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Prolongation of solutions and Lyapunov stability for Stieltjes dynamical systems

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Abstract. In this article, we present Lyapunov-type results to study the stability of an equilibrium of a Stieltjes dynamical system. We utilize prolongation results to establish the global existence of the maximal solution. Using Lyapunov's second method, we establish results of stability (resp. uniform stability) and asymptotic stability (resp. asymptotic uniform stability). Finally, we present examples and real-life applications to study asymptotic stability of equilibria in two population dynamics models.

Keywords: Stieltjes differential equations, Lyapunov stability, Lyapunov function, stable equilibrium, asymptotic stability, dynamical system.

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1 Introduction

In recent years, the field of differential equations has witnessed a notable surge in interest surrounding Stieltjes differential equations. This renewed focus is largely driven by the pursuit of results that not only unify existing findings but also extend those related to classical derivatives [8, 9, 18, 19, 24, 27–30] through the Stieltjes derivative.

Unlike the classical derivative, the Stieltjes derivative consists of differentiating with respect to a derivator $g : \mathbb{R} \to \mathbb{R}$, assumed to be left-continuous and nondecreasing. Stieltjes differential equations permit to obtain a broader range of applications, particularly in contexts where certain processes may exhibit discontinuities and/or stationary periods [1, 8, 9, 18–21, 24, 25]. Such situations are common in various fields, including population dynamics, and physics, where classical differentiation has limitations in capturing the complexities of real-world phenomena.

Typically, investigations into first-order Stieltjes differential equations and systems focus on solutions defined on bounded intervals. However, Larivière in [17, Chapter 4] turned his attention to Stieltjes differential equations on the positive real half-line. In doing so, he provided results related to the prolongation of solutions and the existence of the maximal

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solution. The motivation behind exploring these equations on the positive real half-line lies in the observation that many natural processes evolve over time without any inherent time limit, while some phenomena can exhibit finite-time blow-up, leading to abrupt changes or singularities. By considering Stieltjes differential equations, we aim to capture these nuanced behaviors and enhance our understanding of more complex dynamical systems from a unified perspective [22]. The reader is also referred to [5] for interesting results on prolongation of solutions and the existence of maximal solutions to generalized ordinary equations on Banach spaces relying on Kurzweil integration.

In the study of dynamical systems, the stability of *equilibria* holds significant importance. Here, the term *equilibrium* refers to a state that does not change dynamically, in the sense that if a system starts at an equilibrium point, it will stay in that state indefinitely. In order to maintain the interactions between species within many ecosystems, stability may be suitable for their functioning, and resilience. Although stability is typically desirable, there are some scenarios where stable equilibrium at zero can be critical and raise concerns for several reasons. For instance, in ecological systems, this concern arises from the vulnerability to perturbations of certain species, which may increase the risk of their extinction.

In this paper, we extend Lyapunov stability results from the classical literature to the Stieltjes dynamical system:

$$\mathbf{x}'_g(t) = \mathbf{f}(t, \mathbf{x}(t)) \quad \text{for } g\text{-almost all } t \geq 0, \tag{1.1}$$

where $\mathbf{f} = (f_1, \dots, f_n) : [0, +\infty) \times B \to \mathbb{R}^n$, $B \subset \mathbb{R}^n$.

