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Uniqueness of positive radial solutions for a class of (p,q)-Laplacian problems in a ball

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Abstract. We prove the uniqueness of positive radial solution to the (p,q)-Laplacian problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p > q > 1$, $\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u)$, $\Omega = B(0,1)$ is the open unit ball in $\mathbb{R}^N$, $f : (0,\infty) \to \mathbb{R}$ is q-sublinear at ∞ with possible semipositone structure at 0, and $\lambda > 0$ is a large parameter.

Keywords: (p,q)-Laplacian, positive solutions, uniqueness.

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1 Introduction

In this paper, we study the uniqueness of positive radial solutions for the (p,q)-Laplacian boundary value problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda f(u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is the open unit ball in $\mathbb{R}^N$, $p > q > 1$, $\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u(|\nabla u|^{r-2}\nabla u)$, $f : (0,\infty) \to \mathbb{R}$ is p-sublinear at ∞ i.e. $\lim_{u\to\infty} \frac{f(u)}{u^{p-1}} = 0$ and λ is a positive parameter. The (p,q)-Laplacian problems occurred in a variety of applied areas such as quantum physics, plasma physics, and reaction diffusion to name a few, see e.g. [2, 4, 6, 13, 14, 23]. When f is p-sublinear at ∞, the existence of positive solutions to (1.1) for λ large was recently established in [1] under the mere additional natural condition that $\liminf_{u\to\infty} u^\beta f(u) > 0$ and $\limsup_{u\to 0^+} u^\beta |f(u)| < \infty$ for some $\beta \in [0,1)$, which extended previous results in [10, 20]. When $p = q$, the uniqueness of positive solutions to (1.1) for all $\lambda > 0$ was established in the pioneering work [3] for $p = 2$ when $\frac{f(u)}{u}$ is strictly decreasing on $(0,\infty)$, and subsequently extended to the

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case $p > 1$ in [11]. When this monotonicity condition is assumed only for u large, uniqueness results for λ large were obtained under additional hypotheses in [5,7–9,12,15–19,21,22,24–26], where the semipositone case i.e. $-\infty < f(0) < 0$ is allowed in [18]. Note that previous uniqueness proofs depend on the homogeneity of the p-Laplace and cannot be extended to the (p,q)-Laplace operator. In this paper, we shall establish the uniqueness of positive solutions to (1.1) for λ large when f is q-sublinear at ∞ with possible semipositone structure at 0, which has not been obtained in the literature to the best of our knowledge. Related uniqueness results for (1.1) in a bounded domain of $\mathbb{R}^N$ can be found in [10, 13]. In [13, Theorem 2.2], the uniqueness of positive solutions for all $\lambda > 0$ was established when $z^{1-q}f(z)$ is decreasing for $z > 0$ and the assumption that any two positive solutions u_i, $i = 1,2$, of (1.1) satisfy $\Delta_p u_i \in L^\infty(\Omega)$, and $u_i/u_j \in L^\infty(\Omega), i \neq j$. The uniqueness for λ large in the semipositone case was observed in [10] for the special Dirichlet two-point boundary problem

$$\begin{cases} -((u')^3)' - u'' = \lambda f(u) \ \text{ in } (0,1), \\ \qquad\qquad u(0) = u(1) = 0, \end{cases}$$

where $f(u) = (u+1)^\gamma - 2$ for some $\gamma \in (0,3)$. Note that this observation is generated by *Mathematica* and does not constitute an analytical proof.

Since we are looking for radial solutions for (1.1), it reduces to finding solutions of the ODE problem

$$\begin{cases} -(r^{N-1}\phi(u'))' = \lambda r^{N-1} f(u), \ 0 < r < 1, \\ \qquad u'(0) = 0, \ u(1) = 0, \end{cases} \tag{1.2}$$

where $\phi(x) = \phi_p(x) + \phi_q(x)$, $\phi_r(x) = |x|^{r-2}x$.

We make the following assumptions.

(A1) $f : (0,\infty) \to \mathbb{R}$ is continuous and f is increasing on $[L,\infty)$ for some $L > 0$.

(A2) $\limsup_{z\to\infty} \frac{zf'(z)}{f(z)} < q - 1$.

(A3) There exists $\gamma \in [0,1)$ such that $\limsup_{z\to 0^+} z^{\gamma+1}|f'(z)| < \infty$ and $f(0^+) < \infty$ if $\gamma = 0$.

By a positive solution of (1.2), we mean a function $u \in C^1[0,1]$ with $u > 0$ on $[0,1)$ and satisfying (1.2). Our main result is

Theorem 1.1. *Let (A1)–(A3) hold and suppose either*

(A4) $f(z) > 0$ *for* $z > 0$ *with* $\liminf_{z\to 0^+} \frac{f(z)}{z^{q-1}} > 0$.

or

(A5) $-\infty < f(0^+) < 0$ *and there exists* $\beta > 0$ *such that* $(z-\beta)f(z) > 0$ *for* $z \neq \beta$.

holds. Then there exists a constant $\bar{\lambda} > 0$ *such that* (1.2) *has a unique positive solution for* $\lambda > \bar{\lambda}$.

