\right\|_{L^2(\Omega)}^{\theta_4\frac{2(p+\alpha+\gamma(\gamma_1+\gamma_2)-1)}{p+m_2-1}} \left\|(u+1)^{\frac{p+m_2-1}{2}}\right\|_{L^{\frac{2}{p+m_2-1}}(\Omega)}^{(1-\theta_4)\frac{2(p+\alpha+\gamma(\gamma_1+\gamma_2)-1)}{p+m_2-1}} \\
&\qquad + c_{26}\left\|(v+1)^{\frac{p+m_2-1}{2}}\right\|_{L^{\frac{2}{p+m_2-1}}(\Omega)}^{\frac{2(p+\alpha+\gamma(\gamma_1+\gamma_2)-1)}{p+m_2-1}} \\
&\quad \le c_{27}\left(\int_\Omega \left|\nabla(v+1)^{\frac{p+m_2-1}{2}}\right|^2\right)^{\theta_4\frac{p+\alpha+\gamma(\gamma_1+\gamma_2)-1}{p+m_2-1}} + c_{28}
\end{aligned} \tag{3.26}$$

for all $t \in (0, T_{\max})$ and $\theta_4 = \frac{\frac{p+m_2-1}{2} - \frac{p+m_2-1}{2(p+\alpha+\gamma(\gamma_1+\gamma_2)-1)}}{\frac{p+m_2-1}{2} - \frac{1}{2} + \frac{1}{n}} \in (0,1)$.
Due to $m_2 + \frac{2}{n} > \alpha + \gamma(\gamma_1+\gamma_2)$ implies $\theta_4 \frac{p+\alpha+\gamma(\gamma_1+\gamma_2)-1}{p+m_2-1} < 1$, we get by Young's inequality that

$$\hat{c}\int_\Omega (v+1)^{p+\alpha+\gamma(\gamma_1+\gamma_2)-1} \le \frac{2(p-1)}{(p+m_2-1)^2}\int_\Omega \left|\nabla(v+1)^{\frac{p+m_2-1}{2}}\right|^2 + c_{30} \tag{3.27}$$

for all $t \in (0, T_{\max})$. Combing (3.18), (3.23), (3.25), and (3.27), we obtain

$$\begin{aligned}
\frac{1}{p}e^t\int_\Omega u^p + \frac{1}{p}e^t\int_\Omega v^p &\le \frac{1}{p}e^t\int_\Omega (u+1)^p + \frac{1}{p}e^t\int_\Omega (v+1)^p \\
&= e^t\varphi(t) + e^t\tilde{\varphi}(t) \\
&\le \varphi(0) + \tilde{\varphi}(0) - \frac{\lambda_1}{2^{r_1}}\int_0^t e^s\int_\Omega (u+1)^{p+r_1-1} \\
&\quad + c_{18}\int_0^t e^s\int_\Omega (u+1)^{\frac{p+r_2-1}{r_2-1}} + c_{31}\int_0^t e^s ds
\end{aligned} \tag{3.28}$$

for all $t \in (0, T_{\max})$. Since $r_2 > 2, p > \frac{(2-r_1)(r_2-1)}{r_2-2}$, it is not difficult to compute that

$$\begin{aligned}
p + r_1 - 1 - \frac{p+r_2-1}{r_2-1} &= \frac{(p+r_1-1)(r_2-1) - (p+r_2-1)}{r_2-1} \\
&= \frac{p(r_2-2) + (r_1-1)(r_2-1) - (r_2-1)}{r_2-1} \\
&= \frac{p(r_2-2)}{r_2-1} + r_1 - 2 > 0.
\end{aligned} \tag{3.29}$$

Thus, we can conclude from Young's inequality that

$$c_{18}\int_0^t e^s\int_\Omega (u+1)^{\frac{p+r_2-1}{r_2-1}} \le \frac{\lambda_1}{2^{r_1}}\int_0^t e^s\int_\Omega (u+1)^{p+r_1-1} + c_{32} \tag{3.30}$$

for all $t \in (0, T_{\max})$. Combining (3.28) and (3.30), we have

$$e^t \left(\int_\Omega u^p + \int_\Omega v^p \right) \le c_{33} + c_{34}(e^t - 1) \tag{3.31}$$

for all $t \in (0, T_{\max})$. Thus, we conclude that

$$\int_\Omega u^p + \int_\Omega v^p \le c_{35} \tag{3.32}$$

for all $t \in (0, T_{\max})$. Therefore, the proof of Lemma 3.1 is complete. □

Now, we are in a position to complete the Theorem 1.1.