Within the realm of dynamical systems theory, Lyapunov's Second Method [23] stands as a fundamental approach for assessing the stability properties of a system near an equilibrium. This stability analysis provides insights into whether small perturbations around an equilibrium lead to convergence or divergence of solutions. The core of this method lies in the concept of the *Lyapunov function* V depending on time and state. This function can be understood as an energy representation of the system (1.1), since in numerous applications, the function considered is the total energy of the system (1.1) through time, see for instance [3] for an example of an energy-based Lyapunov function for physical systems. This function was the subject of numerous works in the classical literature starting from the works [12,13,15,16,23,35] and references therein. In our context, the derivator g takes into account the relevance of each moment during the process by means of the changes of the slopes of g accordingly. Put differently, g amplifies an alternative measurement for time, which may differ from the linear time line typically used in the classical case where $g \equiv \mathrm{id}_{\mathbb{R}}$, see for instance the works [18–21,24,25] where g represents the life cycle of some populations, also we refer to [1,9] for more applications. Nevertheless, in the context of Stieltjes differentiation, we still can rely on Lyapunov's function which will then permit a better understanding of how the energy of the system (1.1) changes, but with respect to this new observed time described by g. Our stability study is inspired by results from classical theory and works such as [14, 16, 33, 34], which address dynamic equations on time scales and impulsive differential equations. To the best of our knowledge, this is the first work to introduce Lyapunov's method adapted to Stieltjes differential equations.

This paper is organized as follows: we present the theoretical framework and some preliminaries in Section 2. In Section 3, we focus on prolongation of solutions and the characterization of the maximal solution. Then, based on a generalized version of the Grönwall lemma [11, 17] for the Stieltjes derivative, we establish the existence of global solutions over the whole positive real half-line. Section 4 is devoted to Lyapunov-like stability results using

Lyapunov's second method. We establish stability results inspired by the works [6, 10], to extend some classical results from [14, 16].

In the last section of this paper, we present two applications to dynamics of population to study the asymptotic stability of some critical equilibria. In the first application, we model the dynamics of a population with Allee's effect [2, 31] negatively impacted by train vibrations. The second application concerns the dynamics of a population of *Cyanobacteria* in a cultured environment, keeping track of ammonia levels in the process.

2 Preliminaries

For $[a, b] \subset \mathbb{R}$, and $\mathbf{u} : [a, b] \to \mathbb{R}^n$ a regulated function, the symbols $\mathbf{u}(t^+)$ and $\mathbf{u}(t^-)$ will be used to denote

$$\mathbf{u}(t^+) = \lim_{\varepsilon \to 0^+} \mathbf{u}(t + \varepsilon), \text{ for all } t \in [a, b),$$
$$\mathbf{u}(t^-) = \lim_{\varepsilon \to 0^-} \mathbf{u}(t - \varepsilon), \text{ for all } t \in (a, b].$$

Throughout this work, we will consider $g : \mathbb{R} \to \mathbb{R}$ a monotone, nondecreasing and left-continuous function, also known as a *derivator*. We denote the set of discontinuity points of g by

$$D_g = \{t \in \mathbb{R} : g(t^+) - g(t) > 0\}.$$

In addition, we denote

$$C_g = \{s \in \mathbb{R} : g \text{ is constant on } (s - \varepsilon, s + \varepsilon) \text{ for some } \varepsilon > 0\}.$$

The set C_g is an open in the usual topology and it can be written as a countable union of disjoint intervals

$$C_g = \bigcup_{n \in \Lambda} (a_n, b_n),$$

with $\Lambda \subset \mathbb{N}$, $a_n, b_n \in \mathbb{R}$. We set $N_g = \{a_n, b_n : n \in \Lambda\} \setminus D_g$.

The function g defines a Lebesgue–Stieltjes measure μ_g such that $\mu_g([a, b)) = g(b) - g(a)$ for any interval $[a, b)$, so that $\mu_g(\{t\}) = g(t^+) - g(t)$ for all $t \in \mathbb{R}$, and $\mu_g(C_g) = \mu_g(N_g) = 0$. The reader is referred to [22] for more details. We refer to the measurability with respect to μ_g by g-measurability. We denote by $\mathcal{L}^1_g([a, b), \mathbb{R})$ the space of μ_g-integrable real-valued functions on $[a, b)$ endowed with the norm

$$\|f\|_{\mathcal{L}^1_g([a,b),\mathbb{R})} := \int_{[a,b)} |f(t)| \, d\mu_g(t), \quad \text{for every } f \in \mathcal{L}^1_g([a, b), \mathbb{R}).$$

