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Existence of solutions for singular quasilinear elliptic problems with dependence of the gradient

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Abstract. In this paper we establish existence of solutions to the following boundary value problem involving a p-gradient term

$$-\Delta_p u + g(u)|\nabla u|^p = \lambda u^\sigma + \Psi(x), \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where $\Delta_p := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is p-Laplacian operator, $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, $1 < p < N$, $0 < \sigma < p^* - 1$ with $p^* := pN/(N-p)$, Ψ is a measurable function and $g(s) \geq 0$ is a continuous function on the interval $(0, +\infty)$ which may have a singularity at the origin, i.e. $g(s) \to +\infty$ as $s \to 0$. Using the topological degree theory, under certain assumptions on Ψ, we prove the existence of a continuum of positive solutions.

Keywords: p-gradient term, singular equations, elliptic equations.

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1 Introduction

In this paper we establish existence of a continuum of positive solutions to the following class of singular quasilinear elliptic equations with a p-gradient term,

$$\begin{cases} -\Delta_p u + g(u)|\nabla u|^p = \lambda u^\sigma + \Psi(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$(P)_{\lambda\sigma}$}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, $1 < p < N$, $0 < \sigma < p^* - 1$ with $p^* := pN/(N-p)$, $g : (0, \infty) \to \mathbb{R}^+$ is a continuous measurable function in a neighborhood of zero and $\Psi : \Omega \to \mathbb{R}^+$ is a L^q integrable function with $q \in \big[\frac{pN}{N(p-1)+p}, \frac{pN}{N-p}\big)$.

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This type of equations involving singular nonlinearities appears in the models of several physical phenomena, such as in theory of electric conductivity [19], in study of pseudoplastic fluids [14], in minimal surfaces with isolated singularities [13] and several other models.

The classic references involving the problem $(P)_{\lambda\sigma}$ were published by Leray and Lions [29] in 1965 and by Ladyzenskaya and Uraltseva [26] in 1968.

For this class of equations involving the term gradient we quote [1–5,7–9,22,31] and without the gradient term we quote [16–18,20,21,23,24,27,30,36]. For existence results involving quasilinear and parabolic elliptic problems with quadratic gradient term we quote [12] and for existence of a continuum of solutions for a quasilinear singular elliptic problem we quote [15].

In 2009, Arcoya, Barile and Martínez-Aparicio [3] studied the quasilinear elliptic boundary value problem

$$\begin{cases} -\Delta u + g(x,u)|\nabla u|^2 = a(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary and $g(x,s)$ is a Carathéodory function on $\Omega \times (0,\infty)$ which may have a singularity at $s = 0$ and may change of sign. Assuming that $a \in L^q(\Omega)$, with $q > N/2$, satisfies the following inequality

$$\inf\{a(x) \mid x \in \Omega_0\} > 0; \qquad \forall \Omega_0 \subset\subset \Omega$$

they proved that if there exist a increasing function $b : (0,+\infty) \to (0,+\infty)$ and a parameter $\mu \in (0,1)$ such that

$$-\mu \leq sg(x,s) \leq b(s); \qquad \forall s > 0; \;\; a.e. \;\; x \in \Omega,$$

then the previous problem has at least one positive solution.

In 2015, Y. Wang and M. Wang [35] extended the result obtained by Arcoya, Barile and Martínez-Aparicio [3] to the case involving the p-Laplacian operator.

Arcoya, Carmona and Martínez-Aparicio [5] studied the boundary value problem with a power type nonlinearity

$$\begin{cases} -\Delta u + g(u)|\nabla u|^2 = \lambda u^p + f_0(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is an open and bounded domain, $\lambda \geq 0$, $0 \leq p < \frac{N+2}{N-2}$, $0 \lneqq f_0 \in L^{\frac{2N}{N+2}}(\Omega)$ and $g \geq 0$ is continuous in $[0,+\infty)$ or $g \geq 0$ is continuous in $(0,+\infty)$, decreasing and integrable in a neighborhood of zero with $\lim_{s\to 0} g(s) = +\infty$. Using the Leray–Schauder degree, the authors showed the existence of "continua of solutions" of (1.2), i.e., connected and closed subsets in the solution set

$$Q := \left\{(\lambda, u) \in \mathbb{R} \times H_0^1(\Omega) : u = K(\lambda, u)\right\}$$

where $K : \mathbb{R} \times H_0^1(\Omega) \to H_0^1(\Omega)$ is an operator such that, for every $\lambda \in \mathbb{R}$ and for every $w \in H_0^1(\Omega)$, $K(\lambda, w)$ is the unique solution $u \in H_0^1(\Omega)$ of an auxiliary problem.

In this paper, we generalize the equation studied by Arcoya, Carmona and Martínez-Aparicio [5] for the p-Laplacian operator $\Delta_p := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, with a non negative continuous function g which may have a singularity at the origin and a measurable function Ψ. To state our results, we say that $u \in W_0^{1,p}(\Omega)$ is a positive solution for $(P)_{\lambda\sigma}$ if $u > 0$ a.e $x \in \Omega$, $g(u)\,|\nabla u|^p \in L^1(\Omega)$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \varphi dx + \int_\Omega g(u)|\nabla u|^p \varphi dx = \lambda \int_\Omega u^\sigma \varphi dx + \int_\Omega \Psi \varphi dx \tag{1.3}$$

for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Our main results read as follows.

Theorem 1.1. *Let $g : (0, +\infty) \to \mathbb{R}^+$ be a continuous and integrable function in a neighborhood of zero such that $\lim_{s\to 0} g(s) = +\infty$. If $\Psi \in L^q(\Omega)$, be a function not identically zero with $q \in \left[\frac{pN}{N(p-1)+p}, \frac{pN}{N-p}\right)$, then problem $(P)_{\lambda\sigma}$ has a unique solution $u \in W_0^{1,p}(\Omega)$ for $\lambda = 0$.*

For $q = \frac{pN}{N(p-1)+p}$, Y. Wang and M. Wang [35, Theorem 3.1] proved the existence of a solution to Theorem 1.1. Before we establish the next result we need some definitions. In this way, we consider the auxiliary problem

$$\begin{cases} -\Delta_p u + g(u)|\nabla u|^p = \lambda^+ w^+(x)^\sigma + \Psi(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\lambda \geq 0$, $w \in W_0^{1,p}(\Omega)$ and $w^+ := \max\{0, w\}$.

By Theorem 1.1, for every $(\lambda, w) \in [0, +\infty) \times W_0^{1,p}(\Omega)$, the problem (1.4) has a unique solution $u = T(\lambda, w) \in W_0^{1,p}(\Omega)$. Thus, by following ideas of Arcoya, Carmona and Martínez-Aparicio [5], we define an operator $K : \times W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ such that

$$K(\lambda, w) := \begin{cases} T(\lambda, w), & \text{if } \lambda \geq 0; \\ T(0, w), & \text{if } \lambda < 0, \end{cases}$$

and a set

$$S := \left\{(\lambda, u) \in \mathbb{R} \times W_0^{1,p}(\Omega) : u = K(\lambda, u)\right\}.$$

Notice that the function $\lambda w^+(x)^\sigma + \Psi(x)$ is in $L^q(\Omega)$, for $q = \frac{Np}{N(p-1)+p}$. Thus $K(\lambda, w)$ is well defined. Indeed,

$$\int_\Omega |\lambda w^+(x)^\sigma + \Psi(x)|^q dx \leq 2^{q-1} \int_\Omega |\lambda w^+(x)^\sigma|^q dx + 2^{q-1} \int_\Omega |\Psi(x)|^q dx,$$

and $\sigma q = \frac{N(p-1)+p}{N-p} \frac{pN}{N(p-1)+p} = \frac{pN}{N-p} = p^*$.

Therefore, with this notation, $(P)_{\lambda\sigma}$ can be rewritten as a fixed point problem, namely,

$$u = K(\lambda, u). \tag{1.5}$$

The next result is related to the case $\Psi \gneqq 0$ and it states the existence of global continua in solution set S emanating from the unique solution of Theorem 1.1.

Theorem 1.2. *Consider $0 \lneqq \Psi \in L^{\frac{pN}{N(p-1)+p}}(\Omega)$ and assume that $g \geq 0$ is continuous in $[0, +\infty)$ or $g \geq 0$ is continuous in $(0, +\infty)$ and integrable in an neighborhood of zero with $\lim_{s\to 0} g(s) = +\infty$. Then there exists an unbounded continuum $\Sigma \subset S$ of positive solutions which contains $(0, u_0)$, where u_0 is the unique solution of $(P)_{\lambda\sigma}$ for $\lambda = 0$.*

To prove Theorem 1.1 we follow some general ideas of [3,5], i.e., we construct a infinite sequence of auxiliary problems $(P)_n$ with $n \in \mathbb{N}$, such that $(P)_{\lambda\sigma}$ for $\lambda = 0$ has at least one solution as $n \to +\infty$. To prove the uniqueness of the solution we follows some general ideas of [6]. Finally, to prove Theorem 1.2 we show that K is a compact operator and using the Leray–Schauder degree theory we prove the existence of "continua of solutions" of $(P)_{\lambda\sigma}$, i.e., connected and closed subsets in the solution set S.

The main difficulties found in the proof of these results are the existence of a singularity g, the presence of the term gradient and the non-linearity of the operator in the case where $p \in (1, N)$. The case where $p = 2$ has been extensively studied by several researches. For example, in the case where g is continuous at zero the existence is due to [11] and the uniqueness to [4]. Moreover, in the case where g is singular at zero the existence is due to [10] and the uniqueness to [4].

This paper is organized as follows: in section 2 we introduce an approximated problem, whose solutions are also solutions to problem (1.4); and we prove some auxiliary lemmas. In section 3, we prove the integrability of $g(u)\,|\nabla u|^p\,\varphi$ and the compactness for the operator K defined in (1.5). After, we prove the Theorem 1.2 using the topological Leray–Schauder degree.

2 Proof of Theorem 1.1

To prove Theorem 1.1 first we give some preliminary considerations and lemmas. In this way, motivated by [2,3], we define

$$g_n(s) = \begin{cases} 0, & \text{if } s \le 0; \\ n^p s^p T_n(g(s)), & \text{if } 0 < s < \frac{1}{n}; \\ T_n(g(s)), & \text{if } \frac{1}{n} \le s; \end{cases} \tag{2.1}$$

where $T_n(s)$ is the truncate function given by

$$T_n(s) = \begin{cases} s, & \text{if } |s| < n; \\ -n, & \text{if } s \le -n; \\ n, & \text{if } s \ge n. \end{cases} \tag{2.2}$$

It is easy to verify that g_n satisfies the following properties

(a) $|g_n(s)| \le \min\{n, g(s)\}$;

(b) $g_n(s) \le n^p s^{p-1}$, for all $s > 0$;

(c) $\lim_{n\to\infty} g_n(s) = g(s)$, for all $s > 0$.