Proof of Theorem 1.1. For any $p > \max\{1 - m_1, 1 - m_2, 1 - \alpha, 1 - \beta, 1 - \alpha + (1 - \gamma)(\gamma_1 + \gamma_2), 1 - \beta + (1 - \gamma)(\gamma_1 + \gamma_2), \frac{(2-r_1)(r_2-1)}{r_2-2}, n\gamma_1, n\gamma_2, 1\}$ with $\gamma, \gamma_1, \gamma_2 > 0, m_1, m_2, \alpha, \beta \in \mathbb{R}$ and $n \ge 1$, by the elliptic L^p-estimate to the fourth equation in system (1.1) ensures the existence of $c_{36} > 0$ fulfilling

$$\|z(\cdot, t)\|_{W^{2, \frac{p}{\max\{\gamma_1, \gamma_2\}}}(\Omega)} \le c_{36}, \qquad t \in (0, T_{\max}).$$

Thus

$$\|z(\cdot, t)\|_{W^{1,\infty}(\Omega)} \le c_{37}, \quad t \in (0, T_{\max}), \quad \text{with } c_{37} > 0,$$

by the Sobolev imbedding theorem. Applying the parabolic regularity to the third equation in system (1.1), we have $\|\omega(\cdot, t)\|_{W^{1,\infty}(\Omega)} \le c_{38}$ for all $t \in (0, T_{\max})$, with $c_{38} > 0$. From Lemma 3.1, we apply Moser iteration technique [1,36] ensures

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \le c_{39},$$

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \le c_{40}$$

for all $t \in (0, T_{\max})$, with $c_{39}, c_{40} > 0$. Recalling Lemma 2.1, we infer that $T_{\max} = \infty$. Thus, we complete the proof of Theorem 1.1. □

Acknowledgements

The author is very grateful to the editor and anonymous reviewers for their careful reading and valuable comments, which greatly improve the work. This work was supported by the National Natural Science Foundation of China (Grant No. 11961037).

References

[1] N. D. Alikakos, L^p bounds of solutions of reaction-diffusion equations, *Comm. Partial Differ. Equations* **4**(1979), No. 8, 827–868. https://doi.org/10.1080/03605307908820113; MR537465; Zbl 0421.35009

[2] X. Bai, M. Winkler, Equilibration in a fully parabolic two-species chemotaxis system with competitive kinetics, *Indiana Univ. Math. J.* **65**(2016), No. 2, 553–583. https://doi.org/10.1512/iumj.2016.65.5775; MR3498177; Zbl 1345.35117

[3] X. Cao, Superlinear transmission in an indirect signal production chemotaxis system, *Appl. Math. Lett.* **158**(2024), 1–6. https://doi.org/10.1016/j.aml.2024.109235; MR4778748; Zbl 1547.35066

[4] X. Cao, X. Gao, Critical mass in a quasilinear parabolic-elliptic Keller–Segel model, *J. Differential Equations* **361**(2023), No. 2, 449–471. https://doi.org/10.1016/j.jde.2023.03.005; MR4564930; Zbl 1519.35204

[5] T. Cieślak, P. Laurençot, C. M. Rodrigo, Global existence and convergence to steay states in a chemorepulsion system, *Banach Center Publ, Polish Acad. Sci.* **81**(2008), 105–117. https://doi.org/10.4064/bc81-0-7; MR2549326; Zbl 1156.35325