Remark 1.2.

(i) Note that (A2) implies $\frac{f(z)}{z^{q-1}}$ *is decreasing for z large and* $\lim_{z\to\infty} \frac{f(z)}{z^{q-1}} = 0$.

(ii) It is easily seen from (A3) that $\limsup_{z\to 0^+} z^\gamma |f(z)| < \infty$. *Since* $\liminf_{z\to\infty} z^\gamma f(z) > 0$ *and* $\limsup_{z\to 0^+} z^\gamma |f(z)| < \infty$ *, it follows from [1, Theorem 1.1] that* (1.2) *has a positive solution for* λ *large under the assumptions of Theorem 1.1.*

2 Preliminary results

Let $H(z) = \frac{\phi_p(z)}{f(z)}$. Then H is increasing for z large and $\lim_{z\to\infty} H(z) = \infty$ in view of Remark 1.2 (i).

Lemma 2.1. *Let $K > 0$. Then for $\lambda \gg 1$,*

(i) $H^{-1}(\lambda K) \le K^{\frac{1}{p-q}} H^{-1}(\lambda)$ if $K > 1$.

(ii) $H^{-1}(\lambda K) \ge K^{\frac{1}{p-q}} H^{-1}(\lambda)$ if $K < 1$.

Proof. Let $z_\lambda = H^{-1}(\lambda)$ and $\tilde{K} = K^{\frac{1}{p-q}}$. Then $z_\lambda \to \infty$ as $\lambda \to \infty$.

Suppose $K > 1$. Then $f(\tilde{K} z_\lambda) \le \tilde{K}^{q-1} f(z_\lambda)$ for λ large in view of Remark 1.2 (i), which implies

$$\frac{\phi_p(\tilde{K} z_\lambda)}{f(\tilde{K} z_\lambda)} \ge \tilde{K}^{p-q} \frac{\phi_p(z_\lambda)}{f(z_\lambda)} = \lambda \tilde{K}^{p-q} = \lambda K,$$

i.e. $\tilde{K} H^{-1}(\lambda) = \tilde{K} z_\lambda \ge H^{-1}(\lambda K)$ and (i) holds. If $K < 1$ then by replacing λ by λK and K by K^{-1} in (i), we obtain (ii). □

Lemma 2.2. *Let $\lambda, x, c > 0$. Then*

$$\phi^{-1}(\lambda x) \ge a \phi_p^{-1}(\lambda x)$$

for $\lambda x > c$, provided that $a^{p-1} + a^{q-1} c^{\frac{q-p}{p-1}} \le 1$.

Proof. Let $y = \phi_p^{-1}(\lambda x)$. Then

$$\phi(ay) = (ay)^{p-1} + (ay)^{q-1} = a^{p-1} \lambda x + a^{q-1} (\lambda x)^{\frac{q-1}{p-1}}. \tag{2.1}$$

Since $\lambda x > c$,

$$a^{q-1} (\lambda x)^{\frac{q-1}{p-1}} = a^{q-1} c^{\frac{q-1}{p-1}} \left(\frac{\lambda x}{c}\right)^{\frac{q-1}{p-1}} < a^{q-1} c^{\frac{q-1}{p-1}} \left(\frac{\lambda x}{c}\right) = a^{q-1} c^{\frac{q-p}{p-1}} \lambda x,$$

which together with (2.1) imply

$$\phi(ay) \le \left(a^{p-1} + a^{q-1} c^{\frac{q-p}{p-1}}\right) \lambda x \le \lambda x$$

if $a^{p-1} + a^{q-1} c^{\frac{q-p}{p-1}} \le 1$ i.e. $ay \le \phi^{-1}(\lambda x)$, which completes the proof. □

If (A4) holds then clearly positive solutions to (1.2) are decreasing on (0,1). The next result shows this is also true in the semipositone case under assumption (A5).

Lemma 2.3. *Let (A1) and (A5) hold. Then any positive solution u of (1.2) is decreasing on $(0,1)$ with $u(0) \ge \theta$, where $\theta > \beta$ is such that $(z-\theta)F(z) > 0$ for $z > 0$, $z \ne \theta$. Here $F(z) = \int_0^z f$.*

Proof. Let u be a positive solution of (1.2). Since $f(0^+) < 0$, it follows that $(r^{N-1}\phi(u'))' > 0$ for r near 1 and therefore $u'(r) < u'(1) \le 0$ for such r. Let $r_0 \in [0,1)$ be the smallest number such that $u' < 0$ on $(r_0, 1)$ and suppose $r_0 > 0$. Multiplying the equation in (1.2) by u' gives

$$G'(r) = -(N-1) r^{N-2} \Phi(u') + \lambda (n-1) r^{N-2} F(u) \quad \text{on } (0,1),$$

where $\Phi(z) = \int_0^z \phi$ and $G(r) = r^{N-1}(\phi(u')u' - \Phi(u') + \lambda F(u))$. Note that $G(1) = (1 - 1/p)|u'(1)|^p + (1-1/q)|u'(1)|^q \geq 0$.

