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Attractivity with asymptotic phase of local center manifolds and an application to one-parameter bifurcation for integral equations with infinite delay

Dedicated to Professor Satoru Murakami on the occasion of his 70th birthday

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Abstract. For autonomous C^1-smooth integral equations with infinite delay, exponential attractivity with asymptotic phase of the local center manifolds of the equilibrium 0, together with a reduction principle, is proved by means of a dynamical systems approach based on the variation-of-constants formula in the phase space established in [*Funkcial. Ekvac.* **55**(2012), 479–520]. As its application to one-parameter family of integral equations, it is also shown that saddle-node and pitchfork bifurcations occur when the equilibrium 0 (the zero solution) of the linearized equation changes its stability properties.

Keywords: integral equations with infinite delay, attractivity with asymptotic phase, center manifolds, reduction principle, bifurcation, a variation-of-constants formula.

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1 Introduction

We consider in this paper the integral equation with infinite delay

$$x(t) = \int_{-\infty}^{t} K(t-s)x(s)ds + f(x_t), \tag{E}$$

where K is a measurable $m \times m$ matrix valued function with complex components that satisfies the conditions

$$\int_0^\infty \|K(t)\|e^{\rho t}dt < \infty \quad \text{and} \quad \text{ess sup}\{\|K(t)\|e^{\rho t} : t \geq 0\} < \infty,$$

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and f belongs to the space $C^1(X;\mathbb{C}^m)$, the set of all continuously (Fréchet) differentiable functions mapping X into $\mathbb{C}^m$, with the property that $f(0) = 0$ and $Df(0) = 0$; here, ρ is a positive constant which is fixed throughout the paper. Let $X := L^1_\rho(\mathbb{R}^-;\mathbb{C}^m)$, $\mathbb{R}^- := (-\infty, 0]$, be a Banach space which will be introduced in the next section as the phase space for Eq. (E), and x_t denotes the element in X defined as $x_t(\theta) = x(t+\theta)$ for $\theta \in \mathbb{R}^-$. In [21] the first author et al. established a "variation-of-constants formula" for integral equations (VCF, for short) in the phase space, which allows us to study behaviors of solutions to Eq. (E) in the dynamical systems framework. Indeed, by means of VCF, the present authors have proved center manifold theorem; that is, the existence, the (local) exponential attractivity and so on of the (local) center manifolds of the equilibrium point 0 of Eq. (E). For VCF in the phase space for abstract functional differential equations with infinite delay and its applications to study of such as almost periodic solutions and invariant manifold theory, see [12–14, 25–27]. We should also refer the reader for treatments of Eq. (E) by adjoint semigroup theory to the pioneering works due to Diekmann and Gyllenberg [5], in which several important results, together with the principle of linearized stability for integral equations, are established.

One of the purposes of the paper is to improve a part of our preceding results on center manifolds for Eq. (E) [23, Theorems 4 and 5]; more precisely, to show that the local center manifolds are exponentially attractive with asymptotic phase under the assumption that the nonlinear term f is of class C^1. Thus, it turns out that the behavior of any solution of Eq. (E) in a neighborhood of the local center manifold of the (nonhyperbolic) equilibrium 0 is determined by the dynamics on this manifold which is described by a finite dimensional ordinary differential equation, which we call the *central equation* of Eq. (E). As its application we also intend to discuss one-parameter bifurcation structures for integral equations. These subjects have been extensively studied and now are popular for ordinary differential equations, functional differential equations, parabolic partial differential equations and so on (see [1, 2, 4, 6–11, 15–17, 19, 20, 24, 28, 29] and the literature cited therein). However, to the best of our knowledge, it seems that they have remained to be open problems to integral equations. Another purpose of the paper is to show that bifurcations of equilibria can occur for one-parameter family of integral equations with infinite delay. Our analysis will be thoroughly done in a dynamical systems viewpoint, based on VCF in the phase space.

The paper is organized as follows. In Section 2, we present preliminary results, e.g., VCF, the center manifold theorem for Eq. (E), and formal adjoint theory for the linear part of Eq. (E) ([21–23]), which are necessary for our later arguments. Section 3 proves the exponential attractivity with asymptotic phase of the local center manifolds of the equilibrium 0 for Eq. (E) under the assumption that the unstable subspace (of the phase space) is trivial (Theorem 3.3 (a)). In addition a reduction principle will be obtained, which is also a refinement of [23, Theorem 6]; so that, besides the asymptotic stability of the equilibrium 0 of Eq. (E), its stability also follows from that of the central equation of Eq. (E) (Theorem 3.3 (b)). We discuss in Section 4 one-parameter bifurcation problems for integral equations. For this we introduce an extended system of integral equations including the parameter as a state variable. By an application of Theorem 3.3 (a), combined with the study of the central equation of the extended system, we show that under some additional assumptions on the integral kernel, saddle-node and pitchfork bifurcations can be observed as the parameter takes the critical value (Theorem 4.4). In Section 5, we will give an example of a scalar integral equation to which our results in the previous section are applicable.

2 Notations and preliminary results

Let $\mathbb{N}$, $\mathbb{R}^-$, $\mathbb{R}^+$, $\mathbb{R}$ and $\mathbb{C}$ be the set of natural numbers, nonpositive real numbers, nonnegative real numbers, real numbers and complex numbers, respectively. For an $m \in \mathbb{N}$, we denote by $\mathbb{C}^m$ the space of all m-column vectors whose components are complex numbers, with the Euclidean norm $|\cdot|$.

Given Banach spaces $(U, \|\cdot\|_U)$ and $(V, \|\cdot\|_V)$ and positive integer n, we denote by $\mathcal{L}^n(U;V)$ the space of bounded n-linear mappings from U to W with norm

$$\|Q\|_{\mathcal{L}^n(U;V)} := \sup\left\{\|Q(u_1,\dots,u_n)\|_V : \|u_j\|_U \le 1,\, u_j \in U,\, j = 1,\dots,n\right\}$$

for $Q \in \mathcal{L}^n(U;V)$. We use the symbol $\mathcal{L}(U;V)$, the space of bounded linear operators, rather than $\mathcal{L}^1(U;V)$; and simply write $\mathcal{L}(U)$ in place of $\mathcal{L}(U;U)$. In particular, for $m \times m$-matrix M with complex components, $\|M\|$ means its operator norm $\|M\|_{\mathcal{L}(\mathbb{C}^m)}$.

For arbitrary linear operator A, the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ stand for the null space and the range of A, respectively.

Given an interval $J \subset \mathbb{R}$ and a Banach space U, we denote by $C(J;U)$ the space of U-valued continuous functions on J, and by $BC(J;U)$ its subspace of bounded continuous functions on J. We also use the notation $B_U(u_0;r)$ which stands for the open ball in U at the center u_0 with radius $r > 0$, that is, $B_U(u_0;r) = \{u \in U : \|u - u_0\|_U < r\}$. If $u_0 = 0$, we simply write $B_U(r)$ rather than $B_U(0;r)$. Also, $\overline{B}_U(u_0;r)$ denotes the closure of $B_U(u_0;r)$.

2.1 Phase space and initial value problems

Let ρ be a fixed positive constant, and let us introduce the function space

$$X := L^1_\rho(\mathbb{R}^-;\mathbb{C}^m) = \{\phi : \mathbb{R}^- \to \mathbb{C}^m : \phi(\theta)e^{\rho\theta} \text{ is integrable on } \mathbb{R}^-\}.$$

X is a Banach space endowed with norm

$$\|\phi\|_X := \int_{-\infty}^0 |\phi(\theta)|e^{\rho\theta}d\theta, \quad \phi \in X.$$

For any function $x : (-\infty, a) \to \mathbb{C}^m$ and $t < a$, we define a function $x_t : \mathbb{R}^- \to \mathbb{C}^m$ by $x_t(\theta) := x(t+\theta)$ for $\theta \in \mathbb{R}^-$; the function x_t is called the t-segment of $x(t)$.

Consider the integral equations

$$x(t) = \int_{-\infty}^t K(t-s)x(s)ds + p(t) \tag{2.1}$$

and

$$x(t) = \int_{-\infty}^t K(t-s)x(s)ds + f(x_t), \tag{E}$$

where we assume, throughout the paper, that the kernel K is a measurable $m \times m$-matrix valued function with complex components satisfying the condition

$$\|K\|_{1,\rho} := \int_0^\infty \|K(t)\|e^{\rho t}dt < \infty, \quad \|K\|_{\infty,\rho} := \operatorname{ess\,sup}\{\|K(t)\|e^{\rho t} : t \ge 0\} < \infty,$$

$p \in C(\mathbb{R};\mathbb{C}^m)$ and $f : X \to \mathbb{C}^m$ is of class C^1. Then Eq. (2.1) (resp. (E)) can be formulated as an abstract equation on the space X of the form

$$x(t) = F(t, x_t), \tag{2.2}$$

with $F(t,\phi) = L(\phi) + p(t)$ (resp. $L(\phi) + f(\phi)$) for $(t,\phi) \in \mathbb{R} \times X$, where

$$L(\phi) := \int_{-\infty}^{0} K(-\theta)\phi(\theta)d\theta, \quad \phi \in X.$$

Note that, in each case, $F(t,\phi)$ is well-defined because of

$$|L(\phi)| \le \int_{-\infty}^{0} \|K(-\theta)\|e^{-\rho\theta}|\phi(\theta)|e^{\rho\theta}d\theta \le \|K\|_{\infty,\rho}\|\phi\|_X.$$

Thus, X may be viewed as the phase space for Eq.'s (2.1) and (E); in what follows we will call X the phase space.

Now let $F : [b,\infty) \times X \to \mathbb{C}^m$ be any continuous function, and consider the equation (2.2) with the initial condition

$$x_\sigma = \phi, \text{ that is, } x(\sigma+\theta) = \phi(\theta) \text{ for } \theta \in \mathbb{R}^-, \tag{2.3}$$

where $(\sigma,\phi) \in [b,\infty) \times X$ is given arbitrarily. A function $x : (-\infty,a) \to \mathbb{C}^m$ is said to be a solution of the initial value problem (2.2)–(2.3) on the interval (σ,a) if x satisfies the following conditions:

(i) $x_\sigma = \phi$, that is, $x(\sigma+\theta) = \phi(\theta)$ for $\theta \in \mathbb{R}^-$;

(ii) $x \in L^1_{\text{loc}}[\sigma,a)$, x is locally integrable on $[\sigma,a)$;

(iii) $x(t) = F(t,x_t)$ for $t \in (\sigma,a)$.

If $F(t,\phi)$ is locally Lipschitz continuous in ϕ, by [21, Proposition 1] the initial value problem (2.2)–(2.3) has a unique (local) solution, which is defined globally if $F(t,\phi)$ is globally Lipschitz continuous in ϕ ([21, Proposition 3]). So for any $(\sigma,\phi) \in \mathbb{R} \times X$ (2.1)–(2.3) has a unique global solution, denoted $x(t;\sigma,\phi,p)$, which is called the solution of Eq. (2.1) through (σ,ϕ). Similarly, (E)–(2.3) has a unique (local) solution, which is denoted by $x(t;\sigma,\phi,f)$. Note also that if $x(t)$ is a solution of Eq. (2.2) on (σ,a), then x_t is an X-valued continuous function on $[\sigma,a)$ (cf. [21, Lemma 1]).

When J is an interval in $\mathbb{R}$, a $\mathbb{C}^m$-valued function $\xi(t)$ is called a solution of Eq. (2.1) on J, if $\xi_t \in X$ is defined for all $t \in J$ and if it satisfies $x(t;\sigma,\xi_\sigma,p) = \xi(t)$ for all t and σ in J with $t \ge \sigma$; and, similarly, a $\mathbb{C}^m$-valued function $\xi(t)$ is called a solution of Eq. (E) on J whenever $\xi_t \in X$ for $t \in J$, and $x(t;\sigma,\xi_\sigma,f) = \xi(t)$ holds for all t and σ in J with $t \ge \sigma$.

2.2 A variation-of-constant formula and decomposition of the phase space

Now, for any $t \ge 0$ and $\phi \in X$, we define $T(t)\phi \in X$ by

$$[T(t)\phi](\theta) := x_t(\theta;,0,\phi,0) = \begin{cases} x(t+\theta;0,\phi,0), & -t < \theta \le 0, \\ \phi(t+\theta), & \theta \le -t. \end{cases}$$

Then $T(t)$ defines a bounded linear operator on X. We call $T(t)$ the solution operator of the homogeneous integral equation

$$x(t) = \int_{-\infty}^{t} K(t-s)x(s)ds. \tag{2.4}$$

$\{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup of bounded linear operators on X, called a solution semigroup for Eq. (2.4).

Given a positive integer n, we introduce a continuous function $\Gamma^n : \mathbb{R}^- \to \mathbb{R}^+$ which is of compact support with $\operatorname{supp}\Gamma^n \subset [-1/n, 0]$ and satisfies $\int_{-\infty}^0 \Gamma^n(\theta)d\theta = 1$. Obviously, $\Gamma^n x \in X$ for $x \in \mathbb{C}^m$ and the inequality $\|\Gamma^n x\|_X \leq |x|$ holds.

The following theorem plays a crucial role throughout the paper, which gives a representation formula for solutions of Eq. (2.1) in the phase space X, and is called the variation-of-constants formula (VCF, for short) in the phase space.

Theorem 2.1 ([21, Theorem 3])**.** *The segment $x_t(\sigma,\phi,p)$ of solution $x(\cdot;\sigma,\phi,p)$ of Eq. (2.1) satisfies the following relation in X:*

$$x_t(\sigma,\phi,p) = T(t-\sigma)\phi + \lim_{n\to\infty}\int_\sigma^t T(t-s)(\Gamma^n p(s))ds, \quad t \geq \sigma.$$

Let $\overline{X}$ be a subset of X of elements $\phi \in X$ which are continuous on $[-\varepsilon_\phi, 0]$ for some $\varepsilon_\phi > 0$, and set

$$X_0 := \{\psi \in X : \psi = \phi \text{ a.e. on } \mathbb{R}^- \text{ for some } \phi \in \overline{X}\}.$$

Then for any $\psi \in X_0$ we can define the value of ψ at $\theta = 0$ by $\psi[0] := \phi(0)$, where ϕ is an element in $\overline{X}$ satisfying $\psi = \phi$ a.e. on $\mathbb{R}^-$. $\psi[0]$ is well-defined, and X_0 is a normed linear space with norm

$$\|\psi\|_{X_0} := \|\psi\|_X + |\psi[0]|, \quad \psi \in X_0.$$

By [21, Lemma 1], we note that the solution $x(\cdot;\sigma,\phi,p)$ of Eq. (2.1) through $(\sigma,\phi) \in \mathbb{R}\times X$ satisfies $x_t(\sigma,\phi,p) \in X_0$ with $(x_t(\sigma,\phi,p))[0] = x(t;\sigma,\psi,p)$ for $t > \sigma$.

The next theorem describes an intimate relation between solutions of Eq. (2.1) and X-valued functions satisfying an integral equation which arises from the variation-of-constants formula in the phase space.

Theorem 2.2 ([21, Theorem 4])**.** *Let $p \in C(\mathbb{R};\mathbb{C}^m)$.*

(i) *If $x(t)$ is a solution of Eq. (2.1) on the entire $\mathbb{R}$, then the X-valued function $\xi(t) := x_t$ satisfies the relations*

  (a) $\xi(t) = T(t-\sigma)\xi(\sigma) + \lim_{n\to\infty}\int_\sigma^t T(t-s)(\Gamma^n p(s))ds$, *$\forall\,(t,\sigma) \in \mathbb{R}^2$ with $t \geq \sigma$, in X;*

  (b) $\xi \in C(\mathbb{R}; X_0)$.

(ii) *Conversely, if a function $\xi : \mathbb{R} \to X$ satisfies the relation*

$$\xi(t) = T(t-\sigma)\xi(\sigma) + \lim_{n\to\infty}\int_\sigma^t T(t-s)(\Gamma^n p(s))ds, \quad \forall\,(t,\sigma) \in \mathbb{R}^2 \text{ with } t \geq \sigma,$$

then

  (c) $\xi \in C(\mathbb{R}; X_0)$;

  (d) *if we set $u(t) = (\xi(t))[0]$ for $t \in \mathbb{R}$, then $u \in C(\mathbb{R};\mathbb{C}^m)$, $u_t = \xi(t)$ (in X) for any $t \in \mathbb{R}$ and u is a solution of Eq. (2.1) on $\mathbb{R}$.*

Based on spectral analysis of the generator A of the solution semigroup $\{T(t)\}_{t\geq 0}$, we also have established the decomposition theorem of the phase space X ([21]); let $\sigma(A)$ and $P_\sigma(A)$

be the spectrum and the point spectrum of the generator A, respectively. Then the following relation holds between the spectrum of A and the characteristic roots of Eq. (2.4)

$$\sigma(A) \cap \mathbb{C}_{-\rho} = P_\sigma(A) \cap \mathbb{C}_{-\rho} = \{\lambda \in \mathbb{C}_{-\rho} : \det \Delta(\lambda) = 0\},$$

where $\mathbb{C}_{-\rho} := \{z \in \mathbb{C} : \operatorname{Re} z > -\rho\}$, and $\Delta(\lambda)$ is the characteristic operator of Eq. (2.4), i.e., $\Delta(\lambda) := E_m - \int_0^\infty K(t)e^{-\lambda t}dt$, E_m being the $m \times m$-unit matrix ([21, Proposition 4]). Moreover, for $\operatorname{ess}(A)$, the essential spectrum of A, we have $\sup_{\lambda \in \operatorname{ess}(A)} \operatorname{Re} \lambda \leq -\rho$ ([21, Corollary 2]). Now set $\Sigma^u := \{\lambda \in \sigma(A) : \operatorname{Re} \lambda > 0\}$, $\Sigma^c := \{\lambda \in \sigma(A) : \operatorname{Re} \lambda = 0\}$, and $\Sigma^s := \sigma(A) \backslash (\Sigma^c \cup \Sigma^u)$. The decomposition theorem of X is the following.

Theorem 2.3 ([21, Theorem 2]). *Let $\{T(t)\}_{t \geq 0}$ be the solution semigroup of Eq. (2.4). Then X is decomposed as a direct sum of closed subspaces E^u, E^c, and E^s*

$$X = E^u \oplus E^c \oplus E^s$$

with the following properties:

(i) $\dim(E^u \oplus E^c) < \infty$,

(ii) $T(t)E^u \subset E^u$, $T(t)E^c \subset E^c$, *and* $T(t)E^s \subset E^s$ *for* $t \in \mathbb{R}^+$,

(iii) $\sigma(A|_{E^u}) = \Sigma^u$, $\sigma(A|_{E^c}) = \Sigma^c$ *and* $\sigma(A|_{E^s \cap \mathcal{D}(A)}) = \Sigma^s$,

(iv) $T^u(t) := T(t)|_{E^u}$ *and* $T^c(t) := T(t)|_{E^c}$ *are extendable for* $t \in \mathbb{R}$ *as groups of bounded linear operators on* E^u *and* E^c*, respectively,*

(v) $T^s(t) := T(t)|_{E^s}$ *is a strongly continuous semigroup of bounded linear operators on* E^s*, and its generator is identical with* $A|_{E^s \cap \mathcal{D}(A)}$*,*

(vi) *there exist positive constants* α, ε *with* $\alpha > \varepsilon$ *and a constant* $C \geq 1$ *such that*

$$\|T^s(t)\|_{\mathcal{L}(X)} \leq Ce^{-\alpha t}, \quad t \in \mathbb{R}^+,$$
$$\|T^u(t)\|_{\mathcal{L}(X)} \leq Ce^{\alpha t}, \quad t \in \mathbb{R}^-,$$
$$\|T^c(t)\|_{\mathcal{L}(X)} \leq Ce^{\varepsilon|t|}, \quad t \in \mathbb{R}.$$

Note that in (vi) of the theorem above C is a constant depending only on the positive numbers α and ε, and that ε can be taken arbitrarily small. We will use the notations $E^{cu} = E^c \oplus E^u$, $E^{su} = E^s \oplus E^u$ etc, and denote by Π^s the projection from X onto E^s along E^{cu}, and likewise for Π^u, Π^{cu} etc. Also, we set

$$C_1 := \|\Pi^s\|_{\mathcal{L}(X)} + \|\Pi^c\|_{\mathcal{L}(X)} + \|\Pi^u\|_{\mathcal{L}(X)}.$$

By an *equilibrium (or equilibrium point) of the integral equation* (*E*) we mean that of the semi-dynamical system on the phase space X induced by Eq. (*E*); namely, let $U : \mathbb{R}^+ \times X \to X$ be the map defined by

$$U(t, \phi) := x_t(0, \phi, f), \quad (t, \phi) \in \mathbb{R}^+ \times X.$$

We then call $\phi \in X$ an equilibrium (or equilibrium point) of Eq. (*E*) if $U(t, \phi) = \phi$ holds for all $t \in \mathbb{R}^+$. If ϕ is an equilibrium, then $\phi(\theta) = \text{const.}$ a.e. $\theta \in \mathbb{R}^-$. Indeed, put $u(t) := x(t; 0, \phi, f)$, then $U(t, \phi) = \phi$ implies that $u(t + \theta) = \phi(\theta) = u(\theta)$ a.e. on $\mathbb{R}^-$ for all $t \in \mathbb{R}^+$; hence any weak derivative of $u(\theta)$ is 0 a.e., so that $\phi(\theta) = u(\theta) = \text{const.}$ a.e. on $\mathbb{R}^-$. When 0 is an equilibrium, we often call it the zero solution of the integral equation.

If $f \in C^1(X; \mathbb{C}^m)$ satisfies $f(0) = 0$ and $Df(0) = 0$, Eq. (2.4) is the linearized equation of Eq.(*E*) around the equilibrium 0. The equilibrium 0 (or the zero solution) of Eq. (*E*) is said to be *hyperbolic* provided that $\Delta(\lambda)$ is invertible on the imaginary axis; that is, $\Sigma^c = \emptyset$.

2.3 Center manifold for an integral equation with modified nonlinear term

Suppose that $f \in C^1(X;\mathbb{C}^m)$ satisfies $f(0) = 0$ and $Df(0) = 0$. Under the assumption we have established the existence of local center manifolds of the equilibrium 0 of Eq. (E) and proved its exponential attractivity ([23, Theorem 5]). To this end, we discussed the corresponding problems for a modified equation of (E)

$$x(t) = \int_{-\infty}^{t} K(t-s)x(s)ds + f_\delta(x_t), \tag{E_δ}$$

where $f_\delta : X \to \mathbb{C}^m$ ($\delta > 0$) is defined by

$$f_\delta(\phi) := \chi(\|\Pi^{su}\phi\|_X/\delta)\chi(\|\Pi^c\phi\|_X/\delta)f(\phi), \quad \phi \in X,$$

χ being a C^∞-function on $\mathbb{R}$ satisfying $\chi(t) = 1$ ($|t| \le 2$) and $\chi(t) = 0$ ($|t| \ge 3$). Note that f_δ is continuous on X, and is of class C^1 when restricted to the open set $S_\delta := \{\phi \in X : \|\Pi^{su}\phi\|_X < \delta\}$ since we may assume that $\|\Pi^c\phi\|_X$ is of class C^1 for $\phi \neq 0$ because of $\dim E^c < \infty$. Also, by the assumption $f(0) = Df(0) = 0$, there exist a $\delta_1 > 0$ and a nondecreasing continuous function $\zeta_* : (0, \delta_1] \to \mathbb{R}^+$ such that $\zeta_*(+0) = 0$,

$$\|f_\delta(\phi)\|_X \le \delta\zeta_*(\delta), \quad \text{and} \quad \|f_\delta(\phi) - f_\delta(\psi)\|_X \le \zeta_*(\delta)\|\phi - \psi\|_X$$

for $\phi, \psi \in X$ and $\delta \in (0, \delta_1]$. Indeed, we may put

$$\zeta_*(\delta) = \left(\sup_{\|\phi\|_X \le 3\delta} \|Df(\phi)\|_{\mathcal{L}(X;\mathbb{C}^m)} \right) \left(1 + 3 \sup_{0 \le t \le 3} |\chi'(t)| \right)$$

(cf. [4, Lemma 4.1]).

Fix a positive number η such that $\varepsilon < \eta < \alpha$, where ε and α are the constants in Theorem 2.3. For the existence of center manifold for Eq. (E_δ) and its exponential attractivity property, we have established the following:

Theorem 2.4 ([23, Theorem 4])**.** *There exist a positive number δ and a C^1-map $F_{*,\delta} : E^c \to E^{su}$ with $F_{*,\delta}(0) = 0$ such that the following properties hold:*

(i) $W^c_\delta := \operatorname{graph} F_{*,\delta}$ *is tangent to E^c at zero,*

(ii) W^c_δ *is invariant for Eq. (E_δ), that is, if $\xi \in W^c_\delta$, then $x_t(0, \xi, f) \in W^c_\delta$ for $t \in \mathbb{R}$.*

(iii) *Assume moreover that $\Sigma^u = \emptyset$. Then there exists a positive constant β_0 with the property that if x is a solution of Eq. (E_δ) on an interval $J = [t_0, t_1]$, then the inequality*

$$\|\Pi^s x_t - F_{*,\delta}(\Pi^c x_t)\|_X \le C\|\Pi^s x_{t_0} - F_{*,\delta}(\Pi^c x_{t_0})\|_X e^{-\beta_0(t-t_0)}, \quad t \in J$$

holds true. In particular, if x is a solution on an interval $[t_0, \infty)$, x_t tends to W^c_δ exponentially as $t \to \infty$.

It has also been proved that W^c_δ has the same smoothness as the nonlinear term $f(\phi)$ does ([23, Appendix]; see also [27] for details). Let us briefly recall the outline of the proof of the existence part. Take a $\delta_1 > 0$ sufficiently small in such a way that

$$\zeta_*(\delta_1)CC_1 \left(\frac{1}{\eta - \varepsilon} + \frac{2}{\alpha + \eta} + \frac{2}{\alpha - \eta} \right) < \frac{1}{2}$$

holds. Let Y_η be the Banach space

$$Y_\eta := \{y \in C(\mathbb{R}; X) : \sup_{t \in \mathbb{R}} \|y(t)\|_X \, e^{-\eta|t|} < \infty\}$$

with norm

$$\|y\|_{Y_\eta} := \sup_{t \in \mathbb{R}} \|y(t)\|_X \, e^{-\eta|t|}, \quad y \in Y_\eta,$$

and consider the map $\mathcal{F}_\delta : E^c \times Y_\eta \to Y_\eta$ defined by

$$\begin{aligned} \mathcal{F}_\delta(\psi, y)(t) := T^c(t)\psi + \lim_{n\to\infty} \int_0^t T^c(t-s)\Pi^c\Gamma^n f_\delta(y(s))ds \\ - \lim_{n\to\infty} \int_t^\infty T^u(t-s)\Pi^u\Gamma^n f_\delta(y(s))ds + \lim_{n\to\infty} \int_{-\infty}^t T^s(t-s)\Pi^s\Gamma^n f_\delta(y(s))ds \end{aligned} \tag{2.5}$$

for $(\psi, y) \in E^c \times Y_\eta$ and $t \in \mathbb{R}$. Then, for each $\psi \in E^c$, $\mathcal{F}_\delta(\psi, \cdot)$ is a contraction map from Y_η into itself provided that $0 < \delta \le \delta_1$, and therefore has a unique fixed point, say $\Lambda_{*,\delta}(\psi) \in Y_\eta$.

Now define $F_{*,\delta} : E^c \to E^{su}$ by $F_{*,\delta}(\psi) := \Pi^{su}\big(\Lambda_{*,\delta}(\psi)(0)\big)$ for $\psi \in E^c$. One can see from (2.5) and the relation $\Lambda_{*,\delta}(\psi)(0) = \mathcal{F}_\delta(\psi, \Lambda_{*,\delta}(\psi))(0)$ that

$$\begin{aligned} F_{*,\delta}(\psi) = & - \lim_{n\to\infty} \int_0^\infty T^u(-s)\Pi^u\Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s))ds \\ & + \lim_{n\to\infty} \int_{-\infty}^0 T^s(-s)\Pi^s\Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s))ds. \end{aligned} \tag{2.6}$$

The map $F_{*,\delta}$ is the required one and the center manifold is given by

$$W^c_\delta := \operatorname{graph} F_{*,\delta} = \{\psi + F_{*,\delta}(\psi) : \psi \in E^c\}.$$

The definition (2.6) of $F_{*,\delta}$ shall be used in the arguments in Subsection 4.4; W^c_δ will be also denoted by $W^c_\delta(0)$ in later sections.

Let r be a positive number with $r \le \delta$, set $F_* := F_{*,\delta}|_{B_{E^c}(r)}$ and $\Omega_0 := \{\phi \in X : \|\Pi^{su}\phi\|_X < \delta, \ \|\Pi^c\phi\|_X < r\}$. Then, $f \equiv f_\delta$ on Ω_0, and therefore, Theorem 2.4 assures that $W^c_{\mathrm{loc}}(0) := \operatorname{graph} F_*$ is a local center manifold for Eq. (E) (see [23, Theorem 5]).

The following proposition is often used in the subsequent sections.

Proposition 2.5 ([23, Propositions 2 and 3]). *The maps $\Lambda_{*,\delta}$ and $F_{*,\delta}$ are (globally) Lipschitz continuous and have the following properties:*

(i) $\|F_{*,\delta}(\psi_1) - F_{*,\delta}(\psi_2)\|_X \le L(\delta)\|\psi_1 - \psi_2\|_X$ *for* $\psi_1, \psi_2 \in E^c$, *where* $L(\delta)$ *is the constant given by* $L(\delta) := 4C^2C_1\zeta_*(\delta)/(\alpha - \eta)$.

(ii) *For* $\hat{\phi} \in W^c_\delta$ *and* $\tau \in \mathbb{R}$,

$$\Pi^{su}x_t(\tau, \hat{\phi}, f_\delta) = F_{*,\delta}\big(\Pi^e x_t(\tau, \hat{\phi}, f_\delta)\big), \quad t \in \mathbb{R}.$$

In particular W^c_δ *is invariant for* (E_δ), *that is,* $x_t(\tau, \hat{\phi}, f_\delta) \in W^c_\delta$ *for* $t \in \mathbb{R}$, *provided that* $\hat{\phi} \in W^c_\delta$.