[6] T. Cieślak, C. Stinner, New critical exponents in a fully parabolic quasilinear Keller–Segel system and applications to volume filling models, *J. Differential Equations* **258**(2015), No. 6, 2080–2113. https://doi.org/10.1016/j.jde.2014.12.004; MR3302531

[7] A. Columbu, R. D. Fuentes, S. Frassu, Uniform-in-time boundedness in a class of local and nonlinear attraction-repulsion chemotaxis models with logistics, *Nonlinear Anal. Real. World Appl.* **79**(2024), 1–14. https://doi.org/10.1016/j.nonrwa.2024.104135; MR4746566; Zbl 1544.35043

[8] M. Ding, W. Wang, Global boundedness in a quasilinear fully parabolic chemotaxis system with indirect signal production, *Discrete Contin. Dyn. Syst. Ser. B.* **24**(2019), No. 9, 4665–4684. https://doi.org/10.3934/dcdsb.2018328; MR3986213; Zbl 1429.35039

[9] M. Ding, W. Wang, S. Zhou, S. Zheng, Asymptotic stability in a fully parabolic quasilinear chemotaxis model with general logistic source and signal production, *J. Differential Equations* **268**(2020), No. 11, 6729–6777. https://doi.org/10.1016/j.jde.2019.11.052; MR4075558; Zbl 1435.35197

[10] S. Fu, L. Miao, Global existence and asymptotic stability in a predator–prey chemotaxis model, *Nonlinear Anal. Real. World Appl.* **54**(2020), 1–25. https://doi.org/10.1016/j.nonrwa.2019.103079; MR4050858; Zbl 1436.35228

[11] H. Gajewski, K. Zacharias, Global behavior of a reaction-diffusion system modelling chemotaxis, *Math. Nachr.* **195**(1998), 77–114. https://doi.org/10.1002/mana.19981950106; MR1654677; Zbl 0918.35064

[12] L. Hong, M. Tian, S. Zheng, An attraction–repulsion chemotaxis system with nonlinear productions, *J. Math. Anal. Appl.* **484**(2020), No. 1, 1–8. https://doi.org/10.1016./j.jmaa.2019.123703; MR4042624; Zbl 1432.92018

[13] D. Horstmann, G. Wang, Blow-up in a chemotaxis model without symmetry assumptions, *European J. Appl. Math.* **12**(2001), No. 2, 159–177. https://doi.org/10.1017/s0956792501004363; MR1931303; Zbl 1017.92006

[14] D. Horstmann, M. Winkler, Boundedness vs. blow-up in a chemotaxis system, *J. Differential Equations* **215**(2005), No. 1, 52–107. https://doi.org/10.1016/j.jde.2004.10.022; MR2146345; Zbl 1085.35065

[15] S. Ishida, J. Lankeit, G. Viglialoro, A Keller–Segel type taxis model with ecological interpretation and boundedness due to gradient nonlinearities, *Discrete Contin.*

Dyn. Syst. Ser. B **29**(2024), No. 9, 3955–3969. https://doi.org/10.3934/dcdsb.2024029; MR4762553; Zbl 1546.35003

[16] S. Ishida, K. Seki, T. Yokota, Boundedness in quasilinear Keller–Segel systems parabolic-parabolic type on non-convex bounded domains, *J. Differential Equations* **256**(2014), No. 8, 2993–3010. https://doi.org/10.1016/j.jde.2014.01.028; MR3199754; Zbl 1295.35252

[17] E. F. Keller, L. A. Segel, Initiation of slime mold aggregation viewed as an instablity, *J. Theoret. Biol.* **26**(1970), No. 3, 399–415. https://doi.org/10.1016/0022-5193(70)90092-5; MR3925816; Zbl 1170.92306

[18] E. F. Keller, L. A. Segel, Model for chemotaxis, *J. Theoret. Biol.* **30**(1971), No. 2, 225–234. https://doi.org/10.1016/0022-5193(71)90050-6; Zbl 1170.92307

[19] O. A. Ladyženskaja, V. A. Solonnikov, N. N. Ural'ceva, Linear and quasilinear equations of parabolic type, Vol. 23, American Mathematical Society, Providence, Rl, 1968. https://doi.org/10.1090/mmono/023; MR0241822; Zbl 0164.12302