Consider the following approximated problem

$$\begin{cases} -\Delta_p w + \frac{g_n(w)|\nabla w|^p}{1+\frac{1}{n}|\nabla w|^p} = \Psi_n(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{$(P)_n$}$$

where $\Psi_n := T_n(\Psi)$.

Lemma 2.1. *There exists at least one solution $w_n \in W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$ of the approximated problem $(P)_n$.*

Proof. Notice that, since the operator Δ_p^{-1} is an homeomorphism, using the Browder–Minty Theorem we obtain that

$$-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) \quad \text{is a homeomorphism.} \tag{2.4}$$

Furthermore, for every $w \in C^1(\overline{\Omega})$ we define

$$F_n(w) = \Psi_n - \frac{g_n(w)|\nabla w|^p}{1 + \frac{1}{n}|\nabla w|^p},$$

and for every $u \in C^1(\overline{\Omega})$ we define the problem

$$\begin{cases} -\Delta_p w = F_n(u) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Let $G : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the functional defined by

$$G(\varphi) = \int_\Omega F_n(u)\varphi dx.$$

Since $F_n(u)$ is bounded independent of u, then G is well defined, linear and $|G(\varphi)| \le C_n \|\varphi\|_{1,p}$, for some $C_n > 0$, i.e., $G \in W^{-1,p'}(\Omega)$. Hence, by (2.4) there exists a unique function $w \in W_0^{1,p}(\Omega)$ such that $-\Delta_p w = G$, i.e.,

$$\int_\Omega |\nabla w|^{p-2}\nabla w \nabla \varphi dx = \int_\Omega F_n(u)\varphi dx.$$

Thus, since all the assumptions of the regularity result obtained by Hai (see [24, Lemma 3.1]) are satisfied, there are constants $\alpha \in (0,1)$ and $M > 0$ such that $w \in C^{1,\alpha}(\overline{\Omega})$ and $\|w\|_{C^{1,\alpha}(\overline{\Omega})} < M$.

Let $K : C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$ be the operator defined by $K(u) = w$, where w is the unique solution of (2.5). Since $K(C^1(\overline{\Omega})) \subset C^{1,\alpha}(\overline{\Omega})$ and $\|K(u)\|_{1,\alpha} \le M$ for every $u \in C^1(\overline{\Omega})$, then K is compact. Furthermore, K is continuous. Indeed, let $\{u_k\} \subset C^1(\overline{\Omega})$ be a sequence such that $u_k \to u$ in $C^1(\overline{\Omega})$. Define now

$$w_k = K(u_k) \quad \text{and} \quad w = K(u).$$

By definition of K, we have

$$-\Delta_p w_k - (-\Delta_p w) = F_n(u_k) - F_n(u) \quad \text{in } \Omega,$$

and consequently

$$\begin{aligned} \int_\Omega (|\nabla u_k|^{p-2}\nabla u_k - |\nabla u|^{p-2}\nabla u).\nabla(w_k - w)dx &= \int_\Omega (F_n(u_k(x)) - F_n(u(x)))(w_k - w)dx \\ &\le 2M \int_\Omega |F_n(u_k) - F_n(u)|dx. \end{aligned} \tag{2.6}$$

Since $F_n(u_k)$ is bounded and $F_n(u_k(x)) \to F_n(u(x))$ a.e. in Ω, then by the Dominated Convergence Theorem we have

$$\int_\Omega |F_n(u_k) - F_n(u)|dx \to 0,$$

as $k \to \infty$.

Thus, applying in (2.6) the inequality (see [33])

$$\langle |x|^{p-2}x - |y|^{p-2}y, x-y\rangle_e \geq C \begin{cases} \dfrac{|x-y|^2}{(|x|+|y|)^{2-p}} & \text{if } 1<p<2; \\ |x-y|^p & \text{if } p \geq 2, \end{cases} \tag{2.7}$$

where $x, y \in^N$ and $C := C(p)$ is a constant, we obtain

$$w_k \to w \quad \text{in } W_0^{1,p}(\Omega).$$

On the other hand, since $\|w_k\|_{C^1(\overline{\Omega})} < M$, by going to subsequence if necessary, there exists $w_0 \in C^1(\overline{\Omega})$ such that

$$w_k \to w_0 \quad \text{in } C^1(\overline{\Omega}).$$

Hence, by uniqueness of limits, we conclude $K(u_k) \to K(u)$ in $C^1(\overline{\Omega})$; i.e., K is continuous.

How K is a continuous and compact operator and $K(C^1(\overline{\Omega})) \subset B_{\tilde{M}}$, where $B_{\tilde{M}}$ is a ball centered at the origin with radius $\tilde{M}$ in $C^1(\overline{\Omega})$, then by Schauder's Fixed Point Theorem, there exists $u \in B_{\tilde{M}}$, such that $K(u) = u$. Therefore, by definition of K, we have that u is solution of $(P)_n$. □

Lemma 2.2. *If $\Psi \in L^q(\Omega)$ with $q \in \left[\frac{pN}{N(p-1)+p}, \frac{pN}{N-p}\right)$, then the solution w_n of $(P)_n$ satisfies the following statements,*

(i) *$\{w_n\}$ is bounded independent of n in $W_0^{1,p}(\Omega)$;*

(ii) *$w_n(x) > 0$, for all $x \in \Omega$.*

Proof. (i) First, we will prove that $w_n \geq 0$ in Ω. Indeed, multiplying $(P)_n$ by w_n^- and integrating in Ω, we obtain

$$\int_\Omega |\nabla w_n|^{p-2}\nabla w_n.\nabla w_n^- dx + \int_\Omega \frac{g_n(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p} w_n^- dx = \int_\Omega \Psi_n w_n^- dx. \tag{2.8}$$

Since $w_n = w_n^+ - w_n^-$ and $\nabla w_n = \nabla w_n^+ - \nabla w_n^-$, then

$$\begin{aligned} \nabla w_n \nabla w_n^- &= (\nabla w_n^+ - \nabla w_n^-)\nabla w_n^- \\ &= \nabla w_n \nabla w_n^- - (\nabla w_n^-)^2 \\ &\quad - (\nabla w_n^-)^2. \end{aligned} \tag{2.9}$$

Furthermore, we have

$$g_n(w_n)w_n^- = 0, \quad \text{for all } w_n \in W_0^{1,p}(\Omega). \tag{2.10}$$

Then, by relations (2.8), (2.9) and (2.10), we have

$$-\int_\Omega |\nabla w_n^-|^p dx = \int_\Omega \Psi_n w_n^- dx \geq 0,$$

i.e., $w_n^- \equiv 0$. Therefore, $w_n = w_n^+ \geq 0$ in Ω.

Taking w_n as test function in $(P)_n$, we obtain

$$\begin{aligned} \int_\Omega |\nabla w_n|^p dx + \int_\Omega \frac{g_n(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p} w_n dx &= \int_\Omega \Psi_n w_n dx \\ &\leq \int_\Omega \Psi w_n dx. \end{aligned}$$

Since $w_n \geq 0$, we have that $\frac{g_n(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|} w_n \geq 0$. Thus, using Hölder's inequality, we obtain

$$\begin{aligned}\int_\Omega |\nabla w_n|^p dx &\leq \int_\Omega \Psi_n w_n dx \\ &\leq \|\Psi\|_q \|w_n\|_{q'},\end{aligned} \tag{2.11}$$

where $q' := \frac{q}{q-1}$. Since $q \geq \frac{pN}{N(p-1)+p}$, then $q' \leq \frac{pN}{N-p}$ and $L^{p*}(\Omega) \subset L^{q'}(\Omega)$. Hence, there are constants $c_1, c_2 > 0$ such that

$$\|w_n\|_{q'} \leq c_1 \|w_n\|_{p*} \leq c_2 \|w_n\|_{1,p}.$$

By applying the previous inequality in the relation (2.11), we conclude that $\{w_n\}$ is bounded in $W_0^{1,p}(\Omega)$.

(ii) Since $w_n \geq 0$ in Ω, by item (b) of properties of g_n listed previously, we obtain

$$\frac{g_n(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p} \leq n^p |w_n|^{p-1} n = n^{p+1}|w_n|^{p-1}. \tag{2.12}$$

Let the function $\beta : [0, +\infty) \to \mathbb{R}$ be defined by

$$\beta(s) = n^{p+1}(s)^{p-1}.$$

Notice that β is continuous, non-decreasing, $\beta(0) = 0$ and $\beta(s) > 0$, for all $s > 0$. Furthermore,

$$\int_0^1 (\beta(s)s)^{-\frac{1}{p}} ds = \infty.$$

Hence, by relation (2.12) we have

$$-\Delta_p w_n + \beta(w_n) \geq \Psi_n \geq 0 \quad \text{a.e. in } \Omega,$$

i.e., $\Delta_p w_n \leq \beta(w_n)$ a.e. in Ω.

Since $\Psi_n \not\equiv 0$, then by the Strong Principle of Maximum (see [34, Theorem 5]) we conclude that $w_n > 0$ in Ω. □

Lemma 2.3. *There are $w \in W_0^{1,p}(\Omega)$ and $q \in (1, p)$ such that*

(i) $\nabla w_n \to \nabla w$ *in* $L^q(\Omega)$*;*

(ii) $w > 0$ *in* Ω*.*

Proof. (i) Since the sequence $\{w_n\}$ is bounded in $W_0^{1,p}(\Omega)$, going if necessary to a subsequence, there exists $w \in W_0^{1,p}(\Omega)$ such that

(a) $w_n \rightharpoonup w$ in $W_0^{1,p}(\Omega)$;

(b) $w_n \to w$ in $L^p(\Omega)$;

(c) $w_n(x) \to w(x)$ a.e. in Ω.