2.4 The projection onto the center-unstable subspace via formal adjoint theory

Let $\Sigma^{cu} = \{\lambda_1, \ldots, \lambda_r\}$; then each λ_i is a normal eigenvalue of the generator A, and hence, its generalized eigenspace $\mathcal{M}_{\lambda_i}(A)$ is of the form $\mathcal{N}((A-\lambda_i I)^{p_i})$, p_i being the ascent of λ_i. The center-unstable subspace E^{cu} $(= E^c \oplus E^u)$ is then expressed as

$$E^{cu} = \bigoplus_{i=1}^{r} \mathcal{M}_{\lambda_i}(A) = \bigoplus_{i=1}^{r} \mathcal{N}((A-\lambda_i I)^{p_i}) \tag{2.7}$$

([22, Subsection 2.2]); and, in addition, each direct summand is characterized by the following proposition. Given $\lambda \in \mathbb{C}_{-\rho}$ and $k \in \mathbb{N}$, consider a function $w_k(\lambda) : \mathbb{R}^- \to \mathbb{C}$ and a $(km) \times (km)$ matrix $D_k(\lambda)$ defined by

$$w_k(\lambda)(\theta) := \frac{\theta^{k-1}}{(k-1)!} e^{\lambda\theta}, \quad \theta \le 0, \tag{2.8}$$

$$D_k(\lambda) := \begin{pmatrix} \Delta(\lambda) & \Delta'(\lambda) & \cdots & \Delta^{(k-1)}(\lambda)/(k-1)! \\ 0 & \Delta(\lambda) & \cdots & \Delta^{(k-2)}(\lambda)/(k-2)! \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \Delta(\lambda) \end{pmatrix}, \tag{2.9}$$

where $\Delta^{(n)}(z) := (d^n/dz^n)\Delta(z)$ for $n \in \mathbb{N}$.

Proposition 2.6 ([22, Proposition 3.1])**.** *Let $\lambda \in \mathbb{C}_{-\rho}$ and $k \in \mathbb{N}$. Then the following statements are equivalent:*

(i) $\phi \in \mathcal{N}((A-\lambda I)^k)$,

(ii) $\phi = \sum_{j=1}^{k} w_j(\lambda)\eta_j$ *in X, where $\eta_1, \ldots, \eta_k$ belong to $\mathbb{C}^m$ satisfying the relation*

$$D_k(\lambda)\,\mathrm{col}\,(\eta_1, \ldots, \eta_k) = \mathrm{col}\,(0, \ldots, 0).$$

Moreover, in [22] the formal adjoint operator $A^\sharp$ was introduced and some duality properties between A and $A^\sharp$ were observed. $A^\sharp$ is indeed defined as follows. Let $\mathbb{C}^{m*}$ be the space of all m-dimensional row vectors with complex components with the usual operator norm $|\cdot|$ as the dual space of $\mathbb{C}^m$. Let $X^\sharp$ be the Banach space defined by

$$X^\sharp := L^1_\rho(\mathbb{R}^+; \mathbb{C}^{m*}) = \left\{\alpha : \mathbb{R}^+ \to \mathbb{C}^{m*} : \alpha(s)e^{-\rho s} \text{ is integrable on } \mathbb{R}^+\right\}$$

with norm

$$\|\alpha\|_{X^\sharp} = \int_0^\infty |\alpha(s)| e^{-\rho s} ds, \quad \alpha \in X^\sharp,$$

and $\tilde{X}^\sharp$ be the subspace

$$\tilde{X}^\sharp = \Big\{\tilde{\alpha} \in X^\sharp : \tilde{\alpha} \text{ is locally absolutely continuous on } \mathbb{R}^+, \\ \frac{d}{ds}\tilde{\alpha} \in X^\sharp \text{ and } \tilde{\alpha}(0) = \int_0^\infty \tilde{\alpha}(s)K(s)ds\Big\}.$$

We define $A^\sharp : X^\sharp \supset \mathcal{D}(A^\sharp) \to X^\sharp$ by

$$A^\sharp\alpha := -\frac{d}{ds}\tilde{\alpha}, \quad \alpha \in \mathcal{D}(A^\sharp),$$

where $\mathcal{D}(A^\sharp) := \{\alpha \in X^\sharp : \alpha(s) = \tilde{\alpha}(s) \text{ a.e. } s \in \mathbb{R}^+ \text{ for some } \tilde{\alpha} \in \tilde{X^\sharp}\}$. In fact, the operator $A^\sharp$ is identical with the infinitesimal generator of the solution semigroup on $X^\sharp$ induced by an adjoint integral equation of (2.4) in some sense (see [22] for details).

A characterization of the space $\mathcal{N}((A^\sharp - \lambda I)^k)$ is obtained in a similar fashion to Proposition 2.6. For any $\lambda \in \mathbb{C}_{-\rho}$ and $k \in \mathbb{N}$, define $w_k^\sharp(\lambda) : \mathbb{R}^+ \to \mathbb{C}$ by

$$w_k^\sharp(\lambda)(s) := w_k(\lambda)(-s) = \frac{(-s)^{k-1}}{(k-1)!} e^{-\lambda s}, \quad s \geq 0. \tag{2.10}$$

Proposition 2.7 ([22, Proposition 3.4]). *Let $\lambda \in \mathbb{C}_{-\rho}$ and $k \in \mathbb{N}$. Then the following statements are equivalent:*

(i) $\alpha \in \mathcal{N}((A^\sharp - \lambda I)^k)$,

(ii) $\alpha = \sum_{j=1}^k w_j^\sharp(\lambda)\zeta_{k+1-j}$ *in $X^\sharp$, where $\zeta_1, \ldots, \zeta_k$ belong to $\mathbb{C}^{m*}$ satisfying the relation*

$$(\zeta_1, \ldots, \zeta_k) D_k(\lambda) = (0, \ldots, 0).$$

Now, let us consider the bilinear form $\langle\langle \cdot, \cdot \rangle\rangle$ on $X^\sharp \times X$ given by

$$\langle\langle \alpha, \phi \rangle\rangle := \int_{-\infty}^0 \int_\theta^0 \alpha(\xi - \theta) K(-\theta) \phi(\xi) d\xi d\theta, \quad \phi \in X, \ \alpha \in X^\sharp,$$

which is well-defined and bounded. Then $\mathcal{R}((A - \lambda_i I)^{p_i})$ is characterized as the annihilator of $\mathcal{N}((A^\sharp - \lambda_i I)^{p_i})$ with respect to this pairing ([22, Proposition 3.5]). So, by virtue of the fact $X = \mathcal{N}((A - \lambda_i I)^{p_i}) \oplus \mathcal{R}((A - \lambda_i I)^{p_i})$ and the fact that $\dim \mathcal{N}((A - \lambda_i I)^{p_i}) = \dim \mathcal{N}((A^\sharp - \lambda_i I)^{p_i})$, which follows from Propositions 2.6 and 2.7, one may think of $\mathcal{N}((A^\sharp - \lambda_i I)^{p_i})$ as the dual space of $\mathcal{N}((A - \lambda_i I)^{p_i})$. Hence, $\mathcal{N}^\sharp := \bigoplus_{i=1}^r \mathcal{N}((A^\sharp - \lambda_i I)^{p_i})$ plays a role of the dual space of E^{cu} due to (2.7).

So, let $\{\phi_1, \ldots, \phi_d\}$ and $\{\psi_1, \ldots, \psi_d\}$ be bases for E^{cu} and $\mathcal{N}^\sharp$, respectively; set $\Phi = (\phi_1, \ldots, \phi_d)$ and $\Psi = \operatorname{col}(\psi_1, \ldots, \psi_d)$. We denote by $\langle\langle \Psi, \Phi \rangle\rangle$ the $d \times d$ matrix whose (i,j)-component is $\langle\langle \psi_i, \phi_j \rangle\rangle$, and by $\langle\langle \Psi, \phi \rangle\rangle$ the column vector $\operatorname{col}(\langle\langle \psi_1, \phi \rangle\rangle, \ldots, \langle\langle \psi_d, \phi \rangle\rangle)$ for any $\phi \in X$. Then, we have:

Theorem 2.8 ([22, Theorem 3.1]). *Let Φ, Ψ be the ones cited above. Then the matrix $\langle\langle \Psi, \Phi \rangle\rangle$ is nonsingular, and the projection $\Pi^{cu} : X \to E^{cu}$ is given by*

$$\Pi^{cu}\phi = \Phi \langle\langle \Psi, \Phi \rangle\rangle^{-1} \langle\langle \Psi, \phi \rangle\rangle, \quad \phi \in X.$$

3 Exponential attractivity with asymptotic phase of the local center manifold and a reduction principle

In this section we prove the global exponential attractivity with asymptotic phase of the center manifold $W_\delta^c(0)$ and a reduction principle for Eq. (E), which are refinements of Theorem 4 (iii) and [23, Theorem 6], respectively. Thus, behaviors of the solutions of Eq. (E_δ), including stability properties, are completely described by the dynamics on $W_\delta^c(0)$. As a corollary the corresponding results are obtained for Eq. (E).

3.1 Main theorems and preparatory propositions

Consider the integral equation

$$x(t) = \int_{-\infty}^{t} K(t-s)x(s)ds + f_\delta(x_t). \tag{E_δ}$$

Now, assume that $\Sigma^u = \emptyset$. Let d be the dimension of E^c, and Φ, Ψ the ones in the previous subsection. Since $\{T^c(t)\}_{t\geq 0}$ is a strongly continuous semigroup on the d-dimensional space E^c, there exists a $d \times d$ matrix G_c such that $T^c(t)\Phi = \Phi e^{tG_c}$ for $t \geq 0$ and $\sigma(G_c)$, the spectrum of G_c, is identical with Σ^c. Let us consider the ordinary differential equation on $\mathbb{C}^d$

$$z'(t) = G_c z(t) + H_c f_\delta(\Phi z(t) + F_{*,\delta}(\Phi z(t))), \tag{CE_δ}$$

where H_c is the $d \times m$ matrix given by

$$H_c x := \lim_{n\to\infty} \langle\!\langle \Psi, \Gamma^n x \rangle\!\rangle, \quad x \in \mathbb{C}^m.$$

We call Eq. (CE_δ) the *central equation* of Eq. (E_δ) (cf. [23, Subsection 3.2]).

Proposition 3.1. *Let $z(t)$ be a solution of Eq. (CE_δ) with $z(t_0) = z_0$ defined on an interval J, and $x(t)$ the solution of Eq. (E_δ) with $x_{t_0} = \Phi z_0 + F_{*,\delta}(\Phi z_0)$. Then $\Phi z(t) + F_{*,\delta}(\Phi z(t))$ is the segment of $x(t)$, that is,*

$$x_t(t_0, \Phi z_0 + F_{*,\delta}(\Phi z_0), f_\delta) = \Phi z(t) + F_{*,\delta}(\Phi z(t)), \quad t \in J.$$

Proof. Put $\hat{\phi} := \Phi z_0 + F_{*,\delta}(\Phi z_0))$, and let $\hat{z}(t)$ be the function determined by $\Phi\hat{z}(t) = \Pi^c x_t(t_0, \hat{\phi}, f_\delta)$. Then, by virtue of [21, Proposition 7], $\hat{z}(t)$ is a solution of

$$z'(t) = G_c z(t) + H_c f_\delta(\Phi z(t) + \Pi^{su} x_t(t_0, \hat{\phi}, f_\delta)).$$

Moreover, it follows from Proposition 2.5 (ii) that $\Pi^{su} x_t(t_0, \hat{\phi}, f_\delta) = F_{*,\delta}(\Pi^c x_t(t_0, \hat{\phi}, f_\delta)) = F_{*,\delta}(\Phi\hat{z}(t))$. Hence, $\hat{z}(t)$ is the solution of (CE_δ) with $\Phi\hat{z}(t_0) = \Pi^c\hat{\phi} = \Phi z_0$, i.e., $\hat{z}(t_0) = z_0$, so that $z(t) = \hat{z}(t)$ ($t \in J$) due to the uniqueness of solutions of (CE_δ). Consequently, $x_t(t_0, \hat{\phi}, f_\delta) = \Pi^c x_t(t_0, \hat{\phi}, f_\delta) + \Pi^{su} x_t(t_0, \hat{\phi}, f_\delta) = \Phi z(t) + F_{*,\delta}(\Phi z(t))$ ($t \in J$). □

One of our main results is the following reduction principle for the modified integral equation (E_δ).

Theorem 3.2. *Assume that $f \in C^1(X; \mathbb{C}^m)$ with $f(0) = Df(0) = 0$, and furthermore that $\Sigma^u = \emptyset$. Then for small $\delta > 0$ the following statements hold.*

(a) *Let β be a positive number satisfying $\varepsilon < \beta < \alpha$. If $x(t)$ is a solution of Eq. (E_δ) defined on $J := [t_0, \infty)$, then there exists a unique solution $z(t)$ of Eq. (CE_δ) on J satisfying*

$$\begin{aligned} \|\Pi^c x_t - \Phi z(t)\|_X &\leq C_0 \|\Pi^s x_{t_0} - F_{*,\delta}(\Phi z(t_0))\|_X e^{-\beta(t-t_0)}, \\ \|\Pi^s x_t - F_{*,\delta}(\Phi z(t))\|_X &\leq C_0 \|\Pi^s x_{t_0} - F_{*,\delta}(\Phi z(t_0))\|_X e^{-\beta(t-t_0)} \end{aligned} \tag{3.1}$$

for $t \in J$, where $C_0 > C$ is a constant which can be chosen as close to C as one expects by taking δ small. In particular, we have the estimate

$$\|x_t - x_t(t_0, \hat{\phi}, f_\delta)\|_X \leq 2C_0 \|\Pi^s x_{t_0} - F_{*,\delta}(\Phi z(t_0))\|_X e^{-\beta(t-t_0)}, \quad t \in J, \tag{3.2}$$

where $\hat{\phi} = \Phi z(t_0) + F_{*,\delta}(\Phi z(t_0)) \in W^c_\delta(0)$.

(b) *If the zero solution of Eq.* (CE_δ) *is stable (resp. asymptotically stable, unstable), the zero solution of Eq.* (E_δ) *is stable (resp. asymptotically stable, unstable).*

As a corollary, we obtain the following theorem for Eq. (E).

Theorem 3.3. *Assume that $f \in C^1(X;\mathbb{C}^m)$ with $f(0) = Df(0) = 0$, and furthermore that $\Sigma^u = \emptyset$. Then the following statements hold.*

(a) *Let β be a positive number satisfying $\varepsilon < \beta < \alpha$. Then there exists an open neighborhood Ω_0 of 0 in X such that if $x(t)$ is a solution of Eq.* (E) *defined on $J := [t_0, t_1]$ satisfying $x_t \in \Omega_0$ for $t \in J$, then there exists a solution $z(t)$ of Eq. (CE) on J with the property*

$$\begin{aligned} \|\Pi^c x_t - \Phi z(t)\|_X &\le C_0 \|\Pi^s x_{t_0} - F_*(\Phi z(t_0))\|_X e^{-\beta(t-t_0)}, \\ \|\Pi^s x_t - F_*(\Phi z(t))\|_X &\le C_0 \|\Pi^s x_{t_0} - F_*(\Phi z(t_0))\|_X e^{-\beta(t-t_0)} \end{aligned} \tag{3.3}$$

for $t \in J$, where $C_0 > C$ is a constant which can be chosen as close to C as one expects by taking Ω_0 small. In particular, we have the estimate

$$\|x_t - x_t(t_0, \phi^0, f)\|_X \le 2C_0 \|\Pi^s x_{t_0} - F_*(\Phi z(t_0))\|_X e^{-\beta(t-t_0)}, \quad t \in J, \tag{3.4}$$

where $\phi^0 = \Phi z(t_0) + F_*(\Phi z(t_0)) \in W^c_{\mathrm{loc}}(0)$.

(b) *If the zero solution of Eq. (CE) is stable (resp. asymptotically stable, unstable), the zero solution of Eq.* (E) *is stable (resp. asymptotically stable, unstable).*

Remark 3.4. Since Eq. (E_δ) is autonomous, it follows that $x_t(t_0, \phi, f_\delta) = x_{t-t_0}(0, \phi, f_\delta)$ for $t \ge t_0$. So it is sufficient to prove part (a) of Theorem 3.2 in case that $t_0 = 0$; and likewise for that of Theorem 3.3. Also, by the same reasoning stability (resp. asymptotic stability) in statement (b) of each theorem means actually uniform stability (resp. uniform asymptotic stability).

For the proof of part (a) of Theorem 3.2 we need the following lemma, which is a modification of [23, Lemma 1].

Lemma 3.5. *The solutions of Eq.* (E_δ) *satisfying* (3.1) *with $t_0 = 0$ are characterized by a system of integral equations in X; more precisely,*

(i) *Suppose that $x(t)$ is a solution of Eq.* (E_δ) *defined on $\mathbb{R}^+$ such that there exists a solution $z(t)$ of Eq.* (CE_δ) *with the properties*

$$\sup_{t\in\mathbb{R}^+} \|\Pi^c x_t - \Phi z(t)\|_X e^{-\beta t} < \infty, \text{ and } \sup_{t\in\mathbb{R}^+} \|\Pi^s x_t - F_{*,\delta}(\Phi z(t))\|_X e^{-\beta t} < \infty. \tag{3.5}$$

Then the X-valued functions $\xi(t)$ and $y(t)$ defined by

$$\xi(t) := \Pi^c x_t - \Phi z(t) \quad \text{and} \quad y(t) := \Pi^s x_t,$$

respectively satisfy

$$y(t) = T^s(t)\phi^s + \lim_{n\to\infty} \int_0^t T^s(t-s)\Pi^s \Gamma^n f_\delta\big(\Phi z(s) + \xi(t) + y(s)\big) ds, \tag{3.6}$$

$$\xi(t) = -\lim_{n\to\infty} \int_t^\infty T^c(t-s)\Gamma^n \Pi^c g_\delta(s) ds \tag{3.7}$$

for $t \in \mathbb{R}^+$, where $\phi := \Phi z(0) + \Pi^s x_0$ and $g_\delta(t)$ is the function defined by

$$g_\delta(t) := f_\delta\big(\Phi z(t) + \xi(t) + y(t)\big) - f_\delta\big(\Phi z(t) + F_{*,\delta}(\Phi z(t))\big)$$

for $t \in \mathbb{R}^+$. Moreover ξ and y belong to $C((0,\infty); X_0)$.

(ii) *Conversely, suppose that* $z(t)$ *is a solution of Eq.* (CE_δ)*, and that* ξ *and* y *are elements of* $C(\mathbb{R}^+; X)$ *with the properties*

$$\sup_{t\in\mathbb{R}^+} \|\xi(t)\|_X e^{\beta t} < \infty, \quad \textit{and} \quad \sup_{t\in\mathbb{R}^+} \|y(t) - F_{*,\delta}(\Phi z(t))\|_X e^{\beta t} < \infty \tag{3.8}$$

which satisfy (3.6) *and* (3.7)*, where* $\phi := \Phi z(0) + y(0)$*. Then* ξ *and* y *belong to* $C((0,\infty); X_0)$ *and the function* $x(t)$ *defined by*

$$x(t) := \begin{cases} \big(\Phi z(t) + \xi(t) + y(t)\big)[0], & t > 0; \\ \big(\Phi z(0) + \xi(0) + y(0)\big)(t), & t \le 0 \end{cases}$$

is a solution of Eq. (E_δ) *on* $\mathbb{R}^+$ *that satisfies* (3.5) *and* $x_t = \Phi z(t) + \xi(t) + y(t)$ *for* $t \in \mathbb{R}^+$.

Proof. (i) We know by VCF (Theorem 2.1)

$$x_t = T(t-\tau)x_\tau + \lim_{n\to\infty} \int_\tau^t T(t-s)\Gamma^n f_\delta(x_s)ds, \quad t \ge \tau \ge 0, \tag{3.9}$$

and hence it follows that

$$y(t) = T^s(t)y(0) + \lim_{n\to\infty} \int_0^t T^s(t-s)\Pi^s\Gamma^n f_\delta\big(\Phi z(s) + \xi(s) + y(s)\big)ds, \tag{3.10}$$

where we used the relation $x_t = \Phi z(t) + \xi(t) + y(t)$ for $t \in \mathbb{R}^+$. By the definition of ϕ one can readily see $\phi^s = y(0)$ to get (3.6). Also, (3.9) yields

$$\Pi^c x_t = T^c(t-\tau)\Pi^c x_\tau + \lim_{n\to\infty} \int_\tau^t T^c(t-s)\Pi^c\Gamma^n f_\delta(x_s)ds.$$

Since, by Proposition 3.1, $\Phi z(t) + F_{*,\delta}(\Phi z(t))$ is the segment of the solution of Eq. (E_δ) through $(0, \phi^c + F_{*,\delta}(\phi^c))$, we deduce from VCF that

$$\Phi z(t) = T^c(t-\tau)\Phi z(\tau) + \lim_{n\to\infty} \int_\tau^t T^c(t-s)\Pi^c\Gamma^n f_\delta\big(\Phi z(s) + F_{*,\delta}(\Phi z(s)\big)ds. \tag{3.11}$$

So, $\xi(t)$ satisfies

$$\xi(t) = T^c(t-\tau)\xi(\tau) + \lim_{n\to\infty} \int_\tau^t T^c(t-s)\Pi^c\Gamma^n g_\delta(s)ds. \tag{3.12}$$

The group property of $\{T^c(t)\}_{t\in\mathbb{R}}$, then implies

$$\xi(\tau) = T^c(\tau-t)\xi(t) - \lim_{n\to\infty} \int_\tau^t T^c(\tau-s)\Pi^c\Gamma^n g_\delta(s)ds, \quad t \ge \tau \ge 0. \tag{3.13}$$

On the other hand, by the assumption (3.5) there exists a constant $C_* \ge 0$ such that $\|\xi(t)\|_X \le C_* e^{-\beta t}$ and $\|y(t) - F_{*,\delta}(\Phi z(t))\|_X \le C_* e^{-\beta t}$ for $t \in \mathbb{R}^+$; it follows from Theorem 2.3 that

$$\|T^c(\tau-t)\xi(t)\|_X \le Ce^{\varepsilon|\tau-t|}C_* e^{-\beta t} = CC_* e^{-\varepsilon\tau}e^{-(\beta-\varepsilon)t}, \quad t \ge \tau.$$

Letting $t \to \infty$ in (3.13), we see

$$\xi(\tau) = -\lim_{t\to\infty}\lim_{n\to\infty} \int_\tau^t T^c(\tau-s)\Pi^c\Gamma^n g_\delta(s)ds.$$

Moreover, by Proposition 2.5 we have

$$\|g_\delta(s)\|_X \le \zeta_*(\delta)\|\xi(s)+y(s)-F_{*,\delta}(\Phi z(s))\|_X \le 2\zeta_*(\delta)C_*e^{-\beta s},$$

and hence

$$\|T^c(\tau-s)\Pi^c\Gamma^n g_\delta(s)\|_X \le 2CC_1C_*\zeta_*(\delta)e^{-\varepsilon\tau-(\beta-\varepsilon)s}, \quad s\ge\tau.$$

This means that $\int_\tau^\infty T^c(\tau-s)\Pi^c\Gamma^n g_\delta(s)ds$ is convergent in X uniformly in n; and in particular that

$$\left\|\int_t^\infty T^c(\tau-s)\Pi^c\Gamma^n g_\delta(s)ds\right\|_X \le C(\delta)e^{-(\beta-\varepsilon)(t-\tau)}, \quad t\ge\tau,$$

where $C(\delta) := 2CC_1C_*\zeta_*(\delta)/(\beta-\varepsilon)$. Thus, for $n,m\in\mathbb{N}$

$$\begin{aligned}
&\left\|\int_\tau^\infty T^c(\tau-s)\Pi^c\Gamma^n g_\delta(s)ds - \int_\tau^\infty T^c(\tau-s)\Pi^c\Gamma^m g_\delta(s)ds\right\|_X \\
&\le \left\|\int_t^\infty T^c(\tau-s)\Pi^c\Gamma^n g_\delta(s)ds\right\|_X + \left\|\int_t^\infty T^c(\tau-s)\Pi^c\Gamma^m g_\delta(s)ds\right\|_X \\
&\quad + \left\|\int_\tau^t T^c(\tau-s)\Pi^c\Gamma^n g_\delta(s)ds - \int_\tau^t T^c(\tau-s)\Pi^c\Gamma^m g_\delta(s)ds\right\|_X \\
&\le 2C(\delta)e^{-(\beta-\varepsilon)(t-\tau)} + \left\|T^c(\tau-t)\left(\int_\tau^t T^c(t-s)\Pi^c\Gamma^n g_\delta(s)ds - \int_\tau^t T^c(t-s)\Pi^c\Gamma^m g_\delta(s)ds\right)\right\|_X.
\end{aligned}$$

Notice that $\lim_{n\to\infty}\int_\tau^t T^c(t-s)\Pi^c\Gamma^n g_\delta(s)ds = \Pi^c x_t(\tau,0,g_\delta)$ (Theorem 2.1), which yields

$$\limsup_{n,m\to\infty}\left\|\int_\tau^\infty T^c(\tau-s)\Pi^c\Gamma^n g_\delta(s)ds - \int_\tau^\infty T^c(\tau-s)\Pi^c\Gamma^m g_\delta(s)ds\right\|_X \le 2C(\delta)e^{-(\beta-\varepsilon)(t-\tau)}.$$

Since $t\ge\tau$ is arbitrary, $\int_\tau^\infty T^c(\tau-s)\Gamma^n g_\delta(s)ds$ turns out to converge in X as $n\to\infty$.

Consequently, the argument in the last paragraph gives

$$\begin{aligned}
\lim_{n\to\infty}\int_\tau^\infty T^c(\tau-s)\Gamma^n g_\delta(s)ds &= \lim_{n\to\infty}\lim_{t\to\infty}\int_\tau^t T^c(\tau-s)\Gamma^n g_\delta(s)ds \\
&= \lim_{t\to\infty}\lim_{n\to\infty}\int_\tau^t T^c(\tau-s)\Gamma^n g_\delta(s)ds = -\xi(\tau), \quad \tau\in\mathbb{R}^+,
\end{aligned}$$

i.e., (3.7) holds. In view of (3.10) and (3.12), combined with the fact that $\xi\in C(\mathbb{R}^+;E^c)$ and $y\in C(\mathbb{R}^+;E^s)$, the latter part of (i) immediately follows from Theorem 2.2 or [21, Theorem 4].

(ii) We observe from the assumption (3.7) that (3.12) holds for $t\ge\tau\ge0$. So, by the same reasoning as the proof of the latter part of (i), we see that ξ and y belong to $C((0,\infty);X_0)$. Now set $u(t) := \Phi z(t)+\xi(t)+y(t)$. Adding (3.6), (3.11) and (3.12) with $\tau=0$, and using $y(0)=\phi^s$, we have

$$\begin{aligned}
u(t) &= T(t)(\Phi z(0)+\xi(0)+y(0)) + \lim_{n\to\infty}\int_0^t T(t-s)\Gamma^n f_\delta(\Phi z(s)+\xi(s)+y(s))ds \\
&= T(t)u(0) + \lim_{n\to\infty}\int_0^t T(t-s)\Gamma^n f_\delta(u(s))ds, \quad t\in\mathbb{R}^+.
\end{aligned}$$

Then, Theorem 2.2 implies $x(t) = (u(t))[0] \equiv x(t;0,u(0),f_\delta)$ and $x_t = u(t) = \Phi z(t)+\xi(t)+y(t)$ for $t\in\mathbb{R}^+$. Thus, $x(t)$ is a solution of Eq. (E_δ) defined on $\mathbb{R}^+$ with $\Pi^c x_t - \Phi z(t) = \xi(t)$ and $\Pi^s x_t = y(t)$ for $t\in\mathbb{R}^+$. Therefore, the assertion of (ii) directly follows from (3.8). □

Assume that $\delta > 0$ is small enough such that

$$K_\delta := CC_1\zeta_*(\delta) \leq \min\left((\alpha-\beta)/2,\ (\beta-\varepsilon)/4\right) \tag{3.14}$$

holds as well as

$$K_\delta(1+L(\delta)) < \varepsilon. \tag{3.15}$$

We will give preparatory propositions below so as to prove Theorem 3.2. Given $\phi \in X$, denote by $z(t,\phi)$, $t \in \mathbb{R}^+$, the solution of Eq. (CE_δ) with $\Phi z(0) = \phi^c$. One can see $z(t,\phi)$ satisfies

$$\Phi z(t,\phi) = T^c(t)\phi^c + \lim_{n\to\infty}\int_0^t T^c(t-s)\Pi^c\Gamma^n f_\delta\big(\Phi z(s,\phi) + F_{*,\delta}(\Phi z(s,\phi)\big)ds \tag{3.16}$$

for $t \in \mathbb{R}^+$ (see (3.11)). Then we have:

Proposition 3.6. *Let $\phi \in X$ and $\xi \in C(\mathbb{R}^+;E^c)$ with $\sup_{t\in\mathbb{R}^+}\|\xi(t)\|_X e^{\beta t} < \infty$. Then there exists one and only one $y \in C(\mathbb{R}^+;E^s)$ that satisfies*

$$y(t) = T^s(t)\phi^s + \lim_{n\to\infty}\int_0^t T^s(t-s)\Pi^s\Gamma^n f_\delta\big(\Phi z(s,\phi) + \xi(s) + y(s)\big)ds \tag{3.17}$$

for $t \in \mathbb{R}^+$.