We claim that $u(r_0) > \theta$. Indeed, if $u(r_0) \leq \theta$ then $u < \theta$ on $(r_0, 1)$, which implies G is decreasing on $(r_0, 1)$. Since $u'(r_0) = 0$, it follows that

$$0 \leq G(1) \leq G(r) < G(r_0) \leq 0 \quad \text{on } (r_0, 1),$$

a contradiction. Hence $u(r_0) > \theta$ and so $(r^{N-1}(\phi(u'))' < 0$ near r_0. Consequently, $u'(r) > 0$ for r near $r_0, r < r_0$. Let $r_1 \in [0, r_0)$ such that $u' > 0$ on (r_1, r_0) and $u'(r_1) = 0$. Since $r_0^{N-1}\phi(u'(r_0)) = 0 = r_1^{N-1}\phi(u'(r_1))$, there exists $r_2 \in (r_1, r_0)$ such that

$$-(r^{N-1}\phi(u'(r))'(r_2) = \lambda r_2^{N-1} f(u(r_2)) = 0$$

i.e. $u(r_2) = \beta$. Thus $\theta \in u\left((r_2, r_0]\right)$ and there exists $r_3 \in (r_2, r_0)$ such that $u < \theta$ on $[r_2, r_3)$ and $u(r_3) = \theta$. Since $u' > 0$ on (r_1, r_0), $u < \theta$ on $[r_1, r_3)$ i.e. and so $G' < 0$ on $[r_1, r_3)$, which implies

$$0 \leq G(r_3) < G(r) < G(r_1) < 0,$$

a contradiction. Thus $r_0 = 0$, which completes the proof. □

Lemma 2.4. *Let $c > 0$ and suppose (A1), (A4) hold. Then*

$$\limsup_{x \to 0^+} \frac{\phi(cx)}{\check{f}(x)} < \infty,$$

where $\check{f}(x) = \inf_{y \geq x} f(y)$.

Proof. Since $\liminf_{y \to 0^+} \frac{f(y)}{y^{q-1}} > 0$, there exist constants $k, \delta > 0$ such that

$$f(y) \geq k\phi(cy) \ \text{ for } y \in (0, \delta). \tag{2.2}$$

Let $x < \delta$ and $y \geq x$. If $y < \delta$ then (2.2) gives

$$f(y) \geq k\phi(cx), \tag{2.3}$$

while if $y > \delta$ then

$$f(y) \geq k_0 = k_1\phi(c\delta) \geq k_1\phi(cx), \tag{2.4}$$

where $k_0 = \inf_{[\delta,\infty)} f > 0$ and $k_1 = k_0/\phi(c\delta)$.

Combining (2.3) and (2.4), we obtain

$$\check{f}(x) \geq k_2\phi(cx) \quad \text{for } x \in (0, \delta),$$

where $k_2 = \min(k, k_1)$, which completes the proof. □

The next result gives sharp lower and upper bound estimates for positive solution of (1.2) when λ is large.

Lemma 2.5. *Let the assumptions of Theorem 1.1 hold. Then there exist positive constants A_1, A_2, and λ_0 such that for $\lambda > \lambda_0$, any positive solution of (1.2) satisfies*

$$A_1 B_\lambda (1 - r) \leq u(r) \leq A_2 B_\lambda (1 - r) \tag{2.5}$$

for $r \in (0, 1)$, where $B_\lambda = \phi_p^{-1}\left(\lambda f(H^{-1}(\lambda))\right)$.

Proof. In what follows, $\lambda \gg 1$ or λ large means $\lambda > \lambda^*$ for some $\lambda^* > 0$ independent of u and λ.

Case 1. Suppose (A4) holds.

Let $\lambda > 0$ and u be a positive solution of (1.2). Then u is decreasing on $(0,1)$. By integrating, we get

$$-u'(r) = \phi^{-1}\left(\frac{\lambda}{r^{N-1}}\int_0^r s^{N-1}f(u)ds\right), \qquad 0 < r < 1.$$

Recalling that $\check{f}(x) = \inf_{y\geq x} f(y)$ and since $\check{f}$ is nondecreasing, we have

$$-u'(r) \geq \phi^{-1}\left(\frac{\lambda}{r^{N-1}}\int_0^{1/2} s^{N-1}\check{f}(u)ds\right) \geq \phi^{-1}\left(\lambda c_1 \check{f}\left(u\left(1/2\right)\right)\right) \quad \text{for } r > 1/2, \tag{2.6}$$

where $c_1 = \frac{1}{N2^N}$, which implies upon integrating on $(1/2,1)$ that

$$2u(1/2) \geq \phi^{-1}\left(\lambda c_1 \check{f}\left(u\left(1/2\right)\right)\right)$$

i.e.

$$\frac{\phi(2u(1/2))}{\check{f}\left(u\left(1/2\right)\right)} \geq \lambda c_1. \tag{2.7}$$

Since $\limsup_{x\to 0^+} \frac{\phi(2x)}{\check{f}(x)} < \infty$ in view of Lemma 2.4, we deduce from (2.7) that $u(1/2) \to \infty$ as $\lambda \to \infty$.