Given an interval $I \subset \mathbb{R}$, we set

$$\mathcal{L}^1_g(I, \mathbb{R}^n) := \prod_{i=1}^{n} \mathcal{L}^1_g(I, \mathbb{R}),$$

and

$$\mathcal{L}^1_{g,\text{loc}}(I, \mathbb{R}^n) = \{\mathbf{u} : I \to \mathbb{R}^n : \mathbf{u}|_{[a,b]} \in \mathcal{L}^1_g([a, b), \mathbb{R}^n) \text{ for every } [a, b] \subset I\}.$$

The derivator g defines a pseudometric $\rho : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ given by

$$\rho(s, t) = |g(s) - g(t)|, \quad \text{for every } s, t \in \mathbb{R}.$$

We denote τ_g the topology induced by the pseudometric ρ. Notice that the interval $(t-\varepsilon, t]$ is open in τ_g for all $t \in D_g$ and all $\varepsilon > 0$. The reader is referred to [9, Section 2] for more properties of the topology τ_g.

Throughout this paper, let $\|\cdot\|$ denotes the maximum norm in $\mathbb{R}^n$ defined by

$$\|\mathbf{x}\| = \max\{|x_1|, \dots, |x_n|\} \quad \text{for } \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n,$$

and $B_{\mathbb{R}^n}(\mathbf{x}, \delta)$ denotes the open ball centered in $\mathbf{x}$ of radius $\delta > 0$.

Now, we recall the notion of g-continuity introduced in [9].

Definition 2.1. Let $\mathbf{u} : [a, b] \to \mathbb{R}^n$. We say that $\mathbf{u}$ is *g-continuous* at $t \in [a, b]$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall s \in [a, b], \qquad |g(s) - g(t)| < \delta \implies \|\mathbf{u}(s) - \mathbf{u}(t)\| < \varepsilon.$$

The following proposition relates the regularity of f and g, the reader is referred to [9, Proposition 3.2].

Proposition 2.2. *If $\mathbf{u} : [a, b] \to \mathbb{R}^n$ is g-continuous on $[a, b]$, then the following statements hold:*

(1) *$\mathbf{u}$ is left-continuous at every $t \in (a, b]$.*

(2) *If g is continuous at $t \in [a, b)$, then so is $\mathbf{u}$.*

(3) *If g is constant on some $[c, d] \subset [a, b]$, then so is $\mathbf{u}$.*

Let $\mathcal{BC}_g([a, b], \mathbb{R}^n)$ denotes the Banach space of the bounded, g-continuous functions defined on $[a, b]$ with values in $\mathbb{R}^n$, endowed with the supremum norm.

Now, we define the g-derivative of a real-valued function.

Definition 2.3. Let $u : [a, b] \to \mathbb{R}$ be a function. The *derivative of u with respect to g at a point* $t \in [a, b] \setminus C_g$ is defined by:

$$u'_g(t) = \begin{cases} \lim\limits_{s \to t} \dfrac{u(s) - u(t)}{g(s) - g(t)} & \text{if } t \notin D_g, \\ \dfrac{u(t^+) - u(t)}{g(t^+) - g(t)} & \text{if } t \in D_g, \end{cases}$$

provided that the limit exists. In this case, u is said to be *g-differentiable at t* and $u'_g(t)$ is also called the *g-derivative of u at t*.

In the next proposition, we recall the g-derivative of the composition of two functions established in [26, Proposition 3.15]. Another version of this formula can be found in [7, Proposition 4.1].