By fixing compact set $K \subset \Omega$, we take $\phi_K \in C_0^\infty(\Omega)$ with $0 \leq \phi_K \leq 1$ and $\phi_K = 1$ in K. Thus, taking v_n as test function in $(P)_n$ defined by

$$v_n := \phi_K [T_\eta(w_n - w)]^+ \in W_0^{1,p}(\Omega),$$

we have

$$\int_\Omega |\nabla w_n|^{p-2} \nabla w_n \nabla v_n dx + \int_\Omega \frac{g_n(w_n)|\nabla w_n|^p}{1 + \frac{1}{n}|\nabla w_n|^p} v_n dx = \int_\Omega \Psi_n v_n dx. \tag{2.13}$$

By applying in (2.13) the relations

$$\nabla v_n = \nabla \phi_K [T_\eta(w_n - w)]^+ + \phi_K \nabla [T_\eta(w_n - w)]^+$$

and

$$\frac{g_n(w_n)|\nabla w_n|^p}{1 + \frac{1}{n}|\nabla w_n|^p} v_n \geq 0,$$

we obtain

$$\begin{aligned}
&\int_\Omega |\nabla w_n|^{p-2} \nabla w_n \phi_K \nabla [T_\eta(w_n - w)]^+ dx \\
&\quad \leq \int_\Omega \Psi_n v_n dx - \int_\Omega |\nabla w_n|^{p-2} \nabla w_n \nabla \phi_K [T_\eta(w_n - w)]^+ dx.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\int_\Omega \phi_K \big[|\nabla w_n|^{p-2} \nabla w_n - |\nabla w|^{p-2} \nabla w \big] \nabla [T_\eta(w_n - w)]^+ dx \\
&\quad \leq \int_\Omega \Psi_n v_n dx - \int_\Omega |\nabla w_n|^{p-2} \nabla w_n \nabla \phi_K [T_\eta(w_n - w)]^+ dx \\
&\quad\quad - \int_\Omega \phi_K |\nabla w|^{p-2} \nabla w \nabla [T_\eta(w_n - w)]^+ dx.
\end{aligned} \tag{2.14}$$

Since $w_n \rightharpoonup w$ in $W_0^{1,p}(\Omega)$, then we have $[T_\eta(w_n - w)]^+ \rightharpoonup 0$, i.e.,

$$\langle \phi, [T_\eta(w_n - w)]^+ \rangle \to 0 \quad \text{for all } \phi \in W_0^{1,p}(\Omega)^*, \tag{2.15}$$

where $W_0^{1,p}(\Omega)^*$ is the dual space of $W_0^{1,p}(\Omega)$.

Note that, by the Dominated Convergence Theorem and by relation (2.15) we obtain

$$\int_\Omega |\nabla w_n|^{p-2} \nabla w_n \nabla \phi_K [T_\eta(w_n - w)]^+ dx \to 0 \quad \text{as } n \to \infty \tag{2.16}$$

and

$$\int_\Omega \phi_K |\nabla w|^{p-2} \nabla w \nabla [T_\eta(w_n - w)]^+ dx \to 0 \quad \text{as } n \to \infty. \tag{2.17}$$

Indeed, by Hölder's inequality and by (2.15) we have

$$\left| \int_\Omega |\nabla w_n|^{p-2} \nabla w_n \nabla \phi_K [T_\eta(w_n - w)]^+ dx \right| \leq \|w_n\|_{1,p}^{p-1} \left(\int_\Omega |\nabla \phi_K [T_\eta(w_n - w)]^+|^p dx \right)^{\frac{1}{p}},$$

where $\|w_n\|_{1,p}$ and $\nabla \phi_K [T_\eta(w_n - w)]^+$ are bounded and $[T_\eta(w_n - w)]^+ \to 0$ a.e. in Ω. Thus, by the Dominated Convergence Theorem the relation (2.16) holds true.

Again, by Hölder's inequality we have

$$\begin{aligned}
&\left|\int_\Omega \phi_K |\nabla w|^{p-2}\nabla w \nabla [T_\eta(w_n-w)]^+dx\right| \\
&\qquad \le \|w_n\|_{1,p}^{p-1}\left[\int_\Omega |\phi_k[\nabla T_\eta(w_n-w)]^+|^p dx\right]^{\frac{1}{p}} \\
&\qquad = \|w_n\|_{1,p}^{p-1}\left[\int_\Omega |\phi_k|^p.\chi_{\{x\in\Omega:|w_n-w|<\eta\}}|\nabla(w_n-w)]^+|^p dx\right]^{\frac{1}{p}}.
\end{aligned}$$

Since $|\phi_k|^p.\chi_{\{x\in\Omega:|w_n-w|<\eta\}}$ is bounded in $L^p(\Omega)^*$ and $|\phi_k|^p.\chi_{\{x\in\Omega:|w_n-w|<\eta\}} \to |\phi_k|^p$ a.e. in Ω, then by Vitali's Convergence Theorem we have

$$|\phi_K|^p.\chi_{\{x\in\Omega:|w_n-w|<\eta\}} \to |\phi_K|^p \quad \text{in } L^p(\Omega)^*.$$

Hence, since $\left|\nabla(w_n-w)^+\right|^p \rightharpoonup 0$ in $L^p(\Omega)$, then the relation (2.17) holds true.

For fixed η, combining the relations (2.14), (2.16) and (2.17), we have

$$\limsup_{n\to\infty}\int_K \left(|\nabla w_n|^{p-2}\nabla w_n - |\nabla w|^{p-2}\nabla w\right)\nabla[T_\eta(w_n-w)]^+dx \le C_\Psi \eta. \tag{2.18}$$

Let H_n^+ defined by

$$H_n^+(x) := \left[|\nabla w_n|^{p-2}\nabla w_n - |\nabla w|^{p-2}\nabla w\right]\nabla[T_\eta(w_n-w)]^+(x).$$

By relation (2.18) we have that H_n^+ is bounded in $L^1(K)$. Moreover, by inequality (2.7) and by definition of T_η, we have that $H_n^+ \ge 0$.

By defining the sets

$$A_n^\eta := \{x\in K : |w_n(x)-w(x)| \le \eta\} \quad \text{and} \quad B_n^\eta := \{x\in K : |w_n(x)-w(x)| > \eta\};$$

and fixing $\nu \in (0,1)$, we obtain

$$\int_K (H_n{}^+)^\nu dx \le \left(\int_{A_n^\eta}(H_n^+)dx\right)^\nu |A_n^\eta|^{1-\nu} + \left(\int_{B_n^\eta}(H_n^+)dx\right)^\nu |B_n^\eta|^{1-\nu}.$$

For fixed η, we have that $|B_n^\eta| \to 0$ as $n\to\infty$. Moreover, since H_n^+ is bounded in $L^1(K)$, we have

$$\limsup_{n\to\infty}\int_K (H_n^+)^\nu dx \le (C_\Psi\eta)^\nu |\Omega|^{1-\nu}. \tag{2.19}$$

By letting $\eta \to 0$ in the previous inequality, we obtain

$$(H_n^+)^\nu \to 0 \quad \text{in } L^1(K).$$

Now, choose v_n as test function in $(P)_n$ defined by

$$v_n := \phi_K[T_\eta(w_n-w)]^- \in W_0^{1,p}(\Omega),$$

where $s^- := \max\{-s, 0\}$. Hence, repeating the arguments previously used, we can conclude that

$$(H_n^-)^\nu \to 0 \quad \text{in } L^1(K),$$

where

$$H_n^-(x) := \left[|\nabla w_n|^{p-2}\nabla w_n - |\nabla w|^{p-2}\nabla w\right]\nabla[(w_n - w)]^-(x).$$

Therefore, if $H_n := H_n^+ - H_n^-$, then

$$H_n(x) = \left[|\nabla w_n|^{p-2}\nabla w_n - |\nabla w|^{p-2}\nabla w\right]\nabla[(w_n - w)](x)$$

and $H_n \to 0$ a.e. in K.

Consider $\Omega = \bigcup_{j=1}^{\infty}\Omega_j$, such that $\Omega_j \subset\subset \Omega_{j+1} \subset\subset \Omega$.

Thus, for $K = \overline{\Omega}_1$, we have

$$H_1^1(x), H_2^1(x), H_3^1(x), \ldots, H_n^1(x) \to 0 \quad \text{a.e. in } \overline{\Omega}_1.$$

Analogously, for $K = \overline{\Omega}_2$, we have

$$H_1^2(x), H_2^2(x), H_3^2(x), \ldots, H_n^2(x) \to 0 \quad \text{a.e. in } \overline{\Omega}_2.$$

Repeating the previous process, we obtain

$$\begin{array}{ccccccccc}
H_1^1(x) & H_2^1(x) & H_3^1(x) & \ldots & H_n^1(x) & \longrightarrow 0 & \text{a.e. in} & \overline{\Omega}_1 \\
H_1^2(x) & H_2^2(x) & H_3^2(x) & \ldots & H_n^2(x) & \longrightarrow 0 & \text{a.e. in} & \overline{\Omega}_2 \\
H_1^3(x) & H_3^3(x) & H_3^3(x) & \ldots & H_n^3(x) & \longrightarrow 0 & \text{a.e. in} & \overline{\Omega}_3 \\
\vdots & \vdots & \vdots & \ddots & & \vdots & & \vdots \\
H_1^j(x) & H_2^j(x) & H_3^j(x) & \ldots & H_n^j(x) & \longrightarrow 0 & \text{a.e. in} & \overline{\Omega}_j \\
\vdots & \vdots & \vdots & & \vdots & & & \vdots
\end{array}$$

Hence, taking the diagonal sequence $\widehat{H}_j = H_j^j$, we have

$$\widehat{H}_j(x) \to 0 \quad \text{a.e. in } \Omega.$$

So, for the sequence of compact sets Ω_j, there exists a subsequence $\{H_{n'}\}$ such that

$$H_{n'}(x) \to 0 \quad \text{a.e. in } \Omega.$$

By applying again the inequality (2.7), we obtain

$$\nabla w_{n'}(x) \to \nabla w(x) \quad \text{a.e. in } \Omega.$$

Thus, since $\{\nabla w_n\}$ is bounded independent of n, by Vitali's Convergence Theorem we have

$$\nabla w_n \to \nabla w \quad \text{in } L^q(\Omega), \quad q < p.$$

(ii) Now, we will prove that w is strictly positive in Ω. Indeed, we have $w_n > 0$ in Ω with $w_n \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. In analogy to the proof of (i), we have

$$\int_\Omega |\nabla w_n|^{p-2}\nabla w_n \nabla\varphi dx + \int_\Omega \frac{g_n(w_n)|\nabla w_n|^p}{1 + \frac{1}{n}|\nabla w_n|^p}\varphi dx = \int_\Omega \Psi_n \varphi dx, \qquad \varphi \in W_0^{1,p}(\Omega). \tag{2.20}$$