Hence $u(1/2) > 1$ for $\lambda \gg 1$, and so

$$\phi(2u(1/2)) \leq 2\phi_p(2u(1/2)) = 2^p\phi_p(u(1/2)),$$

which together with (2.7) and the fact that $f(z) = \check{f}(z)$ for $z > L$, gives

$$H(u(1/2)) = \frac{\phi_p(u(1/2))}{f\left(u\left(1/2\right)\right)} \geq \lambda c_2$$

where $c_2 = \min(c_1/2^p, 1)$. Hence by Lemma 2.1 (ii),

$$u(1/2) \geq H^{-1}(\lambda c_2) \geq c_3 H^{-1}(\lambda), \tag{2.8}$$

where $c_3 = c_2^{\frac{1}{p-q}}$. From (2.6) and (2.8), we get

$$-u'(r) \geq \phi^{-1}\left(\lambda c_1 f\left(c_3 H^{-1}(\lambda)\right)\right) \geq \phi^{-1}\left(\lambda c_1 c_3^{q-1} f\left(H^{-1}(\lambda)\right)\right) \quad \text{for } r > 1/2, \tag{2.9}$$

For $\lambda \gg 1, \lambda c_1 c_3^{q-1} f\left(H^{-1}(\lambda)\right) > 1$, from which (2.9) and Lemma 2.2 with $c = 1$ and $x = c_1 c_3^{q-1} f\left(H^{-1}(\lambda)\right)$ give

$$-u'(r) \geq a\phi_p^{-1}\left(\lambda c_1 c_3^{q-1} f\left(H^{-1}(\lambda)\right)\right) \equiv a_1\phi_p^{-1}(\lambda f\left(H^{-1}(\lambda)\right) = a_1 B_\lambda \quad \text{for } r > 1/2, \tag{2.10}$$

where $B_\lambda = \phi_p^{-1}\left(\lambda f(H^{-1}(\lambda)\right)$ and $a > 0$ is such that $a^{p-1} + a^{q-1} \leq 1$ and $a_1 = a\phi_p^{-1}(c_1 c_3^{q-1})$.

Note that $B_\lambda \to \infty$ as $\lambda \to \infty$. Integrating (2.10) yields

$$u(r) \geq a_1 B_\lambda (1-r) \tag{2.11}$$

for $r > 1/2$, while for $r < 1/2$,

$$u(r) > u(1/2) \geq \frac{a_1}{2} B_\lambda \geq \frac{a_1}{2} B_\lambda (1-r). \tag{2.12}$$

Combining (2.11) and (2.12), we obtain

$$u(r) \geq A_1 B_\lambda (1-r) \quad \text{for } r \in (0,1), \tag{2.13}$$

where $A_1 = a_1/2$ i.e. the lower bound estimate in (2.5) holds.

Let $C = \sup_{z \in (0,L)} z^\gamma f(z)$ and note that $0 < C < \infty$. Since $f > 0$ on $(0, \infty)$,

$$f(z) \leq \frac{C}{z^\gamma} + f(\max(z, L)) \tag{2.14}$$

for $z > 0$. Since $\|u\|_\infty \to \infty$ as $\lambda \to \infty$, $\|u\|_\infty > L$ for $\lambda \gg 1$ and hence (2.13)–(2.14) yield

$$f(u(\tau)) \leq \frac{C}{u^\gamma(\tau)} + f(\max(u(\tau), L)) \leq \frac{C_1}{B_\lambda^\gamma (1-\tau)^\gamma} + f(\|u\|_\infty),$$

for $\tau \in (0,1)$, where $C_1 = C/A_1^\gamma$. Thus

$$\begin{aligned}
u(r) &= \int_r^1 \phi^{-1}\left(\frac{\lambda}{s^{N-1}} \int_0^s \tau^{N-1} f(u) d\tau\right) ds \\
&\leq \int_r^1 \phi^{-1}\left(\frac{\lambda}{s^{N-1}} \int_0^s \tau^{N-1} \left(\frac{C_1}{B_\lambda^\gamma (1-\tau)^\gamma} + f(\|u\|_\infty)\right) d\tau\right) ds \\
&\leq \int_r^1 \phi^{-1}\left(\lambda \left(C_1 \int_0^s \frac{d\tau}{(1-\tau)^\gamma} + f(\|u\|_\infty)\right)\right) \leq \phi^{-1}\left(\lambda (C_2 + f(\|u\|_\infty))\right)(1-r) \\
&\leq \phi^{-1}(\lambda C_3 f(\|u\|_\infty))(1-r) \leq \phi_p^{-1}(\lambda C_3 f(\|u\|_\infty))(1-r)
\end{aligned} \tag{2.15}$$

for $r \in (0,1)$, where $C_2 = C_1 \int_0^1 \frac{d\tau}{(1-\tau)^\gamma}$ and $C_3 > 1$ is such that $(C_3 - 1) f(L) > C_2$.