Proof. Let η_0 be a positive number with $\eta_0 > 2\varepsilon$, and consider the Banach space

$$Z^s_{\eta_0} := BC^{-\eta_0}(\mathbb{R}^+;E^s) = \left\{y \in C(\mathbb{R}^+;E^s) : \sup_{t\in\mathbb{R}^+}\|y(s)\|_X e^{-\eta_0 t} < \infty\right\}$$

with norm $\|y\|_{Z^s_{\eta_0}} := \sup_{t\in\mathbb{R}^+}\|y(s)\|_X e^{-\eta_0 t}$ for $y \in Z^s_{\eta_0}$. Set

$$r_0 := \max\left(\|\phi\|_X,\ \sup_{t\in\mathbb{R}^+}\|\xi(t)\|_X e^{\beta t}\right),$$

and take an $r > 0$ such that

$$r > 2r_0\left\{CC_1 + K_\delta\left(\frac{1}{\alpha-\beta} + \frac{CC_1}{\alpha+2\varepsilon}\right)\right\}.$$

Now let $A_r := \overline{B}_{Z^s_{\eta_0}}(r)$, and for each $y \in A_r$ define $\mathcal{G}_{\phi,\xi}(y) \in C(\mathbb{R}^+;E^s)$ by

$$\big(\mathcal{G}_{\phi,\xi}(y)\big)(t) := T^s(t)\phi^s + \lim_{n\to\infty}\int_0^t T^s(t-s)\Pi^s\Gamma^n f_\delta\big(\Phi z(s,\phi) + \xi(s) + y(s)\big)ds \tag{3.18}$$

for $t \in \mathbb{R}^+$. Then, $\mathcal{G}_{\phi,\xi} : y \longmapsto \mathcal{G}_{\phi,\xi}(y)$ is a contraction map from A_r to itself. Indeed, by (3.16) and Proposition 2.5 we have

$$\begin{aligned}\|\Phi z(t,\phi)\|_X &\leq Ce^{\varepsilon t}\|\phi^c\|_X + \int_0^t CC_1\zeta_*(\delta)e^{\varepsilon(t-s)}\|\Phi z(s,\phi) + F_{*,\delta}(\Phi z(s,\phi)\|_X ds\\ &\leq Ce^{\varepsilon t}\|\phi^c\|_X + \int_0^t K_\delta(1+L(\delta))e^{\varepsilon(t-s)}\|\Phi z(s,\phi)\|_X ds,\end{aligned}$$

or equivalently, by (3.15)

$$e^{-\varepsilon t}\|\Phi z(t,\phi)\|_X \leq C\|\phi^c\|_X + \int_0^t \varepsilon e^{-\varepsilon s}\|\Phi z(s,\phi)\|_X ds,$$

so that Gronwall's inequality yields

$$\|\Phi z(t,\phi)\|_X \le C\|\phi^c\|_X e^{2\varepsilon t}, \quad t \in \mathbb{R}^+.$$

Thus, for $y \in A_r$,

$$\|(\mathcal{G}_{\phi,\xi}(y))(t)\|_X \le C\|\phi^s\|_X e^{-\alpha t} + \int_0^t K_\delta e^{-\alpha(t-s)}(C\|\phi^c\|_X e^{2\varepsilon s} + r_0 e^{-\beta s} + r e^{\eta_0 s})ds,$$

and hence

$$\begin{aligned}\|(\mathcal{G}_{\phi,\xi}(y))(t)\|_X e^{-\eta_0 t} &\le CC_1 r_0 + K_\delta\Big(\frac{CC_1 r_0}{\alpha+2\varepsilon} + \frac{r_0}{\alpha-\beta} + \frac{r}{\alpha+\eta_0}\Big)\\ &\le \frac{r}{2} + \frac{K_\delta r}{\alpha+\eta_0} \le r, \quad t \in \mathbb{R}^+.\end{aligned}$$

Therefore $\mathcal{G}_{\phi,\xi}(y) \in A_r$; so, we have $\mathcal{G}_{\phi,\xi}(A_r) \subset A_r$. Next, let y_1, $y_2 \in A_r$ be given. Then

$$\begin{aligned}\|(\mathcal{G}_{\phi,\xi}(y_1))(t) - (\mathcal{G}_{\phi,\xi}(y_2))(t)\|_X &\le \int_0^t K_\delta e^{-\alpha(t-s)}\|y_1(s)-y_2(s)\|_X ds\\ \le \int_0^t K_\delta e^{-\alpha(t-s)}\|y_1-y_2\|_{Z^s_{\eta_0}} e^{\eta_0 s}ds &\le \frac{K_\delta}{\alpha+\eta_0}\|y_1-y_2\|_{Z^s_{\eta_0}} e^{\eta_0 t}, \quad t\in\mathbb{R}^+,\end{aligned}$$

which, combined with (3.14), implies

$$\|\mathcal{G}_{\phi,\xi}(y_1) - \mathcal{G}_{\phi,\xi}(y_2)\|_{Z^s_{\eta_0}} \le \frac{1}{2}\|y_1-y_2\|_{Z^s_{\eta_0}}.$$

Consequently, $\mathcal{G}_{\phi,\xi}$ is a contraction map from A_r to itself; hence, it has a unique fixed point in A_r, which is a solution of Eq. (3.17) on $\mathbb{R}^+$. This proves the proposition since r can be chosen arbitrarily large. □

Let us denote by $y(\phi,\xi)(t)$ the solution of Eq. (3.17).

Proposition 3.7. *If $\phi \in X$ and $\xi \in C(\mathbb{R}^+; E^c)$ with $\sup_{t\in\mathbb{R}^+}\|\xi(t)\|_X e^{\beta t} < \infty$, then we have for $t \in \mathbb{R}^+$*

$$\|y(\phi,\xi)(t) - F_{*,\delta}(\Phi z(t,\phi))\|_X \le \Big(C\|\phi^s - F_{*,\delta}(\phi^c)\|_X + \sup_{\tau\in\mathbb{R}^+}\|\xi(\tau)\|_X e^{\beta\tau}\Big)e^{-\beta t}.$$

Proof. By the invariance of the center manifold $W^c_\delta(0)$ (Theorem 2.4) and VCF,

$$\begin{aligned}F_{*,\delta}(\Phi z(t,\phi)) &= \Pi^s x_t(0,\phi^c + F_{*,\delta}(\phi^c), f_\delta)\\ &= T^s(t)F_{*,\delta}(\phi^c) + \lim_{n\to\infty}\int_0^t T^s(t-s)\Pi^s\Gamma^n f_\delta\big(\Phi z(s,\phi) + F_{*,\delta}(\Phi z(s,\phi))\big)ds.\end{aligned}$$

Since $y(\phi,\xi) = \mathcal{G}_{\phi,\xi}(y(\phi,\xi))$, we have

$$\begin{aligned}y(\phi,\xi)(t) - F_{*,\delta}(\Phi z(t,\phi)) = T^s(t)(\phi^s - F_{*,\delta}(\phi^c)) + \lim_{n\to\infty}\int_0^t T^s(t-s)\Pi^s\Gamma^n\big\{f_\delta\big(\Phi z(s,\phi)\\ + \xi(s) + y(\phi,\xi)(s)\big) - f_\delta\big(\Phi z(s,\phi) + F_{*,\delta}(\Phi z(s,\phi))\big)\big\}ds,\end{aligned}$$

and therefore

$$
\begin{aligned}
&\|y(\phi,\xi)(t) - F_{*,\delta}(\Phi z(t,\phi))\|_X \\
&\quad \le Ce^{-\alpha t}\|\phi^s - F_{*,\delta}(\phi^c)\|_X + \int_0^t K_\delta e^{-\alpha(t-s)}(\|\xi(s)\|_X + \|y(\phi,\xi)(s) - F_{*,\delta}(\Phi z(s,\phi))\|_X)ds \\
&\quad \le Ce^{-\alpha t}\|\phi^s - F_{*,\delta}(\phi^c)\|_X + \int_0^t K_\delta e^{-\alpha(t-s)}\big(\sup_{\tau\in\mathbb{R}^+}\|\xi(\tau)\|_X e^{\beta\tau}\big)e^{-\beta s}ds \\
&\qquad + \int_0^t K_\delta e^{-\alpha(t-s)}\|y(\phi,\xi)(s) - F_{*,\delta}(\Phi z(s,\phi))\|_X ds.
\end{aligned}
$$

By an application of Lemma 3.8 below, combined with (3.14), we obtain

$$
\begin{aligned}
&\|y(\phi,\xi)(t) - F_{*,\delta}(\Phi z(t,\phi))\|_X \\
&\quad \le C\|\phi^s - F_{*,\delta}(\phi^c)\|_X e^{-(\alpha-K_\delta)t} + \frac{K_\delta}{\alpha-\beta-K_\delta}\big(\sup_{\tau\in\mathbb{R}^+}\|\xi(\tau)\|_X e^{\beta\tau}\big)e^{-\beta t} \\
&\quad \le \big(C\|\phi^s - F_{*,\delta}(\phi^c)\|_X + \sup_{\tau\in\mathbb{R}^+}\|\xi(\tau)\|_X e^{\beta\tau}\big)e^{-\beta t}, \quad t\in\mathbb{R}^+,
\end{aligned}
$$

which proves the assertion. □

The following lemma is an analogue of [23, Lemma 2] and we omit the proof.

Lemma 3.8. *Let $g, r \in C(\mathbb{R}^+;\mathbb{R})$ and $h \in C^1(\mathbb{R}^+;\mathbb{R})$ satisfy $r(t) \ge 0$ $(t \in \mathbb{R}^+)$ and*

$$
g(t) \le h(t) + \int_0^t r(s)g(s)ds, \quad t\in\mathbb{R}^+.
$$

Then we have

$$
g(t) \le h(0)\exp\left(\int_0^t r(s)ds\right) + \int_0^t h'(s)\exp\left(\int_s^t r(u)du\right)ds, \quad t\in\mathbb{R}^+.
$$

Let Z^c_β be the Banach space defined by

$$
Z^c_\beta := BC^\beta(\mathbb{R}^+;E^c) = \{\xi \in C(\mathbb{R}^+;E^c) : \sup_{t\in\mathbb{R}^+}\|\xi(t)\|_X e^{\beta t} < \infty\},
$$

with norm $\|\xi\|_{Z^c_\beta} := \sup_{t\in\mathbb{R}^+}\|\xi(t)\|_X e^{\beta t}$. Also, set

$$
h_\delta(t,\phi,\xi) := f_\delta\big(\Phi z(t,\phi) + \xi(t) + y(\phi,\xi)(t)\big) - f_\delta\big(\Phi z(t,\phi) + F_{*,\delta}(\Phi z(t,\phi))\big)
$$

for $t\in\mathbb{R}^+$, $\phi\in X$ and $\xi\in Z^c_\beta$.

Proposition 3.9. *For each $\phi \in X$ the equation*

$$
\xi(t) = -\lim_{n\to\infty}\int_t^\infty T^c(t-s)\Pi^c\Gamma^n h_\delta(s,\phi,\xi)ds, \quad t\in\mathbb{R}^+ \tag{3.19}
$$

has a unique solution in Z^c_β.

Proof. Let $r_0 > \|\phi\|_X$, and take a positive number r_1 satisfying

$$
r_1 > \frac{1}{2}CC_1(1+L(\delta))r_0. \tag{3.20}
$$

We define a map $\mathcal{H} : \overline{B}_X(r_0) \times \overline{B}_{Z^c_\beta}(r_1) \to C(\mathbb{R}^+; E^c)$ by

$$(\mathcal{H}(\phi, \xi))(t) := -\lim_{n\to\infty} \int_t^\infty T^c(t-s)\Pi^c\Gamma^n h_\delta(s, \phi, \xi) ds$$

for $t \in \mathbb{R}^+$, $\phi \in \overline{B}_X(r_0)$, and $\xi \in \overline{B}_{Z^c_\beta}(r_1)$. In view of Proposition 3.7,

$$\begin{aligned}\|h_\delta(s,\phi,\xi)\|_X &\le \zeta_*(\delta)\big(\|\xi\|_X + \|y(\phi,\xi)(s) - F_{*,\delta}(\Phi z(s,\phi))\|_X\big)\\ &\le \zeta_*(\delta)\big\{\|\xi\|_{Z^c_\beta} e^{-\beta s} + \big(C\|\phi^s - F_{*,\delta}(\phi^c)\|_X + \|\xi\|_{Z^c_\beta}\big)e^{-\beta s}\big\}\\ &= \zeta_*(\delta)\big(CC_1(1+L(\delta))\|\phi\|_X + 2\|\xi\|_{Z^c_\beta}\big)e^{-\beta s},\end{aligned} \tag{3.21}$$

and therefore

$$\begin{aligned}\|(\mathcal{H}(\phi,\xi))(t)\|_X &\le \int_t^\infty K_\delta e^{-\varepsilon(t-s)}\big(CC_1(1+L(\delta))\|\phi\|_X + 2\|\xi\|_{Z^c_\beta}\big)e^{-\beta s} ds\\ &\le \frac{K_\delta}{\beta-\varepsilon}(CC_1(1+L(\delta))r_0 + 2r_1)e^{-\beta t} \le r_1 e^{-\beta t}\end{aligned} \tag{3.22}$$

for $t \in \mathbb{R}^+$, where we used (3.14) and (3.20). So, $\mathcal{H}(\phi,\xi)$ belongs to Z^c_β with $\|\mathcal{H}(\phi,\xi)\|_{Z^c_\beta} \le r_1$; hence for each $\phi \in \overline{B}_X(r_0)$, $\mathcal{H}(\phi,\cdot)$ defines a map from $\overline{B}_{Z^c_\beta}(r_1)$ to itself. We will prove the proposition by showing that $\mathcal{H}(\phi,\cdot)$ is a contraction map. Indeed, let $\xi_1, \xi_2 \in \overline{B}_{Z^c_\beta}(r_1)$. Then

$$\begin{aligned}&\|(\mathcal{H}(\phi,\xi_1))(t) - (\mathcal{H}(\phi,\xi_2))(t)\|_X\\ &\quad\le \int_t^\infty K_\delta e^{-\varepsilon(t-s)}(\|\xi_1(s)-\xi_2(s)\|_X + \|y(\phi,\xi_1)(s) - y(\phi,\xi_2)(s)\|_X) ds.\end{aligned} \tag{3.23}$$

We know $y(\phi,\xi_i) = \mathcal{G}_{\phi,\xi_i}(y(\phi,\xi_i))$ $(i=1,2)$, so that

$$\begin{aligned}y(\phi,\xi_1)(t) - y(\phi,\xi_2)(t) = \lim_{n\to\infty}\int_0^t T^s(t-s)\Pi^s\Gamma^n\big\{&f_\delta\big(\Phi z(s,\phi) + \xi_1(s) + y(\phi,\xi_1)(s)\big)\\ &- f_\delta\big(\Phi z(s,\phi) + \xi_2(s) + y(\phi,\xi_2(s))\big)\big\} ds.\end{aligned}$$

Hence

$$\begin{aligned}&\|y(\phi,\xi_1)(t) - y(\phi,\xi_2)(t)\|_X\\ &\quad\le \int_0^t K_\delta e^{-\alpha(t-s)}(\|\xi_1(s)-\xi_2(s)\|_X + \|y(\phi,\xi_1)(s) - y(\phi,\xi_2)(s)\|_X) ds\\ &\quad\le \int_0^t K_\delta e^{-\alpha(t-s)}\|\xi_1-\xi_2\|_{Z^c_\beta} e^{-\beta s} ds + \int_0^t K_\delta e^{-\alpha(t-s)}\|y(\phi,\xi_1)(s) - y(\phi,\xi_2)(s)\|_X ds,\end{aligned}$$

that is,

$$\begin{aligned}&e^{\alpha t}\|y(\phi,\xi_1)(t) - y(\phi,\xi_2)(t)\|_X\\ &\qquad\le \int_0^t K_\delta\|\xi_1-\xi_2\|_{Z^c_\beta} e^{(\alpha-\beta)s} ds + \int_0^t K_\delta e^{\alpha s}\|y(\phi,\xi_1)(s) - y(\phi,\xi_2)(s)\|_X ds\end{aligned}$$

holds for $t \in \mathbb{R}^+$. It follows from Lemma 3.8 and (3.14) that

$$\|y(\phi,\xi_1)(t) - y(\phi,\xi_2)(t)\|_X \le \frac{K_\delta}{\alpha-\beta-K_\delta}\|\xi_1-\xi_2\|_{Z^c_\beta} e^{-\beta t} \le \|\xi_1-\xi_2\|_{Z^c_\beta} e^{-\beta t}, \quad t \in \mathbb{R}^+.$$

Consequently, we see from (3.23) that

$$\begin{aligned}\|(\mathcal{H}(\phi,\xi_1))(t)-(\mathcal{H}(\phi,\xi_2))(t)\|_X &\le \int_t^\infty K_\delta e^{-\varepsilon(t-s)}\,2\|\xi_1-\xi_2\|_{Z^c_\beta}e^{-\beta s}ds\\ &= \frac{2K_\delta}{\beta-\varepsilon}\|\xi_1-\xi_2\|_{Z^c_\beta}e^{-\beta t},\quad t\in\mathbb{R}^+,\end{aligned}$$

which, together with (3.14), implies

$$\|\mathcal{H}(\phi,\xi_1)-\mathcal{H}(\phi,\xi_2)\|_{Z^c_\beta}\le\frac{1}{2}\|\xi_1-\xi_2\|_{Z^c_\beta}.$$

Thus, $\mathcal{H}(\phi,\cdot)$ is a contraction map from $\overline{B}_{Z^c_\beta}(r_1)$ to itself; and therefore has a unique fixed point in $\overline{B}_{Z^c_\beta}(r_1)$ for $\phi\in\overline{B}_X(r_0)$. Since one can choose r_1 arbitrarily large, the proposition immediately follows. □

For $\phi\in X$ denote $\xi(\phi)(t)$ the solution of Eq. (3.19).

Proposition 3.10. *Define a map $\mathcal{K}:X\to X$ by $\mathcal{K}(\phi):=\xi(\phi)(0)$, $\phi\in X$. Then $\mathcal{K}$ is continuous.*

Proof. Since Z^c_β is a subspace of the Banach space $BC(\mathbb{R}^+;E^c)$, it is sufficient to show that $\xi:\phi\longmapsto\xi(\phi)$ is continuous as a map from X to $BC(\mathbb{R}^+;E^c)$. So letting ϕ_0 be an element of X and $\{\phi_l\}\subset X$ an arbitrary sequence converging to ϕ_0, we show $\xi(\phi_l)\to\xi(\phi_0)$ in $BC(\mathbb{R}^+;E^c)$ as $l\to\infty$. The proof will be divided into several steps.

Step 1. $\{\xi(\phi_l)\}$ is relatively compact in $BC(\mathbb{R}^+;E^c)$. To confirm the assertion, take positive numbers r_0 and r_1 in such a way that $\phi_l\in B_X(r_0)$ $(l\in\mathbb{N})$ and (3.20) are satisfied. Since $\|\xi(\phi_l)(t)\|_X\le\|\xi(\phi_l)\|_{Z^c_\beta}e^{-\beta t}\le r_1e^{-\beta t}$ $(t\in\mathbb{R}^+)$ follows from the proof of Proposition 3.9, the sequence $\{\xi(\phi_l)(t)\}$ is uniformly bounded in $\mathbb{R}^+$, and is equi-convergent (to 0) as $t\to\infty$. We next verify that $\{\xi(\phi_l)(t)\}$ is equi-continuous in $\mathbb{R}^+$. Let $t_2\ge t_1\ge 0$. In view of the relation $\xi(\phi_l)=\mathcal{H}(\phi_l,\xi(\phi_l))$ (cf. Proposition 3.9),

$$\begin{aligned}\xi(\phi_l)(t_2)-\xi(\phi_l)(t_1) &= \lim_{n\to\infty}\int_{t_1}^{t_2}T^c(t_1-s)\Pi^c\Gamma^n h_\delta(s,\phi_l,\xi(\phi_l))ds\\ &\quad+\lim_{n\to\infty}\int_{t_2}^{\infty}(T^c(t_1-s)-T^c(t_2-s))\Pi^c\Gamma^n h_\delta(s,\phi_l,\xi(\phi_l))ds\\ &= \lim_{n\to\infty}\int_{t_1}^{t_2}T^c(t_1-s)\Pi^c\Gamma^n h_\delta(s,\phi_l,\xi(\phi_l))ds\\ &\quad+\lim_{n\to\infty}\int_{t_2}^{\infty}T^c(t_1-s)(I_{E^c}-T^c(t_2-t_1))\Pi^c\Gamma^n h_\delta(s,\phi_l,\xi(\phi_l))ds,\end{aligned}$$

where I_{E^c} is the identity map of E^c. Hence it follows from (3.21) and (3.20) that

$$\begin{aligned}\|\xi(\phi_l)(t_2)-\xi(\phi_l)(t_1)\|_X &\le \int_{t_1}^{t_2}CC_1e^{\varepsilon(s-t_1)}\|h_\delta(s,\phi_l,\xi(\phi_l))\|_X ds\\ &\quad+\int_{t_2}^{\infty}CC_1e^{\varepsilon(s-t_1)}\|I_{E^c}-T^c(t_2-t_1)\|\|h_\delta(s,\phi_l,\xi(\phi_l))\|_X ds\\ &\le \left(C\|\phi_l^s-F_{*,\delta}(\phi_l^c)\|_X+2\|\xi(\phi_l)\|_{Z^c_\beta}\right)\\ &\quad\times\left(\int_{t_1}^{t_2}K_\delta e^{\varepsilon(s-t_1)}e^{-\beta s}ds+\int_{t_2}^{\infty}K_\delta e^{\varepsilon(s-t_1)}\|T^c(t_2-t_1)-I_{E^c}\|e^{-\beta s}ds\right)\\ &\le \frac{K_\delta}{\beta-\varepsilon}\left(CC_1(1+L(\delta))r_0+2r_1\right)\left(|t_2-t_1|+\|T^c(t_2-t_1)-I_{E^c}\|_{\mathcal{L}(E^c)}\right)\\ &\le r_1\left(|t_2-t_1|+\|T^c(t_2-t_1)-I_{E^c}\|_{\mathcal{L}(E^c)}\right),\end{aligned}$$

which shows the equi-continuity in $\mathbb{R}^+$ of $\{\xi(\phi_l)(t)\}$ since E^c is a finite-dimensional subspace.

By the properties of $\{\xi(\phi_l)(t)\}$ above and the fact that $\dim E^c < \infty$, a variant of the Arzelà–Ascoli theorem yields that $\{\xi(\phi_l)\}$ is relatively compact in $BC(\mathbb{R}^+; E^c) = Z_0^c$. So there is a subsequence of $\{\phi_l\}$, denoted $\{\phi_l\}$ again, such that $\|\xi(\phi_l) - \xi_*\|_{Z_0^c} \to 0$ $(l \to \infty)$ for some $\xi_* \in BC(\mathbb{R}^+; E^c)$.

Step 2. $\{y(\phi_l, \xi(\phi_l))(t)\}$ converges to $y(\phi_0, \xi_*)(t)$ uniformly in any compact set of $\mathbb{R}^+$ as $l \to \infty$. Indeed, since $y(\phi_l, \xi(\phi_l))$ and $y(\phi_0, \xi_*)$ are the fixed points of $\mathcal{G}_{\phi_l, \xi(\phi_l)}$ and $\mathcal{G}_{\phi_0, \xi_*}$ respectively (cf. (3.18)),

$$\begin{aligned} y(\phi_l, \xi(\phi_l))(t) &- y(\phi_0, \xi_*)(t) \\ &= T(t)\phi_l^s - T(t)\phi_0^s + \lim_{n\to\infty} \int_0^t T^s(t-s)\Pi^s\Gamma^n \{ f_\delta(\Phi z(s,\phi_l) + \xi(\phi_l)(s) \\ &\quad + y(\phi_l, \xi(\phi_l)(s)) - f_\delta(\Phi z(s,\phi_0) + \xi_*(s) + y(\phi_0,\xi_*)(s)) \} ds. \end{aligned}$$

So, letting $u_l(t) := y(\phi_l, \xi(\phi_l))(t) - y(\phi_0, \xi_*)(t)$, we have

$$\begin{aligned} \|u_l(t)\|_X \le CC_1 \|\phi_l - \phi_0\|_X e^{-\alpha} + \int_0^t K_\delta e^{-\alpha(t-s)} \{ \|\Phi z(s,\phi_l) - \Phi z(s,\phi_0)\|_X \\ + \|\xi(\phi_l)(s) - \xi_*(s)\|_X + \|u_l(s)\|_X \} ds. \end{aligned} \tag{3.24}$$

Notice from (3.16) that

$$\|\Phi z(t,\phi_l) - \Phi z(t,\phi_0)\|_X \le Ce^{\varepsilon t}\|\phi_l^c - \phi_0^c\|_X + \int_0^t K_\delta e^{\varepsilon(t-s)}(1 + L(\delta))\|\Phi z(s,\phi_l) - \Phi z(s,\phi_0)\|_X ds;$$

then one can see from Gronwall's inequality, combined with (3.15), that

$$\|\Phi z(t,\phi_l) - \Phi z(t,\phi_0)\|_X \le C\|\phi_l^c - \phi_0^c\|_X e^{(\varepsilon + K_\delta(1+L(\delta))t} \le C\|\phi_l^c - \phi_0^c\|_X e^{2\varepsilon t}, \quad t \in \mathbb{R}^+. \tag{3.25}$$

Thus (3.24) implies

$$\begin{aligned} \|u_l(t)\|_X e^{\alpha t} \le CC_1\|\phi_l - \phi_0\|_X + \int_0^t K_\delta e^{\alpha s}(C\|\phi_l^c - \phi_0^c\|_X e^{2\varepsilon s} \\ + \|\xi(\phi_l)(s) - \xi_*(s)\|_X) ds + \int_0^t K_\delta e^{\alpha s}\|u_l(s)\|_X ds. \end{aligned}$$

By Lemma 3.8,

$$\begin{aligned} \|u_l(t)\|_X e^{\alpha t} &\le CC_1\|\phi_l - \phi_0\|_X e^{K_\delta t} + \int_0^t K_\delta e^{\alpha s}(C\|\phi_l^c - \phi_0^c\|_X e^{2\varepsilon s} \\ &\quad + \|\xi(\phi_l)(s) - \xi_*(s)\|_X) e^{K_\delta(t-s)} ds \\ &\le CC_1\|\phi_l - \phi_0\|_X e^{K_\delta t} + \frac{K_\delta C}{\alpha + 2\varepsilon - K_\delta}\|\phi_l^c - \phi_0^c\|_X e^{(\alpha+2\varepsilon)t} \\ &\quad + \frac{K_\delta}{\alpha - K_\delta}\|\xi(\phi_l) - \xi_*\|_{Z_0^c} e^{\alpha t}, \end{aligned}$$

and therefore from (3.14)

$$\|u_l(t)\|_X \le CC_1\|\phi_l - \phi_0\|_X e^{-(\alpha - K_\delta)t} + C\|\phi_l^c - \phi_0^c\|_X e^{2\varepsilon t} + \|\xi(\phi_l) - \xi_*\|_{Z_0^c}$$

holds for $t \in \mathbb{R}^+$. It follows from Step 1 that for any $\tau_* > 0$, the sequence $\{y(\phi_l, \xi(\phi_l))(t)\}$ is uniformly convergent to $y(\phi_0, \xi_*)(t)$ in $[0, \tau_*]$ as $l \to \infty$, which proves Step 2.

Step 3. ξ_* coincides with $\xi(\phi_0)$. Indeed, given positive numbers ε_* and τ, choose a τ_* in such a way that $\tau_* > \tau$ and

$$e^{-(\beta-\varepsilon)\tau_*} < \frac{\varepsilon_*}{2Cr_1} \tag{3.26}$$

hold. Observe that

$$\begin{aligned}\xi(\phi_l)(t) - \mathcal{H}(\phi_0,\xi_*)(t) = &-\lim_{n\to\infty}\int_t^{\tau_*} T^c(t-s)\Pi^c\Gamma^n(h_\delta(s,\phi_l,\xi(\phi_l)) - h_\delta(s,\phi_0,\xi_*))ds \\ &+ T^c(t-\tau_*)(\xi(\phi_l)(\tau_*) - \mathcal{H}(\phi_0,\xi_*)(\tau_*)).\end{aligned}$$

Since neither $\|\xi(\phi_l)(\tau_*)\|_X$ nor $\|\mathcal{H}(\phi_0,\xi_*)(\tau_*)\|_X$ is greater than $r_1e^{-\beta\tau_*}$ (see (3.22)), it follows from (3.26) that

$$\begin{aligned}\big\|T^c(t-\tau_*)\big(\xi(\phi_l)(\tau_*) - \mathcal{H}(\phi_0,\xi_*)(\tau_*)\big)\big\|_X &\le Ce^{\varepsilon(\tau_*-t)}(r_1e^{-\beta\tau_*} + r_1e^{-\beta\tau_*}) \\ &< \varepsilon_*, \quad t\in[0,\tau_*].\end{aligned}$$

Consequently,

$$\begin{aligned}&\|\xi(\phi_l)(t) - \mathcal{H}(\phi_0,\xi_*)(t)\|_X \\ &\quad< \int_t^{\tau_*} K_\delta Ce^{\varepsilon(s-t)}\{(2+L(\delta))\|\Phi z(s,\phi_l) - \Phi z(s,\phi_0)\|_X \\ &\qquad+ \|\xi(\phi_l)(s) - \xi_*(s)\|_X + \|y(\phi_l,\xi(\phi_l))(s) - y(\phi_0,\xi_*)(s)\|_X\}ds + \varepsilon_* \\ &\quad\le \int_t^{\tau_*} K_\delta Ce^{\varepsilon(s-t)}\{C_1(2+L(\delta))\|\phi_l - \phi_0\|_X e^{2\varepsilon s} + \|\xi(\phi_l) - \xi_*\|_{Z_0^c} \\ &\qquad+ \|y(\phi_l,\xi(\phi_l))(s) - y(\phi_0,\xi_*)(s)\|_X\}ds + \varepsilon_*\end{aligned}$$

for $0\le t\le\tau_*$ because of (3.25). Passing to the limit as $l\to\infty$, we deduce from Steps 1 and 2 that

$$\|\xi_*(t) - \mathcal{H}(\phi_0,\xi_*)(t)\|_X = \lim_{m\to\infty}\|\xi(\phi_l)(t) - \mathcal{H}(\phi_0,\xi_*)(t)\|_X \le \varepsilon_*, \quad t\in[0,\tau].$$

Since ε_* is arbitrary, we have $\|\xi_*(t) - \mathcal{H}(\phi_0,\xi_*)(t)\|_X = 0$ for $t\in[0,\tau]$; hence ξ_* must coincide with $\mathcal{H}(\phi_0,\xi_*)$, for τ is also arbitrary. The uniqueness of fixed points of $\mathcal{H}(\phi_0,\cdot)$ then implies $\xi_* = \xi(\phi_0)$.

Step 4. The argument above shows that given a subsequence of $\{\phi_l\}$, one can choose its subsequence, say $\{\phi_{l_k}\}$, such that $\xi(\phi_{l_k})$ is convergent to $\xi(\phi_0)$ with norm $\|\cdot\|_{Z_0^c}$ as $k\to\infty$. It therefore turns out that $\{\xi(\phi_l)\}$ is itself convergent to $\xi(\phi_0)$ in $BC(\mathbb{R}^+;E^c)$. Thus, $\xi:\phi\longmapsto\xi(\phi)$ is continuous as a map from X to $BC(\mathbb{R}^+;E^c)$, which completes the proof. □

3.2 Proofs of Theorems 3.2 and 3.3

We are now able to give a proof of Theorem 3.2.