Thus, taking v_n as test function in (2.20) defined by

$$v_n := e^{-\tilde{H}_n(w_n)}\varphi, \qquad \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \qquad \varphi \geq 0,$$

where $\tilde{H}_n(t) := \int_0^t g_n(s)ds$ and $\tilde{H}_n{}'(t) := g_n(t)$, with $g_n(s) \leq g(s)$, we obtain

$$\begin{aligned} &-\int_\Omega |\nabla w_n|^p \tilde{H}_n{}'(w_n) e^{-\tilde{H}_n(w_n)}\varphi dx + \int_\Omega |\nabla w_n|^{p-2}\nabla w_n e^{-\tilde{H}_n(w_n)}\nabla\varphi dx \\ &= \int_\Omega \Psi_n e^{-\tilde{H}_n(w_n)}\varphi dx - \int_\Omega \frac{g_n(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p} e^{-\tilde{H}_n(w_n)}\varphi dx. \end{aligned}$$

By applying in the previous equation the following inequality

$$\frac{|y|^p}{1+\frac{1}{n}|y|^p} \leq |y|^p \quad \text{for every } \ y \in \mathbb{R}^n,$$

we obtain

$$\begin{aligned} &\int_\Omega |\nabla w_n|^{p-2}\nabla w_n e^{-\tilde{H}_n(w_n)}\nabla\varphi dx - \int_\Omega \Psi_n e^{-\tilde{H}_n(w_n)}\varphi dx \\ &\quad= \int_\Omega |\nabla w_n|^p \tilde{H}_n{}'(w_n) e^{-\tilde{H}_n(w_n)}\varphi dx - \int_\Omega \frac{g_n(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p} e^{-\tilde{H}_n(w_n)}\varphi dx \\ &\quad\geq \int_\Omega g_n(w_n)|\nabla w_n|^p e^{-\tilde{H}_n(w_n)}\varphi dx - \int_\Omega g_n(w_n)|\nabla w_n|^p e^{-\tilde{H}_n(w_n)}\varphi dx \\ &\quad= 0. \end{aligned}$$

Hence,

$$\int_\Omega |\nabla w_n|^{p-2}\nabla w_n \nabla\varphi e^{-\tilde{H}_n(w_n)}dx \geq \int_\Omega \Psi_n e^{-\tilde{H}_n(w_n)}\varphi dx. \tag{2.21}$$

Define $\tilde{H}(w) = \lim_{n\to\infty} \tilde{H}_n(w_n)$. Taking the limit in (2.21) as $n \to \infty$, since $w_n > 0$ and $e^{-\tilde{H}_n(w_n)} < 1$ in Ω, we obtain

$$\begin{aligned} \int_\Omega |\nabla w|^{p-2}\nabla w \nabla\varphi e^{-\tilde{H}(w)}dx &\geq \int_\Omega \Psi e^{-\tilde{H}(w)}\varphi dx \\ &\geq \int_\Omega T_1(\Psi) e^{-\tilde{H}(w)}\varphi dx. \end{aligned} \tag{2.22}$$

Define $v(x) := \psi(w(x)) = \int_0^{w(x)} \left(e^{-\tilde{H}(s)}\right)^{\frac{1}{p-1}} dt$, where $\psi(s) := \int_0^s \left(e^{-\tilde{H}(s)}\right)^{\frac{1}{p-1}} dt$ is strictly increasing. Let z be a solution of problem

$$\begin{cases} -\Delta_p z = \dfrac{T_1(\Psi)}{e^{\tilde{H}(w)}} & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\frac{T_1(\Psi)}{e^{\tilde{H}(w)}} \in L^\infty(\Omega)$, by a result of Lieberman (see [28, Theorem 1]), we have that $z \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Moreover, by strong maximum principle (see [34]), we conclude that $z > 0$ in Ω.

By applying in (2.22) the relation $\nabla v = \nabla w \left(e^{-\tilde{H}(w)}\right)^{\frac{1}{p-1}}$, we obtain

$$\begin{aligned} \int_\Omega |\nabla v|^{p-2}\nabla v \nabla\varphi dx &= \int_\Omega |\nabla w|^{p-2}\nabla w \nabla\varphi e^{-\tilde{H}(w)}dx \\ &\geq \int_\Omega T_1(\Psi) e^{-\tilde{H}(w)}\varphi dx. \end{aligned}$$

Hence, by weak comparison principle (see [34]), we have that $v(x) \geq z(x) > 0$ in Ω. Finally, since $\psi(w(x)) = v(x) > 0$ and ψ is strictly increasing in Ω, then $w(x) > 0$ in Ω. $\square$

The following lemmas concerning with the uniqueness of solution will be useful in the sequel and they can be deduced by using ideas of Bénilan, Boccardo, Gallouët, Gariepy, Pierre and Vazquez [6].

Lemma 2.4. *If w is a solution of $(P)_{\lambda\sigma}$ for $\lambda = 0$, then for every $a, k > 0$*

(i) $\frac{1}{k}\int_{\{|w|<k\}} |\nabla w|^p dx \leq \int_\Omega \Psi dx$;

(ii) $\frac{1}{a}\int_{\{k<|w|<k+a\}} |\nabla w|^p dx \leq \int_\Omega T_{k,a}(w)\Psi dx \leq \int_{\{|w|>k\}} \Psi dx$,

where $T_{k,a}(s) := T_a\big(s - T_k(s)\big)$.

Lemma 2.5. *Let $1 < p < N$. If Ω is a bounded domain in $\mathbb{R}^N$ and $w \in W_0^{1,p}(\Omega)$ satisfies*

$$\frac{1}{k}\int_{\{|w|<k\}} |\nabla w|^p dx \leq M \tag{2.23}$$

for every $k > 0$, then there exists $C = C(N, p)$ such that

$$meas\,\{x \in \Omega \,:\, |w| > k\} \leq CM^{\frac{N}{N-p}} k^{-p_1}, \tag{2.24}$$

where $p_1 = \frac{N(p-1)}{N-p}$.

Completing the proof of Theorem 1.1: Let u and v solutions of $(P)_{\lambda\sigma}$ for $\lambda = 0$, so

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi dx + \int_\Omega g(u)|\nabla u|^p \varphi dx = \int_\Omega \Psi\varphi dx \tag{2.25}$$

and

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla \varphi dx + \int_\Omega g(v)|\nabla v|^p \varphi dx = \int_\Omega \Psi\varphi dx, \tag{2.26}$$

for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

For every $h \geq 0$, choosing $\varphi = T_k(u - T_h v)^+$ and $\varphi = T_k(v - T_h u)^+$ in (2.25) and (2.26), respectively, we obtain

$$\int_{\{|u-T_h v|<k\}} \langle |\nabla u|^{p-2}\nabla u, \nabla(u - T_h v)^+\rangle dx \leq \int_\Omega T_k(u - T_h v)^+ \Psi dx$$

and

$$\int_{\{|v-T_h u|<k\}} \langle |\nabla v|^{p-2}\nabla v, \nabla(v - T_h u)^+\rangle dx \leq \int_\Omega T_k(v - T_h u)^+ \Psi dx.$$

Thus, if we define

$$I := \int_{\{|u-T_h v|<k\}} \langle |\nabla u|^{p-2}\nabla u, \nabla(u - T_h v)^+\rangle dx + \int_{\{|v-T_h u|<k\}} \langle |\nabla v|^{p-2}\nabla v \nabla(v - T_h u)^+\rangle dx, \tag{2.27}$$

the conclusion $u = v$ will be reached after passing to the limit $h \to \infty$ in the previous relations and disregarding some positive terms. We will to split the previous integrals into the contributions corresponding to different integration sets.

Consider the following set

$$A_0 := \{x \in \Omega \,:\, |u - v| < k,\ |u| < h,\ |v| < h\}.$$

Thus, when restricted to A_0 the first member of (2.27) gives the following main contribution

$$\begin{aligned} I_0 &:= \int_{A_0} \langle |\nabla u|^{p-2}\nabla u, \nabla(u-v)^+\rangle dx + \int_{A_0} \langle |\nabla v|^{p-2}\nabla v, \nabla(v-u)^+\rangle dx \\ &= \int_{A_0} \langle |\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla(u-v)^+\rangle dx. \end{aligned}$$

The remaining first member of (2.27) is estimated taking the first term on the set

$$A_1 := \{x \in \Omega \ : \ |u - T_h v| < k, \ |v| > h\},$$

i.e.,

$$\int_{A_1} \langle |\nabla u|^{p-2}\nabla u, \nabla(u - T_h v)^+\rangle dx = \int_{A_1} |\nabla u|^p dx \geq 0.$$

On the remaining set

$$A_2 := \{x \in \Omega \ : \ |u - T_h v| < k, \ |v| < h, \ |u| \geq h\}$$

we have

$$\begin{aligned} \int_{A_2} \langle |\nabla u|^{p-2}\nabla u, \nabla(u - T_h v)^+\rangle dx &= \int_{A_2} \langle |\nabla u|^{p-2}\nabla u, \nabla(u-v)^+\rangle dx \\ &\geq -\int_{A_2} |\nabla u|^{p-2}\nabla u \nabla v dx. \end{aligned}$$

Now, we estimate the second member of (2.27) in the sets A_1' where $|u| \geq h$, and A_2', where $|u| < h$ and $|v| \geq h$. Notice that all these sets and integrals depend of k and h.

Summing up we estimate the first member of (2.27) as follows

$$I \geq I_0 - I_3,$$

where

$$I_3 := \int_{A_2} |\nabla u|^{p-2}\nabla u \nabla v dx + \int_{A_2'} |\nabla v|^{p-2}\nabla v \nabla u dx.$$

Now, we will check that $I_3 \to 0$ as $h \to \infty$. Indeed, the first term of I_3 can be estimated by

$$\begin{aligned} \int_{A_2} |\nabla u|^{p-2}\nabla u \nabla v dx &\leq \left(\int_{A_2} |\nabla u|^{(p-1)\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left(\int_{A_2'} |\nabla v|^p dx \right)^{\frac{1}{p}} \\ &= \left(\int_{\{h \leq |u| \leq h+k\}} |\nabla u|^p dx \right)^{\frac{p-1}{p}} \left(\int_{\{h-k \leq |v| \leq h\}} |\nabla v|^p dx \right)^{\frac{1}{p}} \\ &= \|u\|^{p-1}_{L^p(\{h \leq |u| \leq h+k\})} \|v\|_{L^p(\{h \leq |u| \leq h+k\})}, \end{aligned}$$

which converges to 0 as $h \to \infty$ due to Lemmas 2.4 and 2.5. The treatment of the second term is analogous.