In particular,

$$\|u\|_\infty \leq \phi_p^{-1}(\lambda C_3 f(\|u\|_\infty)),$$

which implies

$$H(\|u\|_\infty) = \frac{\phi_p(\|u\|_\infty)}{f(\|u\|_\infty)} \leq \lambda C_3,$$

and therefore

$$\|u\|_\infty \leq H^{-1}(\lambda C_3) \leq C_4 H^{-1}(\lambda), \tag{2.16}$$

in view of Lemma 2.1 (i), where $C_4 = C_3^{\frac{1}{p-q}}$.

Combining (2.15)–(2.16) and Remark 1.2 (i), we infer that

$$\begin{aligned}
u(r) &\leq \phi_p^{-1}(\lambda C_3 f\left(C_4 H^{-1}(\lambda)\right))(1-r) \leq \phi_p^{-1}(\lambda C_3 C_4^{q-1} f(H^{-1}(\lambda))(1-r) \\
&= A_2 \phi_p^{-1}(\lambda f(H^{-1}(\lambda))(1-r) \equiv A_2 B_\lambda (1-r)
\end{aligned}$$

for $r \in (0,1)$, where $A_2 = \phi_p^{-1}(C_3 C_4^{q-1})$ i.e. the upper bound estimate in (2.5) holds, which completes the proof.

Case 2. Suppose (A5) holds.

By Lemma 2.3, u is decreasing on $(0, 1)$. Recall that $\beta, \theta > 0$ are such that $(z - \beta)f(z) > 0$ for $z \neq \beta$ and $(z - \theta)F(z) > 0$ for for $z \neq \theta$. Let $\rho = \frac{\beta+\theta}{2}$ and $\delta \in (0, 1)$ be such that

$$\frac{\delta^N}{2N} \inf_{[\rho,\infty)} f - (1 - \delta)K > 0, \tag{2.17}$$

where $K > 0$ is such that $f(z) \geq -K$ for all $z \in (0, \infty)$.

We will show that $u(\delta) \to \infty$ as $\lambda \to \infty$. To this end, we need to verify the following claims. Let $\beta_1 \in (0, \beta)$ and δ_0, δ_1 satisfy $\delta < \delta_0 < \delta_1 < 1$.

Claim 1. $u(\delta_1) > \beta_1$ for $\lambda \gg 1$.

Suppose to the contrary that $u(\delta_1) \leq \beta_1$. Then $u < \beta_1$ on $(\delta_1, 1)$, which implies

$$(r^{N-1}\phi(u'))' = -\lambda r^{N-1} f(u) \geq \lambda m r^{N-1} \quad \text{on } (\delta_1, 1), \tag{2.18}$$

where $m = -\sup_{(0,\beta_1)} f > 0$.

Let $\delta_2 \in (\delta_1, 1)$. By the Mean Value Theorem, there exists $\sigma \in (\delta_1, \delta_2)$ such that

$$|u'(\sigma)| = \frac{u(\delta_1) - u(\delta_2)}{\delta_2 - \delta_1} \leq \frac{\beta_1}{\delta_2 - \delta_1} \equiv \bar{\beta}.$$

Hence by integrating (2.18) on $(\sigma, 1)$, we obtain

$$0 \geq \phi(u'(1)) \geq \sigma^{N-1}\phi(u'(\sigma)) + \lambda m \int_\sigma^1 r^{N-1} dr \geq \lambda m \int_{\delta_2}^1 r^{N-1} dr - \phi(\bar{\beta}) > 0,$$

for λ large, a contradiction which proves the claim.

Claim 2. $u(\delta_0) > \rho$ for $\lambda \gg 1$.

Suppose to the contrary that $u(\delta_0) \leq \rho$. Then $u < \rho$ on $(\delta_0, 1)$ and therefore $G' < 0$ on $(\delta_0, 1)$, where G is defined in the proof of Lemma 2.3. Hence $G(r) \geq G(1) \geq 0$ on $(\tau_0, 1)$ i.e.

$$\phi(u')u' - \Phi(u') + \lambda F(u)) \geq 0 \quad \text{on } (\delta_0, 1),$$

or, equivalently,

$$(1 - 1/p)|u'|^p + (1 - 1/q)|u'|^q \geq -\lambda F(u) \quad \text{on } (\delta_0, 1). \tag{2.19}$$

By Claim 1 and the monotonicity of u,

$$\beta_1 \leq u \leq \rho \quad \text{on } (\delta_0, \delta_1),$$

and hence $\kappa \equiv \inf_{[\beta_1,\rho]}(-F(z)) > 0$. Consequently, (2.19) gives

$$|u'|^p + |u'|^q \geq \frac{\lambda q \kappa}{q - 1} \equiv \lambda \kappa_0 \quad \text{on } (\delta_0, \delta_1).$$

For $\lambda\kappa_0 > 2$, this implies $|u'| > 1$ and hence $2|u'|^p \geq \lambda\kappa_0$ on (δ_0, δ_1) follows i.e.

$$-u' = |u'| \geq (\lambda\kappa_0/2)^{1/p} \quad \text{on } (\delta_0, \delta_1). \tag{2.20}$$

Integrating (2.20) on (δ_0, δ_1) gives

$$u(\delta_0) \geq (\lambda\kappa_0/2)^{1/p}(\delta_1 - \delta_0) > \rho$$

for $\lambda \gg 1$, a contradiction which proves Claim 2.