Proposition 2.4. *Let $t \in \mathbb{R} \setminus C_g$, $f : \mathbb{R} \to \mathbb{R}$ and h a real function defined on a neighborhood of $f(t)$. We assume that there exist $h'(f(t))$, $f'_g(t)$ and that the function h is continuous at $f(t^+)$. Then, the composition $h \circ f$ is g-differentiable in t and*

$$(h \circ f)'_g(t) = \begin{cases} h'(f(t)) f'_g(t) & \text{if } t \notin D_g, \\ \dfrac{h(f(t^+)) - h(f(t))}{f(t^+) - f(t)} f'_g(t) & \text{if } t \in D_g. \end{cases}$$

In the particular case where the derivator $g : \mathbb{R} \to \mathbb{R}$ is increasing and continuous on an interval $[a,b] \subset \mathbb{R}$, we obtain easily a generalized version of the Mean Value theorem in the context of Stieltjes differentiation.

Theorem 2.5. *Let $g : \mathbb{R} \to \mathbb{R}$ be a left-continuous and nondecreasing function, continuous and increasing on an interval $[a,b] \subset \mathbb{R}$. Let $f : [a,b] \to \mathbb{R}$ be g-continuous on $[a,b]$ and g-differentiable on (a,b). Then, there exists $c \in (a,b)$ such that $f'_g(c) = \frac{f(b)-f(a)}{g(b)-g(a)}$.*

Proof. Let us consider the function $F : [a,b] \to \mathbb{R}$ defined by

$$F(t) = f(t) - \left(\frac{f(b)-f(a)}{g(b)-g(a)}(g(t)-g(a)) + f(a) \right), \quad \text{for all } t \in [a,b].$$

Clearly F is g-continuous on $[a,b]$, g-differentiable on (a,b), and

$$F'_g(t) = f'_g(t) - \frac{f(b)-f(a)}{g(b)-g(a)} \quad \text{for all } t \in (a,b).$$

It satisfies also $F(a) = F(b)$.

Proposition 2.2 implies that F is continuous on $[a,b]$. We set

$$m = \min_{t\in[a,b]} F(t) \quad \text{and} \quad M = \max_{t\in[a,b]} F(t).$$

If $m = M$, then F is constant on $[a,b]$ and $F'_g(t) = 0$ for all $t \in (a,b)$. Otherwise if $m < M$, we have $F(a) \neq m$ or $F(a) \neq M$. As $F(a) = F(b)$, without loss of generality, we assume that $F(a) = F(b) \neq M$. Thus, there exists $c \in (a,b)$ such that $F(c) = M$. Therefore, there exists $\delta > 0$ such that $(c-\delta, c+\delta) \subset (a,b)$ and $F(s) \le F(c) = M$ for all $s \in (c-\delta, c+\delta)$. Since g is increasing and continuous on (a,b),

$$0 \le F'_g(c) = \lim_{s\to c^+} \frac{F(s)-F(c)}{g(s)-g(c)} = \lim_{s\to c^-} \frac{F(s)-F(c)}{g(s)-g(c)} \le 0.$$

We deduce that $F'_g(c) = 0$. Hence, there exists $c \in (a,b)$ such that $f'_g(c) = \frac{f(b)-f(a)}{g(b)-g(a)}$. □

Now, we recall the notion of g-absolute continuity.

Definition 2.6. A map $F : [a,b] \to \mathbb{R}$ is *g-absolutely continuous*, if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for any family $\{(a_i,b_i)\}_{i=1}^{i=n}$ of pairwise disjoint open subintervals of $[a,b]$,

$$\sum_{i=1}^{n} g(b_i) - g(a_i) < \delta \Rightarrow \sum_{i=1}^{n} |F(b_i) - F(a_i)| < \varepsilon.$$

We denote by $\mathcal{AC}_g([a,b],\mathbb{R})$ the space of g-absolutely continuous functions $F : [a,b] \to \mathbb{R}$ on the interval $[a,b]$ and

$$\mathcal{AC}_g([a,b],\mathbf{B}) = \{\mathbf{u} = (u_1,\dots,u_n) : [a,b] \to \mathbf{B} \subset \mathbb{R}^n : u_i \in \mathcal{AC}_g([a,b],\mathbb{R}) \quad \text{for } i = 1,\dots,n\}.$$

In [22, Theorem 5.4], a Fundamental Theorem of Calculus for Lebesgue–Stieltjes integrals was established.