Now, we will estimate

$$\int_{\Omega} \Psi\left[T_k(u - T_h v)^+ - T_k(v - T_h u)^+\right] dx.$$

The previous integral on the set $B_0 := \{x \in \Omega \ : \ |u| < h, \ |v| < h\}$ gives

$$J_0 := \int_{B_0} \Psi\left[T_k(u - T_h v)^+ - T_k(v - T_h u)^+\right] dx = 0.$$

The integral on the set $B_1 := \{x \in \Omega \,:\, |u| \geq h\}$ is estimate by

$$\begin{aligned}|J_1| &:= \left| \int_{B_1} \Psi\left[T_k(u - T_h v)^+ - T_k(v - T_h u)^+\right] dx \right| \\ &= \left| \int_{B_1} \Psi\left[T_k(u - T_h v)^+ - T_k(v - h)^+\right] dx \right| \\ &\leq 2k \int_{B_1} |\Psi| dx,\end{aligned}$$

while on $B_2 := \{x \in \Omega \,:\, |v| \geq h\}$ we get

$$\begin{aligned}|J_2| &:= \left| \int_{B_2} \Psi\left[T_k(u - T_h v)^+ - T_k(v - T_h u)^+\right] dx \right| \\ &= \left| \int_{B_2} \Psi\left[T_k(u - h)^+ - T_k(v - Thu)^+\right] dx \right| \\ &\leq 2k \int_{B_2} |\Psi| dx.\end{aligned}$$

Since the measure of both sets $B_1(h, k)$ and $B_2(h, k)$ converges to zero as $h \to \infty$ for fixed $k > 0$, then $J_1 + J_2 \to 0$ as $h \to \infty$.

Combining the previous estimates, for fixed $k > 0$, we get from (2.27)

$$\int_{A_0(h,k)} \langle |\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla(u - v)^+\rangle dx \leq \varpi(h),$$

where $\lim_{h\to\infty} \varpi(h) = 0$.

Since the set $A_0(h, k)$ converges to $\{x \in \Omega \,:\, |u - v| < k\}$, then

$$\int_{\{x\in\Omega \,:\, |u-v|<k\}} \langle |\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla(u - v)^+\rangle dx \leq 0, \qquad k > 0 \text{ fixed.}$$

Since the previous inequality is true for all $k > 0$, we conclude by (2.7) that $\nabla u(x) = \nabla v(x)$ a.e. in Ω. Thus, since $u, v \in W_0^{1,p}(\Omega)$ then $u(x) = v(x)$ a.e. in Ω.

Now, we will prove that w satisfies

$$\int_\Omega |\nabla w|^{p-2}\nabla w \nabla \varphi dx + \int_\Omega g(w)|\nabla w|^p \varphi dx = \int_\Omega \Psi \varphi dx, \tag{2.28}$$

for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Indeed, we have

$$\int_\Omega |\nabla w_n|^{p-2}\nabla w_n \nabla \varphi dx + \int_\Omega \frac{g(w_n)|\nabla w_n|^p}{1 + \frac{1}{n}|\nabla w_n|^p} \varphi dx = \int_\Omega \Psi_n \varphi dx \leq \int_\Omega \Psi \varphi dx, \tag{2.29}$$

for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\varphi \geq 0$.

For every $\epsilon > 0$, taking $\varphi = \frac{1}{\epsilon} T_\epsilon(w_n)$ as test function in the previous relation, we have

$$\begin{aligned}\frac{1}{\epsilon}\int_\Omega |\nabla w_n|^{p-2}\nabla w_n \nabla T_\epsilon(w_n) dx + \int_\Omega \frac{g(w_n)|\nabla w_n|^p}{1 + \frac{1}{n}|\nabla w_n|^p}\frac{1}{\epsilon} T_\epsilon(w_n) dx &= \int_\Omega \Psi_n \frac{1}{\epsilon} T_\epsilon(w_n) dx \\ &\leq \int_\Omega \Psi_n dx;\end{aligned}$$

hence,

$$\int_\Omega \frac{g(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p}\frac{T_\epsilon(w_n)}{\epsilon}dx \le \int_\Omega \Psi_n dx. \tag{2.30}$$

Since $\frac{T_\epsilon(w_n(x))}{\epsilon} = \frac{1}{\epsilon}w_n(x)\chi_{\{x\in\Omega\,:\,w_n\le\epsilon\}} + \chi_{\{x\in\Omega\,:\,w_n>\epsilon\}}$ for every $x\in\Omega$, then $w_n(x)>\epsilon$ and $T_\epsilon(w_n(x))=\epsilon$ as $\epsilon\to 0$. Taking the limit in (2.30) as $\epsilon\to 0$, by the Dominated Convergence Theorem, we have

$$\int_\Omega \frac{g(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p}dx \le \int_\Omega \Psi_n dx, \tag{2.31}$$

i.e.,

$$\frac{g(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p}\varphi \in L^1(\Omega), \qquad \forall \varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega).$$

Define $A_n := |\nabla w_n|^{p-2}\nabla w_n\nabla\varphi$ and $B_n := \frac{g(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p}\varphi$. So, using Fatou's lemma in (2.29), we obtain

$$\begin{aligned}\int_\Omega |\nabla w|^{p-2}\nabla w\nabla\varphi dx + \int_\Omega g(w)|\nabla w|^p\varphi dx &\le \liminf_{n\to\infty}\left(\int_\Omega (A_n+B_n)\varphi dx\right)\\ &\le \int_\Omega \Psi\varphi dx,\end{aligned}$$

i.e.,

$$\int_\Omega |\nabla w|^{p-2}\nabla w\nabla\varphi dx + \int_\Omega g(w)|\nabla w|^p\varphi dx \le \int_\Omega \Psi\varphi dx, \tag{2.32}$$

for all $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ with $\varphi\ge 0$.

Now, define $S(t) := \int_0^t \beta(s)ds$, $\beta\ge 0$ measurable, and take $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, $\varphi\ge 0$ and $k>0$. Thus, taking v_n as test function in $(P)_n$ defined by

$$v_n := e^{-S(w_n)}e^{S(T_k(w_n))}\varphi,$$

we obtain

$$\begin{aligned}&\int_\Omega |\nabla w_n|^{p-2}\nabla w_n\nabla\varphi e^{-S(w_n)}e^{S(T_k(w_n))}dx\\ &\qquad+\int_\Omega |\nabla w_n|^{p-2}\nabla w_n\nabla T_k(w_n)\beta(T_k(w_n))e^{-S(w_n)}e^{S(T_k(w_n))}\varphi dx\\ &=\int_\Omega |\nabla w_n|^{p-2}\nabla w_n\nabla w_n\beta(w_n)e^{-S(w_n)}e^{S(w_n)}\varphi dx\\ &\qquad-\int_\Omega \frac{g(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p}e^{-S(w_n)}e^{S(T_k(w_n))}\varphi dx\\ &\qquad+\int_\Omega \Psi_n e^{-S(w_n)}e^{S(T_k(w_n))}\varphi dx\\ &\ge 0,\end{aligned}$$

because

$$\int_\Omega \frac{g(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p}e^{-S(w_n)}e^{S(T_k(w_n))}\varphi dx \le \int_\Omega \Psi_n e^{-S(w_n)}e^{S(T_k(w_n))}\varphi dx.$$

Again, by Fatou's lemma, we have

$$\begin{aligned}\int_\Omega &|\nabla w|^{p-2}\nabla w\nabla\varphi e^{-S(w)}e^{S(T_k(w))}dx\\&+\int_\Omega|\nabla w|^{p-2}\nabla w\nabla T_k(w)\beta(T_k(w))e^{-S(w)}e^{S(T_k(w))}\varphi dx\\&\geq\int_\Omega|\nabla w|^{p-2}\nabla w\nabla w\beta(w)e^{-S(w)}e^{S(w)}\varphi dx\\&-\int_\Omega g(w)|\nabla w|^pe^{-S(w)}e^{S(T_k(w))}\varphi dx\\&+\int_\Omega\Psi e^{-S(w)}e^{S(T_k(w))}\varphi dx,\end{aligned}$$

as $n\to\infty$. Since $0\leq e^{-S(w)}e^{S(T_k(w))}\leq 1$, by letting $k\to\infty$, it follows immediately from the previous inequality that

$$\int_\Omega|\nabla w|^{p-2}\nabla w\nabla\varphi dx+\int_\Omega g(w)|\nabla w|^p\varphi dx\geq\int_\Omega\Psi\varphi dx, \tag{2.33}$$

for all $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ with $\varphi\geq 0$.

Hence, using the relations (2.32) and (2.33), we conclude that the equality (2.28) holds for all $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ with $\varphi\geq 0$.

Thus, since $\varphi:=\varphi^+-\varphi^-$ and $\varphi^+,\varphi^-\geq 0$, we obtain

$$\int_\Omega|\nabla w|^{p-2}\nabla w\nabla\varphi dx+\int_\Omega g(w)|\nabla w|^p\varphi dx=\int_\Omega\Psi\varphi dx$$

for all $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Therefore, $(P)_{\lambda\sigma}$ has unique solution in $W_0^{1,p}(\Omega)$ for $\lambda=0$. □

3 Proof of Theorem 1.2

In this section, first we prove some results which are used in the proof of our main theorem. Notice that our definition of solution of $(P)_{\lambda\sigma}$ includes the integrability of $g(u)|\nabla u|^p$. Using some ideas of Arcoya, Carmona and Martínez-Aparicio [5], we will see in the following result that a consequence is the integrability of $g(u)|\nabla u|^p\varphi$ for all $\varphi\in W_0^{1,p}(\Omega)$.