By Claim 2, $u > \rho$ on $(0, \delta_0)$ and hence

$$f(u) \geq \kappa_1 \quad \text{on } (0, \delta_0),$$

where $\kappa_1 = \inf_{[\rho,\infty)} f > 0$. This implies

$$u(\delta) = u(\delta_0) + \int_{\delta}^{\delta_0} \phi^{-1}\left(\frac{\lambda}{s^{N-1}} \int_0^s \tau^{N-1} f(u) d\tau\right) ds \geq (\delta_0 - \delta)\phi^{-1}\left(\lambda\kappa_1 \int_0^{\delta} \tau^{N-1} d\tau\right) ds,$$

and thus $u(\delta) \to \infty$ as $\lambda \to \infty$.

Hence for λ large, $u(\delta) > \rho$ and for $r > \delta$, we have

$$\int_0^r s^{N-1} f(u) ds = \int_0^{\delta} s^{N-1} f(u) ds + \int_{\delta}^r s^{N-1} f(u) ds \geq \frac{\delta^N}{N} f(u(\delta)) - (1-\delta)K \geq \frac{\delta^N}{2N} f(u(\delta))$$

in view of (2.17). Hence

$$-u'(r) = \phi^{-1}\left(\frac{\lambda}{r^{N-1}} \int_0^r s^{N-1} f(u) ds\right) \geq \phi^{-1}\left(\frac{\lambda\delta^N f(u(\delta))}{2N}\right), \tag{2.21}$$

for $r > \delta$, and upon integrating on $(\delta, 1)$, we get

$$u(\delta) \geq (1-\delta)\phi^{-1}\left(\frac{\lambda\delta^N f(u(\delta))}{2N}\right)$$

i.e.

$$\frac{\phi(cu(\delta))}{f(u(\delta))} \geq \lambda D_1, \tag{2.22}$$

where $c = (1-\delta)^{-1}$ and $D_1 = \frac{\delta^N}{2N}$.

For $\lambda \gg 1$, $cu(\delta > 1$ and $\phi(cu(\delta)) \leq 2\phi_p(cu(\delta)) = 2\phi_p(c)\phi_p(\delta)$ and (2.22) becomes

$$H(u(\delta)) = \frac{\phi_p(u(\delta))}{f(u(\delta))} \geq \lambda D_2,$$

where $D_2 = \frac{D_1}{2\phi_p(c)} < 1$, which implies

$$u(\delta) \geq H^{-1}(\lambda D_2) \geq D_3 H^{-1}(\lambda) \tag{2.23}$$

follows, where $D_3 = D_2^{\frac{1}{p-q}}$.

Combining (2.21), (2.23), Remark 1.2 (i), and Lemma 2.2, we obtain for $\lambda \gg 1$,

$$\begin{aligned}
-u'(r) &\geq \phi^{-1}\left(\frac{\lambda\delta^N f(D_3 H^{-1}(\lambda))}{2N}\right) \geq \phi^{-1}\left(\frac{\lambda\delta^N D_3^{q-1} f(H^{-1}(\lambda))}{2N}\right), \\
&\geq a\phi_p^{-1}\left(\frac{\lambda\delta^N D_3^{q-1} f(H^{-1}(\lambda))}{2N}\right) = D_4\phi_p^{-1}\left(\lambda f(H^{-1}(\lambda))\right),
\end{aligned}$$

for $r \in (\delta, 1)$, where $D_4 = a\phi_p^{-1}(\frac{\delta^N D_3^{q-1}}{2N})$. Integrating this inequality gives

$$u(r) \geq D_4\phi_p^{-1}(\lambda f(H^{-1}(\lambda))(1-r) \quad \text{for } r \in (\delta, 1),$$

which, together with the monotonicity of u, gives the lower bound estimate in (2.5).

For the upper bound estimate, by increasing L if necessary, we can assume $L > \beta$ with $f(L) > 0$. Hence

$$f(z) \le C_1 + f(\max(z, L))$$

for $z > 0$, where $C_1 = \max_{[\beta,L]} f$.

Suppose $\|u\|_\infty > L$ and $C_2 > 1$ be such that

$$C_1 < (C_2 - 1) f(L) < (C_2 - 1) f(\|u\|_\infty).$$

Then

$$\begin{aligned} u(r) &= \int_r^1 \phi^{-1}\left(\frac{\lambda}{s^{N-1}}\int_0^s \tau^{N-1} f(u) d\tau\right) ds \le \int_r^1 \phi^{-1}\left(\frac{\lambda}{s^{N-1}}\int_0^s \tau^{N-1}\left(C_1 + f(\|u\|_\infty)\right) d\tau\right) ds \\ &\le \int_r^1 \phi^{-1}\left(\frac{\lambda}{s^{N-1}}\int_0^s \tau^{N-1}\left(C_2 f(\|u\|_\infty)\right) d\tau\right) ds \le \phi^{-1}\left(\lambda (C_2 f(\|u\|_\infty)\right)(1-r) \\ &\le \phi_p^{-1}(\lambda C_2 f(\|u\|_\infty))(1-r) \end{aligned}$$

for $r \in (0,1)$. The rest of the proof uses the same arguments as in the upper bound estimate for Case 1, which completes the proof. □