[20] J. Lankeit, Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source, *J. Differential Equations* **258**(2015), No. 4, 1158–1191. https://doi.org/10.1016/j.jde.2014.10.016; MR3294344; Zbl 1319.35085

[21] X. Li, Global existence and boundedness of a chemotaxis model with indirect production and general kinetic function, *Z. Angew. Math. Phys.* **71**(2020), No. 4, 1–22. https://doi.org/10.1007/s00033-020-01339-z; MR4119568; Zbl 1450.35066

[22] G. Li, Y. Tao, M. Winkler, Large time behavior in a predator–prey system with indirect pursuit–evasion interaction, *Discrete Contin. Dyn. Syst., Ser. B* **25**(2020), No. 11, 4383–4396. https://doi.org/10.3934/dcdsb.2020102; MR4160112; Zbl 1453.92250

[23] K. Lin, T. Xiang, Strong damping effect of chemo-repulsion prevents blow-up, *J. Math. Phys.* **62**(2021), No. 4, 1–29. https://doi.org/10.1063/5.0032829; MR4240598; Zbl 1466.92018

[24] D. Liu, Y. Tao, Boundedness in a chemotaxis system with nonlinear signal production, *Appl. Math. J. Chinese Univ. Ser. B* **31**(2016), No. 4, 379–388. https://doi.org/10.1007/s11766-016-3386-z; MR3579264; Zbl 1374.35009

[25] X. Liu, J. Zheng, Convergence rates of solutions in a predator–prey system with indirect pursuit-evasion interaction in domains of arbitrary dimension, *Discrete Contin. Dyn. Syst. Ser. B* **28**(2023), No. 3, 2269–2293. https://doi.org/10.3934/dcdsb.2022168; MR4512334; Zbl 1502.35019

[26] A. Lotka, *Elements of mathematical biology*, Dover Publications, Inc., New York, 1956. https://doi.org/10.1002/bimj.19650070323; MR0094263; Zbl 0074.14404

[27] H. Lv, S. Zheng, Boundedness for the chemotaxis system in flux limitation with indirect signal production, *J. Math. Anal. Appl.* **538**(2024), No. 1, 1–9. https://doi.org/10.1016/j.jmaa.2024.128364; MR4728604 Zbl 1540.35075

[28] Y. Ma, C. Mu, S. Qiu, Boundedness and asymptotic stability in a two-species predator–prey chemotaxis model, *Discrete Contin. Dyn. Syst. Ser. B* **27**(2022), No. 7, 4077–4095. https://doi.org/10.3934/dcdsb.2021218; MR4430650; Zbl 1500.35045

[29] L. Miao, H. Yang, S. Fu, Global boundedness in a two-species predator–prey chemotaxis model, *Appl. Math. Lett.* **111**(2021), 1–8. https://doi.org/10.1016/j.aml.2020.106639; MR4128585; Zbl 1450.35087

[30] K. Osaki, A. Yagi, Finite dimensional attractors for one-dimensional Keller–Segel equations, *Funkcial. Ekvac.* **44**(2001), 441–469. https://doi.org/10.1016/0022-2364(85)90127-1; MR1893940; Zbl 1145.37337

[31] X. Pan, C. Mu, W. Tao, On the strongly competitive case in a fully parabolic two-species chemotaxis system with Lotka–Volterra competitive kinetics, *J. Differential Equations* **354**(2023), 90–132. https://doi.org/10.1016/j.jde.2023.01.008; MR4537693; Zbl 1515.35053

[32] T. Senba, T. Suzuki, Parabolic system of chemotaxis: blow up in a finite and the infinite time, *Methods Appl. Anal.* **8**(2001), No. 2, 349–367. https://doi.org/10.4310/MAA.2001.v8.n2.a9; MR1904534; Zbl 1056.92007

[33] G. Shanmugasundaram, G. Arumugam, A. Erhardt, N. Nagarajan, Global existence of solutions to a two-species predator–prey parabolic chemotaxis system, *Int. J. Biomath.* **15**(2022), No. 8, 1–22. https://doi.org/10.1142/S1793524522500541; MR4480357; Zbl 1498.92035