Lemma 3.1. *If $0<u\in W_0^{1,p}(\Omega)$ is a solution for $(P)_{\lambda\sigma}$, then $g(u)|\nabla u|^p\varphi$ is integrable in Ω for all $\varphi\in W_0^{1,p}(\Omega)$. Moreover, we have*

$$\int_\Omega|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi dx+\int_\Omega g(u)|\nabla u|^p\varphi dx=\lambda\int_\Omega u^\sigma\varphi dx+\int_\Omega\Psi\varphi dx. \tag{3.1}$$

Proof. Since $\sigma\leq p^*-1$, note that u^σ, $u^\sigma\varphi\in L^1(\Omega)$. Indeed, since $u\in L^p(\Omega)$ and $W_0^{1,p}(\Omega)\hookrightarrow L^{p^*}(\Omega)$ for $p<N$, we have

$$\int_\Omega u^\sigma dx\leq\int_\Omega|u|^\sigma dx<\infty.$$

On the other hand, we have

$$\sigma\frac{p^*}{p^*-1}\leq(p^*-1)\frac{p^*}{p^*-1}=p^*;$$

thus, by Hölder's inequality we obtain

$$\int_\Omega u^\sigma \varphi dx \le \left[\left(\int_\Omega |u|^{\sigma\frac{p^*}{p^*-1}}dx\right)^{\frac{p^*-1}{\sigma p^*}}\right]^\sigma \left(\int_\Omega |\varphi|^{p^*}dx\right)^{\frac{1}{p^*}} \le C\|u\|_{1,p}^\sigma \|\varphi\|. \tag{3.2}$$

Hence, by previous relations, we conclude that $u^\sigma,\ u^\sigma\varphi \in L^1(\Omega)$.

By taking $T_k(\varphi^+)$ as test function in (1.3) and using Hölder's inequality we have

$$\begin{aligned}\int_\Omega g(u)|\nabla u|^p T_k(\varphi^+)dx &= -\int_\Omega |\nabla u|^{p-2}\nabla u \nabla T_k(\varphi^+)dx + \int_\Omega (\lambda u^\sigma + \Psi)T_k(\varphi^+)dx \\ &\le \int_\Omega |\nabla u|^{p-1}|\nabla T_k(\varphi^+)|dx + \int_\Omega (\lambda u^\sigma + \Psi)T_k(\varphi^+)dx \\ &\le \left(\int_\Omega (|\nabla u|^{p-1})^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla T_k(\varphi^+)|^p dx\right)^{\frac{1}{p}} \\ &\quad + \int_\Omega (\lambda u^\sigma + \Psi)T_k(\varphi^+)dx \\ &\le \|u\|_{1,p}^{p-1}\|\varphi\|_{1,p} + \int_\Omega (\lambda u^\sigma + \Psi)T_k(\varphi^+)dx.\end{aligned}$$

Now, by taking limit as $k \to \infty$ and using Fatou's lemma, we deduce that $g(u)|\nabla u|^p\varphi^+ \in L^1(\Omega)$ with

$$\int_\Omega g(u)|\nabla u|^p \varphi^+ dx \le \|u\|_{1,p}\|\varphi\|_{1,p} + \int_\Omega (\lambda u^\sigma + \Psi)\varphi^+ dx. \tag{3.3}$$

Similarly, by taking $T_k(-\varphi^-)$ as test function in (1.3), we obtain that $g(u)|\nabla u|^p\varphi^- \in L^1(\Omega)$ with

$$-\int_\Omega g(u)|\nabla u|^p \varphi^- dx \le \|u\|_{1,p}\|\varphi^-\|_{1,p} + \int_\Omega (\lambda u^\sigma - \Psi)\varphi^- dx. \tag{3.4}$$

By combining the relations (3.3) and (3.4), we conclude that $g(u)|\nabla u|^p\varphi \in L^1(\Omega)$ for all $\varphi \in W_0^{1,p}(\Omega)$ with

$$\int_\Omega g(u)|\nabla u|^p \varphi dx \le \|u\|_{1,p}\|\varphi\|_{1,p} + \int_\Omega (\lambda u^\sigma + \Psi)\varphi dx.$$

Lastly, note that this integrability of $g(u)|\nabla u|^p\varphi$ allows to use a density argument to conclude (3.1) from (1.3). □

The next result will be related with the compactness for the operator $K(\lambda, w)$ defined in (1.5).

Lemma 3.2. *Assume that $\Psi \in L^q(\Omega)$ with $q = \frac{pN}{N(p-1)+p}$, $g \ge 0$ is continuous in $[0,+\infty)$ or $g \ge 0$ is continuous in $(0,+\infty)$ and integrable in an neighborhood of zero with $\lim_{s\to 0} g(s) = +\infty$. If the sequences $\{t_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$ are convergent, respectively, to t^* and λ, and $\{w_n\} \subset W_0^{1,p}(\Omega)$ weakly convergent to w, then the sequence of (uniquely defined) solutions $\{u_n\} \subset W_0^{1,p}(\Omega)$ of*

$$\begin{cases} -\Delta_p u_n + t_n g(u_n)|\nabla u_n|^p = \lambda_n (w_n^+(x))^\sigma + \Psi(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

is strongly convergent in $W_0^{1,p}(\Omega)$ to the solution u of

$$\begin{cases} -\Delta_p u + t^* g(u)|\nabla u|^p = \lambda (w^+(x))^\sigma + \Psi(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

Proof. Since the functions $\lambda_n(w_n^+(x))^\sigma + \Psi(x)$ and $\lambda(w^+(x))^\sigma + \Psi(x)$ are in $L^q(\Omega)$ with $q = \frac{pN}{N(p-1)+p}$, then by Theorem 1.1 the existence of an unique solution of (3.5) and (3.6) it is holds. Indeed, we have that

$$\int_\Omega |\lambda w_n^+(x)^\sigma + \Psi(x)|^q dx \le 2^{q-1}\int_\Omega |\lambda w_n^+(x)^\sigma|^q dx + 2^{q-1}\int_\Omega |\Psi(x)|^q dx$$

and $\sigma q \le \frac{N(p-1)+p}{N-P}\frac{pN}{N(p-1)+p} = \frac{pN}{N-p} = p^*$.

In order to prove the compactness of $K(\lambda, u)$ is suffices to prove that every subsequence of $\{u_n\}$ possesses a subsequence converging to the unique solution $u \in W_0^{1,p}(\Omega)$ of (3.6). First we will prove that $\{u_n\}$ is a bounded sequence in $W_0^{1,p}(\Omega)$. Indeed, choosing u_n as test function in (3.5) and using that t_n and $g(u_n)$ are nonnegative, we have

$$\begin{aligned}\int_\Omega |\nabla u_n|^p dx &\le \int_\Omega |\nabla u_n|^p dx + t_n\int_\Omega g(u_n)|\nabla u_n|^p u_n dx\\ &= \lambda_n\int_\Omega (w_n^+(x))^\sigma u_n dx + \int_\Omega \Psi u_n dx.\end{aligned} \tag{3.7}$$

Since $W_0^{1,p}(\Omega) \hookrightarrow L^{p*}(\Omega)$, by Hölder's inequality we obtain

$$\int_\Omega \Psi u_n dx \le C\|\Psi\|_{\frac{pN}{N(p-1)+p}}\|u_n\|_{1,p}. \tag{3.8}$$

Furthermore, since $\sigma\frac{N(p-1)+p}{pN} = \frac{p^*}{p^*-1} \le p^*$, again by Hölder's inequality we have

$$\begin{aligned}\lambda_n\int_\Omega (w_n^+(x))^\sigma u_n dx &\le C\left[\left(\int_\Omega |w_n|^{\sigma\frac{N(p-1)+p}{pN}}dx\right)^{\frac{pN}{\sigma[N(p-1)+p]}}\right]^\sigma\left(\int_\Omega |u_n|^{\frac{pN}{N-p}}dx\right)^{\frac{N-p}{pN}}\\ &\le \overline{C}\|w_n\|_{p^*}^\sigma\|u_n\|_{p^*}\\ &\le \overline{\overline{C}}\|w_n\|_{1,p}^\sigma\|u_n\|_{1,p}.\end{aligned} \tag{3.9}$$

By combining the relations (3.7), (3.8) and (3.9) we conclude that

$$\|u_n\|_{1,p} \le \left[C\|w_n\|_{1,p}^\sigma + \|\Psi\|_{\frac{pN}{N(p-1)+p}}\right]^{\frac{1}{p-1}}.$$

Therefore, $\{u_n\}$ is a bounded sequence in $W_0^{1,p}(\Omega)$. Thus, going if necessary to a subsequence, still denoted by $\{u_n\}$, there exists $\overline{u} \in W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup \overline{u}$ weakly in $W_0^{1,p}(\Omega)$.

Repeating the arguments used in the proof Lemma 2.3, we obtain that

$$\nabla u_n(x) \to \nabla\overline{u}(x) \quad \text{a.e in } \Omega \quad \text{and} \quad \nabla u_n \to \nabla\overline{u} \quad \text{in } L^q(\Omega), \quad q < p.$$

Now, we will prove that $\overline{u}$ satisfies the following equality

$$\int_\Omega |\nabla\overline{u}|^{p-2}\nabla\overline{u}\cdot\nabla\varphi dx + t^*\int_\Omega g(\overline{u})|\nabla\overline{u}|^p\varphi dx = \int_\Omega (\lambda(w^+)^\sigma + \Psi)\varphi dx, \quad \varphi \in W_0^{1,p}(\Omega). \tag{3.10}$$

First, we will show that $\overline{u} > 0$ in Ω. Indeed, we have

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\varphi dx + t_n\int_\Omega g(u_n)|\nabla u_n|^p\varphi dx = \int_\Omega (\lambda_n(w_n^+)^\sigma + \Psi)\varphi dx, \quad \varphi \in W_0^{1,p}(\Omega). \tag{3.11}$$

Thus, choosing v_n as test function in the previous equality such that

$$v_n := e^{-H(u_n)}\varphi, \quad \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \quad \varphi \geq 0,$$

we obtain

$$\begin{aligned} -\int_\Omega |\nabla u_n|^p H'(u_n) e^{-H(u_n)} \varphi dx &+ \int_\Omega |\nabla u_n|^{p-2} \nabla u_n e^{-H(u_n)} \nabla \varphi dx \\ &= \int_\Omega (\lambda_n (w_n^+)^\sigma + \Psi) e^{-H(u_n)} \varphi dx - t_n \int_\Omega g(u_n) |\nabla u_n|^p e^{-H(u_n)} \varphi dx, \end{aligned}$$

where $H(t) := \int_0^t g(s)ds$.