Proof of Theorem 3.2. (a) Without loss of generality, we may prove part (a) in case $t_0 = 0$ (see Remark 3.4). For each $\phi\in X$, we set $y(\phi)(t) := y(\phi,\xi(\phi))(t)$ for $t\in\mathbb{R}^+$. Then it follows from Propositions 3.6, 3.9 and 3.7 that the X-valued functions $\xi(\phi)$ and $y(\phi)$ satisfy (3.6) and (3.7) together with (3.8). Hence, by virtue of Lemma 3.5, the $\mathbb{C}^m$-valued function $x(\phi)(t)$ on $\mathbb{R}$, defined by

$$x(\phi)(t) := \begin{cases}(\Phi z(t,\phi) + \xi(\phi)(t) + y(\phi)(t))[0], & t>0,\\ (\Phi z(0,\phi) + \xi(\phi)(0) + y(\phi)(0))(t), & t\le 0,\end{cases}$$

is a solution of Eq. (E_δ) on $\mathbb{R}^+$, which satisfies both of

$$\|\Pi^c(x(\phi))_t - \Phi z(t,\phi)\|_X \le C_* e^{-\beta t},$$
$$\|\Pi^s(x(\phi))_t - F_{*,\delta}(\Phi z(t,\phi))\|_X \le C_* e^{-\beta t}$$

and $(x(\phi))_t = \Phi z(t,\phi) + \xi(\phi)(t) + y(\phi)(t)$ for $t \in \mathbb{R}^+$, where C_* is a nonnegative constant. So, if one can find a $\phi \in X$ with $x_0 = (x(\phi))_0$, the uniqueness of solutions for Eq. (E_δ) ensures that $x(t) \equiv x(\phi)(t)$ for $t \in \mathbb{R}^+$; so that $z(t) = z(t,\phi)$ is a solution of Eq. (CE_δ) satisfying

$$\|\Pi^c x_t - \Phi z(t)\|_X \le C_* e^{-\beta t} \quad \text{and} \quad \|\Pi^s x_t - F_{*,\delta}(\Phi z(t))\|_X \le C_* e^{-\beta t} \tag{3.27}$$

for $t \in \mathbb{R}^+$. Now consider the map $\widehat{g} : X \to X$ defined by

$$\widehat{g}(\phi) := (x(\phi))_0, \quad \phi \in X. \tag{3.28}$$

Claim 1. $\widehat{g}$ is a bijection from X to itself if $\delta > 0$ is sufficiently small.

Proof of Claim 1. We first verify the surjectivity. Note, by definition, that

$$(x(\phi))_0 = \Phi z(0,\phi) + \xi(\phi)(0) + y(\phi)(0) = \phi^c + \mathcal{K}(\phi) + \phi^s \tag{3.29}$$

for $\phi \in X$ (see also (3.16) and (3.17)), and therefore that $\widehat{g}(\phi) = \phi + \mathcal{K}(\phi)$.

Let $M(\delta) := 2CK_\delta/(\beta - \varepsilon)$ and take a δ so small that

$$C_1 M(\delta)(1 + L(\delta)) < \frac{1}{2}. \tag{3.30}$$

In the same way as (3.22), we get

$$\|\xi(\phi)\|_{Z^c_\beta} \le \frac{K_\delta}{\beta - \varepsilon}(C\|\phi^s - F_{*,\delta}(\phi^c)\|_X + 2\|\xi(\phi)\|_{Z^c_\beta})$$

and in particular by (3.14)

$$\|\mathcal{K}(\phi)\|_X \le \|\xi(\phi)\|_{Z^c_\beta} \le M(\delta)\|\phi^s - F_{*,\delta}(\phi^c)\|_X \le C_1 M(\delta)(1 + L(\delta))\|\phi\|_X. \tag{3.31}$$

Now given $\bar{\phi} \in \operatorname{graph} F_{*,\delta}$ and $r_0 > 0$, define $H : \overline{B}_X(\bar{\phi}; r_0) \times [0,1] \to X$ by

$$H(\phi, \lambda) := \phi + \lambda \mathcal{K}(\phi), \quad \phi \in \overline{B}_X(\bar{\phi}; r_0),\ \lambda \in [0,1].$$

Since $\mathcal{K}$ is continuous (Proposition 3.10), bounded on each bounded set of X, and the range of $\mathcal{K}$ is contained in E^c, which is finite-dimensional, we deduce that the map $\mathcal{K}$ is compact. In addition, observe from (3.31) and the fact $\bar{\phi}^s = F_{*,\delta}(\bar{\phi}^c)$ that

$$\begin{aligned} \|\mathcal{K}(\phi)\|_X &\le M(\delta)(\|\phi^s - \bar{\phi}^s\|_X + \|F_{*,\delta}(\bar{\phi}^c) - F_{*,\delta}(\phi^c)\|_X) \\ &\le M(\delta)(\|\phi^s - \bar{\phi}^s\|_X + L(\delta)\|\phi^c - \bar{\phi}^c\|_X) \\ &\le C_1 M(\delta)(1 + L(\delta))\|\phi - \bar{\phi}\|_X. \end{aligned} \tag{3.32}$$

Let $r_0 > 0$ and $\psi \in \overline{B}_X(\bar{\phi}; r_0/2)$, then it follows from (3.30) and (3.32) that for $\phi \in \partial B_X(\bar{\phi}; r_0)$ and $\lambda \in [0,1]$,

$$\begin{aligned} \|H(\phi,\lambda) - \psi\|_X &\ge \|\phi - \bar{\phi}\|_X - \|\psi - \bar{\phi}\|_X - \|\mathcal{K}(\phi)\|_X \\ &\ge \frac{r_0}{2} - C_1 M(\delta)(1 + L(\delta)) r_0 > 0. \end{aligned}$$

Hence, deg $(H(\cdot,\lambda), B_X(\bar\phi;r_0),\psi)$, the Leray–Schauder degree ([3]) of $H(\cdot,\lambda)$, is well-defined and is independent of λ. Since H is a homotopy between $\widehat{g}$ and the identity I_X as maps from $\overline{B}_X(\bar\phi;r_0)$ to X, it follows that

$$\deg(\widehat{g}, B_X(\bar\phi;r_0),\psi) = \deg(I_X, B_X(\bar\phi;r_0),\psi) = 1,$$

which shows that $\widehat{g}^{-1}(\psi)\cap B_X(\bar\phi;r_0)\neq\emptyset$ for each $\psi\in B_X(\bar\phi;r_0/2)$; in other words,

$$B_X(\bar\phi;r_0/2)\subset \widehat{g}(B_X(\bar\phi;r_0)). \tag{3.33}$$

Thus, $\widehat{g}$ is a surjection from X onto itself since $r_0>0$ is arbitrary.

We next prove the injectivity. Let $\delta>0$ be so small that

$$\mu(\delta) := \frac{K_\delta(1+L(\delta))}{\beta-\varepsilon} < 1, \tag{3.34}$$

and assume that $\widehat{g}(\phi)=\widehat{g}(\tilde\phi)$ for some ϕ and $\tilde\phi$ in X. Then by the definition of $\widehat{g}$ we obtain

$$\|\Pi^c x_t(0,\widehat{g}(\phi), f_\delta) - \Phi_c z(t,\phi)\|_X \le C_* e^{-\beta t}$$

and

$$\|\Pi^c x_t(0,\widehat{g}(\tilde\phi), f_\delta) - \Phi z(t,\tilde\phi)\|_X \le \tilde C_* e^{-\beta t}$$

for $t\in\mathbb{R}^+$, where C_* and $\tilde C_*$ are some nonnegative constant. In particular,

$$\|\Phi z(t,\phi)-\Phi z(t,\tilde\phi)\|_X \le C^* e^{-\beta t}, \quad t\in\mathbb{R}^+ \tag{3.35}$$

with $C^* := C_*+\tilde C_*$. We know by (3.16) that $\Phi z(t,\phi)$ satisfies

$$\Phi z(t,\phi) = T^c(t-\tau)\Phi z(\tau,\phi) + \lim_{n\to\infty}\int_\tau^t T^c(t-s)\Pi^c f_\delta(\Phi z(s,\phi)+F_{*,\delta}(\Phi z(s,\phi)))ds$$

for $t\ge\tau\ge 0$. So, by the group property of $\{T^c(t)\}_{t\in\mathbb{R}}$, we see

$$\Phi z(\tau,\phi) = T^c(\tau-t)\Phi z(t,\phi) - \lim_{n\to\infty}\int_\tau^t T^c(\tau-s)\Pi^c f_\delta(\Phi z(s,\phi)+F_{*,\delta}(\Phi z(s,\phi)))ds$$

and likewise for $\Phi_c z(\tau,\tilde\phi)$. Hence it follows from (3.35) that

$$\begin{aligned}
&\|\Phi z(\tau,\phi)-\Phi z(\tau,\tilde\phi) - T^c(\tau-t)\big(\Phi z(t,\phi)-\Phi z(t,\tilde\phi)\big)\|_X \\
&\quad\le \int_\tau^t K_\delta(1+L(\delta))e^{\varepsilon|\tau-s|}\|\Phi z(s,\phi)-\Phi z(s,\tilde\phi)\|_X ds \\
&\quad\le \int_\tau^t K_\delta(1+L(\delta))e^{\varepsilon(s-\tau)}\cdot C^* e^{-\beta s} ds \le \frac{C^* K_\delta(1+L(\delta))}{\beta-\varepsilon} e^{-\beta\tau},
\end{aligned}$$

so that

$$\|\Phi z(\tau,\phi)-\Phi z(\tau,\tilde\phi)\|_X \le CC^* e^{-\varepsilon\tau-(\beta-\varepsilon)t} + C^*\mu(\delta)e^{-\beta\tau}, \quad t\ge\tau\ge 0.$$

Passing to the limit as $t\to\infty$, we get

$$\|\Phi z(\tau,\phi)-\Phi z(\tau,\tilde\phi)\|_X \le C^*\mu(\delta)e^{-\beta\tau}, \quad \tau\in\mathbb{R}^+. \tag{3.36}$$

Thus, (3.35) implies (3.36). Applying the same argument to (3.36) (in place of (3.35)), we have $\|\Phi z(t,\phi)-\Phi z(t,\tilde\phi)\|_X \le C^*\mu(\delta)^2 e^{-\beta t}$ $(t\in\mathbb{R}^+)$. By the repetition of this procedure, one reaches $\|\Phi z(t,\phi) - \Phi z(t,\tilde{\phi})\|_X \le C^* \mu(\delta)^n e^{-\beta t}$ ($t \in \mathbb{R}^+, n = 1,2,\dots$), which yields $\|\Phi z(t,\phi) - \Phi z(t,\tilde{\phi})\|_X = 0$ ($t \in \mathbb{R}^+$) because of (3.34). In particular, $\phi^c = \Phi z(0,\phi) = \Phi(0,\tilde{\phi}) = \tilde{\phi}^c$. Moreover, since $\Pi^s \mathcal{K}(\phi) = \Pi^s \mathcal{K}(\tilde{\phi}) = 0$, it also follows that $\phi^s = \Pi^s \widehat{g}(\phi) = \Pi^s \widehat{g}(\tilde{\phi}) = \tilde{\phi}^s$, and therefore that $\phi = \tilde{\phi}$. Consequently the claim is proved. □

In view of Claim 1, given any solution $x(t)$ of Eq. (E_δ) defined on $\mathbb{R}^+$, there exists a unique solution $z(t)$ of Eq. (CE_δ) satisfying (3.27), that is, $z(t) = z(t,\phi)$ with $\widehat{g}(\phi) = x_0$. We will estimate the constant C_* in (3.27). Since $x_0 = \widehat{g}(\phi)$ for some $\phi \in X$, $x_t = (x(\phi))_t = \Phi z(t,\phi) + \xi(\phi)(t) + y(\phi)(t)$ for $t \in \mathbb{R}^+$. Hence (3.29) yields $\phi^s = \Pi^s x_0$. It then follows from Proposition 3.7 and (3.31) that

$$\begin{aligned} \|y(\phi)(t) - F_{*,\delta}(\Phi z(t,\phi))\|_X &\le (C\|\phi^s - F_{*,\delta}(\phi^c)\|_X + \|\xi(\phi)\|_{Z^c_\beta})e^{-\beta t} \\ &\le C_0 \|\Pi^s x_0 - F_{*,\delta}(\Phi z(0,\phi))\|_X e^{-\beta t} \end{aligned} \tag{3.37}$$

for $t \in \mathbb{R}^+$, where $C_0 := C + M(\delta)$. Thus we obtain

$$\begin{aligned} \|\Pi^c x_t - \Phi z(t,\phi)\|_X = \|\xi(\phi)(t)\|_X &\le M(\delta)\|\phi^s - F_{*,\delta}(\phi^c)\|_X e^{-\beta t} \\ &\le C_0 \|\Pi^s x_0 - F_{*,\delta}(\Phi z(0,\phi))\|_X e^{-\beta t}, \end{aligned}$$

and by (3.37)

$$\|\Pi^s x_t - F_{*,\delta}(\Phi z(t,\phi))\|_X \le C_0 \|\Pi^s x_0 - F_{*,\delta}(\Phi z(0,\phi))\|_X e^{-\beta t}$$

for $t \in \mathbb{R}^+$. Obviously, C_0 can be chosen as close to C as one expects by taking $\delta > 0$ small. Moreover, we know from Proposition 3.1 that $\Phi z(t,\phi) + F_{*,\delta}(\Phi z(t,\phi)) = x_t(0,\hat{\phi}, f_\delta)$ with $\hat{\phi} = \Phi z(0,\phi) + F_{*,\delta}(\Phi z(0,\phi))$, so that (3.2) readily follows. This proves part (a).

(b) Given $t_0 \in \mathbb{R}$ and $z^0 \in \mathbb{C}^d$, write the solution of Eq. (CE_δ) through (t_0, z^0) as $z(t; t_0, z^0)$; in other words, $z(t; t_0, z^0) := z(t - t_0, \Phi z^0)$, $t \in [t_0, \infty)$. Moreover we set

$$\|\Phi\|^* := \left(\sum_{j=1}^d \|\phi_j\|_X^2 \right)^{1/2}, \quad \text{and} \quad \|\Phi\|_* := \inf\left\{ \|\Phi z\|_X : |z| = 1,\, z \in \mathbb{C}^d \right\}.$$

Suppose that the zero solution of Eq. (CE_δ) is stable. Then for arbitrary $\hat{\varepsilon} > 0$ there exists a $\delta_0 > 0$ such that $|z(t; t_0, z^0)| < \hat{\varepsilon}/(2(1 + L(\delta))\|\Phi\|^*)$ for every $t \in [t_0, \infty)$ and $z^0 \in \mathbb{C}^d$ with $|z^0| < \delta_0$. Take a $\delta_* > 0$ such that

$$\delta_* < \min\left(\frac{\|\Phi\|_* \delta_0}{2C_1}, \frac{\hat{\varepsilon}}{4C_0 C_1 (1 + 2L(\delta))} \right).$$

Now let $\psi \in B_X(\delta_*)$. Then by Claim 1 and (3.33) there exists a (unique) $\phi \in B_X(2\delta_*)$ such that $\psi = \widehat{g}(\phi)$. Let z^0 be the element of $\mathbb{C}^d$ determined by $\phi^c = \Phi z^0$. Since $\|\Phi\|_* |z^0| \le \|\phi^c\|_X \le C_1 \|\phi\|_X < 2C_1 \delta_*$, i.e., $|z^0| < \delta_0$, we see

$$\|\Phi z(t - t_0, \phi)\|_X \le \|\Phi\|^* |z(t; t_0, z^0)| < \frac{\hat{\varepsilon}}{2(1 + L(\delta))}, \quad t \in [t_0, \infty). \tag{3.38}$$

Consider the solution $x(t) = x(t + t_0; t_0, \psi, f_\delta)$ of Eq. (E_δ). Since $x_0 = x_{t_0}(t_0, \psi, f_\delta) = \psi$ and $\Phi z(0,\phi) = \phi^c$, we have

$$\begin{aligned} \|\Pi^s x_0 - F_{*,\delta}(\Phi z(0,\phi))\|_X = \|\psi^s - F_{*,\delta}(\phi^c)\|_X &\le \|\psi^s\|_X + L(\delta)\|\phi^c\|_X \\ &\le C_1(1 + 2L(\delta))\delta_* < \frac{\hat{\varepsilon}}{4C_0}. \end{aligned} \tag{3.39}$$

So, by letting $\hat{\phi} := \Phi z^0 + F_{*,\delta}(\Phi z^0)$, it follows from Proposition 3.1 and (3.38) that

$$\begin{aligned}\|x_t(t_0,\hat{\phi},f_\delta)\|_X &= \|\Phi z(t-t_0,\phi) + F_{*,\delta}(\Phi z(t-t_0,\phi)\|_X \\ &\le (1+L(\delta))\|\Phi z(t-t_0,\phi)\|_X < \frac{\hat{\varepsilon}}{2},\end{aligned}$$

and hence from part (a) and (3.39) that

$$\begin{aligned}\|x_t(t_0,\psi,f_\delta)\|_X &\le \|x_t(t_0,\psi,f_\delta) - x_t(t_0,\hat{\phi},f_\delta)\|_X + \|x_t(t_0,\hat{\phi},f_\delta)\|_X \\ &< \frac{\hat{\varepsilon}}{2}e^{-\beta(t-t_0)} + \frac{\hat{\varepsilon}}{2} < \hat{\varepsilon}\end{aligned} \tag{3.40}$$

for $t \in [t_0,\infty)$. Hence the zero solution of Eq. (E_δ) is stable.

Although we had proved the asymptotic stability part in [23, Theorem 6] via construction of a Liapunov function, we will give another proof below. Let us next assume that the zero solution of Eq. (CE_δ) is asymptotically stable. Then the stability of the zero solution of Eq. (E_δ) follows from the argument above. By the attractivity of the zero solution of Eq. (CE_δ) there exists an $R_0 > 0$ with the property that given $\hat{\varepsilon} > 0$, there exists a $\tau > 0$ such that $|z(t;t_0,z^0)| < \hat{\varepsilon}/(2(1+L(\delta))\|\Phi\|^*)$ for every $t \ge t_0 + \tau$ and $z^0 \in \mathbb{C}^d$ with $|z^0| \le R_0$. Now choose an $R_* > 0$ and a $\tau_* > 0$ in such a way that $R_* < \|\Phi\|_* R_0/2$ and

$$\tau_* > \max\left(\tau,\, \beta^{-1}\log\frac{4C_0C_1(1+L(\delta))R_*}{\hat{\varepsilon}}\right).$$

If $\psi \in B_X(R_*)$, then, in a similar fashion to the last paragraph, ψ can be written as $\psi = \widehat{g}(\phi)$ with some $\phi \in B_X(2R_*)$. Corresponding to (3.39), we get this time

$$\|\Pi^s x_0 - F_{*,\delta}(\Phi z(0,\phi))\|_X \le C_1(1+2L(\delta))R_* < \frac{\hat{\varepsilon}}{4C_0}e^{\beta\tau_*}.$$

and hence, in a similar way to (3.40),

$$\begin{aligned}\|x_t(t_0,\psi,f_\delta)\|_X &\le \frac{\hat{\varepsilon}}{2}e^{-\beta(t-t_0-\tau_*)} + (1+L(\delta))\|\Phi\|^*|z(t-t_0,\phi)| \\ &< \frac{\hat{\varepsilon}}{2} + \frac{\hat{\varepsilon}}{2} = \hat{\varepsilon}\end{aligned}$$

for $t \ge t_0 + \tau_*$. Thus, the zero solution of Eq. (E_δ) is also asymptotically stable.

The instability part immediately follows from the invariance of $W^c_\delta(0)$. □

Proof of Theorem 3.3. (a) Let us take a $\delta > 0$ sufficiently small such that $\|F_{*,\delta}(\psi)\|_X < \delta$ for any $\psi \in B_{E^c}(\delta)$. This is possible by Proposition 2.5 and $\zeta_*(+0) = 0$ (in fact, $F_{*,\delta}(0) = 0$ and $DF_{*,\delta}(0) = 0$). Set $F_* := F_{*,\delta}|_{B_{E^c}(\delta)}$, and let Ω and Ω_0 be open neighborhoods of 0 in X defined by $\Omega := \{\phi \in X : \|\Pi^s\phi\|_X < \delta,\ \|\Pi^c\phi\|_X < \delta\}$ and $\Omega_0 := \{\phi \in X : \|\Pi^s\phi\|_X < \delta/3,\ \|\Pi^c\phi\|_X < \delta/3\}$, respectively. Observe that

$$f(\phi) = f_\delta(\phi) \quad \text{for } \phi \in \Omega. \tag{3.41}$$

Since $x(t)$ is a solution of (E) satisfying $x_t \in \Omega_0$ ($t \in J$), by virtue of VCF, one can readily see that $x(t)$ is also a solution of (E_δ) on J. Then by Theorem 3.2 there exists a solution of (CE_δ), say $z(t)$, satisfying (3.1). So, by Claim 1 in the proof of Theorem 3.2, there exists a (unique) $\phi \in X$ such that $x_0 = \widehat{g}(\phi)$; hence $x(t) = x(\phi)(t)$, $z(t) = z(t,\phi)$, and $x_t = \Phi z(t) + \xi(\phi)(t) + y(\phi)(t)$ hold for $t \in J$. In view of (3.30) and (3.31), note that $\|\xi(\phi)\|_{Z^c_\beta} \le (1/2)\|\phi\|_X$. Also, by the relation $B_X(r_0/2) \subset \widehat{g}(B_X(r_0))$ for $r_0 > 0$ (cf. (3.33)), one can see $\|\phi\|_X \le 2\|x_0\|_X$, which is valid even if $x_0 = 0$. Since $\Pi^c x_t = \Phi z(t) + \xi(\phi)(t)$, we have

$$\|\Phi z(t)\|_X \le \|\Pi^c x_t\|_X + \|\xi(\phi)\|_{Z^c_\beta} < \frac{\delta}{3} + \frac{1}{2}\|\phi\|_X \le \frac{\delta}{3} + \|x_0\|_X.$$

Besides, noting that

$$\|x_0\|_X \le \|\Pi^c x_0\|_X + \|\Pi^s x_0\|_X < \frac{\delta}{3} + \frac{\delta}{3} = \frac{2\delta}{3},$$

we obtain $\|\Phi z(t)\|_X < \delta$ $(t \in J)$; and therefore $\|F_{*,\delta}(\Phi z(t))\|_X < \delta$ for $t \in J$ by the choice of δ. In particular, $\Phi z(t) + F_{*,\delta}(\Phi z(t)) \in \Omega$ for $t \in J$. So, $F_{*,\delta}(\Phi z(t)) = F_*(\Phi z(t))$, and

$$f_\delta(\Phi z(t) + F_{*,\delta}(\Phi z(t))) = f(\Phi z(t) + F_*(\Phi z(t)))$$

for $t \in J$, which implies that $z(t)$ is also a solution of (CE) on J. Hence, (3.3) and (3.4) directly follow from the estimates (3.1) and (3.2). This proves part (a).

(b) Suppose that the zero solution of (CE) is stable (asymptotically stable, unstable). Then one can see from (3.41) that the zero solution of (CE_δ) is stable (asymptotically stable, unstable); so is the zero solution of (E_δ) by Theorem 3.2 (b). By (3.41) again, so is that of (E). This completes the proof. □

4 One-parameter bifurcation structures

4.1 Statement of the results

Employing Theorem 3.3, we shall in this section discuss bifurcation structures of equilibria for the parametrized integral equation of the form

$$x(t) = \lambda \int_{-\infty}^{t} P(t-s)x(s)ds + f(x_t), \quad \lambda \in \mathbb{R}, \tag{PE}$$

where P is a measurable $m \times m$-matrix valued function on $\mathbb{R}^+$ with real components that satisfies

$$\int_0^\infty \|P(t)\|e^{\rho t}dt < \infty \quad \text{and} \quad \operatorname{ess\,sup}\{\|P(t)\|e^{\rho t} : t \in \mathbb{R}^+\} < \infty, \tag{4.1}$$

and $f \in C^1(X;\mathbb{R}^m)$ satisfies $f(0) = 0$ and $Df(0) = 0$. Also, in this section we set $X := L^1_\rho(\mathbb{R}^-;\mathbb{R}^m)$, and put

$$P_0 := \int_0^\infty P(t)dt, \quad P_1 := \int_0^\infty tP(t)dt.$$

More specifically, we assume the following conditions (A_1) through (A_4) throughout this section, and show that Eq. (PE) possesses a saddle-node bifurcation structure of equilibria as well as pitchfork one when the parameter λ varies in a neighborhood of 1.

(A_1) 1 is a simple eigenvalue of P_0;

(A_2) $\mathcal{R}(E_m - P_0) \oplus P_1(\mathcal{N}(E_m - P_0)) = \mathbb{R}^m$;

(A_3) $\det \Delta(\sigma + i\omega) \ne 0$ for $(\sigma,\omega) \in \mathbb{R}^+ \times \mathbb{R}^+ \setminus \{(0,0)\}$;

(A$_4$) $f : X \to \mathbb{R}^m$ is given by

$$f(\phi) = Q(\phi) + g(\phi), \quad \phi \in X, \tag{4.2}$$

where $Q : X \to \mathbb{R}^m$ is defined by

$$Q(\phi) = Q_*(\phi, \phi, \ldots, \phi), \quad \phi \in X \tag{4.3}$$

with Q_* a bounded n-linear map from X^n to $\mathbb{R}^m$, and $g \in C^1(X; \mathbb{R}^m)$ satisfies

$$g(\phi) = o(\|\phi\|_X^n) \quad \text{and} \quad Dg(\phi) = o(\|\phi\|_X^{n-1}) \quad \text{as } \phi \to 0 \text{ in } X \tag{4.4}$$

for some integer n greater than 1.

Remark 4.1. The assumption (A$_2$) is equivalent to $\mathcal{R}(E_m - P_0) + P_1\big(\mathcal{N}(E_m - P_0)\big) = \mathbb{R}^m$ because of $\dim \mathcal{N}(E_m - P_0) + \dim \mathcal{R}(E_m - P_0) = m$.

Remark 4.2. (A$_3$) holds for instance under the following conditions:

(a) $\int_0^\infty \|P(t)\| dt \le 1$;

(b) $\det \Delta(i\omega) \neq 0$ for $\omega > 0$.

Indeed, (a) implies

$$\|E_m - \Delta(\sigma + i\omega)\| = \Big\| \int_0^\infty P(t) e^{-(\sigma + i\omega)t} dt \Big\| \le \int_0^\infty \|P(t)\| e^{-\sigma t} dt < 1$$

for $\sigma > 0$; hence $\Delta(\sigma + i\omega)$ is invertible for $\sigma > 0$. Thus, (A$_3$) follows from (a) and (b).

Remark 4.3. A typical example of Q_* in (A$_4$) is of the form

$$Q_*(\phi_1, \phi_2, \ldots, \phi_n) = \sum_{j=1}^{l} \Big(\prod_{k=1}^{n} \int_{-\infty}^{0} q_k^{(j)}(-\theta) \phi_k(\theta) d\theta \Big) v_j,$$

with $\phi_k \in X$ ($k = 1, 2, \ldots, n$), where each $q_k^{(j)}$ is a measurable $\mathbb{R}^{m*}$-valued function on $\mathbb{R}^+$ that satisfies

$$\int_0^\infty |q_k^{(j)}(t)| e^{\rho t} dt < \infty \quad \text{and} \quad \operatorname{ess\,sup}\{|q_k^{(j)}(t)| e^{\rho t} : t \ge 0\} < \infty$$

($k = 1, 2, \ldots, n$) and $v_j \in \mathbb{R}^m$ ($j = 1, 2, \ldots, l$).

Now let η_* be an eigenvector of P_0 associated with eigenvalue 1 and ζ^* an eigenvector of P_0^*, the adjoint of P_0; and put

$$q_* := -\frac{\langle \zeta^*, P_1 \eta_* \rangle}{\langle \zeta^*, \eta_* \rangle}, \quad c_0 := \langle \zeta^*, Q(w_1(0)\eta_*) \rangle. \tag{4.5}$$

Note that $\langle \zeta^*, \eta_* \rangle \neq 0$, and q_* does not depend on the choice of η_* and ζ^* since both $\mathcal{N}(E_m - P_0)$ and $\mathcal{N}(E_m - P_0^*)$ are one-dimensional spaces due to (A$_1$).

Our main theorem on bifurcation structure is as follows:

Theorem 4.4. *Suppose that the assumptions* (A$_1$) *through* (A$_4$) *are satisfied. Suppose furthermore that* $c_0 \neq 0$. *Then, the following statements hold:*

(i) *Let n be even.*

(ia) *If* $\lambda = 1$, *then there exists an open neighborhood* $\mathcal{W}_0$ *of* 0 *in* X *such that Eq.* (*PE*) *has no equilibria in* $\mathcal{W}_0$ *other than* 0. *The equilibrium* 0 *is also unstable. Moreover, there exist an* $\varepsilon^* > 0$ *and a continuous map* $\phi_* : (1-\varepsilon^*, 1+\varepsilon^*)\backslash\{1\} \to X\backslash\{0\}$ *such that for each* λ, ϕ_*^λ *is an equilibrium of Eq.* (*PE*), *and* $\phi_*^\lambda \to 0$ *as* $\lambda \to 1$. *(Here we used the notation* ϕ_*^λ *rather than* $\phi_*(\lambda)$ *for* $\lambda \in (1-\varepsilon^*, 1+\varepsilon^*)$.*)*

(ib) *If* $q_* < 0$, *then the equilibria* ϕ_*^λ *and* 0 *are asymptotically stable (resp. unstable) in case* $\lambda < 1$ *(resp.* $\lambda > 1$*); and if* $q_* > 0$, *then* ϕ_*^λ *and* 0 *are unstable (resp. asymptotically stable) in case* $\lambda < 1$ *(resp.* $\lambda > 1$*).*

(ic) *There exists an open neighborhood* $\mathcal{W}$ *of* 0 *in* X *such that if* $\phi_* \in \mathcal{W}$ *is an equilibrium of Eq.* (*PE*) *with* $|\lambda - 1|$ *small, then* $\phi_* = 0$ *or* $\phi_* = \phi_*^\lambda$.

(ii) *Let* n *be odd and* c_0 *positive.*

(iia) *If* $\lambda = 1$, *then there exists an open neighborhood* $\mathcal{W}_0$ *of* 0 *in* X *such that Eq.* (*PE*) *has no equilibria in* $\mathcal{W}_0$ *other than* 0. *The equilibrium* 0 *is also unstable. Moreover, there exist an* $\varepsilon^* > 0$ *and continuous maps* $\phi_\pm : (1, 1+\varepsilon^*) \to \mathcal{W}_0\backslash\{0\}$ *such that for each* λ, ϕ_+^λ *and* ϕ_-^λ *are two distinct equilibria of Eq.* (*PE*), *and* $\phi_\pm^\lambda \to 0$ *as* $\lambda \to 1+0$.