[34] C. Stinner, J. I. Tello, M. Winkler, Competitive exclusion in a two-species chemotaxis model, *J. Math. Biol.* **68**(2014), No. 7, 1607–1626. https://doi.org/10.1007/s00285-013-0681-7; MR3201907; Zbl 1319.92050

[35] Y. Tao, Global dynamics in a higher-dimensional repulsion chemotaxis model with nonlinear sensitivity, *Discrete Contin. Dyn. Syst. Ser. B* **18**(2013), No. 10, 2705–2722. https://doi.org/10.3934/dcdsb.2013.18.2705; MR3124759; Zbl 1282.35189

[36] Y. Tao, M. Winkler, Boundedness in a quasilinear parabolic-parabolic Keller–Segel system with subcritical sensitivity, *J. Differential Equations* **252**(2012), No. 1, 692–715. https://doi.org/10.1016/j.jde.2011.08.019; MR2852223; Zbl 1382.35127

[37] J. I. Tello, M. Winkler, A chemotaxis system with logistic source, *Comm. Partial Differential Equations* **32**(2007), No. 6, 849–877. https://doi.org/10.1080/03605300701319003; MR2334836; Zbl 1121.37068

[38] J. I. Tello, M. Winkler, Stabilization in a two-species chemotaxis system with a logistic source, *Nonlinearity* **25**(2012), No. 5, 1413–1425. https://doi.org/10.1088/0951-7715/25/5/1413; MR2914146; Zbl 1260.92014

[39] M. Tian, S. Zheng, Global boundedness versus finite-time blow-up of solutions to a quasilinear fully parabolic Keller–Segel system of two species, *Commun. Pure Appl. Anal.* **15**(2016), No. 1, 243–260. https://doi.org/10.3934/CPAA.2016.15.243; MR3437543; Zbl 1356.92014

[40] C. Wang, C. Ke, Global classical solutions to a predator–prey model with nonlinear indirect chemotaxis mechanism, *Acta. Appl. Math.* **190**(2024), Paper No. 11, 14 pp. https://doi.org/10.1007/s10440-024-00648-z; MR4730373; Zbl 1537.35012

[41] D. Wang, F. Zeng, M. Jiang, Global existence and boundedness of solutions to a two-species chemotaxis-competition system with singular sensitivity and indirect signal production, *Z. Angew. Math. Phys.* **74**(2023), No. 1, 1–23. https://doi.org/10.1007/s00033-022-01921-7; MR4530208; Zbl 1504.35081

[42] C. Wang, Y. Zhu, X. Zhu, Long time behavior of the solution to a chemotaxis system with nonlinear indirect signal production and logistic source, *Electron. J. Qual. Theory Differ. Equ.* **2023**(2023), No. 11, 1–21. https://doi.org/10.14232/ejqtde.2023.1.11; MR4577527; Zbl 1538.35068

[43] M. Winkler, Does a 'volume-filling effect' always prevent chemotaxis collapse?, *Math. Methods Appl. Sci.* **33**(2010), No. 1, 12–24. https://doi.org/10.1002/mma.1146; MR2591220; Zbl 1182.35220

[44] M. Winkler, Boundedness in the higher-dimensional parabolic–parabolic chemotaxis system with logistic source, *Comm. Partial Differential Equations* **35**(2010), No. 8, 1516–1537. https://doi.org/10.1080/03605300903473426; MR2754053; Zbl 1290.35139

[45] M. Winkler, Aggregation vs. Global diffusive behavior in the higher-dimensional Keller–Segel model, *J. Differential Equations* **248**(2010), No. 12, 2889–2905. https://doi.org/10.1016/j.jde.2010.02.008; MR2644137; Zbl 1190.92004

[46] M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller–Segel system, *J. Math. Pures Appl.* **100**(2013), No. 5, 748–767. https://doi.org/10.1016/J.MATPUR.2013.01.020; MR3115832; Zbl 1326.35053