By ordering the terms of the previous equation, by using $H_n'(t) = g(t)$, we obtain

$$\begin{aligned} \int_\Omega |\nabla u_n|^{p-2} \nabla u_n e^{-H(u_n)} \nabla \varphi dx &- \int_\Omega (\lambda_n (w_n^+)^\sigma + \Psi) e^{-H(u_n)} \varphi dx \\ &= (1 - t_n) \int_\Omega g(u_n) |\nabla u_n|^p e^{-H(u_n)} \varphi dx \\ &\geq 0, \end{aligned}$$

i.e.,

$$\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi e^{-H(u_n)} dx \geq \int_\Omega (\lambda_n (w_n^+)^\sigma + \Psi) e^{-H(u_n)} \varphi dx.$$

Thus, by taking limit as $n \to \infty$, we have

$$\int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \nabla \varphi e^{-H(\overline{u})} dx \geq \int_\Omega (\lambda (w^+)^\sigma + \Psi) e^{-H(\overline{u})} \varphi dx. \tag{3.12}$$

Define $v(x) := \psi(\overline{u}(x)) = \int_0^{\overline{u}(x)} \left(e^{-H(s)}\right)^{\frac{1}{p-1}} dt$, where $\psi(s) := \int_0^s \left(e^{-H(s)}\right)^{\frac{1}{p-1}} dt$ is strictly increasing. Let z be a solution of problem

$$\begin{cases} -\Delta_p z = \dfrac{T_1((w^+)^\sigma + \Psi)}{e^{H(\overline{u})}} & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\frac{T_1(\Psi)}{e^{H(\overline{u})}} \in L^\infty(\Omega)$, by a result of Lieberman (see [28, Theorem 1]), we have that $z \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Furthermore, by strong maximum principle, we conclude that $z > 0$ in Ω. By applying in (3.12) the relation $\nabla v = \nabla \overline{u} \left(e^{-H(\overline{u})}\right)^{\frac{1}{p-1}}$, we obtain

$$\int_\Omega |\nabla v|^{p-2} \nabla v \nabla \varphi dx \geq \int_\Omega T_1(\Psi) e^{-H(\overline{u})} \varphi dx.$$

Thus, by weak comparison principle, we have $v(x) \geq z(x) > 0$ in Ω. However, since $\psi(\overline{u}(x)) := v(x) > 0$ and ψ is strictly increasing in Ω, then $\overline{u}(x) > 0$ in Ω.

Now, we resume the proof of (3.10). For every $\epsilon > 0$, taking $\varphi := \frac{1}{\epsilon} T_\epsilon(u_n)$ as test function in (3.10), we obtain

$$\begin{aligned} \frac{1}{\epsilon} \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla T_\epsilon(u_n) dx + t_n \int_\Omega g(u_n) |\nabla u_n|^p \frac{1}{\epsilon} T_\epsilon(u_n) dx &= \int_\Omega (\lambda_n (w_n^+)^\sigma + \Psi) \frac{1}{\epsilon} T_\epsilon(u_n) dx \\ &\leq \int_\Omega (\lambda_n (w_n^+)^\sigma + \Psi) dx; \end{aligned}$$

hence,

$$t_n \int_\Omega g(u_n) |\nabla u_n|^p \frac{T_\epsilon(u_n)}{\epsilon} dx \leq \int_\Omega (\lambda_n (w_n^+)^\sigma + \Psi) dx. \tag{3.13}$$

Since $\frac{T_\epsilon(u_n(x))}{\epsilon} = \frac{1}{\epsilon}u_n(x)\chi_{\{x\in\Omega\,:\,w_n\le\epsilon\}} + \chi_{\{x\in\Omega\,:\,u_n>\epsilon\}}$ for every $x \in \Omega$, then $u_n(x) > \epsilon$ and $T_\epsilon(u_n(x)) = \epsilon$ as $\epsilon \to 0$. Taking the limit in (3.13) as $\epsilon \to 0$, by the Dominated Convergence Theorem, we have

$$t_n \int_\Omega g(u_n)|\nabla u_n|^p dx \le \int_\Omega (\lambda_n (w_n^+)^\sigma + \Psi)dx, \tag{3.14}$$

i.e.,

$$g(u_n)|\nabla u_n|^p\varphi \in L^1(\Omega), \qquad \forall \varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega).$$

Define $A_n := |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi$ and $B_n := g(u_n)|\nabla u_n|^p\varphi$. Hence, using Fatou's lemma in (3.14), we have

$$\begin{aligned}\int_\Omega |\nabla \overline{u}|^{p-2}\nabla\overline{u}\nabla\varphi dx + t^*\int_\Omega g(\overline{u})|\nabla\overline{u}|^p\varphi dx &\le \liminf_{n\to\infty}\left(\int_\Omega (A_n + B_n)\varphi dx\right)\\ &= \int_\Omega (\lambda(w^+)^\sigma + \Psi)\varphi dx,\end{aligned}$$

i.e.,

$$\int_\Omega |\nabla \overline{u}|^{p-2}\nabla\overline{u}\nabla\varphi dx + t^*\int_\Omega g(\overline{u})|\nabla\overline{u}|^p\varphi dx \le \int_\Omega (\lambda(w^+)^\sigma + \Psi)\varphi dx, \tag{3.15}$$

for all $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ with $\varphi \ge 0$.

Now, define $S(t) := \int_0^t \beta(s)ds$, $\beta \ge 0$ measurable, and take $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, $\phi \ge 0$ and $k > 0$. Hence, taking v_n as test function in (3.12) defined by

$$v_n := e^{-S(u_n)}e^{S(T_k(u_n))}\varphi,$$

we obtain

$$\begin{aligned}&\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\varphi e^{-S(u_n)}e^{S(T_k(u_n))}dx\\ &\qquad + \int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla T_k(u_n)\beta(T_k(u_n))e^{-S(u_n)}e^{S(T_k(u_n))}\varphi dx\\ &= \int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla u_n\beta(u_n)e^{-S(u_n)}e^{S(u_n)}\varphi dx\\ &\qquad - t_n\int_\Omega g(u_n)|\nabla u_n|^p e^{-S(u_n)}e^{S(T_k(u_n))}\varphi dx\\ &\qquad + \int_\Omega (\lambda(w_n^+)^\sigma + \Psi)\varphi e^{-S(u_n)}e^{S(T_k(u_n))}\varphi dx\\ &\ge 0,\end{aligned}$$

because

$$t_n\int_\Omega g(u_n)|\nabla u_n|^p e^{-S(u_n)}e^{S(T_k(u_n))}\varphi dx \le \int_\Omega (\lambda(w_n^+)^\sigma + \Psi)\varphi e^{-S(u_n)}e^{S(T_k(u_n))}\varphi dx.$$

Again, by Fatou's lemma, we have

$$\begin{aligned}&\int_\Omega |\nabla\overline{u}|^{p-2}\nabla\overline{u}\nabla\varphi e^{-S(\overline{u})}e^{S(T_k(\overline{u}))}dx\\ &\qquad + \int_\Omega |\nabla\overline{u}|^{p-2}\nabla\overline{u}\nabla T_k(\overline{u})\beta(T_k(\overline{u}))e^{-S(\overline{u})}e^{S(T_k(\overline{u}))}\varphi dx\\ &\ge \int_\Omega |\nabla\overline{u}|^{p-2}\nabla\overline{u}\nabla\overline{u}\beta(\overline{u})e^{-S(\overline{u})}e^{S(\overline{u})}\varphi dx\\ &\qquad - t^*\int_\Omega g(\overline{u})|\nabla\overline{u}|^p e^{-S(\overline{u})}e^{S(T_k(\overline{u}))}\varphi dx\\ &\qquad + \int_\Omega (\lambda(w^+)^\sigma + \Psi)\varphi e^{-S(\overline{u})}e^{S(T_k(\overline{u}))}\varphi dx.\end{aligned} \tag{3.16}$$

as $n \to \infty$. Since $0 \leq e^{-S(\overline{u})} e^{S(T_k(\overline{u}))} \leq 1$, by letting $k \to \infty$, it follows from (3.16) that

$$\int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \nabla \varphi dx + t^* \int_\Omega g(\overline{u}) |\nabla \overline{u}|^p \varphi dx \geq \int_\Omega (\lambda (w^+)^\sigma + \Psi) \varphi dx, \tag{3.17}$$

for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\varphi \geq 0$.

Thus, by (3.15) and (3.17), we conclude that (3.10) holds for all for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\varphi \geq 0$.

Hence, since $\varphi := \varphi^+ - \varphi^-$ and $\varphi^+, \varphi^- \geq 0$, we have

$$\int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \nabla \varphi dx + t^* \int_\Omega g(\overline{u}) |\nabla \overline{u}|^p \varphi dx = \int_\Omega (\lambda (w^+)^\sigma + \Psi) \varphi dx, \tag{3.18}$$

for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Since problem (3.6) has an unique solution, then $u = \overline{u}$.

We still need to prove that $u_n \to u$ in $W_0^{1,p}(\Omega)$. For fixed $k > 0$, by taking $u_n = G_k(u_n) + T_k(u_n)$, we have

$$\begin{aligned} \|u_n - u\|_{1,p} &= \|u_n - T_k(u) + T_k(u) - u\|_{1,p} \\ &\leq \|u_n - T_k(u)\|_{1,p} + \|T_k(u) - u\|_{1,p} \\ &= \|G_k(u_n) + T_k(u_n) - T_k(u)\|_{1,p} + \|T_k(u) - u\|_{1,p} \\ &\leq \|G_k(u_n)\|_{1,p} + \|T_k(u_n) - T_k(u)\|_{1,p} + \|T_k(u) - u\|_{1,p}. \end{aligned} \tag{3.19}$$

Hence, the strong convergence of $\{u_n\}$ in $W_0^{1,p}(\Omega)$ is stated provided that we show the strong convergence of $\{T_k(u_n)\}$ to $T_k(u)$ in $W_0^{1,p}(\Omega)$ and that for every $\delta > 0$ there exists $k_0 = k_0(\delta)$ such that $k \geq k_0$ implies

$$\|G_k(u_n)\|_{1,p} < \delta, \qquad \text{for all } n \in \mathbb{N}.$$

This is done in two steps.

Step 1. For fixed $k > 0$, we have that $T_k(u_n) \to T_k(u)$ in $W_0^{1,p}(\Omega)$. Indeed, by fixing compact set $K \subset \Omega$ we take $\varphi_K \in C_0^\infty(\Omega)$ with $0 \leq \varphi_K \leq 1$ and $\varphi = 1$ in K. Thus, taking v_n as test function in (3.5) defined by

$$v_n := \big(T_k(u_n) - T_k(u)\big)^+ \varphi_K \in W_0^{1,p}(\Omega),$$

we have

(i) $\nabla v_n = \nabla \varphi_K \big(T_k(u_n) - T_k(u)\big)^+ + \nabla \big(T_k(u_n) - T_k(u)\big)^+ \varphi_K$;

(ii) $t_n g(u_n) |\nabla u_n|^p v_n \geq 0$;

(iii) $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$.