(iib) *If* $q_* < 0$, *then the equilibria* ϕ_+^λ *and* ϕ_-^λ *are unstable for* $\lambda > 1$, *whereas* 0 *is asymptotically stable for* $\lambda > 1$; *and if* $q_* > 0$, *then* ϕ_+^λ *and* ϕ_-^λ *are asymptotically stable and* 0 *is unstable for* $\lambda > 1$.

(iic) *There exists an open neighborhood* $\mathcal{W}$ *of* 0 *in* X *such that if* $\phi_* \in \mathcal{W}$ *is an equilibrium of Eq.* (*PE*) *with* $\lambda - 1 > 0$ *small, then* ϕ^* *coincides with one of* 0, ϕ_+^λ *and* ϕ_-^λ.

(iii) *Let* n *be odd and* c_0 *negative.*

(iiia) *If* $\lambda = 1$, *then there exists an open neighborhood* $\mathcal{W}_0$ *of* 0 *in* X *such that Eq.* (*PE*) *has no equilibria in* $\mathcal{W}_0$ *other than* 0. *The equilibrium* 0 *is also asymptotically stable. Moreover, there exist an* $\varepsilon^* > 0$ *and continuous maps* $\phi_\pm : (1-\varepsilon^*, 1) \to \mathcal{W}_0\backslash\{0\}$ *such that for each* λ, ϕ_+^λ *and* ϕ_-^λ *are two distinct equilibria of Eq.* (*PE*), *and* $\phi_\pm^\lambda \to 0$ *as* $\lambda \to 1-0$.

(iiib) *If* $q_* < 0$, *then the equilibria* ϕ_+^λ *and* ϕ_-^λ *are asymptotically stable for* $\lambda < 1$, *whereas* 0 *is unstable for* $\lambda < 1$; *and if* $q_* > 0$, *then* ϕ_+^λ *and* ϕ_-^λ *are unstable and* 0 *is asymptotically stable for* $\lambda < 1$.

(iiic) *There exists an open neighborhood* $\mathcal{W}$ *of* 0 *in* X *such that if* $\phi_* \in \mathcal{W}$ *is an equilibrium of Eq.* (*PE*) *with* $1 - \lambda > 0$ *small, then* ϕ^* *coincides with one of* 0, ϕ_+^λ *and* ϕ_-^λ.

4.2 Extended system and the projection onto its center subspace

For the convenience, put $\lambda := 1 + \varepsilon$. Then, Eq. (*PE*) becomes

$$x(t) = \int_{-\infty}^{t} P(t-s)x(s)ds + \varepsilon \int_{-\infty}^{t} P(t-s)x(s)ds + f(x_t). \tag{4.6}$$

For the proof of Theorem 4.4, we will treat an extended system of (4.6), and apply Theorem 3.2. To do so, consider the system of integral equations

$$\begin{aligned} \varepsilon(t) &= \int_{-\infty}^{t} p_0(t-s)\varepsilon(s)ds, \\ x(t) &= \int_{-\infty}^{t} P(t-s)x(s)ds + \left(\int_{-\infty}^{t} p_0(t-s)\varepsilon(s)ds\right)\left(\int_{-\infty}^{t} P(t-s)x(s)ds\right) + f(x_t), \end{aligned} \tag{4.7}$$

where $p_0(t)$ is the function on $\mathbb{R}^+$ defined by

$$p_0(t) := \rho_0 e^{-\rho_0 t}, \quad t \in \mathbb{R}^+,$$

and ρ_0 is a positive number satisfying $\rho_0 > \rho$. Note that

$$\int_0^\infty p_0(t)dt = 1 \tag{4.8}$$

and that

$$\int_0^\infty |p_0(t)|e^{-\rho t}dt < \infty \quad \text{and} \quad \text{ess sup}\{|p_0(t)|e^{\rho t} : t \geq 0\} < \infty.$$

Let us denote

$$\widetilde{P}(t) := \text{diag}\,(p_0(t), P(t)) = \begin{pmatrix} p_0(t) & 0 \\ 0 & P(t) \end{pmatrix}, \quad t \in \mathbb{R}^+.$$

Set $\widetilde{X} := L^1_\rho(\mathbb{R}^-; \mathbb{R}^{m+1})$ and $X_1 := L^1_\rho(\mathbb{R}^-; \mathbb{R})$ (see Subsection 2.1); then $\widetilde{X}$ can be naturally identified with $X_1 \times X$. We denote by Π_1 and Π_2 the projections from $\widetilde{X}$ to X_1 and X, respectively. Moreover, define $\widetilde{G} : \widetilde{X} \to \mathbb{R}^{m+1}$ by

$$\widetilde{G}(\tilde{\phi}) := \text{col}\,(0, G^{(2)}(\tilde{\phi})), \quad \tilde{\phi} \in \widetilde{X},$$

where $G^{(2)}$ is an $\mathbb{R}^m$-valued function given by

$$G^{(2)}(\tilde{\phi}) := \left(\int_{-\infty}^0 p_0(-\theta)\phi^{(1)}(\theta)d\theta \right) \left(\int_{-\infty}^0 P(-\theta)\phi^{(2)}(\theta)d\theta \right) + f(\phi^{(2)}), \tag{4.9}$$

that is, $G^{(2)}(\tilde{\phi}) = \Pi_2 \widetilde{G}(\tilde{\phi})$. Here we used the notation

$$\tilde{\phi} = \text{col}\,(\phi^{(1)}, \phi^{(2)}) \quad \text{with } \phi^{(1)} \in X_1 \text{ and } \phi^{(2)} \in X. \tag{4.10}$$

By letting $\tilde{x}(t) := \text{col}\,(\varepsilon(t), x(t))$, Eq. (4.7) can then be rewritten as

$$\tilde{x}(t) = \int_{-\infty}^t \widetilde{P}(t-s)\tilde{x}(s)ds + \widetilde{G}(\tilde{x}_t). \tag{$\widetilde{E}$}$$

In connection with Eq. ($\widetilde{E}$) we will also consider the integral equation

$$\tilde{x}(t) = \int_{-\infty}^t \widetilde{P}(t-s)\tilde{x}(s)ds + \widetilde{G}_\delta(\tilde{x}_t), \tag{$\widetilde{E}_\delta$}$$

where $\widetilde{G}_\delta : \widetilde{X} \to \mathbb{R}^{m+1}$ is defined by

$$\widetilde{G}_\delta(\tilde{\phi}) := \chi(\|\Pi^s\tilde{\phi}\|_{\widetilde{X}}/\delta)\chi(\|\Pi^c\tilde{\phi}\|_{\widetilde{X}}/\delta)G(\tilde{\phi}), \quad \tilde{\phi} \in \widetilde{X}$$

(see Subsection 2.3). By (4.2), (4.3) and (4.4), it is clear that $\widetilde{G}$ belongs to $C^1(\widetilde{X}; \mathbb{R}^{m+1})$ with $\widetilde{G}(0) = 0$ and $D\widetilde{G}(0) = 0$. Hence Theorem 2.4 implies that there exists the (global) center manifold of the equilibrium 0 of Eq. ($\widetilde{E}_\delta$), denoted $\widetilde{W}^c_\delta(0)$, which is given as the graph of some C^1 map $\widetilde{F}_{*,\delta} : \widetilde{E}^c \to \widetilde{E}^s$:

$$\widetilde{W}^c_\delta(0) := \{\psi + \widetilde{F}_{*,\delta}(\psi) : \psi \in \widetilde{E}^c\}. \tag{4.11}$$

Recall that $\widetilde{F}_{*,\delta}$ is defined by

$$\widetilde{F}_{*,\delta}(\psi) = \lim_{n\to\infty} \int_{-\infty}^0 \widetilde{T}^s(-s)\Pi^s\Gamma^n\widetilde{G}_\delta(\Lambda_{*,\delta}(\psi)(s))ds, \quad \psi \in \widetilde{E}^c, \tag{4.12}$$

where $\{\widetilde{T}(t)\}_{t\geq 0}$ is the solution semigroup of the integral equation

$$\widetilde{x}(t) = \int_{-\infty}^{t} \widetilde{P}(t-s)\widetilde{x}(s)ds \tag{4.13}$$

(see (2.6)). Let $\widetilde{F}_* : B_{\widetilde{E}^c}(\delta) \to \widetilde{X}$ be the restriction of $\widetilde{F}_{*,\delta}$ to $B_{\widetilde{E}^c}(\delta)$. The local center manifold $\widetilde{W}^c_{\mathrm{loc}}(0)$ of the equilibrium 0 of Eq. ($\widetilde{E}$) is then given by $\widetilde{W}^c_{\mathrm{loc}}(0) := \operatorname{graph}\widetilde{F}_* \subset \widetilde{W}^c_\delta(0)$ (see [23, Theorem 5]). Henceforth, we set

$$\widetilde{\Omega} := \{\tilde{\phi} \in \widetilde{X} : \|\Pi^c\tilde{\phi}\|_{\widetilde{X}} < \delta,\ \|\Pi^s\tilde{\phi}\|_{\widetilde{X}} < \delta\},$$

and

$$\widetilde{\Omega}_0 := \{\tilde{\phi} \in \widetilde{X} : \|\Pi^c\tilde{\phi}\|_{\widetilde{X}} < \delta/3,\ \|\Pi^s\tilde{\phi}\|_{\widetilde{X}} < \delta/3\}.$$

Proposition 4.5. *Given any $t_0 \in \mathbb{R}$ and $\tilde{\phi} \in \widetilde{X}$, let $\widetilde{x}(t; t_0, \tilde{\phi}, \widetilde{G}_\delta)$ be the solution of Eq. ($\widetilde{E}_\delta$) with $\widetilde{x}_{t_0} = \tilde{\phi}$; and $\widetilde{x}(t; t_0, \tilde{\phi}, \widetilde{G})$ the one of Eq. ($\widetilde{E}$) with $\widetilde{x}_{t_0} = \tilde{\phi}$. Let $\varepsilon \in \mathbb{R}$ and $j_\varepsilon : X \to \widetilde{X}$ be the map defined by*

$$j_\varepsilon(\phi) := \operatorname{col}\big(w_1(0)\varepsilon, \phi\big), \quad \phi \in X \tag{4.14}$$

where $w_1(0)$ is the function defined in (2.8) (see Subsection 2.4). Then we have

(i) *Given any $\varepsilon \in \mathbb{R}$ and $\phi \in X$, $\widetilde{x}_t(t_0, j_\varepsilon(\phi), \widetilde{G}_\delta) \in j_\varepsilon(X)$ holds for $t \geq t_0$.*

(ii) *Let $x(t; t_0, \phi, f)$ be the solution on $J = [t_0, t_1]$ of Eq. (PE) satisfying $x_{t_0} = \phi \in X$. Then*

$$\widetilde{x}(t; t_0, j_\varepsilon(\phi), \widetilde{G}) = \operatorname{col}\big(\varepsilon, x(t; t_0, \phi, f)\big), \quad t \in J;$$

so that $\widetilde{x}_t(t_0, j_\varepsilon(\phi), \widetilde{G}) = j_\varepsilon(x_t(t_0, \phi, f))$ holds for every $t \in J$ and $\varepsilon \in \mathbb{R}$.

In particular, $j_\varepsilon(X)$ is positively invariant for both Eq. ($\widetilde{E}_\delta$) and Eq. ($\widetilde{E}$).

Proof. (i) Set $\chi_0(\tilde{\phi}) := \chi\big(\|\Pi^s\tilde{\phi}\|_{\widetilde{X}}/\delta\big)\chi\big(\|\Pi^c\tilde{\phi}\|_{\widetilde{X}}/\delta\big)$ for $\tilde{\phi} \in \widetilde{X}$ and let $u(t)$ be the solution of

$$u(t) = \big(1 + \varepsilon\chi_0(j_\varepsilon(u_t))\big)\int_{-\infty}^{t} P(t-s)u(s)ds + \chi_0(j_\varepsilon(u_t))f(u_t)$$

with $u_{t_0} = \phi$. The existence and uniqueness of $u(t)$ is due to [21, Proposition 3]. Define $\widetilde{x} : \mathbb{R} \to \widetilde{X}$ by

$$\widetilde{x}(t) = \operatorname{col}\big(x^{(1)}(t), x^{(2)}(t)\big) := \operatorname{col}\big(\varepsilon, u(t)\big) \in X_1 \times X, \quad t \in \mathbb{R}.$$

Then, it readily follows from (4.8) that

$$x^{(1)}(t) = \int_{-\infty}^{t} p_0(t-s)x^{(1)}(s)ds.$$

Besides, since $\widetilde{x}_t = j_\varepsilon(u_t)$ $(t \geq t_0)$, $x^{(2)}(t)$ satisfies

$$x^{(2)}(t) = \int_{-\infty}^{t} P(t-s)x^{(2)}(s)ds + \chi_0(\widetilde{x}_t)\left(\int_{-\infty}^{t} p_0(t-s)\varepsilon ds\right)\left(\int_{-\infty}^{t} P(t-s)x^{(2)}(s)ds\right) + \chi_0(\widetilde{x}_t)f\big(x^{(2)}_t\big)$$

for $t \geq t_0$. Therefore $\widetilde{x}(t)$ is the solution of Eq. ($\widetilde{E}_\delta$) with $\widetilde{x}_{t_0} = j_\varepsilon(u_{t_0}) = j_\varepsilon(\phi)$. Thus, we obtain $\widetilde{x}(t) = \widetilde{x}(t; t_0, j_\varepsilon(\phi), \widetilde{G}_\delta)$; hence

$$\widetilde{x}_t(t_0, j_\varepsilon(\phi), \widetilde{G}_\delta) = \widetilde{x}_t = j_\varepsilon(u_t) \in j_\varepsilon(X) \quad \text{for } t \geq t_0.$$

(ii) One can easily see that the argument above is valid with $\chi_0 = 1$, $\widetilde{G}_\delta = \widetilde{G}$ and $u(t) = x(t; t_0, \phi, f)$. Hence, it follows that $\tilde{x}(t; t_0, j_\varepsilon(\phi), \widetilde{G}) = \mathrm{col}\,(\varepsilon, x(t; t_0, \phi, f))$. So,

$$\tilde{x}_t(t_0, j_\varepsilon(\phi), \widetilde{G}) = \mathrm{col}\,(w_1(0)\varepsilon, x_t(t_0, \phi, f)) = j_\varepsilon(x_t(t_0, \phi, f)) \in j_\varepsilon(X)$$

for $t \in J$. Thus, $j_\varepsilon(X)$ is positively invariant for Eq. ($\widetilde{E}_\delta$) and Eq. ($\widetilde{E}$). □

The bilinear form, induced in the formal adjoint theory (Subsection 2.4; see also [22]) associated with Eq. (4.13), the linear part of Eq. ($\widetilde{E}$), is given by

$$\langle\!\langle \widetilde{\psi}, \widetilde{\phi} \rangle\!\rangle := \int_{-\infty}^{0} \Big(\int_{\theta}^{0} \widetilde{\psi}(\xi - \theta)\widetilde{P}(-\theta)\widetilde{\phi}(\xi) d\xi \Big) d\theta, \quad \widetilde{\phi} \in \widetilde{X}, \ \widetilde{\psi} \in \widetilde{X}^\sharp, \tag{4.15}$$

where $\widetilde{X}^\sharp := L^1_\rho(\mathbb{R}^+; \mathbb{R}^{m+1})$. Using the notations (4.10) and $\widetilde{\psi} = (\psi^{(1)}, \ \psi^{(2)})$ with $\psi^{(1)} \in X_1^\sharp := L^1_\rho(\mathbb{R}^+; \mathbb{R})$ and $\psi^{(2)} \in X^\sharp := L^1_\rho(\mathbb{R}^+; \mathbb{R}^m)$, we have

$$\widetilde{\psi}(\xi - \theta)\widetilde{P}(-\theta)\widetilde{\phi}(\eta) = (\psi^{(1)}(\xi - \theta), \ \psi^{(2)}(\xi - \theta)) \begin{pmatrix} p_0(-\theta) & 0 \\ 0 & P(-\theta) \end{pmatrix} \begin{pmatrix} \phi^{(1)}(\theta) \\ \phi^{(2)}(\theta) \end{pmatrix};$$

hence (4.15) becomes

$$\langle\!\langle \widetilde{\psi}, \widetilde{\phi} \rangle\!\rangle = \int_{-\infty}^{0} \Big(\int_{\theta}^{0} (\psi^{(1)}(\xi - \theta)p_0(-\theta)\phi^{(1)}(\xi) + \psi^{(2)}(\xi - \theta)P(-\theta)\phi^{(2)}(\xi)) d\xi \Big) d\theta. \tag{4.16}$$

The characteristic operator of Eq. (4.13) is

$$\begin{aligned} \widetilde{\Delta}(z) &= E_{m+1} - \int_0^\infty \widetilde{P}(t)e^{-zt}dt \\ &= \begin{pmatrix} 1 - \int_0^\infty p_0(t)e^{-zt}dt & 0 \\ 0 & E_m - \int_0^\infty P(t)e^{-zt}dt \end{pmatrix} = \begin{pmatrix} z/(z+\rho_0) & 0 \\ 0 & \Delta(z) \end{pmatrix}. \end{aligned}$$

So, it follows that

$$\det \widetilde{\Delta}(z) = \frac{z}{z + \rho_0} \det \Delta(z),$$

and in view of $\Delta(0) = E_m - P_0$ and (A_1), 0 is a characteristic root of Eq. (4.13) whose order as a zero of $\det \widetilde{\Delta}(z)$ is at least greater than 1 (Recall that $\det \Delta(z)$ is analytic in the domain $\mathbb{C}_{-\rho}$).

Proposition 4.6. $\det \Delta(z)$ *has* 0 *as a zero of order* 1 *(i.e.,* $\det \widetilde{\Delta}(z)$ *has* 0 *as the one of order* 2*) if and only if* (A_1) *and* (A_2) *hold.*

Proof. Let us denote $\Delta(0) = E_m - P_0 = (a_1 \, a_2 \, \cdots \, a_m)$ and $P_1 = (p_1 \, p_2 \, \cdots \, p_m)$ with $a_j, p_j \in \mathbb{R}^m$ $(j = 1, 2, \ldots, m)$. By virtue of the first inequality of (4.1), we have

$$\Delta^{(k)}(z) = -\frac{d^k}{dz^k} \int_0^\infty P(t)e^{-zt}dt = (-1)^{k-1} \int_0^\infty t^k P(t)e^{-zt}dt, \quad z \in \mathbb{C}_{-\rho},$$

$(k = 1, 2, \ldots)$ and hence $\Delta(z)$ is expressed as

$$\Delta(z) = \Delta(0) + z\Delta'(0) + o(z) = E_m - P_0 + zP_1 + o(z).$$

Put $d(z) := \det \Delta(z)$. Then, $d(0) = 0$ and

$$\begin{aligned} d'(0) &= \frac{d}{dz}\Big|_{z=0} \det \big(a_1 + zp_1 + o(z) \quad a_2 + zp_2 + o(z) \quad \cdots \quad a_m + zp_m + o(z) \big) \\ &= \sum_{j=1}^{m} \det \big(a_1 \ \cdots \ \underset{\widehat{j}}{p_j} \ \cdots \ a_m \big). \end{aligned}$$

Suppose that $\det\Delta(z)$ has 0 as a zero of order 1, that is, $d'(0) \neq 0$. Let $k := \dim \mathcal{R}(E_m - P_0)$ and assume that $k \leq m-2$. Without loss of generality, we may assume that $\mathcal{R}(E_m - P_0) = \operatorname{span}\{a_1, a_2, \dots, a_k\}$. Since $\operatorname{span}\{a_1, \dots, a_{j-1}, p_j, a_{j+1}, \dots, a_m\} = \operatorname{span}\{a_1, \dots, a_k, p_j\}$, we have $\dim\operatorname{span}\{a_1, \dots, a_{j-1}, p_j, a_{j+1}, \dots, a_m\} \leq k+1 \leq m-1$ so that

$$\det\begin{pmatrix} a_1 & \cdots & \underset{\widehat{j}}{p_j} & \cdots & a_m \end{pmatrix} = 0, \quad j = 1, 2, \dots, m.$$

Hence, we get $d'(0) = 0$, contradicting to our assumption. Thus, we obtain $k = m-1$, and therefore (A_1) holds. Moreover, in this case, we may assume that a_m can be written as $a_m = \sum_{k=1}^{m-1} c_k a_k$ with some $c_k \in \mathbb{R}$ $(k = 1, 2, \dots, m-1)$. Then,

$$\begin{aligned} d'(0) &= \sum_{j=1}^{m}\sum_{k=1}^{m-1} \det\begin{pmatrix} a_1 & \cdots & \underset{\widehat{j}}{p_j} & \cdots & c_k a_k \end{pmatrix} \\ &= \det\begin{pmatrix} a_1 & \cdots & a_{m-1} & p_m \end{pmatrix} + \sum_{j=1}^{m-1} \det\begin{pmatrix} a_1 & \cdots & \underset{\widehat{j}}{p_j} & \cdots & a_{m-1} & c_j a_j \end{pmatrix} \\ &= \det\begin{pmatrix} a_1 & \cdots & a_{m-1} & p_m \end{pmatrix} + \sum_{j=1}^{m-1} \det\begin{pmatrix} a_1 & \cdots & \underset{\widehat{j}}{a_j} & \cdots & a_{m-1} & -c_j p_j \end{pmatrix} \\ &= \det\left(a_1 \;\; a_2 \;\; \cdots \;\; a_{m-1} \;\; p_m - \sum_{j=1}^{m-1} c_j p_j \right). \end{aligned}$$

Since $\operatorname{col}(c_1, \dots, c_{m-1}, -1) \in \mathcal{N}(E_m - P_0)$, we see $p_m - \sum_{j=1}^{m-1} c_j p_j$ belongs to $P_1(\mathcal{N}(E_m - P_0))$. Consequently, it follows from $d'(0) \neq 0$ that (A_2) is also valid.

Conversely, suppose that (A_1) and (A_2) hold. Then $d'(0) \neq 0$ is clear from the argument above. □

Note that Eq. (4.13) has no characteristic roots with positive real parts. Indeed, (A_3) implies $\det\Delta(z) \neq 0$ for $\operatorname{Re} z > 0$ since $\Delta(z) = \overline{\Delta(\bar{z})}$. Thus, $\Sigma^u = \emptyset$, and therefore we have

$$\widetilde{X} = \widetilde{E}^c \oplus \widetilde{E}^s, \tag{4.17}$$

where $\widetilde{E}^c$ and $\widetilde{E}^s$ are the center subspace and the stable subspace of the equilibrium 0 for Eq. (4.13), respectively. Also, notice from (A_3) that $\Sigma^c = \{0\}$.

By (2.7) of Subsection 2.4, $\widetilde{E}^c$ is given by the generalized eigenspace of $\widetilde{A}$ associated with eigenvalue 0:

$$\widetilde{E}^c = \mathcal{M}_0(\widetilde{A}) = \bigcup_{k\geq 1} \mathcal{N}(\widetilde{A}^k),$$

where $\widetilde{A}$ is the generator of the solution semigroup $\{\widetilde{T}(t)\}_{t\geq 0}$ of Eq. (4.13).

Proposition 4.7. *We have $\mathcal{M}_0(\widetilde{A}) = \mathcal{N}(\widetilde{A}) = \operatorname{span}\{w_1(0)e_1, w_1(0)\tilde{\eta}_*\}$, where e_1 and $\tilde{\eta}_*$ are the elements of $\mathbb{R}^{m+1}$ given by $e_1 := \operatorname{col}(1, 0, \dots, 0)$, $\tilde{\eta}_* = \operatorname{col}(0, \eta_*)$, η_* being an eigenvector of P_0 corresponding to eigenvalue 1, and $w_1(0)$ is the one given by (2.8).*

Proof. By virtue of Proposition 2.6 we know that $\tilde{\phi} \in \mathcal{N}(\widetilde{A})$ if and only if $\tilde{\phi} = w_1(0)\eta_1$ for some $\tilde{\eta}_1 \in \mathbb{R}^{m+1}$ with $\widetilde{\Delta}(0)\tilde{\eta}_1 = 0$. Since

$$D_1(0) = \widetilde{\Delta}(0) = \begin{pmatrix} 0 & 0 \\ 0 & E_m - P_0 \end{pmatrix}$$

(see ((2.9)), one can see from (A_1) that $\mathcal{N}(\widetilde{A}) = \text{span}\,\{w_1(0)e_1,\, w_1(0)\tilde{\eta}_*\}$.

Recall the fact that

$$\mathcal{N}(\widetilde{A}^2) = \left\{ w_1(0)\tilde{\eta}_1 + w_2(0)\tilde{\eta}_2 : D_2(0)\begin{pmatrix}\tilde{\eta}_1\\ \tilde{\eta}_2\end{pmatrix} = 0,\ \tilde{\eta}_1, \tilde{\eta}_2 \in \mathbb{R}^{m+1} \right\},$$

where $D_2(0)$ is the matrix defined by

$$D_2(0) = \begin{pmatrix}\widetilde{\Delta}(0) & \widetilde{\Delta}'(0)\\ 0 & \widetilde{\Delta}(0)\end{pmatrix}$$

(see (2.9)). So, $w_1(0)\tilde{\eta}_1 + w_2(0)\tilde{\eta}_2 \in \mathcal{N}(\widetilde{A}^2)$ is equivalent to

$$\widetilde{\Delta}(0)\tilde{\eta}_1 + \widetilde{\Delta}'(0)\tilde{\eta}_2 = 0 \quad \text{and} \quad \widetilde{\Delta}(0)\tilde{\eta}_2 = 0. \tag{4.18}$$

It is also easily seen that

$$\widetilde{\Delta}'(z) = \begin{pmatrix}\rho_0/(z+\rho_0)^2 & 0\\ 0 & \int_0^\infty tP(t)e^{-zt}dt\end{pmatrix}; \quad \text{hence} \quad \widetilde{\Delta}'(0) = \begin{pmatrix}1/\rho_0 & 0\\ 0 & P_1\end{pmatrix}.$$

(4.18) implies $\widetilde{\Delta}'(0)\tilde{\eta}_2 \in \widetilde{\Delta}'(0)\big(\mathcal{N}(\widetilde{\Delta}(0))\big)$, so that $\widetilde{\Delta}(0)\tilde{\eta}_1 = 0$ and $\widetilde{\Delta}'(0)\tilde{\eta}_2 = 0$ because

$$\mathcal{R}(\widetilde{\Delta}(0)) \oplus \widetilde{\Delta}'(0)\big(\mathcal{N}(\widetilde{\Delta}(0))\big) = \mathbb{R}^{m+1} \tag{4.19}$$

follows from (A_3) immediately. In particular, $\tilde{\eta}_2 = 0$. Indeed, assume that $\tilde{\eta}_2 \neq 0$. Since $\tilde{\eta}_2$ belongs to $\mathcal{N}(\widetilde{\Delta}(0))$, which is a two-dimensional subspace spanned by e_1 and $\tilde{\eta}_*$, the fact that $\widetilde{\Delta}'(0)\tilde{\eta}_2 = 0$ means $\dim \widetilde{\Delta}'(0)\big(\mathcal{N}(\widetilde{\Delta}(0))\big) \leq 1$. On the other hand, in view of (A_1), $\dim \mathcal{R}(\widetilde{\Delta}(0)) = \dim \mathcal{R}(\Delta(0)) = m-1$, which contradicts to (4.19). Hence, we obtain $\tilde{\eta}_2 = 0$, and therefore (4.18) is equivalent to $\widetilde{\Delta}(0)\tilde{\eta}_1 = 0$ and $\tilde{\eta}_2 = 0$. Consequently, $\mathcal{N}(\widetilde{A}^2) = \{w_1(0)\tilde{\eta}_1 : \widetilde{\Delta}(0)\tilde{\eta}_1 = 0,\ \tilde{\eta}_1 \in \mathbb{R}^{m+1}\} = \mathcal{N}(\widetilde{A})$ and the proof is completed. □

Remark 4.8. By virtue of Proposition 4.6, together with [22, Cororally 3.1], one may also conclude that $\dim \mathcal{M}_0(\widetilde{A}) = 2$ under the assumptions (A_1) and (A_3).

Similarly, the generalized eigenspace of the formal adjoint operator $\widetilde{A}^\sharp$ is identical with its eigenspace.

Proposition 4.9. *We have* $\mathcal{N}((\widetilde{A}^\sharp)^k) = \mathcal{N}(\widetilde{A}^\sharp)$ $(k = 2, 3, \dots)$ *and*

$$\mathcal{N}(\widetilde{A}^\sharp) = \text{span}\,\{w_1^\sharp(0)e_1^*,\, w_1^\sharp(0)\tilde{\zeta}^*\},$$

where e_1^* *and* $\tilde{\zeta}^*$ *are the elements of* $\mathbb{R}^{m+1*}$ *given by* $e_1^* := (1, 0, \dots, 0)$, $\tilde{\zeta}^* = (0, \zeta^*)$ *being an eigenvector of* P_0^* *associated with eigenvalue* 1, *that is,* $\zeta^* \in \mathcal{N}(\Delta(0)^*)\backslash\{0\}$, *and* $w_1^\sharp(0)$ *is the one given in* (2.10). *Here* $\Delta(0)^*$ *and* P_0^* *denote the adjoint operators of* $\Delta(0)$ *and* P_0, *respectively.*

Proof. Recall from Proposition 2.7 that $\mathcal{N}(\widetilde{A}^\sharp) = \text{span}\,\{w_1^\sharp(0)\tilde{\zeta} : \tilde{\zeta}D_1(0) = 0,\ \tilde{\zeta} \in \mathbb{R}^{m+1*}\}$ and

$$\mathcal{N}((\widetilde{A}^\sharp)^2) = \{w_1^\sharp(0)\tilde{\zeta}_2 + w_2^\sharp(0)\tilde{\zeta}_1 : (\tilde{\zeta}_1, \tilde{\zeta}_2)D_2(0) = 0,\ \tilde{\zeta}_1, \tilde{\zeta}_2 \in \mathbb{R}^{m+1*}\}.$$

One can readily verify that $\mathcal{N}(\widetilde{A}^\sharp) = \text{span}\,\{w_1^\sharp(0)e_1^*,\, w_1^\sharp(0)\tilde{\zeta}_*\}$. Also, $(\tilde{\zeta}_1, \tilde{\zeta}_2)D_2(0) = 0$ implies

$$\tilde{\zeta}_1\widetilde{\Delta}(0) = 0 \quad \text{and} \quad \tilde{\zeta}_1\widetilde{\Delta}'(0) + \tilde{\zeta}_2\widetilde{\Delta}(0) = 0. \tag{4.20}$$

Hence, $\tilde{\zeta}_1$ vanishes on the subspace $\mathcal{R}(\widetilde{\Delta}(0))$ of $\mathbb{R}^{m+1}$ and on $\widetilde{\Delta}'(0)(\mathcal{N}(\widetilde{\Delta}(0)))$, as well. Therefore, by (4.19), we must have $\tilde{\zeta}_1 = 0$. Thus, (4.20) is equivalent to $\tilde{\zeta}_1 = 0$ and $\tilde{\zeta}_2\widetilde{\Delta}(0) = 0$, so that $\mathcal{N}((\widetilde{A}^\sharp)^2) = \{w_1^\sharp(0)\tilde{\zeta}_2 : \tilde{\zeta}_2\widetilde{\Delta}(0) = 0,\ \tilde{\zeta}_2 \in \mathbb{R}^{m+1*}\} = \mathcal{N}(\widetilde{A}^\sharp)$. This completes the proof. □

By Theorem 2.8 the projection $\Pi^c : \widetilde{X} \to \widetilde{E}^c$ along the decomposition (4.17) is given as follows:

Proposition 4.10.

$$\Pi^c\widetilde{\phi} = w_1(0)\begin{pmatrix} \rho_0 \int_{-\infty}^0 p_0(-\theta)\Big(\int_\theta^0 \phi^{(1)}(\xi)d\xi\Big)d\theta \\ (1/r)\int_{-\infty}^0 \zeta^* P(-\theta)\Big(\int_\theta^0 \phi^{(2)}(\xi)d\xi\Big)d\theta \cdot \eta_* \end{pmatrix}$$

for $\widetilde{\phi} = \mathrm{col}\,(\phi^{(1)}, \phi^{(2)}) \in \widetilde{X}$, where r denotes the constant $\langle \zeta^*, P_1\eta_*\rangle$.