[47] M. Winkler, A critical blow-up exponent in a chemotaxis system with nonlinear signal production, *Nonlinearity* **31**(2018), No. 5, 2031–2056. https://doi.org/10.1088/1361-6544/aaaa0e; MR3816663; Zbl 1391.35240

[48] M. Winkler, Global classical solvability and generic infinite-time blow-up in quasilinear Keller–Segel systems with bounded sensitivities, *J. Differential Equations* **266**(2019), No. 12, 8034–8066. https://doi.org/10.1016/j.jde.2018.12.019; MR3944248; Zbl 1415.35052

[49] M. Winkler, A family of mass-critical Keller–Segel systems, *Proc. Lond. Math. Soc.* **124**(2022), No. 2, 133–181. https://doi.org/10.1112/plms.12425; MR4400933

[50] S. Wu, Boundedness in a quasilinear chemotaxis model with logistic growth and indirect signal production, *Acta. Appl. Math.* **176**(2021), No. 9, 1–14. https://doi.org/10.1007/s10440-021-00454-x; MR4340636; Zbl 1477.35290

[51] T. Xiang, How strong a logistic damping can prevent blow-up for the minimal Keller–Segel chemotaxis system?, *J. Math. Anal. Appl.* **459**(2018), No. 2, 1172–1200. https://doi.org/10.1016/j.jmaa.2017.11.022; MR3732580; Zbl 1380.35032

[52] Y. Xiang, P. Zheng, On a two-species chemotaxis-competition system with indirect signal consumption, *Z. Angew. Math. Phys.* **73**(2022), No. 2, 1–19. https://doi.org/10.1007/s00033-022-01680-5; MR4380107; Zbl 1485.35041

[53] Y. Xiang, P. Zheng, J. Xing, Boundedness and stabilization in a two-species chemotaxis-competition system with indirect signal production, *J. Math. Anal. Appl.* **507**(2022), No. 2, 1–40. https://doi.org/10.1016/j.jmaa.2021.125825; MR4343779; Zbl 1480.35384

[54] L. Yang, L. Xu, X. Pan, Improvement of conditions for global boundedness in a predator–prey system with pursuit-evasion interaction, *Discrete Contin. Dyn. Syst. Ser. B* **29**(2024), No. 3, 1187–1205. https://doi.org/10.3934/dcdsb.2023128; MR4717069; Zbl 1532.35009

[55] J. Zheng, Boundedness of solutions to a quasilinear parabolic-parabolic Keller–Segel system with a logistic source, *J. Math. Anal. Appl.* **431**(2015), No. 2, 867–888. https://doi.org/10.1016/j.jmaa.2015.05.071; MR3365844; Zbl 1333.35098

[56] W. Zheng, P. Niu, S. Liu, Large time behavior in a chemotaxis model with logistic growth and indirect signal production, *Nonlinear Anal. Real World Appl.* **50**(2019), 484–497. https://doi.org/10.1016/j.nonrwa.2019.05.002; MR3959244; Zbl 1435.35068

[57] S. Zhou, T. Gong, J. Yang, Boundedness in a fully parabolic quasilinear repulsion chemotaxis model of higher dimension, *Appl. Math. J. Chinese Univ. Ser. B* **35**(2020), No. 2, 244–252. https://doi.org/10.1007/s11766-020-3994-5; MR4113282; Zbl 1463.35008

[58] Z. Zhu, Asymptotic stability in a predator–prey system with density-dependent diffusion and indirect pursuit-evasion interaction in 2D, *Bound. Value Probl.* **2024**(2024), No. 147, 1-16. https://doi.org/10.1186/s13661-024-01960-1; MR4821877; Zbl 1557.35053

[59] M. Zhuang, W. Wang, S. Zheng, Boundedness in a fully parabolic chemotaxis system with logistic-type source and nonlinear production, *Nonlinear Anal., Real World Appl.* **47**(2019), 473–483. https://doi.org/10.1016/j.nonrwa.2018.12.001; MR3894330; Zbl 1415.35067