Theorem 2.7 (Fundamental Theorem of Calculus for the Lebesgue–Stieltjes integral)**.** *Let $a, b \in \mathbb{R}$ be such that $a < b$, and let $F : [a,b] \to \mathbb{R}$. The following assumptions are equivalent.*

(1) *The function F is g-absolutely continuous,*

(2) *The function F satisfies the following conditions:*

(a) *there exists $F'_g(t)$ for g-almost all $t \in [a,b)$;*

(b) $F'_g \in \mathcal{L}^1_g([a,b),\mathbb{R})$;

(c) *for each $t \in [a,b]$, we have*

$$F(t) = F(a) + \int_{[a,t)} F'_g(s)\, d\mu_g(s).$$

The following lemma provides conditions ensuring the g-absolute continuity of the composition of two functions, the proof is based on arguments as in [9, Proposition 5.3].

Lemma 2.8. *Let $\mathbf{u} : [a,b] \to B \subset \mathbb{R}^n$ be a g-absolutely continuous function, and let $v : B \to \mathbb{R}$ be a Lipschitz continuous function on B. Then, the composition $v \circ \mathbf{u} \in \mathcal{AC}_g([a,b],\mathbb{R})$.*

Now, we define the partial Stieltjes derivative as follows.

Definition 2.9. Given a function $V : [a,b] \times B_{\mathbb{R}^n}(\mathbf{0},r_0) \to \mathbb{R}$. The *partial g-derivative of V with respect to the first argument* at a point $(t,\mathbf{x}) \in ([a,b) \setminus C_g) \times B_{\mathbb{R}^n}(\mathbf{0},r_0)$ is defined as:

$$\frac{\partial V}{\partial_g t}(t,\mathbf{x}) = \begin{cases} \lim\limits_{s\to t} \frac{V(s,\mathbf{x})-V(t,\mathbf{x})}{g(s)-g(t)}, & \text{if } t \in [a,b] \setminus D_g, \\ \dfrac{V(t^+,\mathbf{x}) - V(t,\mathbf{x})}{g(t^+) - g(t)}, & \text{if } t \in [a,b] \cap D_g, \end{cases}$$

provided that the limit exists.

The following proposition gives a formula related to the g-derivative of the composition involving a function with two variables. Formulae of this fashion were stated without proof in [29, Lemma 11] for $t \notin D_g$. Here, we derive formulae in the case where D_g and N_g do not have accumulation points.

Proposition 2.10. *Let $g : \mathbb{R} \to \mathbb{R}$ be a left-continuous and nondecreasing function such that D_g and N_g do not have accumulation points. Given a function $V : [a,b] \times B_{\mathbb{R}^n}(\mathbf{0},r_0) \to \mathbb{R}$ satisfying the following assumptions:*

(1) *$V(\cdot,\mathbf{u})$ is g-differentiable on $[a,b] \setminus (D_g \cup C_g)$ for all $\mathbf{u} \in B_{\mathbb{R}^n}(\mathbf{0},r_0)$;*

(2) *$V(t,\cdot) \in C^1(B_{\mathbb{R}^n}(\mathbf{0},r_0),\mathbb{R})$ for all $t \in [a,b]$;*

(3) *$\frac{\partial V}{\partial_g t}(\cdot,\mathbf{u})$ is continuous on $[a,b] \setminus (D_g \cup C_g)$ for all $\mathbf{u} \in B_{\mathbb{R}^n}(\mathbf{0},r_0)$;*

(4) *$V(\cdot,\mathbf{x}(\cdot)) \in \mathcal{AC}_g([a,b],\mathbb{R})$ for every $\mathbf{x} \in \mathcal{AC}_g\big([a,b], B_{\mathbb{R}^n}(\mathbf{0},r_0)\big)$.*