Lemma 2.6. *Let $\alpha \in (0,1)$ and $\tau > 0$. Then*

$$\min(\tau, 1)(1-\alpha) \le 1 - \alpha^\tau \le \max(\tau, 1)(1-\alpha). \tag{2.24}$$

Proof. Suppose $\tau < 1$. Then $1 - \alpha^\tau \le 1 - \alpha$. By the Mean Value Theorem with $g(\alpha) = (1-\alpha)^\tau$,

$$(1-\alpha)^\tau = |g(\alpha) - g(1)| = (1-\alpha)\frac{\tau}{(1-\xi)^{1-\tau}} \ge \tau(1-\alpha),$$

for some $\xi \in (\alpha, 1)$ i.e. (2.24) holds. The case $\tau > 1$ is proved in the same way. □

The existence of a positive solution to (1.1) in a general bounded domain is based on the following result:

Lemma 2.7 ([1, Theorem 1.1])**.** *Suppose*

(A1) $f : (0,\infty) \to \mathbb{R}$ *is continuous and* $\lim_{u\to\infty} \frac{f(u)}{u^{p-1}} = 0$.

(A2) There exist constants $A, L > 0$ and $0 \le \beta < 1$ such that

$$f(u) \ge \frac{A}{u^\beta} \quad \textit{for } u > L$$

and

$$\limsup_{u\to 0^+} u^\beta |f(u)| < \infty.$$

Then there exists a constant $\lambda_0 > 0$ such that (1.1) has a positive solution u_λ for $\lambda > \lambda_0$ with $\inf_\Omega \frac{u_\lambda}{d} \to \infty$ as $\lambda \to \infty$, where $d(x)$ denotes the distance from x to $\partial\Omega$.

3 Proof of Theorem 1.1

By Lemma 2.7, (1.2) has a positive solution for λ large. Let u, v be such solutions. In view of Lemma 2.5, there exists a maximum constant $\alpha \in (0,1]$ such that $\alpha v \leq u \leq \alpha^{-1} v$ on $(0,1)$. Then $\alpha \geq A_1/A_2 \equiv \alpha_0$ by (2.5). Recalling that $L > 0$ is such that f is increasing on (L, ∞). By using (A2) and increasing L if necessary, we can assume that $f(L) > 0$ and $z^{-q_0} f(z)$ is decreasing on $[L, \infty)$ for some $q_0 \in (0, q-1)$. We will show that $\alpha \geq 1$. Suppose to the contrary that $\alpha < 1$. Suppose $\lambda > \lambda_0$, where λ_0 is defined in Lemma 2.5.

Let $L_1 = \alpha_0^{-1} L$. Then

$$v(1/2) > L_1$$

for $\lambda > \lambda^*$ for some $\lambda^* > 0$ independent of u, v.

For $r \leq 1/2$,

$$u(r) \geq \alpha v(r) \geq \alpha_0 v(1/2) > \alpha_0 L_1 = L,$$

which implies

$$f(u(r)) \geq f(\alpha v(r)) \geq \alpha^{q_0} f(v(r)). \tag{3.1}$$

Suppose $r > 1/2$.

Case 1. $v(r) \geq L_1$

Then $u(r) > L$ and therefore (3.1) holds.

Case 2. $v(r) < L_1$.

Then

$$u(r) \leq \alpha_0^{-1} v(r) < \alpha_0^{-1} L_1 \equiv L_2.$$

Let $K_0 = \sup_{z \in (0, L_2)} z^\gamma |f(z)|$ and $K_1 = \sup_{z \in (0, L_2)} z^{\gamma+1} |f'(z)|$. Then $K_0, K_1 < \infty$ by (A3) and (A5). Then the lower bound estimate in Lemma 2.5 gives,

$$|f(v(r))| \leq \frac{K_0}{v^\gamma(r)} \leq \frac{K_0}{A_1^\gamma B_\lambda^\gamma (1-r)^\gamma}, \tag{3.2}$$

which implies in view of Lemma 2.6,

$$f(v(r)) \geq \alpha^{q_0} f(v(r)) - (1-\alpha^{q_0})|f(v(r))| \geq \alpha^{q_0} f(v(r)) - \frac{K_0 c_{q_0}(1-\alpha)}{A_1^\gamma B_\lambda^\gamma (1-r)^\gamma}, \tag{3.3}$$

where $c_{q_0} = \max(1, q_0)$, and

$$\begin{aligned} |f(u(r)) - f(v(r))| = |u(r) - v(r)||f'(\xi)| &\leq \frac{K_1(1-\alpha)v(r)}{\alpha \xi^{\gamma+1}} \\ &\leq \frac{K_1(1-\alpha)}{\alpha_0^{2+\gamma} v^\gamma(r)} \leq \frac{K_1(1-\alpha)}{\alpha_0^{2+\gamma} A_1^\gamma B_\lambda^\gamma (1-r)^\gamma} \end{aligned} \tag{3.4}$$

for some ξ between $u(r)$ and $v(r)$. From (3.3) and (3.4), we deduce that

$$f(u(r)) \geq f(v(r)) - |f(u(r)) - f(v(r))| \geq \alpha^{q_0} f(v(r)) - \frac{K_2(1-\alpha)}{B_\lambda^\gamma (1-r)^\gamma}, \tag{3.5}$$

where $K_2 = (K_0 c_{q_0} + K_1 \alpha_0^{-2-\gamma})/A_1^\gamma$. Hence (3.5) holds for $r > 1/2$ in both cases.