In addition, we also have

$$\begin{aligned} &\int_\Omega \varphi_K |\nabla u_n|^{p-2} \nabla u_n \nabla \big(T_k(u_n) - T_k(u)\big)^+ dx \\ &\quad \leq \int_\Omega (\lambda_(w_n^+)^\sigma + \Psi) v_n dx - \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla \phi_K \big(T_k(u_n) - T_k(u)\big)^+ dx. \end{aligned} \tag{3.20}$$

Thus, by Kavian (see [25, Lemma 4.8]), we conclude that

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{in } W_0^{1,p}(\Omega). \tag{3.21}$$

For fixed $k > 0$, combining the relations (3.20) and (3.21), we obtain

$$\int_\Omega \varphi_K |\nabla u_n|^{p-2}\nabla u_n \nabla\big(T_k(u_n) - T_k(u)\big)^+ dx \to 0, \quad \text{as } n \to \infty. \tag{3.22}$$

By define the sets

$$S_n := \{x \in \Omega \,:\, |u_n(x)| \le k\}| \quad \text{and} \quad G_n := \{x \in \Omega \,:\, |u_n(x)| > k\},$$

and denote by χ_{G_n} the characteristic function of G_n. Moreover, take

$$\begin{aligned} E_n^+ :=& \int_\Omega \varphi_K \big[|\nabla T_k(u_n)|^{p-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p-2}\nabla T_k(u)\big]\nabla\big(T_k(u_n) - T_k(u)\big)^+ dx \\ =& \int_\Omega \varphi_K |\nabla u_n|^{p-2}\nabla u_n \nabla\big(T_k(u_n) - T_k(u)\big)^+ dx \\ &- \int_\Omega \varphi_K \big[|\nabla u_n|^{p-2}\nabla u_n - |\nabla T_k(u)|^{p-2}\nabla T_k(u)\big]\nabla\big(T_k(u_n) - T_k(u)\big)^+ dx \\ &- \int_\Omega \varphi_K |\nabla T_k(u)|^{p-2}\nabla T_k(u)\nabla\big(T_k(u_n) - T_k(u)\big)^+ dx. \end{aligned} \tag{3.23}$$

By using the relations (3.21) and (3.22), we conclude that the first and third term of (3.23) tends to zero as $n \to \infty$.

With respect to the second term of (3.23), we have

$$\begin{aligned} &- \int_\Omega \varphi_K \big[|\nabla u_n|^{p-2}\nabla u_n - |\nabla T_k(u)|^{p-2}\nabla T_k(u)\big]\nabla\big(T_k(u_n) - T_k(u)\big)^+ dx \\ &\quad = - \int_{G_n} \varphi_K \big[|\nabla u_n|^{p-2}\nabla u_n - |\nabla T_k(u)|^{p-2}\nabla T_k(u)\big]\nabla\big(T_k(u_n) - T_k(u)\big)^+ dx \\ &\qquad - \int_{S_n} \varphi_K \big[|\nabla u_n|^{p-2}\nabla u_n - |\nabla T_k(u)|^{p-2}\nabla T_k(u)\big]\nabla\big(T_k(u_n) - T_k(u)\big)^+ dx \\ &\quad = - \int_{G_n} \varphi_K |\nabla u_n|^{p-2}\nabla u_n \nabla\big(T_k(u_n) - T_k(u)\big)^+ dx. \end{aligned}$$

Furthermore, by the Dominated Convergence Theorem, we have

$$\begin{aligned} \int_{G_n} \varphi_K |\nabla u_n|^{p-2}\nabla u_n \nabla\big(T_k(u_n) - T_k(u)\big)^+ dx &= \int_\Omega \varphi_K |\nabla u_n|^{p-2}\nabla u_n . \chi_{G_n}\big(\nabla\big(T_k(u_n) - T_k(u)\big)^+\big) dx \\ &\le \|u_n\|_{1,p}^{p-1}\left[\int_\Omega \chi_{G_n}|\nabla T_k(u)|^p dx\right]^{\frac{1}{p}} \to 0, \end{aligned}$$

as $n \to \infty$.

Therefore, we conclude that $E_n^+ \to 0$ as $n \to \infty$ and thus,

$$\int_K \big[|\nabla T_k(u_n)|^{p-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p-2}\nabla T_k(u)\big]\nabla\big(T_k(u_n) - T_k(u)\big)^+ dx \to 0. \tag{3.24}$$

Now, taking v_n as test function in (3.5) defined by

$$v_n = \big(T_k(u_n) - T_k(u)\big)^- \varphi_K \in W_0^{1,p}(\Omega)$$

and repeating the previous arguments, we obtain

$$\int_K \big[|\nabla T_k(u_n)|^{p-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p-2}\nabla T_k(u)\big]\nabla\big(T_k(u_n) - T_k(u)\big)^- dx \to 0. \tag{3.25}$$

By combining the relations (3.24) and (3.25), we have

$$\int_K \left[|\nabla T_k(u_n)|^{p-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p-2}\nabla T_k(u)\right]\nabla\big(T_k(u_n) - T_k(u)\big)dx \to 0.$$

Thus, by inequality (2.7) we conclude that $\nabla T_k(u_n) \to \nabla T_k(u)$ in $L^p(K)$. Since $T_k(u_n) \in W_0^{1,p}(\Omega)$, then

$$\nabla T_k(u_n) \to \nabla T_k(u) \quad \text{in } L^p(\Omega).$$

Therefore, the sequence $\{T_k(u_n)\}$ converges strongly to $T_k(u)$ in $W_0^{1,p}(\Omega)$.

Step 2. By taking $v_n := G_k(u_n)$ as test function in (3.5) we have

$$\begin{aligned}\int_\Omega |\nabla G_k(u_n)|^p dx &= \int_{\{u_n \geq k\}} |\nabla u_n|^{p-2}\nabla u_n \nabla G_k(u_n)dx \\ &\leq \left(\int_{\{u_n\geq k\}} (\lambda_n (w_n^+)^\sigma + \Psi)dx\right)^{\frac{pN}{N(p-1)+p}} \|G_k(u_n)\|_{p^*}.\end{aligned}$$

Since $\frac{pN}{N(p-1)+p}\sigma < p^*$, $\{w_n\}$ is strongly convergent in $L^{\frac{pN}{N(p-1)+p}\sigma}(\Omega)$, $\{\lambda_n\}$ is bounded and $\Psi \in L^{\frac{pN}{N(p-1)+p}}(\Omega)$, the right-hand side of the previous inequality tends uniformly in n to zero as k_0 diverges, i.e., for every $\delta > 0$ there exists $k_0 = k_0(\delta)$ such that $k \geq k_0$ implies

$$\|\nabla G_k(u_n)\|_{1,p} < \delta, \quad \text{for all } n \in \mathbb{N}.$$

Therefore, by step 1 and 2 and by inequality (3.19), we conclude that $\{u_n\}$ converges strongly to u in $W_0^{1,p}(\Omega)$. □

Completing the proof of Theorem 1.2: Let $u_0 \in W_0^{1,p}(\Omega)$ be solution of $(P)_{\lambda\sigma}$ for $\lambda = 0$. For every isolated solution $u_\lambda \in W_0^{1,p}(\Omega)$ of $(P)_{\lambda\sigma}$ for some $\lambda \in \mathbb{R}$, we denote by $i(K_\lambda, , u_\lambda)$ the index of such a solution, that is, the topological Leray–Schauder degree $\deg(I - K_\lambda, B_\epsilon(u_\lambda), 0)$ of the operator $I - K_\lambda$ in a ball $B_\epsilon(u_\lambda)$ centered at u_λ with radius $\epsilon > 0$ small enough.

We will prove that $\deg(I - K_\lambda, B_\epsilon(u_\lambda), 0) \neq 0$ for $\lambda = 0$. Indeed, we denote by $U(t)$ the unique solution for

$$\begin{cases} -\Delta_p u + tg(u)|\nabla u|^p = \Psi(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and we define the following homotopy

$$\begin{aligned} H : [0,1] \times W_0^{1,p}(\Omega) &\longrightarrow W_0^{1,p}(\Omega) \\ (t, w) &\longmapsto H(t, w) := U(t). \end{aligned}$$

Hence,

$$H(1, w) = U(1) = K_0(w) = K(0, w) = u_0$$

and

$$H(0, w) = U(0) = (-\Delta_p^{-1})(\Psi(x)).$$

Since $i\big((-\Delta_p^{-1})(\Psi(x)), U(0)\big) \neq 0$, by Lemma 3.2 we deduce that H is compact. Observing the first part of the proof of Lemma 3.2 we obtain $R > 0$ such that

$$\|U(t)\|_{1,p} < R, \quad \text{for all } t \in [0, 1].$$

If $u \in W_0^{1,p}(\Omega)$ and $\|u\|_{1,p} \geq R$, then $u \neq H(t,u)$. Thus, by the homotopy invariance of the degree, we conclude that

$$\begin{aligned} i(K_0,u_0) &= i\big(H(1,\cdot),U(1)\big) \\ &= i\big(H(0,\cdot),U(0)\big) \\ &= i\big((-\Delta_p^{-1})(\Psi(x)),U(0)\big) \\ &\neq 0. \end{aligned} \tag{3.26}$$

Hence, we have that $K : \times\overline{B}_R \to B_R$ is continuous and compact and u_0 is an isolated solution of $(P)_{\lambda\sigma}$ in the ball $B_\epsilon(u_\lambda)$ for $\lambda = 0$. Thus, for $\lambda_0 > 0$ small enough, we have

$$K \ : \ [0,\lambda_0] \times \overline{B_\epsilon(u_0)} \longrightarrow B_\epsilon(u_0).$$

If $\Phi(\lambda,u) := u - K(\lambda,u)$, then $\deg\big(\Phi(\lambda,\cdot),B_\epsilon(u_0),0\big)$ is well defined for $\lambda \leq \lambda_0$. Hence, by applying the homotopy invariance of the degree, we have

$$\deg\big(\Phi(\lambda,\cdot),B_\epsilon(u_0),0\big) = \text{constant}, \qquad \lambda \leq \lambda_0.$$

Thus, by relation (3.26), we conclude that

$$\deg\big(\Phi(\lambda,\cdot),B_\epsilon(u_0),0\big) \neq 0, \qquad |\lambda| \leq \lambda_0.$$

The theorem follows now from the Rabinowitz Theorem 3.2 in [32]. □

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