Then, for every $\mathbf{x} \in \mathcal{AC}_g\big([a,b], B_{\mathbb{R}^n}(\mathbf{0},r_0)\big)$ and for g-almost all $t \in [a,b] \setminus (D_g \cup C_g)$, one has that

$$V'_g(t,\mathbf{x}(t)) = \frac{\partial V}{\partial_g t}(t,\mathbf{x}(t)) + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(t,\mathbf{x}(t))(x_i)'_g(t). \tag{2.1}$$

Moreover, if $t \in [a,b) \cap D_g$, then

$$V'_g(t,\mathbf{x}(t)) = \frac{V\big(t^+,\mathbf{x}(t) + \mu_g(\{t\})\mathbf{x}'_g(t)\big) - V(t,\mathbf{x}(t))}{g(t^+) - g(t)}. \tag{2.2}$$

Proof. Let $\mathbf{x} \in \mathcal{AC}_g([a,b], B_{\mathbb{R}^n}(\mathbf{0}, r_0))$. By (4), $V(\cdot, \mathbf{x}(\cdot)) \in \mathcal{AC}_g([a,b], \mathbb{R})$. For g-almost every $t \in [a,b] \setminus (D_g \cup C_g)$:

$$\begin{aligned}
V_g'(t, \mathbf{x}(t)) &= \lim_{s\to t} \frac{V(s,\mathbf{x}(s)) - V(t,\mathbf{x}(t))}{g(s)-g(t)} \\
&= \lim_{s\to t} \frac{V(s,\mathbf{x}(s)) - V(t,\mathbf{x}(s))}{g(s)-g(t)} + \frac{V(t,\mathbf{x}(s)) - V(t,\mathbf{x}(t))}{g(s)-g(t)} \\
&= \lim_{s\to t} \frac{V(s,\mathbf{x}(s)) - V(t,\mathbf{x}(s))}{g(s)-g(t)} \\
&\quad + \sum_{i=1}^{n} \Bigg(\frac{V(t,(x_1(t),\dots,x_{i-1}(t),x_i(s),\dots,x_n(s)))}{g(s)-g(t)} \\
&\qquad - \frac{V(t,(x_1(t),\dots,x_i(t),x_{i+1}(s),\dots,x_n(s)))}{g(s)-g(t)} \Bigg).
\end{aligned}$$

For s sufficiently close to t, and since D_g and N_g do not have accumulation points, g is continuous and increasing on the interval with endpoint points s and t. By applying Theorem 2.5 to the function $V(\cdot, \mathbf{x}(s))$, we obtain that there exists c between s and t such that

$$\frac{V(s,\mathbf{x}(s)) - V(t,\mathbf{x}(s))}{g(s)-g(t)} = \frac{\partial V}{\partial_g t}(c, \mathbf{x}(s)).$$

As $s \to t$, $c \to t$, and using Condition (3) we obtain

$$\frac{V(s,\mathbf{x}(s)) - V(t,\mathbf{x}(s))}{g(s)-g(t)} = \frac{\partial V}{\partial_g t}(c, \mathbf{x}(s)) \to \frac{\partial V}{\partial_g t}(t, \mathbf{x}(t)).$$

Therefore,

$$V_g'(t, \mathbf{x}(t)) = \frac{\partial V}{\partial_g t}(t, \mathbf{x}(t)) + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(t, \mathbf{x}(t)) (x_i)_g'(t).$$

For $t \in [a,b) \cap D_g$, we obtain immediately that

$$\begin{aligned}
V_g'(t, \mathbf{x}(t)) &= \frac{V(t^+, \mathbf{x}(t^+)) - V(t, \mathbf{x}(t))}{g(t^+) - g(t)} \\
&= \frac{V\big(t^+, \mathbf{x}(t) + \mu_g(\{t\})\mathbf{x}_g'(t)\big) - V(t, \mathbf{x}(t))}{g(t^+) - g(t)}.
\end{aligned}$$

□

In [9, Definition 6.1], an exponential function was introduced.