Proof. We first verify $r \neq 0$. Suppose by contradiction that $r = 0$; then $\zeta^*|_{\Delta'(0)(\mathcal{N}(\Delta(0)))} = 0$ because of $\mathcal{N}(\Delta(0)) = \mathrm{span}\,\{\eta_*\}$. Note that $\zeta^* \in \mathcal{N}(\Delta(0)^*) = \mathcal{R}(\Delta(0))^\perp$, where for any subspace W of $\mathbb{R}^{m+1}$, $W^\perp$ stands for the annihilator of W. By the same reasoning as the proof of Proposition 4.9, it follows from (A_3) that $\zeta^* = 0$, contradicting to the fact that ζ^* is an eigenvector of P_0^* corresponding to eigenvalue 1.

We next consider the representation of Π^c. In virtue of (4.16), it is easy to see

$$\begin{aligned}
\langle\!\langle w_1^\sharp(0)e_1^*, w_1(0)e_1\rangle\!\rangle &= \int_{-\infty}^0 \Big(\int_\theta^0 p_0(-\theta)d\xi\Big)d\theta = \int_{-\infty}^0 (-\theta)p_0(-\theta)d\theta = \int_0^\infty tp_0(t)dt = \frac{1}{\rho_0}, \\
\langle\!\langle w_1^\sharp(0)e_1^*, w_1(0)\tilde{\eta}_*\rangle\!\rangle &= \langle\!\langle w_1^\sharp(0)\tilde{\zeta}^*, w_1(0)e_1\rangle\!\rangle = 0, \\
\langle\!\langle w_1^\sharp(0)\tilde{\zeta}^*, w_1(0)\tilde{\eta}_*\rangle\!\rangle &= \int_{-\infty}^0 \Big(\int_\theta^0 \zeta^* P(-\theta)\eta_* d\xi\Big)d\theta = \zeta^*\Big(\int_{-\infty}^0 (-\theta)P(-\theta)d\theta\Big)\eta_* \\
&= \zeta^*\Big(\int_0^\infty tP(t)dt\Big)\eta_* = \langle \zeta^*, P_1\eta_*\rangle = r.
\end{aligned}$$

Therefore, by letting

$$\Phi_c := (w_1(0)e_1, w_1(0)\eta_*), \quad \Psi_c := \begin{pmatrix} w_1^\sharp(0)e_1^* \\ w_1^\sharp(0)\zeta^* \end{pmatrix},$$

we have

$$\langle\!\langle \Phi_c, \Psi_c\rangle\!\rangle = \begin{pmatrix} 1/\rho_0 & 0 \\ 0 & r \end{pmatrix},$$

and so,

$$\langle\!\langle \Phi_c, \Psi_c\rangle\!\rangle^{-1}\langle\!\langle \Psi_c, \widetilde{\phi}\rangle\!\rangle = \begin{pmatrix} \rho_0 & 0 \\ 0 & 1/r \end{pmatrix}\begin{pmatrix} \langle\!\langle w_1^\sharp(0)e_1^*, \widetilde{\phi}\rangle\!\rangle \\ \langle\!\langle w_1^\sharp(0)\zeta^*, \widetilde{\phi}\rangle\!\rangle \end{pmatrix} = \begin{pmatrix} \langle\!\langle w_1^\sharp(0)(\rho_0 e_1^*), \widetilde{\phi}\rangle\!\rangle \\ \langle\!\langle w_1^\sharp(0)((1/r)\zeta^*), \widetilde{\phi}\rangle\!\rangle \end{pmatrix}.$$

Hence, Theorem 2.8 yields

$$\Pi^c\widetilde{\phi} = \Phi_c\langle\!\langle \Phi_c, \Psi_c\rangle\!\rangle^{-1}\langle\!\langle \Psi_c, \widetilde{\phi}\rangle\!\rangle = w_1(0)\begin{pmatrix} \langle\!\langle w_1^\sharp(0)(\rho_0 e_1^*), \widetilde{\phi}\rangle\!\rangle \\ \langle\!\langle w_1^\sharp(0)((1/r)\zeta^*), \widetilde{\phi}\rangle\!\rangle\eta_* \end{pmatrix}.$$

Since

$$\langle\!\langle w_1^\sharp(0)(\rho_0 e_1^*), \widetilde{\phi}\rangle\!\rangle = \int_{-\infty}^0 \Big(\int_\theta^0 \rho_0 p_0(-\theta)\phi^{(1)}(\xi)d\xi\Big)d\theta$$

and

$$\langle\!\langle w_1^\sharp(0)((1/r)\zeta^*), \widetilde{\phi}\rangle\!\rangle = \int_{-\infty}^0 \Big(\int_\theta^0 (1/r)\zeta^* P(-\theta)\phi^{(2)}(\xi)d\xi\Big)d\theta,$$

we have

$$\Pi^c\widetilde{\phi} = w_1(0)\begin{pmatrix} \rho_0\int_{-\infty}^0 p_0(-\theta)\Big(\int_\theta^0 \phi^{(1)}(\xi)d\xi\Big)d\theta \\ (1/r)\int_{-\infty}^0 \zeta^* P(-\theta)\Big(\int_\theta^0 \phi^{(2)}(\xi)d\xi\Big)d\theta\cdot\eta_* \end{pmatrix}.$$

This completes the proof. □

4.3 Central equation for the extended system

Now we shall derive the central equation for Eq. ($\widetilde{E}$). Let $\widetilde{x}(t) = \mathrm{col}\,(\varepsilon(t), x(t))$ be a solution of Eq. ($\widetilde{E}$). Then, by Proposition 4.10,

$$\Pi^c\widetilde{x}_t = \Pi^c\begin{pmatrix}\varepsilon_t\\ x_t\end{pmatrix} = w_1(0)\begin{pmatrix} \rho_0\int_{-\infty}^0 p_0(-\theta)\Big(\int_\theta^0 \varepsilon(t+\xi)d\xi\Big)d\theta \\ (1/r)\int_{-\infty}^0 \zeta_* P(-\theta)\Big(\int_\theta^0 x(t+\xi)d\xi\Big)d\theta\,\cdot\eta_* \end{pmatrix}.$$

Since

$$\begin{aligned}
&\int_{-\infty}^0 p_0(-\theta)\Big(\int_\theta^0 \varepsilon(t+\xi)d\xi\Big)d\theta \\
&\quad= \int_{-\infty}^0 p_0(-\theta)\Big(\int_{t+\theta}^t \varepsilon(s)ds\Big)d\theta = \int_{-\infty}^t p_0(t-\tau)\Big(\int_\tau^t \varepsilon(s)ds\Big)d\tau \\
&\quad= \int_{-\infty}^t \Big(\int_{-\infty}^s p_0(t-\tau)d\tau\Big)\varepsilon(s)ds = \int_{-\infty}^t \Big(\int_{t-s}^\infty p_0(w)dw\Big)\varepsilon(s)ds
\end{aligned}$$

and similarly

$$\int_{-\infty}^0 \zeta^* P(-\theta)\Big(\int_\theta^0 x(t+\xi)d\xi\Big)d\theta = \int_{-\infty}^t \Big(\int_{t-s}^\infty \zeta^* P(w)dw\Big)x(s)ds,$$

it follows that

$$\begin{aligned}
\Pi^c\widetilde{x}_t &= w_1(0)\begin{pmatrix} \rho_0\int_{-\infty}^t \Big(\int_{t-s}^\infty p_0(w)dw\Big)\varepsilon(s)ds \\ (1/r)\int_{-\infty}^t \Big(\int_{t-s}^\infty \zeta^* P(w)dw\Big)x(s)ds\cdot\eta_* \end{pmatrix} \\
&= \Phi_c\begin{pmatrix} \rho_0\int_{-\infty}^t \Big(\int_{t-s}^\infty p_0(w)dw\Big)\varepsilon(s)ds \\ (1/r)\int_{-\infty}^t \Big(\int_{t-s}^\infty \zeta^* P(w)dw\Big)x(s)ds \end{pmatrix}.
\end{aligned}$$

Hence, in view of $\Pi^c\widetilde{x}_t = \Phi_c z_c(t)$, we obtain

$$z_c(t) = \begin{pmatrix} \rho_0\int_{-\infty}^t \Big(\int_{t-s}^\infty p_0(w)dw\Big)\varepsilon(s)ds \\ (1/r)\int_{-\infty}^t \Big(\int_{t-s}^\infty \zeta^* P(w)dw\Big)x(s)ds \end{pmatrix},$$

which, together with Eq. ($\widetilde{E}$) (or (4.7)), leads to

$$\begin{aligned}
\frac{d}{dt}z_c(t) &= \begin{pmatrix} \rho_0\Big(\varepsilon(t) - \int_{-\infty}^t p_0(t-s)\varepsilon(s)ds\Big) \\ (1/r)\zeta^*\Big(x(t) - \int_{-\infty}^t P(t-s)x(s)ds\Big) \end{pmatrix} \\
&= \frac{1}{r}\begin{pmatrix} 0 \\ \zeta^* G^{(2)}(\widetilde{x}_t) \end{pmatrix} = \frac{1}{r}\begin{pmatrix} 0 \\ \zeta^* G^{(2)}\big(\Phi_c z_c(t) + \widetilde{F}_*(\Phi_c z_c(t))\big) \end{pmatrix}.
\end{aligned}$$

Summarizing, we have:

Proposition 4.11. *The central equation of the equilibrium* 0 *of Eq.* ($\widetilde{E}$) *is described as the* 2*-dimensional ODE*

$$z' = H(z), \tag{4.21}$$

where H *is an* $\mathbb{R}^2$*-valued function defined on a neighborhood of* 0 *of* $\mathbb{R}^2$ *by*

$$H(w) := H_c\widetilde{G}\big(\Phi_c w + \widetilde{F}_*(\Phi_c w)\big) \quad \textit{for small } |w|,$$

and H_c *is the* $2 \times (m+1)$ *matrix given by* $H_c := r^{-1}\mathrm{col}\,(0, \tilde{\zeta}^*)$.

Let us denote $w = \mathrm{col}\,(\varepsilon, s) \in \mathbb{R}^2$. Then $\Phi_c w = w_1(0)(\varepsilon e_1 + s\eta_*) = \mathrm{col}\,\big(w_1(0)\varepsilon,\, w_1(0)(s\eta_*)\big)$ and hence by (4.9),

$$\begin{aligned}
G^{(2)}(\Phi_c w) &= \left(\int_{-\infty}^{0} p_0(-\theta)\big(w_1(0)\varepsilon\big)(\theta)d\theta\right)\left(\int_{-\infty}^{0} P(-\theta)\big(w_1(0)(s\eta_*)\big)(\theta)d\theta\right) \\
&\quad + Q(w_1(0)(s\eta_*)) + g(w_1(0)(s\eta_*)) \\
&= \left(\int_{-\infty}^{0} p_0(-\theta)\varepsilon d\theta\right)\left(\int_{-\infty}^{0} P(-\theta)(s\eta_*)d\theta\right) + Q(w_1(0)\eta_*)s^n \\
&\quad + o(\|w_1(0)(s\eta_*)\|_X^n) \\
&= \varepsilon s\eta_* + s^n v_0 + o(|s|^n) \quad \text{as } (\varepsilon, s) \to (0,0),
\end{aligned} \tag{4.22}$$

where we used the fact $P_0\eta_* = \eta_*$ and put $v_0 := Q(w_1(0)\eta_*)$.

Proposition 4.12. $\widetilde{F}_*\big(w_1(0)(\varepsilon e_1)\big) = 0$ *for sufficiently small* $|\varepsilon|$.

Proof. $\tilde{x}^\varepsilon(t) := \varepsilon e_1$ $(t \in \mathbb{R})$ is obviously a solution on $\mathbb{R}^+$ of ($\widetilde{E}$) with $\tilde{x}^\varepsilon_t = w_1(0)(\varepsilon e_1)$ $(= \Phi_c(\varepsilon e_1))$. So, if $|\varepsilon| < \rho\delta/3$, we have $\tilde{x}^\varepsilon_t \in \widetilde{\Omega}_0$ for $t \in \mathbb{R}^+$, and hence Theorem 3.3 means $\mathrm{dist}\,\big(\tilde{x}^\varepsilon_t, W^c_{\mathrm{loc}}(0)\big) \to 0$ as $t \to \infty$. Thus, $w_1(0)(\varepsilon e_1) \in W^c_{\mathrm{loc}}(0) = \mathrm{graph}\,\widetilde{F}_*$, that is,

$$w_1(0)(\varepsilon e_1) = \Pi^c(w_1(0)(\varepsilon e_1)) + \widetilde{F}_*\big(\Pi^c(w_1(0)(\varepsilon e_1)\big),$$

which proves the proposition because $\Pi^c(w_1(0)(\varepsilon e_1)) = w_1(0)(\varepsilon e_1)$. □

Proposition 4.13. $G^{(2)}(\Phi_c w + \widetilde{F}_*(\Phi_c w))$ *satisfies*

(i) $G^{(2)}(\Phi_c w + \widetilde{F}_*(\Phi_c w)) = \varepsilon s\eta_* + s^n v_0 + o(|s|^n) + o(|\varepsilon s|)$,

(ii) $(\partial/\partial s)G^{(2)}(\Phi_c w + \widetilde{F}_*(\Phi_c w)) = \varepsilon + o(|\varepsilon|) + ns^{n-1}v_0 + o(|s|^{n-1})$

as $w = \mathrm{col}\,(\varepsilon, s) \to (0,0)$.

Proof. We assume that $\|\Phi_c w\|_{\widetilde{X}} < \delta$ so that $\widetilde{F}_*(\Phi_c w)$ is well-defined. Let us denote $\widetilde{F}_*(\Phi_c w) = \mathrm{col}\,(\phi^w, \psi^w) \in \widetilde{X}$ with $\phi^w \in X_1$ and $\psi^w \in X$. By Proposition 4.12,

$$\begin{aligned}
\widetilde{F}_*(\Phi_c w) &= \widetilde{F}_*(w_1(0)(\varepsilon e_1 + s\eta_*)) - \widetilde{F}_*(w_1(0)(\varepsilon e_1)) \\
&= \int_0^1 \frac{d}{d\tau}\widetilde{F}_*(w_1(0)(\varepsilon e_1 + \tau s\eta_*))d\tau \\
&= \int_0^1 D\widetilde{F}_*\big(w_1(0)(\varepsilon e_1 + \tau s\eta_*)\big)(w_1(0)(s\eta_*))d\tau,
\end{aligned}$$

and hence

$$\|\widetilde{F}_*(\Phi_c w)\|_{\widetilde{X}} \le \rho^{-1}|s||\eta_*| \sup_{0\le\tau\le1} \big\|D\widetilde{F}_*\big(w_1(0)(\varepsilon e_1 + \tau s\eta_*)\big)\big\|_{\mathcal{L}(\widetilde{E}^c;\widetilde{E}^s)}.$$

This yields $\widetilde{F}_*(\Phi_c w) = o(|s|)$ as $(\varepsilon, s) \to (0,0)$ since $\widetilde{F}_*$ is of class C^1 with $D\widetilde{F}_*(0) = 0$; so, in view of $\|\psi^w\|_X \le \|\widetilde{F}_*(\Phi_c w)\|_{\widetilde{X}}$,

$$\psi^w = o(|s|), \quad (\varepsilon, s) \to (0,0). \tag{4.23}$$

In addition, since

$$\frac{\partial}{\partial s}\widetilde{F}_*(\Phi_c w) = D\widetilde{F}_*(\Phi_c w)(w_1(0)\eta_*) = o(1) \quad \text{as } (\varepsilon, s) \to (0,0)$$

and $\|\partial \psi^w/\partial s\|_X \le \|(\partial/\partial s)\widetilde{F}_*(\Phi_c w)\|_{\widetilde{X}}$, it follows that

$$\frac{\partial}{\partial s}\psi^w = o(1), \quad (\varepsilon, s) \to (0,0). \tag{4.24}$$

(i) We first observe that $\Pi^c \widetilde{F}_*(\Phi_c w) = 0$ because of the fact $\widetilde{F}_*(\Phi_c w) \in \widetilde{E}^s$; so Proposition 4.10 yields

$$\begin{aligned} 0 &= \rho_0 \int_{-\infty}^0 p_0(-\theta)\left(\int_\theta^0 \phi^w(\xi)d\xi\right)d\theta \\ &= \int_{-\infty}^0 \left(\int_{-\infty}^\xi \rho_0 p_0(-\theta)d\theta\right)\phi^w(\xi)d\xi = \int_{-\infty}^0 p_0(-\xi)\phi^w(\xi)d\xi. \end{aligned}$$

Hence, one can see from (4.9) that

$$\begin{aligned} &G^{(2)}(\Phi_c w + \widetilde{F}_*(\Phi_c w)) - \widetilde{G}^{(2)}(\Phi_c w) \\ &= \left(\int_{-\infty}^0 p_0(-\theta)(\phi^w(\theta) + \varepsilon)d\theta\right)\left(\int_{-\infty}^0 P(-\theta)(\psi^w(\theta) + s\eta_*)d\theta\right) + f(\psi^w + w_1(0)(s\eta_*)) \\ &\quad - \left(\int_{-\infty}^0 p_0(-\theta)\varepsilon d\theta\right)\left(\int_{-\infty}^0 P(-\theta)(s\eta_*)d\theta\right) - f(w_1(0)(s\eta_*)) \\ &= \varepsilon \int_{-\infty}^0 P(-\theta)\psi^w(\theta)d\theta + f(\psi^w + w_1(0)(s\eta_*)) - f(w_1(0)(s\eta_*)). \end{aligned} \tag{4.25}$$

Notice from (4.23) that the first term is estimated by

$$\left|\varepsilon \int_{-\infty}^0 P(-\theta)\psi^w(\theta)d\theta\right| \le |\varepsilon|\|P\|_{\infty,\rho}\|\psi^w\|_X = o(|\varepsilon s|) \tag{4.26}$$

as $(\varepsilon, s) \to (0,0)$. On the other hand, we have

$$\begin{aligned} |f(\psi^w + w_1(0)(s\eta_*)) - f(w_1(0)(s\eta_*))| &\le |Q(\psi^w + w_1(0)(s\eta_*)) - Q(w_1(0)(s\eta_*))| \\ &\quad + |g(\psi^w + w_1(0)(s\eta_*)) - g(w_1(0)(s\eta_*))| =: d_1 + d_2. \end{aligned}$$

Since

$$d_2 \le \int_0^1 \|Dg(\psi^w + \tau w_1(0)(s\eta_*))\|_{\mathcal{L}(X;\mathbb{C}^m)}\|\psi^w\|_X d\tau$$

and, by (4.23) again,

$$\begin{aligned} \|\psi^w + \tau w_1(0)(s\eta_*)\|_X &\le \|\psi^w\|_X + \|w_1(0)(s\eta_*)\|_X \\ &= o(|s|) + \rho^{-1}|s||\eta_*| \le C_3|s|, \quad \tau \in [0,1] \end{aligned} \tag{4.27}$$

hold with some positive constant C_3, it follows from the assumption (4.4) that

$$d_2 = o(|s|^{n-1})\, o(|s|) = o(|s|^n) \quad \text{as} \quad (\varepsilon, s) \to (0,0).$$

Also, by the multi-linearity of Q_* and (4.27),

$$\begin{aligned}
d_1 &= |Q_*(\psi^w + w_1(0)(s\eta_*), \dots, \psi^w + w_1(0)(s\eta_*)) - Q_*(w_1(0)(s\eta_*), \dots, w_1(0)(s\eta_*))| \\
&\leq \sum_{k=1}^{n} \big| Q_*(\psi^w, \dots, \psi^w, \underbrace{\psi^w + w_1(0)(s\eta_*)}_{k}, \dots, \psi^w + w_1(0)s) \\
&\quad - Q_*(\psi^w, \dots, \psi^w, \underbrace{w_1(0)(s\eta_*)}_{k}, \psi^w + w_1(0)(s\eta_*), \dots, \psi^w + w_1(0)(s\eta_*)) \big| \\
&= \sum_{k=1}^{n} \big| Q_*(\psi^w, \dots, \psi^w, \underbrace{\psi^w}_{k}, \psi^w + w_1(0)(s\eta_*), \dots, \psi^w + w_1(0)(s\eta_*)) \big| \\
&\leq \sum_{k=1}^{n} \|Q_*\|_{\mathcal{L}^n(X;\mathbb{R}^m)} \|\psi^w\|_X^k \, \|\psi^w + w_1(0)(s\eta_*)\|_X^{n-k} \\
&\leq \|Q_*\|_{\mathcal{L}^n(X;\mathbb{R}^m)} \sum_{k=1}^{n} \|\psi^w\|_X^k \, (C_3|s|)^{n-k},
\end{aligned}$$

and hence (4.23) implies $d_1 = o(|s|^n)$ as $(\varepsilon, s) \to (0,0)$. By (4.25), (4.26) and the estimates of d_i $(i = 1,2)$, we obtain

$$G^{(2)}(\Phi_c w + \widetilde{F}_*(\Phi_c w)) - G^{(2)}(\Phi_c w) = o(|\varepsilon s|) + o(|s|^m),$$

and so, by (4.22),

$$G^{(2)}(\Phi_c w + \widetilde{F}_*(\Phi_c w)) = \varepsilon s + \mu s^n v_0 + o(|s|^n) + o(\varepsilon s|) \quad \text{as} \;\; (\varepsilon, s) \to (0,0).$$

(ii) We know that

$$\begin{aligned}
&G^{(2)}(\Phi_c w + \widetilde{F}_*(\Phi_c w)) \\
&\quad = \left(\int_{-\infty}^{0} p_0(\theta)(\phi^w(\theta) + \varepsilon) d\theta \right) \left(\int_{-\infty}^{0} P(\theta)(\psi^w(\theta) + s\eta_*) d\theta \right) + f(\psi^w + w_1(0)(s\eta_*)) \\
&\quad = \varepsilon s\eta_* + \varepsilon \int_{-\infty}^{0} P(\theta)\psi^w(\theta) d\theta + \mu Q(\psi^w + w_1(0)(s\eta_*)) + g(\psi^w + w_1(0)(s\eta_*)),
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{\partial}{\partial s} G^{(2)}(\Phi_c w + \widetilde{F}_*(\Phi_c w)) &= \varepsilon\eta_* + \varepsilon \int_{-\infty}^{0} P(\theta) \frac{\partial}{\partial s}\psi^w(\theta) d\theta + \mu \frac{\partial}{\partial s} Q(\psi^w + w_1(0)(s\eta_*)) \\
&\quad + Dg(\psi^w + w_1(0)(s\eta_*)) \left(\frac{\partial}{\partial s}\psi^w + w_1(0)\eta_* \right).
\end{aligned} \tag{4.28}$$

In view of (4.24),

$$\left| \varepsilon \int_{-\infty}^{0} P(\theta) \frac{\partial}{\partial s} \psi^w(\theta) d\theta \right| \leq |\varepsilon| \|P\|_{\infty,\rho} \left\| \frac{\partial \psi^w}{\partial s} \right\|_X = o(|\varepsilon|), \quad (\varepsilon, s) \to (0,0).$$

Moreover, observe that

$$Q(\psi^w + w_1(0)(s\eta_*)) = s^n v_0 + \sum_{(\phi_1, \dots, \phi_m) \in B} Q_*(\phi_1, \dots, \phi_m),$$

where we used the fact that $Q(w_1(0)(s\eta_*)) = s^n v_0$, and denoted by B the subset of X^n, $\{\psi^w, w_1(0)(s\eta_*)\}^n \setminus \{(w_1(0)(s\eta_*), \ldots, w_1(0)(s\eta_*))\}$. Hence,

$$\frac{\partial}{\partial s} Q(\psi^w + w_1(0)(s\eta_*)) = ns^{n-1}v_0 + \sum_{(\phi_1,\ldots,\phi_n)\in B} \sum_{k=1}^{n} Q_*(\phi_1, \ldots, \partial\phi_k/\partial s, \ldots, \phi_n).$$

One can readily deduce from (4.23) and (4.24) that for any $(\phi_1, \ldots, \phi_n) \in B$,

$$Q_*(\phi_1, \ldots, \partial\phi_k/\partial s, \ldots, \phi_n) = o(|s|^{n-1}) \quad \text{as} \quad (\varepsilon, s) \to (0,0),$$

so that

$$\frac{\partial}{\partial s} Q(\psi^w + w_1(0)(s\eta_*)) = ns^{n-1}v_0 + o(|s|^{n-1}).$$

Also, by (4.24) again, there exists a positive constant C_4 such that

$$\left\| \frac{\partial}{\partial s}\psi^w + w_1(0)\eta_* \right\|_X \le C_4 \quad \text{for small } |\varepsilon| \text{ and } |s|,$$

which, together with (4.27) and the assumption (4.4), means that

$$\left\| Dg(\psi^w + w_1(0)(s\eta_*)) \left(\frac{\partial}{\partial s}\psi^w + w_1(0)\eta_* \right) \right\|_X \le C_4 \| Dg(\psi^w + w_1(0)(s\eta_*) \|_X = o(|s|^{n-1}).$$

Consequently (4.28), together with the estimates above, leads to

$$\frac{\partial}{\partial s} G^{(2)}(\Phi_c w + \widetilde{F}_*(\Phi_c w)) = \varepsilon\eta_* + o(|\varepsilon|) + ns^{n-1}v_0 + o(|s|^{n-1})$$

as $(\varepsilon, s) \to (0,0)$, and this completes the proof. □

In virtue of Proposition 4.13, the central equation (4.21) turns out to be of the form

$$\begin{aligned} \varepsilon' &= 0, \\ s' &= h(\varepsilon, s), \end{aligned} \tag{$C\widetilde{E}$}$$

where $h(\varepsilon, s) = (1/r)\langle \zeta^*, G^{(2)}(\Phi_c w + \widetilde{F}_*(\Phi_c w)) \rangle$ is of class C^1 and satisfies

$$\begin{aligned} h(\varepsilon, s) &= \frac{1}{r} \langle \zeta^*, \varepsilon s\eta_* + s^n v_0 + o(|s|^n) + o(\varepsilon s|) \rangle \\ &= -\frac{1}{q_*}\varepsilon s + c_0 s^n + o(|s|^n) + o(|\varepsilon s|) \end{aligned} \tag{4.29}$$

as $(\varepsilon, s) \to (0,0)$. Furthermore, its derivative with respect to s has the property

$$\frac{\partial h}{\partial s}(\varepsilon, s) = -\frac{1}{q_*}\varepsilon + nc_0 s^{n-1} + o(|s|^{n-1} + |\varepsilon|), \quad (\varepsilon, s) \to (0,0), \tag{4.30}$$

where q_* and c_0 are constants introduced in (4.5), that is,

$$q_* = -\langle \zeta^*, P_1\eta_* \rangle / \langle \zeta^*, \eta_* \rangle, \text{ and } c_0 = \langle \zeta^*, Q(w_1(0)\eta_*) \rangle.$$

4.4 Proof of Theorem 4.4

In this subsection we will prove Theorem 4.4. Under the conditions (4.29) and (4.30), the behaviors of solutions of ($C\widetilde{E}$) are determined by the following lemma, which is a slight refinement of [11, Lemma 6.3.1].

Lemma 4.14. *Let $h(\varepsilon, s)$ be a real valued function of class C^1 defined in a neighborhood U_0 of $(0,0)$ of $\mathbb{R}^2$ that satisfies*

$$h(\varepsilon, s) = -\alpha s\varepsilon + \beta s^m + o(|s|^m + |\varepsilon s|), \tag{4.31}$$

and

$$\frac{\partial h}{\partial s}(\varepsilon, s) = -\alpha\varepsilon + m\beta s^{m-1} + o(|s|^{m-1} + |\varepsilon|) \tag{4.32}$$

as $(s, \varepsilon) \to (0,0)$, where $\alpha > 0$, $\beta \neq 0$ and $m = 2, 3, \ldots$ Then we have:

(a) *If m is even, then*

- (a$_1$) *there exists an $s^* > 0$ such that $h(0,s)$ is positive (resp. negative) definite for $0 < |s| < s^*$ in the case of $\beta > 0$ (resp. $\beta < 0$). Moreover, there exist an $\varepsilon^* > 0$ and a continuous function $s : (-\varepsilon^*, \varepsilon^*)\backslash\{0\} \to \mathbb{R}\backslash\{0\}$ such that $h(\varepsilon, s_\varepsilon) = 0$ for $0 < |\varepsilon| < \varepsilon^*$, $s_\varepsilon \to 0$ as $\varepsilon \to 0$, and*

$$\frac{\partial h}{\partial s}(\varepsilon, s_\varepsilon) \begin{cases} < 0 & (\varepsilon < 0), \\ > 0 & (\varepsilon > 0) \end{cases} \tag{4.33}$$

  holds for small $|\varepsilon|$.