Let $q_1 \in (q_0, q-1)$. Then for $r > 1/2$, it follows from (3.2), (3.5) and Lemmas 2.5–2.6 that

$$\begin{aligned} f(u(r)) - \alpha^{q_1} f(v(r)) &\geq (\alpha^{q_0} - \alpha^{q_1}) f(v(r)) - \frac{K_2(1-\alpha)}{B_\lambda^\gamma (1-r)^\gamma} \\ &\geq -\frac{(1-\alpha^{q_1-q_0})K_0}{A_1^\gamma B_\lambda^\gamma (1-r)^\gamma} - \frac{K_2(1-\alpha)}{B_\lambda^\gamma (1-r)^\gamma} \geq -\frac{K_3(1-\alpha)}{B_\lambda^\gamma (1-r)^\gamma}, \end{aligned} \tag{3.6}$$

where $K_3 = K_0 A_1^{-\gamma} \max(1, q_1 - q_0) + K_2$.

We claim next that

$$\phi(u') \leq \alpha^{q_1} \phi(v') \tag{3.7}$$

on $(0,1)$. Using the formula

$$-r^{N-1}[\phi(u'(r))) - \alpha^{q_1}\phi(v'(r)] = \lambda \int_0^r s^{N-1}(f(u) - \alpha^{q_1} f(v))ds, \tag{3.8}$$

for $r \in (0,1)$, we see from (3.1) that for $r \leq 1/2$,

$$f(u(s)) - \alpha^{q_1} f(v(s)) \geq (\alpha^{q_0} - \alpha^{q_1}) f(v(s)) > 0 \tag{3.9}$$

for $s \leq 1/2$ i.e. (3.7) holds on $(0, 1/2]$. For $r > 1/2$, it follows from (3.6), (3.8), (3.9), and Lemma 2.6 that

$$\begin{aligned} -r^{N-1}(\phi(u') - \alpha^{q_1}\phi(v')) &= \lambda \left(\int_0^{1/2} s^{N-1}(f(u) - \alpha^{q_1} f(v))ds + \int_{1/2}^r s^{N-1}(f(u) - \alpha^{q_1} f(v))ds \right) \\ &\geq \lambda \left[(\alpha^{q_0} - \alpha^{q_1}) \int_0^{1/2} s^{N-1} f(v) ds - \frac{K_3(1-\alpha)}{B_\lambda^\gamma} \int_{1/2}^1 \frac{1}{(1-s)^\gamma} ds \right] \\ &\geq \lambda(1-\alpha) \left[K_4 f(v(1/2)) - \frac{K_3}{B_\lambda^\gamma} \int_\delta^1 \frac{1}{(1-s)^\gamma} ds \right] > 0, \end{aligned}$$

where $K_4 = \frac{\alpha_0^{q_0} \min(1, q_1 - q_0)}{N 2^N}$, provided that $\lambda > \bar{\lambda}$, where $\bar{\lambda} > \lambda_0$ is such that

$$K_4 f(L_1) - \frac{K_3}{B_\lambda^\gamma} \int_{1/2}^1 \frac{1}{(1-s)^\gamma} ds > 0.$$

Note that this is possible since $f(L_1) > 0$ and $B_\lambda \to \infty$ as $\lambda \to \infty$. Hence (3.7) holds on $(0,1)$, which implies

$$\phi(u') + \alpha^{q_1} \phi(|v'|) \leq 0. \tag{3.10}$$

Since

$$\phi\left(\alpha^{\frac{q_1}{q-1}} |v'|\right) = \alpha^{q_1} |v'|^{q-1} + \alpha^{\frac{q_1(p-1)}{q-1}} |v'|^{p-1} \leq \alpha^{q_1}(|v'|^{q-1} + |v'|^{p-1}) = \alpha^{q_1} \phi(|v'|),$$

it follows from (3.10) that

$$\phi(u') + \phi\left(\alpha^{\frac{q_1}{q-1}} |v'|\right) \leq 0,$$

i.e.

$$\phi(u') \leq -\phi\left(\alpha^{\frac{q_1}{q-1}} |v'|\right) = \phi\left(\alpha^{\frac{q_1}{q-1}} v'\right).$$

Hence $\left(u - \alpha^{\frac{q_1}{q-1}} v\right)' \leq 0$ on $(0,1)$ and since $u(1) = v(1) = 0$, it follows that $u \geq \alpha^{\frac{q_1}{q-1}} v$ on $(0,1)$, a contradiction with the maximality of α since $q_1 < q-1$. This completes the proof of Theorem 1.1. □

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