Definition 2.11. Let $p \in \mathcal{L}^1_g([a,b), \mathbb{R})$ be such that

$$1 + p(t)\big(g(t^+) - g(t)\big) > 0 \text{ for every } t \in [a,b) \cap D_g. \tag{2.3}$$

The *exponential function* $e_p(\cdot, a) : [a,b] \to (0, \infty)$ is defined by

$$e_p(t,a) = e^{\int_{[a,t)} \tilde{p}(s)\, d\mu_g(s)} \quad \text{for every } t \in [a,b],$$

where

$$\tilde{p}(t) = \begin{cases} p(t) & \text{if } t \in [a,b] \setminus D_g, \\ \dfrac{\log\big(1 + p(t)(g(t^+) - g(t))\big)}{g(t^+) - g(t)} & \text{if } t \in [a,b) \cap D_g. \end{cases} \tag{2.4}$$

In particular, given $p \in \mathcal{L}^1_g([a,b),\mathbb{R})$ a function satisfying Condition (2.3), then $\tilde{p} \in \mathcal{L}^1_g([a,b),\mathbb{R})$ and $e_p(\cdot,a) \in \mathcal{AC}_g([a,b),\mathbb{R})$. The reader is referred to [9, Lemmas 6.2 and 6.3] and [27, Theorem 3.2] for more details.

Now, we recall the generalization of the Grönwall Lemma to the Stieltjes derivative obtained by Larivière in [17, Proposition 4.1.4], and further generalized in [11, Theorem 5.4]. This lemma will play a crucial role in establishing global solutions defined on the positive real half-line as we shall prove in the following section.

Lemma 2.12. *Let $u \in \mathcal{AC}_g([a,b],\mathbb{R})$. Assume that there exist functions $k,p \in \mathcal{L}^1_g([a,b),\mathbb{R})$, satisfying $1+p(t)\mu_g(\{t\})>0$ for all $t \in [a,b)\cap D_g$, such that*

$$u'_g(t) \le k(t)+p(t)u(t) \quad \text{for } g\text{-almost all } t \in [a,b).$$

Then,

$$u(t) \le e_p(t,a)\left(\int_{[a,t)} \frac{e_p^{-1}(s,a)k(s)}{1+p(s)\mu_g(\{s\})}\,d\mu_g(s)+u(a)\right), \quad t \in [a,b].$$

3 Prolongation of solutions and maximal interval of existence

Let O be a nonempty open set of $\mathbb{R}^n$ and $I \subset \mathbb{R}$ an open interval in the topology τ_g. We set $\Omega := I \times O$. Let $(t_0,\mathbf{x}_0) \in \Omega$ be such that $t_0 < \sup I$.

Let us consider the Stieltjes dynamical system:

$$\begin{aligned} \mathbf{x}'_g(t) &= \mathbf{f}(t,\mathbf{x}(t)) \quad \text{for } g\text{-almost all } t \ge t_0,\ t \in I,\\ \mathbf{x}(t_0) &= \mathbf{x}_0, \end{aligned} \tag{3.1}$$

where $\mathbf{f}=(f_1,\dots,f_n) : \Omega \cap \big([t_0,\infty)\times\mathbb{R}^n\big) \to \mathbb{R}^n$ satisfies the following assumptions:

($\mathrm{H}_{\mathbf{x}_0,t_0}$) $\big(t_0, \mathbf{x}_0+\mu_g(\{t_0\})\mathbf{f}(t_0,\mathbf{x}_0)\big) \in \Omega$.