- (a$_2$) *There exists an open subset U_1 of U_0 containing $(0,0)$ such that $h(\hat{\varepsilon}, \hat{s}) = 0$ with some $(\hat{\varepsilon}, \hat{s}) \in U_1$ implies $\hat{s} = 0$ or $\hat{s} = s_{\hat{\varepsilon}}$.*

(b) *If m is odd and $\beta > 0$, then*

- (b$_1$) *there exists an $s^* > 0$ such that $sh(0,s) > 0$ for $0 < |s| < s^*$. Moreover, if $\varepsilon \leq 0$ with $|\varepsilon|$ small, $h(\varepsilon, \cdot)$ has no small zeros other than 0; on the other hand, there exist an $\varepsilon^* > 0$ and continuous functions $s^\pm : (0, \varepsilon^*) \to \mathbb{R}\backslash\{0\}$ such that $h(\varepsilon, s_\varepsilon^+) = h(\varepsilon, s_\varepsilon^-) = 0$ for $0 < \varepsilon < \varepsilon^*$, $s_\varepsilon^\pm \to 0$ as $\varepsilon \to +0$, and*

$$\frac{\partial h}{\partial s}(\varepsilon, s_\varepsilon^\pm) > 0 \quad \text{for small } \varepsilon > 0. \tag{4.34}$$

- (b$_2$) *There exists an open subset U_1 of U_0 containing $(0,0)$ such that $h(\hat{\varepsilon}, \hat{s}) = 0$ with $(\hat{\varepsilon}, \hat{s}) \in U_1$ implies that $\hat{s}$ coincides with 0, $s_{\hat{\varepsilon}}^+$ or $s_{\hat{\varepsilon}}^-$.*

(c) *If m is odd and $\beta < 0$, then*

- (c$_1$) *there exists an $s^* > 0$ such that $sh(0,s) < 0$ for $0 < |s| < s^*$. Moreover, if $\varepsilon \geq 0$ with $|\varepsilon|$ small, $h(\varepsilon, \cdot)$ has no small zeros other than 0; on the other hand, there exist an $\varepsilon^* > 0$ and continuous functions $s^\pm : (-\varepsilon^*, 0) \to \mathbb{R}\backslash\{0\}$ such that $h(\varepsilon, s_\varepsilon^+) = h(\varepsilon, s_\varepsilon^-) = 0$ for $-\varepsilon^* < \varepsilon < 0$, $s_\varepsilon^\pm \to 0$ as $\varepsilon \to -0$, and*

$$\frac{\partial h}{\partial s}(\varepsilon, s_\varepsilon^\pm) < 0 \quad \text{for } \varepsilon < 0 \text{ with } |\varepsilon| \text{ small.}$$

- (c$_2$) *There exists an open subset U_1 of U_0 containing $(0,0)$ s such that $h(\hat{\varepsilon}, \hat{s}) = 0$ with $(\hat{\varepsilon}, \hat{s}) \in U_1$ implies that $\hat{s}$ coincides with 0, $s_{\hat{\varepsilon}}^+$, or $s_{\hat{\varepsilon}}^-$.*

We will first prove Theorem 4.4 and, for completeness, give a proof of Lemma 4.14 at the end of this subsection. Note that any equilibrium in $\widetilde{\Omega}_0$ of Eq. ($\widetilde{E}$) must be contained in $\widetilde{W}^c_{\rm loc}(0)$ because of the attractivity of $\widetilde{W}^c_{\rm loc}(0)$ (see [23, Theorem 5], or Theorem 3.3); so in view of Proposition 3.1, $\tilde{\phi}_* \in \widetilde{\Omega}$ is an equilibrium of Eq. ($\widetilde{E}$) if and only if there exists an equilibrium z_* of the central equation ($C\widetilde{E}$) (or (4.21)) such that

$$\tilde{\phi}_* = \Phi_c z_* + \widetilde{F}_*(\Phi_c z_*). \tag{4.35}$$

Remark 4.15. If $\tilde{\phi}_*$ is an equilibrium of Eq. ($\widetilde{E}$) (or (4.7)), then $\phi_* := \Pi_2\tilde{\phi}_*$ is an equilibrium of Eq. (PE) with $\lambda = 1+\varepsilon$, where ε is given by $w_1(0)\varepsilon = \Pi_1\tilde{\phi}_*$. Conversely, if ϕ_* is an equilibrium of Eq. (PE), then $\tilde{\phi}_* := j_{\lambda-1}(\phi_*)$ is an equilibrium of Eq. ($\widetilde{E}$) (or (4.7)), where $j_\varepsilon : X \to \widetilde{X}$ is the map defined in (4.14) (see Proposition 4.5).

Moreover, throughout this subsection, we assume that

$$L(\delta) < 1. \tag{4.36}$$

by taking $\delta > 0$ sufficiently small.

Proof of Theorem 4.4. We know by (4.29) and (4.30) that $h(\varepsilon, s)$ in Eq. ($C\widetilde{E}$) satisfies the assumptions (4.31) and (4.32).

(i) Let $\lambda = 1$, that is, $\varepsilon := \lambda - 1 = 0$, and $s^* > 0$ be the one in Lemma 4.14 (a_1). Then, $s' = h(0,s)$, the second equation of ($C\widetilde{E}$) with $\varepsilon = 0$, has no equilibria in $(-s^*, s^*)\backslash\{0\}$. By taking s^* small if necessary, we may assume that

$$s^* < \frac{\rho\delta}{3|\eta_*|}. \tag{4.37}$$

(ia) Note that $\|\Phi_c z\|_{\widetilde{X}} = \|s w_1(0)\tilde{\eta}_*\|_{\widetilde{X}} = \rho^{-1}|s||\eta_*|$ for $z = \mathrm{col}\,(0,s)$. So such a z with $s \in (-s^*, s^*)$ satisfies $\|\Phi_c z\|_{\widetilde{X}} < \delta/3$, and hence

$$\|\Phi_c z + \widetilde{F}_*(\Phi_c z)\|_{\widetilde{X}} \geq (1 - L(\delta))\|\Phi_c z\|_{\widetilde{X}} = (1-L(\delta))\rho^{-1}|s||\eta_*.$$

Let $\rho_1 := (1-L(\delta))\rho^{-1}s^*|\eta_*|$; then $0 < \rho_1 < \delta/3$ due to (4.36) and (4.37). Therefore, one can see from (4.35) that Eq. ($\widetilde{E}$) has no equilibria in $B_{\widetilde{X}}(\rho_1)\backslash\{0\}$. So, by Remark 4.15, Eq. (PE) has no equilibria in the set $j_0^{-1}(B_{\widetilde{X}}(\rho_1)\backslash\{0\}) = B_X(\rho_1)\backslash\{0\}$, where we used the fact that j_ε is an isometry for $\varepsilon \in \mathbb{R}$. Thus, the former part of (ia) is valid with $\mathcal{W}_0 = B_X(\rho_1)$.

Let $s : (-\varepsilon^*, \varepsilon^*)\backslash\{0\} \to \mathbb{R}\backslash\{0\}$ be the function in Lemma 4.14 (a_1) and set

$$\tilde{\phi}^\lambda_* := \Phi_c z^{\lambda-1} + \widetilde{F}_*(\Phi_c z^{\lambda-1}), \quad \lambda \in (1-\varepsilon^*, 1+\varepsilon^*)\backslash\{1\}, \tag{4.38}$$

where we used the notation $z^\varepsilon := \mathrm{col}\,(\varepsilon, s_\varepsilon)$ for $\varepsilon \in (-\varepsilon^*, \varepsilon^*)\backslash\{0\}$. We may assume

$$\|\Phi_c z^\varepsilon\|_{\widetilde{X}} < \delta/3 \quad \text{for } \varepsilon \in (-\varepsilon^*, \varepsilon^*)\backslash\{0\}. \tag{4.39}$$

Since z^ε is an equilibrium of Eq. ($C\widetilde{E}$), $\tilde{\phi}^\lambda_*$ is also an equilibrium of Eq. ($\widetilde{E}$) by (4.35).

Note that $\widetilde{T}(t) = \mathrm{diag}\,(T_0(t), T(t))$, where $\{T_0(t)\}_{t\geq 0}$ is the solution semigroup of the first equation of (4.7). So, $\widetilde{T}^s(t) = \mathrm{diag}\,(T^s_0(t), T^s(t))$ because $X_1 \times \{0\}$ and $\{0\} \times X$ are $\widetilde{T}(t)$-invariant subspaces of $\widetilde{X}$ for $t \geq 0$. Since $\Gamma^n\widetilde{G}_\delta(\Lambda_{*,\delta}(\psi)(s)) = \mathrm{col}\,(0, \Gamma^n G^{(2)}_\delta(\Lambda_{*,\delta}(\psi)(s)))$ with $G^{(2)}_\delta(\tilde{\phi}) := \Pi_2\widetilde{G}_\delta(\tilde{\phi})$, we see $\Pi_1(\Pi^s\Gamma^n\widetilde{G}_\delta(\Lambda_{*,\delta}(\psi)(s))) = 0$. Hence, (4.12) means that for $\psi \in \widetilde{E}^c$,

$$\Pi_1(\widetilde{F}_{*,\delta}(\psi)) = \lim_{n\to\infty}\int_{-\infty}^0 T^s_0(-s)\Pi_1(\Pi^s\Gamma^n\widetilde{G}_\delta(\Lambda_{*,\delta}(\psi)(s)))ds = 0. \tag{4.40}$$

It also follows from Proposition 4.10 that $\Phi_c z^{\lambda-1} = w_1(0)\big((\lambda-1)e_1 + s_{\lambda-1}\tilde{\eta}_*\big)$; so that

$$\Pi_1(\tilde{\phi}_*^\lambda) = \Pi_1(\Phi_c z^{\lambda-1}) + \Pi_1(\widetilde{F}_*(\Phi_c z^{\lambda-1})) = w_1(0)(\lambda-1)$$

because of (4.38) and the fact that $\widetilde{F}_* \equiv \widetilde{F}_{*,\delta}$ on $B_{\widetilde{E}^c}(\delta)$; in other words, $\Pi_1(\tilde{\phi}_*^\lambda)(\theta) = \lambda - 1$ for $\theta \in \mathbb{R}^-$.

Define $\phi_* : (1-\varepsilon^*, 1+\varepsilon^*)\backslash\{1\} \to X\backslash\{0\}$ by

$$\phi_*^\lambda := \Pi_2(\tilde{\phi}_*^\lambda), \quad \lambda \in (1-\varepsilon^*, 1+\varepsilon^*)\backslash\{1\}.$$

Then, ϕ_*^λ is seen to be an equilibrium of Eq. (PE) by Remark 4.15. In addition, ϕ_*^λ is continuous in λ and satisfies $\phi_*^\lambda \to 0$ $(\lambda \to 1)$ since $z^{\lambda-1} \to 0$ as $\lambda \to 1$.

We next show the instability of the equilibrium 0. When $\varepsilon = 0$, the second equation of ($C\widetilde{E}$) is $s' = c_0 s^n + o(|s|^n)$; so its zero solution, and hence the equilibrium 0 of ($C\widetilde{E}$), is unstable because n is even. Given $t_0 \in \mathbb{R}$ and $s_0 \in \mathbb{R}\backslash\{0\}$, let $z(t)$ be the solution of ($C\widetilde{E}$) with $z(t_0) = \mathrm{col}\,(0, s_0)$ and set $\widetilde{\psi} := \Phi_c z(t_0) + \widetilde{F}_*(\Phi_c z(t_0))$, where we assume $|s_0|$ is sufficiently small in such a way that $\|\Phi_c z(t_0)\|_X < \delta$ holds. Then it follows from (4.40) that $\Pi_1 \widetilde{\psi} = 0$, so that $\widetilde{\psi} = j_0(\psi)$ with $\psi = \Pi_2 \widetilde{\psi}$. So we deduce from Proposition 4.5 that

$$j_0(x_t(t_0, \psi, f)) = \tilde{x}_t(t_0, \widetilde{\psi}, \widetilde{G}) = \Phi_c z(t) + \widetilde{F}_*(\Phi_c z(t))$$

for $t \geq t_0$ with $t - t_0$ small. Hence the instability of the equilibrium 0 of (PE) with $\lambda = 1$ follows from that of the equilibrium 0 of ($C\widetilde{E}$). Thus, (ia) holds.

(ib) The central equation of the equilibrium 0 of Eq. ($\widetilde{E}_\delta$) is of the form

$$z' = H_c \widetilde{G}_\delta(\Phi_c z + \widetilde{F}_{*,\delta}(\Phi_c z)) \tag{$C\widetilde{E}_\delta$}$$

(cf. Proposition 4.11 and (4.11)). Let $h_\delta(z)$ be the second component of $H_c \widetilde{G}_\delta(\Phi_c z + \widetilde{F}_{*,\delta}(\Phi_c z))$. In view of (4.36), $\|\Phi_c z\|_{\widetilde{X}} < \delta$ implies

$$\|\widetilde{F}_{*,\delta}(\Phi_c z)\|_{\widetilde{X}} \leq L(\delta)\|\Phi_c z\|_{\widetilde{X}} < \delta,$$

and hence $\Phi_c z + \widetilde{F}_{*,\delta}(\Phi_c z) \in \widetilde{\Omega}$. So, $h(z) = h_\delta(z)$ for $\|\Phi_c z\|_{\widetilde{X}} < \delta$; in particular, by (4.39), $z^{\lambda-1}$ is also an equilibrium of Eq. ($C\widetilde{E}_\delta$) for $\lambda \in (1-\varepsilon^*, 1+\varepsilon^*)$.

Suppose that $q_* < 0$. Notice from Lemma 4.14 (a_1) and the principle of linearized stability that $s_{\lambda-1}$ $(\lambda \neq 1)$ is the nonzero equilibrium of $s' = h(\lambda-1, s)$, hence of $s' = h_\delta(\lambda-1, s)$, which is asymptotically stable (resp. unstable) if $\lambda < 1$ (resp. $\lambda > 1$).

Now let $\lambda < 1$. Since $\{\lambda-1\} \times \mathbb{R}$ is an invariant set for Eq. ($C\widetilde{E}_\delta$), the stability of $s_{\lambda-1}$ implies that for arbitrary $t_0 \in \mathbb{R}^+$ and $\hat{\varepsilon} > 0$, there exists a $\delta_0(\hat{\varepsilon}) > 0$ such that if $z^* \in \{\lambda - 1\} \times \mathbb{R}$ satisfies $|z^* - z^{\lambda-1}| < \delta_0(\hat{\varepsilon})$, then $z(t; t_0, z^*) \in \{\lambda-1\} \times \mathbb{R}$ and $|z(t; t_0, z^*) - z^{\lambda-1}| < \hat{\varepsilon}$ hold for $t \geq t_0$, where $z(t; t_0, z^*)$ denotes the solution of Eq. ($C\widetilde{E}_\delta$) with $z(t_0) = z^*$.

Let us denote by Π_X^c and Π_X^s the projections defined by $\Pi_X^c := \Pi_2 \circ \Pi^c \circ j_0$ and $\Pi_X^s := \Pi_2 \circ \Pi^s \circ j_0$, respectively. It is easy to see that $\Pi^c \circ j_\varepsilon = j_\varepsilon \circ \Pi_X^c$ and $\Pi^s \circ j_\varepsilon = j_0 \circ \Pi_X^s$ for $\varepsilon \in \mathbb{R}$. Set

$$\phi_*^{\lambda,c} := \Pi_X^c \phi_*^\lambda, \quad \text{and} \quad \phi_*^{\lambda,s} := \Pi_X^s \phi_*^\lambda.$$

Similarly, given $\phi \in X$ and $\tilde{\phi} \in \widetilde{X}$, we put $\phi^c := \Pi_X^c \phi$ and $\tilde{\phi}^c := \Pi^c \tilde{\phi}$; and likewise for ϕ^s and $\tilde{\phi}^s$. Given $\phi \in X$, let z^* be the point of $\mathbb{R}^2$ satisfying $\Phi_c z^* = (j_{\lambda-1}(\phi))^c = j_{\lambda-1}(\phi^c)$. Then $z(t; t_0, z^*) = z(t - t_0, j_{\lambda-1}(\phi))$ for $t \geq t_0$ (cf. (3.16)). Observe that $\Phi_c z^{\lambda-1} = \Pi^c \tilde{\phi}_*^\lambda = \Pi^c(j_{\lambda-1}(\phi_*^\lambda)) = j_{\lambda-1}(\phi_*^{\lambda,c})$ and

$$\begin{aligned}\|\Phi_c\|_*|z^* - z^{\lambda-1}| &\leq \|\Phi_c z^* - \Phi_c z^{\lambda-1}\|_{\widetilde{X}} \\ &= \|j_{\lambda-1}(\phi^c) - j_{\lambda-1}(\phi_*^{\lambda,c})\|_{\widetilde{X}} = \|\phi^c - \phi_*^{\lambda,c}\|_X,\end{aligned}$$

where the last equality is due to the fact that j_ε is an isometry. So, if $\phi \in X$ satisfies $\|\phi^c - \phi_*^{\lambda,c}\|_X < \|\Phi_c\|_* \delta_0(\hat{\varepsilon}/\|\Phi_c\|^*)$, then

$$\begin{aligned}\|\Phi_c z(t-t_0, j_{\lambda-1}(\phi)) - j_{\lambda-1}(\phi_*^{\lambda,c})\|_{\widetilde{X}} &= \|\Phi_c z(t;t_0,z^*) - \Phi_c z^{\lambda-1}\|_{\widetilde{X}} \\ &\le \|\Phi_c\|^* |z(t;t_0,z^*) - z^{\lambda-1}| < \hat{\varepsilon}\end{aligned} \tag{4.41}$$

for $t \ge t_0$. For $\hat{\varepsilon} > 0$, set

$$\hat{\delta}(\hat{\varepsilon}) := \min\left(\frac{\hat{\varepsilon}}{3C_0C_1(1+2L(\delta))}, \frac{\|\Phi_c\|_*}{2C_1}\delta_0\left(\frac{\hat{\varepsilon}}{3(1+L(\delta))\|\Phi_c\|^*}\right)\right). \tag{4.42}$$

Given any $\hat{\varepsilon}$ with $0 < \hat{\varepsilon} < \delta$, let $\psi \in B_X(\phi_*^\lambda; \hat{\delta}(\hat{\varepsilon}))$. Then $j_{\lambda-1}(\psi) \in B_{\widetilde{X}}(j_{\lambda-1}(\phi_*^\lambda); \hat{\delta}(\hat{\varepsilon}))$, and hence, it follows from Claim 1 (Subsection 3.2) and (3.33) that there exists a $\tilde{\phi} \in B_{\widetilde{X}}(j_{\lambda-1}(\phi_*^\lambda); 2\hat{\delta}(\hat{\varepsilon}))$ such that $\widehat{g}(\tilde{\phi}) = j_{\lambda-1}(\psi)$, where $\widehat{g}$ is the one in (3.28) applied for Eq. ($\widetilde{E}_\delta$). In particular,

$$\begin{aligned}&\|\Pi^c \widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G}_\delta) - \Phi_c z(t-t_0, \tilde{\phi})\|_{\widetilde{X}} \\ &\quad \le C_0 \|\Pi^s \widetilde{x}_{t_0}(t_0, j_{\lambda-1}(\psi), \widetilde{G}_\delta) - \widetilde{F}_{*,\delta}(\Phi_c z(0,\tilde{\phi}))\|_{\widetilde{X}} e^{-\beta(t-t_0)}\end{aligned} \tag{4.43}$$

holds for $t \ge t_0$ because of Theorem 3.2 (a) and its proof. By Proposition 4.5 we have

$$\Pi^c \widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G}_\delta) = \Pi^c j_{\lambda-1}(x_t(t_0,\psi,f_\delta)) = j_{\lambda-1}\big(\Pi_X^c x_t(t_0,\psi,f_\delta)\big) \in j_{\lambda-1}(X). \tag{4.44}$$

Moreover, let $\tilde{z} = \operatorname{col}(\tilde{\varepsilon}, \tilde{s})$ be the point of $\mathbb{R}^2$ determined by $\Pi^c \tilde{\phi} = \Phi_c \tilde{z}$. Then it readily follows from Eq. ($C\widetilde{E}$) that $\Phi_c z(t-t_0, \tilde{\phi}) \in j_{\tilde{\varepsilon}}(X)$. Consequently, we must have $\lambda - 1 = \tilde{\varepsilon}$ due to (4.43) and (4.44), that is, $\tilde{\phi} = j_{\lambda-1}(\phi)$ for some $\phi \in X$ with $\|\phi - \phi_*^\lambda\|_X < 2\hat{\delta}(\hat{\varepsilon})$.

Noting that $\tilde{\phi}_*^{\lambda,s} = \widetilde{F}_{*,\delta}(\tilde{\phi}_*^{\lambda,c})$ follows from $\tilde{\phi}_*^\lambda \in \widetilde{W}_\delta^c(0)$, and the fact that $\tilde{\phi}_*^\lambda = j_{\lambda-1}(\phi_*^\lambda)$, one can see $(j_{\lambda-1}(\phi_*^\lambda))^s = \widetilde{F}_{*,\delta}(j_{\lambda-1}(\phi_*^{\lambda,c}))$, so that

$$\begin{aligned}&\|\Pi^s \widetilde{x}_{t_0}(t_0, j_{\lambda-1}(\psi), \widetilde{G}_\delta) - \widetilde{F}_{*,\delta}(\Phi_c z(0, j_{\lambda-1}(\phi)))\|_{\widetilde{X}} \\ &\quad = \big\|(j_{\lambda-1}(\psi))^s - \widetilde{F}_{*,\delta}\big((j_{\lambda-1}(\phi))^c\big)\big\|_{\widetilde{X}} \\ &\quad \le \|(j_{\lambda-1}(\psi))^s - (j_{\lambda-1}(\phi_*^\lambda))^s\|_{\widetilde{X}} + \|\widetilde{F}_{*,\delta}(j_{\lambda-1}(\phi_*^{\lambda,c})) - \widetilde{F}_{*,\delta}(j_{\lambda-1}(\phi^c))\|_{\widetilde{X}} \\ &\quad \le C_1 \|j_{\lambda-1}(\psi) - j_{\lambda-1}(\phi_*^\lambda)\|_{\widetilde{X}} + L(\delta)\|(j_{\lambda-1}(\phi_*^\lambda))^c - (j_{\lambda-1}(\phi))^c\|_{\widetilde{X}} \\ &\quad \le C_1(\|\psi - \phi_*^\lambda\|_X + L(\delta)\|\phi_*^\lambda - \phi\|_X) < C_1(1+2L(\delta))\hat{\delta}(\hat{\varepsilon}).\end{aligned} \tag{4.45}$$

Since

$$\|\phi^c - \phi_*^{\lambda,c}\|_X \le C_1\|\phi - \phi_*^\lambda\|_X < 2C_1\hat{\delta}(\hat{\varepsilon}) \le \|\Phi_c\|_* \delta_0\left(\frac{\hat{\varepsilon}}{3(1+L(\delta))\|\Phi_c\|^*}\right), \tag{4.46}$$

it follows from (4.41), combined with $j_{\lambda-1}(\phi_*^{\lambda,c}) = \tilde{\phi}_*^{\lambda,c}$, (4.38) and (4.39), that

$$\|\Phi_c z(t-t_0, j_{\lambda-1}(\phi))\|_{\widetilde{X}} < \|\tilde{\phi}_*^{\lambda,c}\|_{\widetilde{X}} + \frac{\hat{\varepsilon}}{3} < \frac{\delta}{3} + \frac{\delta}{3} = \frac{2\delta}{3}. \tag{4.47}$$

Hence, by (4.43) and (4.45), together with $\tilde{\phi} = j_{\lambda-1}(\phi)$,

$$\begin{aligned}\|\Pi^c \widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G}_\delta)\|_{\widetilde{X}} &\le \|\Phi_c z(t-t_0, j_{\lambda-1}(\phi))\|_{\widetilde{X}} + C_0C_1(1+2L(\delta))\hat{\delta}(\hat{\varepsilon})e^{-\beta(t-t_0)} \\ &< \frac{2\delta}{3} + \frac{\hat{\varepsilon}}{3} < \delta, \quad t \ge t_0.\end{aligned}$$

Moreover, (3.1), (4.36), (4.45) and (4.47) yield

$$\begin{aligned}\|\Pi^s \widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G}_\delta)\|_{\widetilde{X}} &\le \|\widetilde{F}_{*,\delta}(\Phi_c z(t-t_0, j_{\lambda-1}(\phi)))\|_{\widetilde{X}} + C_0C_1(1+2L(\delta))\hat{\delta}(\hat{\varepsilon})e^{-\beta(t-t_0)} \\ &\le \|\Phi_c z(t-t_0, j_{\lambda-1}(\phi))\|_{\widetilde{X}} + \frac{\hat{\varepsilon}}{3} < \delta, \quad t \ge t_0.\end{aligned}$$

Thus, we deduce $\widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G}_\delta) \in \widetilde{\Omega}$ $(t \ge t_0)$. Noting that $\widetilde{G}_\delta \equiv \widetilde{G}$ on $\widetilde{\Omega}$, we have $\widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G}_\delta) = \widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G})$. Therefore, Proposition 4.5 ensures that the solution $x(t; t_0, \psi, f)$ is defined on $[t_0, \infty)$ and satisfies $j_{\lambda-1}(x_t(t_0, \psi, f)) = \widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G})$ there.

We also know from (3.2) and (4.45) that

$$\begin{aligned}\|\widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G}_\delta) - \widetilde{x}_t(t_0, j_{\lambda-1}(\phi), \widetilde{G}_\delta)\|_{\widetilde{X}} &< 2C_0C_1(1+2L(\delta))\hat{\delta}(\hat{\varepsilon})e^{-\beta(t-t_0)} \\ &\le \frac{2\hat{\varepsilon}}{3}e^{-\beta(t-t_0)}.\end{aligned} \tag{4.48}$$

So, in view of (4.48), (4.41), (4.46) and Proposition 3.1, combined with the fact that

$$j_{\lambda-1}(\phi_*^\lambda) = \tilde{\phi}_*^\lambda = \tilde{\phi}_*^{\lambda,c} + \tilde{\phi}_*^{\lambda,s} = j_{\lambda-1}(\phi_*^{\lambda,c}) + \widetilde{F}_{*,\delta}(j_{\lambda-1}(\phi_*^{\lambda,c})),$$

we conclude that if $\psi \in B_X(\phi_*^\lambda; \hat{\delta}(\hat{\varepsilon}))$, then

$$\begin{aligned}\|x_t(t_0, \psi, f) - \phi_*^\lambda\|_X &= \|j_{\lambda-1}(x_t(t_0, \psi, f)) - j_{\lambda-1}(\phi_*^\lambda)\|_{\widetilde{X}} = \|\widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G}) - \tilde{\phi}_*^\lambda\|_{\widetilde{X}} \\ &\le \|\widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G}_\delta) - \widetilde{x}_t(t_0, j_{\lambda-1}(\phi), \widetilde{G}_\delta)\|_{\widetilde{X}} \\ &\quad + \|\Phi_c z(t-t_0, j_{\lambda-1}(\phi)) - j_{\lambda-1}(\phi_*^{\lambda,c})\|_{\widetilde{X}} \\ &\quad + \|\widetilde{F}_{*,\delta}(\Phi_c z(t-t_0, j_{\lambda-1}(\phi))) - \widetilde{F}_{*,\delta}(j_{\lambda-1}(\phi_*^{\lambda,c}))\|_{\widetilde{X}} \\ &< \frac{2\hat{\varepsilon}}{3}e^{-\beta(t-t_0)} + (1+L(\delta))\|\Phi_c z(t-t_0, j_{\lambda-1}(\phi)) - j_{\lambda-1}(\phi_*^{\lambda,c})\|_{\widetilde{X}} < \hat{\varepsilon}\end{aligned}$$

for $t \ge t_0$. The equilibrium ϕ_*^λ is therefore stable.

We next verify the attractivity of ϕ_*^λ. By the asymptotic stability of $s_{\lambda-1}$ as an equilibrium of $s' = h_\delta(\lambda-1, s)$, together with the invariance of $\{\lambda-1\} \times \mathbb{R}$ for Eq. ($C\widetilde{E}_\delta$), there exist an $R_0 > 0$ with the property that to any $\hat{\varepsilon} > 0$, there corresponds a $\tau(\hat{\varepsilon}) > 0$ such that $|z(t; t_0, z^*) - z^{\lambda-1}| < \hat{\varepsilon}$ for every $t \ge t_0 + \tau(\hat{\varepsilon})$ and $z^* \in \{\lambda-1\} \times \mathbb{R}$ with $|z^* - z^{\lambda-1}| \le R_0$. Let $\hat{R} := \min(\hat{\delta}(\delta), \|\Phi_c\|_* R_0/(2C_1))$, where $\hat{\delta}(\cdot)$ is the one in (4.42). Given any $\hat{\varepsilon} \in (0, \delta)$, set

$$\hat{\tau}(\hat{\varepsilon}) := \max\left(\left(\frac{1}{\beta}\log\frac{4C_0C_1(1+2L(\delta))\hat{R}}{\hat{\varepsilon}}, \tau\left(\frac{\hat{\varepsilon}}{2(1+L(\delta))\|\Phi_c\|^*}\right)\right)\right)$$

and assume that $\psi \in B_X(\phi_*^\lambda, \hat{R})$. Then $j_{\lambda-1}(\psi) \in B_{\widetilde{X}}(j_{\lambda-1}(\phi_*^\lambda); \hat{R})$, and there exists a $\tilde{\phi} \in B_{\widetilde{X}}(j_{\lambda-1}(\phi_*^\lambda); 2\hat{R})$ with $\widehat{g}(\tilde{\phi}) = j_{\lambda-1}(\psi)$. Since $\hat{R} \le \hat{\delta}(\delta)$, we know from the former part that $\widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G}_\delta) \in \widetilde{\Omega}$ $(t \ge t_0)$. So, in the same way as above, $\hat{\phi}$ can be written as $\hat{\phi} = j_{\lambda-1}(\phi)$ for some $\phi \in X$ with $\|\phi - \phi_*^\lambda\|_X < 2\hat{\delta}(\delta)$ and the following estimate is valid:

$$\begin{aligned}\|x_t(t_0, \psi, f) - \phi_*^\lambda\|_X &\le \|\widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G}_\delta) - \widetilde{x}_t(t_0, j_{\lambda-1}(\phi), \widetilde{G}_\delta)\|_{\widetilde{X}} \\ &\quad + (1+L(\delta))\|\Phi_c z(t-t_0, j_{\lambda-1}(\phi)) - j_{\lambda-1}(\phi_*^{\lambda,c})\|_{\widetilde{X}} \\ &\le 2C_0C_1(1+2L(\delta))\hat{R}e^{-\beta(t-t_0)} + (1+L(\delta))\|\Phi_c\|^*|z(t; t_0, z^*) - z^{\lambda-1}| \\ &< \frac{\hat{\varepsilon}}{2} + \frac{\hat{\varepsilon}}{2} = \hat{\varepsilon} \quad \text{for } t \ge t_0 + \hat{\tau}(\hat{\varepsilon}),\end{aligned}$$

where z^* is again the point of $\mathbb{R}^2$ determined by $\Phi_c z^* = j_{\lambda-1}(\phi^c)$. Thus, ϕ_*^λ is an attractive equilibrium of Eq. (*PE*), and hence, is asymptotically stable.

Similarly, the equilibrium 0 of Eq. $s' = h(\lambda - 1, s)$ is asymptotically stable because

$$\frac{\partial h}{\partial s}(\varepsilon, 0) = -\frac{1}{q_*}\varepsilon + o(|\varepsilon|) < 0$$

for $\varepsilon < 0$ with $|\varepsilon|$ small. By the same reasoning as above, one can see the asymptotic stability of the equilibrium 0 of Eq. (*PE*).

Suppose now that $\lambda > 1$. Then $z^{\lambda-1} = \operatorname{col}(\lambda - 1, s_{\lambda-1})$ is a unstable equilibrium of Eq. ($C\widetilde{E}$). So, by Theorem 3.3, $\tilde{\phi}_*^\lambda$ is a unstable equilibrium of Eq. ($\widetilde{E}$). The invariance of $j_{\lambda-1}(X)$ then implies the instability of ϕ_*^λ, and likewise for the equilibrium 0.

If $q_* > 0$, then consider the function $\check{h}(\varepsilon, s) := h(\varepsilon, -s)$ ($(\varepsilon, s) \in \mathbb{R}^2$) in place of $h(\varepsilon, s)$ in Eq. ($C\widetilde{E}$). One can readily verify that $\check{h}(\varepsilon, s)$ satisfies the conditions (4.31) and (4.32) of Lemma 4.14. Applying the lemma to $\check{h}(\varepsilon, s)$ and noting that $(\partial h/\partial s)(\varepsilon, s) = -(\partial\check{h}/\partial s)(\varepsilon, -s)$, we see that the equilibrium $s_{\lambda-1}$ of the scalar equation $s' = h(\lambda - 1, s)$ is unstable (resp. asymptotically stable) if $\lambda < 1$ (resp. $\lambda > 1$). So, the argument above yields the desired stability properties of ϕ_*^λ; and similarly for the equilibrium 0.

(ic) Let U_1 be the open set of $\mathbb{R}^2$ in Lemma 4.14 (a_2). We may assume that U_1 is a rectangle $U(r) = (-r, r) \times (-r, r)$, where $r > 0$ satisfies $r < \rho\delta/3 - s^*|\eta_*|$ (see the proof of the lemma and notice (4.37)). Set

$$\mathcal{W} := \Pi_2\big(\widetilde{\Omega}_0 \cap (\Pi^c)^{-1}(\Phi_c U_1)\big);$$

then $\mathcal{W}$ is an open set of X since Φ_c is a homeomorphism as a map from $\mathbb{R}^2$ to $\widetilde{E}^c$, and Π_2 is an open map. Now assume that $\phi_* \in \mathcal{W}\backslash\{0\}$ is an equilibrium of Eq. (*PE*) with $|\lambda - 1| < r$. Since $\phi_* \in \mathcal{W}$, ϕ_* can be written as $\phi_* = \Pi_2\tilde{\phi}^\circ$, where $\tilde{\phi}^\circ \in \widetilde{\Omega}_0$ satisfies $\Pi^c\tilde{\phi}^\circ = \Phi_c z^\circ$ for some $z^\circ = \operatorname{col}(\lambda^\circ - 1, s^\circ) \in U_1$. Hence, $\Pi_X^c\phi_* = \Pi_X^c\Pi_2\tilde{\phi}^\circ = \Pi_2\Pi^c\tilde{\phi}^\circ = \Pi_2\Phi_c z^\circ = w_1(0)(s^\circ\eta_*)$.

On the other hand, in view of Remark 4.15, $\tilde{\phi}_* := j_{\lambda-1}(\phi_*)$ is an equilibrium of Eq. ($\widetilde{E}$). Since $\Pi^c\tilde{\phi}_* = j_{\lambda-1}(\Pi_X^c\phi_*) = w_1(0)\big(\lambda - 1)e_1 + s^\circ\eta_*\big)$ and $\Pi^s\tilde{\phi}_* = j_0(\Pi_X^s\phi_*) = j_0(\Pi_2\Pi^s\tilde{\phi}_\circ)$, we have

$$\|\Pi^c\tilde{\phi}_*\|_{\widetilde{X}} \le \rho^{-1}(|\lambda - 1| + |s^\circ||\eta_*|) < \rho^{-1}(r + s^*|\eta_*|) < \frac{\delta}{3},$$

and

$$\|\Pi^s\tilde{\phi}_*\|_{\widetilde{X}} = \|\Pi_2\Pi^s\tilde{\phi}^\circ\|_X \le \|\Pi^s\tilde{\phi}^\circ\|_{\widetilde{X}} < \frac{\delta}{3}$$

(because of $\phi^\circ \in \widetilde{\Omega}_0$). Hence $\tilde{\phi}_*$ is an equilibrium of Eq. ($\widetilde{E}$) in $\widetilde{\Omega}_0$, and therefore, by (4.35), $\tilde{\phi}_* = \Phi_c z_* + \widetilde{F}_*(\Phi_c z_*)$ for some equilibrium z_* of Eq. ($C\widetilde{E}$). Thus, we have $\Pi_1\tilde{\phi}_* = \Pi_1 j_{\lambda-1}(\phi_*) = w_1(0)(\lambda - 1)$, and $\Pi_1\tilde{\phi}_* = \Pi_1\Phi_c z_*$ as well (see (4.40)); in particular, z_* is of the form $z_* = \operatorname{col}(\lambda - 1, s_*) \in \mathbb{R}^2$. One can also obtain that $\Pi_X^c\phi_* = \Pi_X^c\Pi_2\tilde{\phi}_* = \Pi_2\Pi^c\tilde{\phi}_* = \Pi_2\Phi_c z_* = w_1(0)(s_*\eta_*)$, so that $s^\circ = s_*$. Consequently, $z_* = \operatorname{col}(\lambda - 1, s^\circ) \in U_1$. It therefore follows from Lemma 4.14 (a_2) that $z_* = z^{\lambda-1}$, and hence that $\tilde{\phi}_* = \Phi_c z^{\lambda-1} + \widetilde{F}_*(\Phi_c z^{\lambda-1}) = \tilde{\phi}_*^\lambda$, which yields $\phi_* = \phi_*^\lambda$.

(ii) (iia) Let $\lambda = 1$ and $s^* > 0$ be the one in Lemma 4.14 (b_1). In a similar manner to the proof of (ia), one can see that $\mathcal{W}_0 := B_X(\rho_1)$ is a required open set. Now let s_+^ε and s_-^ε the functions from $(0, \varepsilon^*)$ to $\mathbb{R}\backslash\{0\}$ given in Lemma 4.14 (b_1), and set $z_\pm^\varepsilon := \operatorname{col}(\varepsilon, s_\pm^\varepsilon)$. Define the functions $\phi_\pm : (1, 1 + \varepsilon^*) \to X$ by

$$\phi_\pm^\lambda := \Pi_2\big(\Phi_c z_\pm^{\lambda-1} + \widetilde{F}_*(\Phi_c z_\pm^{\lambda-1})\big), \quad \lambda \in (1, 1 + \varepsilon^*),$$

where we choose ε^* small if necessary. Then ϕ_+^λ and ϕ_-^λ are distinct equilibria of Eq. (*PE*) which tends to 0 as $\lambda \to 1+0$.

The proof of the instability of the equilibrium 0 is quite similar to that of (1a).

(iib) By the same reasoning as (iia), the stability properties of $\phi_\pm^\lambda$ and 0 directly follow from Lemma 4.14 (b_1).

(iic) Let U_1 be the open set of $\mathbb{R}^2$ in Lemma 4.14 (b_2), which may also be thought of a rectangle $U(r) = (-r, r) \times (-r, r)$ with $r > 0$ small. Then, $\mathcal{W} := \Pi_2\big(\widetilde{\Omega}_0 \cap (\Pi^c)^{-1}(\Phi_c U_1)\big)$ is a desired open set of X; the proof being similar to that of (ic).

(iii) The proof is quite the same as that of (ii), and hence we will omit it. □

Now we prove Lemma 4.14. For this, we will employ the following lemma, which is the implicit function theorem for Lipschitz continuous maps.

Lemma 4.16. *Let U and V be open sets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and (u_0, v_0) a point of $U \times V$. Suppose that $G : U \times V \to \mathbb{R}^n$ is a continuous map satisfying the followings:*

(i) *For each $v \in V$, $G(\cdot, v)$ is a Lipschitz continuous map from U to $\mathbb{R}^n$; and in addition there exists an $l_* \in [0, 1)$ such that the Lipschitz constant of $E_n - G(\cdot, v)$ does not exceed l_* for $v \in V$, E_n being the $n \times n$ unit matrix.*

(ii) $G(u_0, v_0) = 0$.

Then there exist an $r > 0$ and a continuous map $\varphi : B_{\mathbb{R}^n}(u_0; r) \to V$ with the properties that $\varphi(u_0) = v_0$ and

$$G(u, \varphi(u)) = 0, \quad u \in B_{\mathbb{R}^n}(u_0; r).$$

Moreover, if $r > 0$ and $\tilde{r} > 0$ are sufficiently small, then $G(u, v) = 0$ with $(u, v) \in B_{\mathbb{R}^n}(u_0; r) \times B_{\mathbb{R}^m}(v_0; \tilde{r})$ implies $v = \varphi(u)$.

Proof of Lemma 4.14. Set $\Delta(\varepsilon, s) := h(\varepsilon, s) + \alpha s\varepsilon - \beta s^m$. By letting $s := \sigma\varepsilon^{1/(m-1)}$, $h(\varepsilon, s)$ can be written as

$$\widetilde{h}(\varepsilon, \sigma) := h(\varepsilon, \sigma\varepsilon^{1/(m-1)}) = \varepsilon^{m/(m-1)}(-\alpha\sigma + \beta\sigma^m) + \Delta(\varepsilon, \sigma\varepsilon^{1/(m-1)}).$$

So,

$$\widetilde{h}(\varepsilon, \sigma) = \varepsilon^{m/(m-1)}\big(-\alpha\sigma + \beta\sigma^m + R(\varepsilon, \sigma)\big), \quad (\varepsilon, \sigma) \in \widetilde{U}_0 := \widetilde{\tau}^{-1}(U_0), \tag{4.49}$$

where $R(\varepsilon, \sigma)$ is the function defined by

$$R(\varepsilon, \sigma) := \begin{cases} \varepsilon^{-m/(m-1)}\Delta(\varepsilon, \sigma\varepsilon^{1/(m-1)}), & \varepsilon \neq 0, \\ 0, & \varepsilon = 0, \end{cases}$$

and $\widetilde{\tau}$ is the one defined by $\widetilde{\tau}(\varepsilon, \sigma) := (\varepsilon, \sigma\varepsilon^{m/(m-1)})$. Then we have:

Claim 2. $R(\varepsilon, \sigma)$ has the following properties.

(R_1) $R(\varepsilon, \sigma)$ is of class C^1 in $\widetilde{U}_0 \backslash \{\varepsilon = 0\}$.

(R_2) $R(\varepsilon, \sigma)$ and the partial derivative $R_\sigma(\varepsilon, \sigma)$ are continuous in $\widetilde{U}_0$.

Proof of Claim 2. (R_1) is evident.

(R_2) Since m is even,

$$R(\varepsilon,\sigma) = \frac{\Delta(\varepsilon,\sigma\varepsilon^{1/(m-1)})}{|\sigma\varepsilon^{1/(m-1)}|^m + |\varepsilon\sigma\varepsilon^{1/(m-1)}|}\left(|\sigma|^m + |\sigma|\right), \quad \varepsilon \neq 0. \tag{4.50}$$

(4.31) therefore implies that $R(\varepsilon,\sigma)$ converges to 0 uniformly for $|\sigma| \le \sigma^*$ as $\varepsilon \to 0$, where $\sigma^* > 0$ is arbitrary; and in particular $R(\varepsilon,\sigma)$ is continuous at the points in $\widetilde{U}_0 \cap \{\varepsilon = 0\}$, which, together with ($R_1$), yields the continuity in $\widetilde{U}_0$ of $R(\varepsilon,\sigma)$. In virtue of (4.32) and the fact that

$$R_\sigma(\varepsilon,\sigma) = \varepsilon^{-1}\frac{\partial\Delta}{\partial s}(\varepsilon,\sigma\varepsilon^{1/(m-1)}) = \frac{\Delta_s(\varepsilon,\sigma\varepsilon^{1/(m-1)})}{|\sigma\varepsilon^{1/(m-1)}|^{m-1} + |\varepsilon|}\left(|\sigma|^{m-1} + 1\right)$$

for $\varepsilon \neq 0$, $R_\sigma(\sigma,\varepsilon)$ converges to $0 = R_\sigma(0,\sigma)$ uniformly for $|\sigma| \le \sigma^*$ as $\varepsilon \to 0$; hence, by the same reasoning as the former part, $R_\sigma(\varepsilon,\sigma)$ is continuous in $\widetilde{U}_0$. This proves Claim 2.

Now let

$$F(\varepsilon,\sigma) := -\alpha\sigma + \beta\sigma^m + R(\varepsilon,\sigma), \quad (\varepsilon,\sigma) \in \widetilde{U}_0.$$

(a) (a_1) The first part is obvious. Since m is even, $F(0,\sigma)$ has always two zeros, i.e., $\sigma = 0$ and

$$\sigma_0 := \left(\frac{\alpha}{\beta}\right)^{1/(m-1)}. \tag{4.51}$$

Notice that $F_\sigma(0,\sigma_0) = -\alpha + m\beta\sigma_0^{m-1} = (m-1)\alpha \neq 0$, and define $G(\varepsilon,\sigma)$ by

$$G(\varepsilon,\sigma) := \frac{1}{F_\sigma(0,\sigma_0)}F(\varepsilon,\sigma), \quad (\varepsilon,\sigma) \in \widetilde{U}_0.$$

G is a continuous function with $G(0,\sigma_0) = 0$ and $G(\varepsilon,\cdot)$ is locally Lipschitz continuous for each ε. In view of (R_2) and the fact that $G_\sigma(0,\sigma_0) = 1$, there exist open intervals U and V containing 0 and σ_0, respectively, and an $l_0 \in [0,1)$ such that $\mathrm{Lip}\,(1 - G(\varepsilon,\cdot))$, the Lipschitz constant of the restriction $(1 - G(\varepsilon,\cdot))\big|_V$, satisfies

$$\mathrm{Lip}\,(1 - G(\varepsilon,\cdot)) = \sup_{(\varepsilon,\sigma)\in U\times V} |1 - G_\sigma(\varepsilon,\sigma)| \le l_0, \quad \varepsilon \in U.$$

Therefore it follows from Lemma 4.16 that there exist an $\varepsilon^* > 0$ and a continuous function $\sigma : (-\varepsilon^*,\varepsilon^*) \to V$ with the properties that $\sigma(0) = \sigma_0$ and $G(\varepsilon,\sigma(\varepsilon)) = 0$ for $\varepsilon \in (-\varepsilon^*,\varepsilon^*)$. Hence, $\widetilde{h}(\varepsilon.\sigma(\varepsilon)) = \varepsilon^{m/(m-1)}F(\varepsilon,\sigma(\varepsilon)) = 0$ for $\varepsilon \in (-\varepsilon^*,\varepsilon^*)\backslash\{0\}$. We may also assume that $\sigma(\varepsilon) \neq 0$ for $\varepsilon \in (-\varepsilon^*,\varepsilon^*)$ since $\sigma_0 \neq 0$ and $\sigma(\varepsilon)$ is continuous. Set $s_\varepsilon := \sigma(\varepsilon)\varepsilon^{1/(m-1)}$. Then $s : (-\varepsilon^*,\varepsilon^*) \to \mathbb{R}$ is a continuous function which maps $(-\varepsilon^*,\varepsilon^*)\backslash\{0\}$ into $\mathbb{R}\backslash\{0\}$, and satisfies

$$h(\varepsilon,s_\varepsilon) = \widetilde{h}(\varepsilon,\sigma(\varepsilon)) = 0, \quad \varepsilon \in (-\varepsilon^*,\varepsilon^*)\backslash\{0\}$$

together with $s_\varepsilon \to 0$ as $\varepsilon \to 0$. Moreover, it follows from (4.32) that

$$\begin{aligned}\frac{\partial h}{\partial s}(\varepsilon,s_\varepsilon) &= -\alpha\varepsilon + m\beta\sigma(\varepsilon)^{m-1}\varepsilon + o(\varepsilon)\\ &= -\alpha\varepsilon + m\beta\sigma_0^{m-1}\varepsilon + m\beta(\sigma(\varepsilon)^{m-1} - \sigma(0)^{m-1})\varepsilon + o(\varepsilon)\\ &= (m-1)\alpha\varepsilon + o(\varepsilon) \quad \text{as } \varepsilon \to 0,\end{aligned} \tag{4.52}$$

where we used (4.51) and the continuity of $\sigma(\varepsilon)$; so that (4.33) holds.

Also, by Lemma 4.16 $G(\varepsilon,\sigma)=0$ with $(\varepsilon,\sigma)\in(-\varepsilon^*,\varepsilon^*)\times(\sigma_0-\sigma^*,\sigma_0+\sigma^*)$ yields $\sigma=\sigma(\varepsilon)$, provided that $\varepsilon^*>0$ and $\sigma^*>0$ are small enough.

We next consider $\sigma=0$ as the other zero of $F(0,\sigma)$. Since $F_\sigma(0,0)=-\alpha\neq 0$, by applying the same argument as above to the function

$$\widetilde{G}(\varepsilon,\sigma):=\frac{1}{F_\sigma(0,0)}F(\varepsilon,\sigma),\quad (\varepsilon,\sigma)\in\widetilde{U}_0,$$

one can see that there exist an $\tilde{\varepsilon}^*>0$ and a continuous function $\tilde{\sigma}:(-\tilde{\varepsilon}^*,\tilde{\varepsilon}^*)\to V$ with the properties that $\widetilde{\sigma}(0)=0$ and $\widetilde{G}(\varepsilon,\widetilde{\sigma}(\varepsilon))=0$ for $\varepsilon\in(-\tilde{\varepsilon}^*,\tilde{\varepsilon}^*)$. On the other hand, it follows from (4.31) that $\Delta(\varepsilon,0)=0$, and hence $\widetilde{G}(\varepsilon,0)=0$. By the same reasoning as the last paragraph we obtain $\widetilde{\sigma}(\varepsilon)=0$ for $\varepsilon\in(-\tilde{\varepsilon}^*,\tilde{\varepsilon}^*)$ with $\tilde{\varepsilon}^*>0$ small.

(a$_2$) Let $U(r):=(-r,r)\times(-r,r)$ and

$$d_r:=\sup\left\{|\Delta(\varepsilon,s)|/(|s|^m+|s\varepsilon|):(\varepsilon,s)\in U(r)\backslash\{s=0\}\right\}\tag{4.53}$$

for $r>0$. Then $d_r\to 0$ as $r\to 0$ by (4.31). Put

$$c_*:=\inf_{|\sigma-\sigma_0|\geq\sigma^*}|-\alpha+\beta\sigma^{m-1}|\tag{4.54}$$

and take an $r>0$ small enough so that $r<\varepsilon^*$ and

$$d_r<\min\left(\frac{|\beta|}{2},\frac{c_*}{2(|\sigma_0|^{m-1}+1)}\right).\tag{4.55}$$

Now suppose that $h(\hat{\varepsilon},\hat{s})=0$ for some $(\hat{\varepsilon},\hat{s})\in U(r)$ with $\hat{s}\neq 0$, and set $\hat{\sigma}:=\hat{s}\hat{\varepsilon}^{-1/(m-1)}$. (Notice that $\hat{\varepsilon}\neq 0$ due to (4.55).) Since $F(\hat{\varepsilon},\hat{\sigma})=0$, it follows from (4.50) that

$$|-\alpha\hat{\sigma}+\beta\hat{\sigma}^m|=|R(\hat{\varepsilon},\hat{\sigma})|\leq d_r(|\hat{\sigma}|^m+|\hat{\sigma}|).\tag{4.56}$$

Hence, in view of $\hat{\sigma}\neq 0$ and (4.55),

$$\frac{|\beta|}{2}|\hat{\sigma}|^{m-1}\leq(|\beta|-d_r)|\hat{\sigma}|^{m-1}\leq\alpha+d_r\leq\alpha+\frac{|\beta|}{2},$$

so that $|\hat{\sigma}|^{m-1}\leq 2|\sigma_0|^{m-1}+1$, and therefore (4.56) and (4.55) imply

$$|-\alpha+\beta\hat{\sigma}^{m-1}|\leq d_r(|\hat{\sigma}|^{m-1}+1)\leq 2d_r(|\sigma_0|^{m-1}+1)<c_*,$$

from which and (4.54) we deduce that $\hat{\sigma}\in(\sigma_0-\sigma^*,\sigma_0+\sigma^*)$. Thus, we arrive at $G(\hat{\varepsilon},\hat{\sigma})=0$ with $(\hat{\varepsilon},\hat{\sigma})\in(-\varepsilon^*,\varepsilon^*)\times(\sigma_0-\sigma^*,\sigma_0+\sigma^*)$. The argument in the proof of (a$_1$) then yields $\hat{\sigma}=\sigma(\hat{\varepsilon})$ and therefore $\hat{s}=s_{\hat{\varepsilon}}$. Consequently, (a$_2$) holds true with $U_1=U(r)$.

(b) (b$_1$) The first part is obvious. Let us take $r>0$ so small that $d_r<\min(\alpha,\beta)$, d_r being the one in (4.53). Then, it follows that for $(\varepsilon,s)\in U(r)$ with $s\neq 0$ and $\varepsilon\leq 0$,

$$\begin{aligned}sh(\varepsilon,s)&=s(-\alpha s\varepsilon+\beta s^m+\Delta(\varepsilon,s))\\&\geq\alpha|\varepsilon|s^2+\beta|s|^{m+1}-|s||\Delta(\varepsilon,s)|\\&\geq(\alpha-d_r)|\varepsilon|s^2+(\beta-d_r)|s|^{m+1}>0.\end{aligned}$$

Thus, $h(\varepsilon,\cdot)$ has no zeros in $(-r,r)$ other than $s=0$, provided that $-r<\varepsilon\leq 0$.

Assume now that $\varepsilon > 0$. Since m is odd, $F(0,\sigma) = 0$ has three zeros, i.e., $\sigma = 0$, $\sigma_0^+ := (\alpha/\beta)^{1/(m-1)}$ and $\sigma_0^- := -(\alpha/\beta)^{1/(m-1)}$. Define $h^+ : U_0 \to \mathbb{R}$ by

$$h^+(\varepsilon,s) := -\alpha s\varepsilon + \beta|s|^m + \Delta(\varepsilon,s), \quad (\varepsilon,s) \in U_0.$$

Then, h^+ is of class C^1 and by (4.32),

$$\frac{\partial h^+}{\partial s}(\varepsilon,s) = -\alpha\varepsilon + m\beta|s|^{m-1}\operatorname{sgn} s + o(|s|^{m-1} + |\varepsilon|),$$

where $\operatorname{sgn} s$ denotes the sign of $s \in \mathbb{R}$ with $\operatorname{sgn} 0 := 0$. Moreover, by letting $s := \sigma|\varepsilon|^{1/(m-1)}\operatorname{sgn}\varepsilon$, we have

$$\widetilde{h}^+(\varepsilon,\sigma) := h^+(\varepsilon,\sigma|\varepsilon|^{1/(m-1)}\operatorname{sgn}\varepsilon) = |\varepsilon|^{m/(m-1)}\big(-\alpha\sigma + \beta|\sigma|^m + R^+(\varepsilon,\sigma)\big),$$

according to (4.49), where

$$R^+(\varepsilon,\sigma) := \begin{cases} |\varepsilon|^{-m/(m-1)}\Delta\big(\varepsilon,\sigma|\varepsilon|^{1/(m-1)}\operatorname{sgn}\varepsilon\big), & \varepsilon \neq 0, \\ 0, & \varepsilon = 0. \end{cases}$$

Consequently, the arguments in the proof of Claim 2 (resp. (a)) are valid for R^+ (resp. h^+ with σ_0^+ in place of σ_0), and hence there exists a function $s : (-\varepsilon^*,\varepsilon^*) \to \mathbb{R}$ satisfying the properties (a_1) and (a_2). In particular, it follows from $\sigma_0^+ > 0$ that $s_\varepsilon > 0$ for $\varepsilon \in (0,\varepsilon^*)$, which, combined with the fact that $h^+(\varepsilon,s) = h(\varepsilon,s)$ for $s \geq 0$, implies that $s^+ := s|_{(0,\varepsilon^*)}$ is one of the desired functions.

For the existence of $s^- : (0,\varepsilon^*) \to \mathbb{R}$, we have only to consider the function

$$h^-(\varepsilon,s) := -\alpha s\varepsilon - \beta|s|^m + \Delta(\varepsilon,s), \quad (\varepsilon,s) \in U_0.$$

In the same way as (4.52), we also obtain

$$\begin{aligned}\frac{\partial h^\pm}{\partial s}(\varepsilon,s_\varepsilon^\pm) &= -\alpha\varepsilon + m\beta|s_\varepsilon^\pm|^{m-1} + o(\varepsilon) = -\alpha\varepsilon + m\beta|\sigma_0^\pm|^{m-1}\varepsilon + o(\varepsilon) \\ &= (m-1)\alpha\varepsilon + o(\varepsilon) \quad \text{as } \varepsilon \to +0,\end{aligned}$$

and (4.34) follows from

$$\frac{\partial h^+}{\partial s}(\varepsilon,s) = \frac{\partial h}{\partial s}(\varepsilon,s) \quad (s \geq 0), \quad \text{and} \quad \frac{\partial h^-}{\partial s}(\varepsilon,s) = \frac{\partial h}{\partial s}(\varepsilon,s) \quad (s \leq 0).$$

(b_2) Since the statement of (a_2) holds for both $h^+(\varepsilon,s)$ and $h^-(\varepsilon,s)$, the definition of s^+ and s^-, together with the argument of the first paragraph, imply the existence of a desired open set of U_0.

(c) The proof is quite similar to that of (b), and is omitted. □

5 Examples

We will show an example on one-parameter bifurcation to illustrate our Theorem 4.4.

Example 5.1. Let us consider a nonlinear scalar integral equation

$$x(t) = \lambda \int_{-\infty}^{t} p(t-s)x(s)ds + f(x_t), \tag{5.1}$$

where λ is a nonnegative parameter and p is a nonnegative continuous function on $\mathbb{R}^+$ that satisfies

$$\int_0^\infty p(t)dt = 1 \tag{5.2}$$

together with

$$\|p\|_{1,\rho} = \int_0^\infty p(t)e^{\rho t}dt < \infty \quad \text{and} \quad \|p\|_{\infty,\rho} = \operatorname{ess\,sup}\{p(t)e^{\rho t} : t \geq 0\} < \infty$$

for some positive constant ρ. Let $X := L^1_\rho(\mathbb{R}^-;\mathbb{R})$ and $f : X \to \mathbb{R}$ be of the form

$$f(\phi) = \left(\int_{-\infty}^{0} q(-\theta)\phi(\theta)d\theta \right)^3 + g(\phi),$$

where $q : \mathbb{R}^+ \to \mathbb{R}$ satisfies

$$\int_0^\infty q(t)dt > 0, \quad \int_0^\infty |q(t)|e^{\rho t}dt < \infty \quad \text{and} \quad \operatorname{ess\,sup}\{|q(t)|e^{\rho t} : t \geq 0\} < \infty$$

and $g \in C^1(X;\mathbb{R})$ with $g(\phi) = o(\|\phi\|_X^3)$ as $\|\phi\|_X \to 0$.

Since $f \in C^1(X;\mathbb{R})$ with $f(0) = 0$ and $Df(0) = 0$, so [23, Proposition 8] implies the following fact:

- If $0 \leq \lambda < 1$, then the zero solution of Eq. (5.1) is exponentially stable (in L^1_ρ);
- If $\lambda > 1$, then the zero solution of Eq. (5.1) is unstable (in L^1_ρ).

We will verify that $\lambda = 1$ is a bifurcation point and a pitchfork bifurcation occurs when λ exceeds 1. Notice that $P_0 = \int_0^\infty p(t)dt = 1$ and $P_1 = \int_0^\infty tp(t)dt > 0$. So 0 is a simple eigenvalue of P_0 and $P_1\big(\mathcal{N}(1-P_0)\big) = \mathbb{R}$; hence ($A_1$) and ($A_2$) hold together with ($A_4$). In addition, since the characteristic operator is $\Delta(z) = 1 - \lambda \int_0^\infty p(t)e^{-zt}dt$, we see from (5.2) that

$$\Delta(i\omega) = \int_0^\infty p(t)(1-\cos\omega t)dt + i\int_0^\infty p(t)\sin\omega t dt, \quad \omega \in \mathbb{R}.$$

So, if $\det\Delta(i\omega) = \Delta(i\omega) = 0$ for some $\omega > 0$, then

$$\int_0^\infty p(t)(1-\cos\omega t)dt = \int_0^\infty p(t)\sin\omega t dt = 0.$$

Hence, $p(t) = 0$ on $\mathbb{R}^+$ follows, for $p(t)$ is continuous and nonnegative on $\mathbb{R}^+$ and $1-\cos\omega t > 0$ ($t \in \mathbb{R}\backslash(2\pi\omega^{-1}\mathbb{Z})$); see also [22, Example 4.1]. This contradicts to (5.2), so that $\det\Delta(i\omega) \neq 0$ for $\omega > 0$; and therefore (A_3) is also valid due to Remark 4.2 and (5.2).

Since one can choose $\eta_* = 1$ and $\zeta^* = 1$, so $q_* = -P_1 < 0$ and

$$c_0 = \left(\int_{-\infty}^{0} q(-\theta)(w_1(0))(\theta)d\theta \right)^3 = \left(\int_0^\infty q(t)dt \right)^3 > 0$$

(see (4.5)). Consequently by Theorem 4.4 (iib) a pitchfork bifurcation is observed when the parameter λ exceeds 1; the equilibrium 0 becomes unstable and two more equilibria appear in a neighborhood of 0, each of which is asymptotically stable for $\lambda > 1$.

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