($\mathrm{H}_{\mathbf{f},t_0}$) (a) for all $\mathbf{u} \in O$, $\mathbf{f}(\cdot,\mathbf{u})$ is g-measurable;

(b) $\mathbf{f}(\cdot,\mathbf{u}_0) \in \mathcal{L}^1_{g,\mathrm{loc}}([t_0,\infty),\mathbb{R}^n)$ for some $\mathbf{u}_0 \in O$;

(c) $\mathbf{f}$ is *g-integrally locally Lipschitz continuous*, i.e. for every $r>0$, there exists a function $L_r \in \mathcal{L}^1_{g,\mathrm{loc}}([t_0,\infty),[0,\infty))$ such that

$$\|\mathbf{f}(t,\mathbf{u})-\mathbf{f}(t,\mathbf{v})\| \le L_r(t)\|\mathbf{u}-\mathbf{v}\|,$$

for g-almost all $t \in I\cap[t_0,\infty)$ and all $\mathbf{u},\mathbf{v} \in \overline{B_{\mathbb{R}^n}(\mathbf{x}_0,r)}\cap O$.

(H_{Ω,t_0}) For every $(t,\mathbf{u}) \in \Omega$ with $t \ge t_0$,

(a) one of the following conditions hold:

(a1) there exists $\delta>0$ such that $(t-\delta,t+\delta)\times B_{\mathbb{R}^n}(\mathbf{u},\delta) \subset \Omega$;

(a2) if for every $\delta>0$, $(t-\delta,t+\delta)\times B_{\mathbb{R}^n}(\mathbf{u},\delta) \not\subset \Omega$, then $t \in D_g$ and there exists $\varepsilon>0$ such that $(t-\varepsilon,t]\times B_{\mathbb{R}^n}(\mathbf{u},\delta) \subset \Omega$;

(b) if $(t,\mathbf{u}) \in \Omega\cap(D_g\times\mathbb{R}^n)$ is such that $(t,\mathbf{u}^+_{\mathbf{f},t}) \in \Omega$, then $(t,\mathbf{u}^+_{\mathbf{f},t})$ satisfies Condition (H_{Ω,t_0})(a)(a1), where

$$\mathbf{u}^+_{\mathbf{f},t} := \mathbf{u}+\mu_g(\{t\})\mathbf{f}(t,\mathbf{u}).$$

We recall the local existence result established in [9, Theorem 7.4].

Theorem 3.1. *Assume that* $(\mathrm{H}_{\mathbf{x}_0,t_0})$ *and* $(\mathrm{H}_{\mathbf{f},t_0})$ *holds. Then, there exists* $\delta > 0$ *such that the system* (3.1) *has a unique solution* $\mathbf{x} \in \mathcal{AC}_g([t_0, t_0 + \delta], \mathbb{R}^n)$.

It should be noticed that Condition $(\mathrm{H}_{\mathbf{x}_0,t_0})$ permits to consider the problem (3.1) even in the case where $t_0 \in D_g$.

In the sequel, as shown in [17, Section 4.2], the solution given by Theorem 3.1 can be extended up to a maximal interval of existence if Condition $(\mathrm{H}_{\Omega,t_0})$ is satisfied. Indeed, this condition ensures that the solution $\mathbf{x}$ can be extended on some larger interval $[t_0, t_0 + \varepsilon]$, $\varepsilon > \delta$ if

$$t_0 + \delta \in D_g \quad \text{and} \quad \big(\mathbf{x}(t_0 + \delta)\big)^+_{\mathbf{f},t_0+\delta} \in O.$$

Let us define the set

$$\begin{aligned}\mathcal{S}(t_0, \mathbf{x}_0) := \{\mathbf{x} : I_\mathbf{x} = J_\mathbf{x} \cap [t_0, \infty) \to \mathbb{R}^n : \mathbf{x} \text{ is a solution of (3.1)},\\ J_\mathbf{x} \text{ is an open interval of } \tau_g \text{ such that } \sup J_\mathbf{x} > t_0 \in J_\mathbf{x}\}.\end{aligned}$$

For $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$, it is worth mentioning that $\mathbf{x} \in \mathcal{AC}_g([a, b], \mathbb{R}^n)$ for every interval $[a, b] \subset I_\mathbf{x}$. In the sequel, for $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$, we denote $\bar{t}_\mathbf{x} := \sup I_\mathbf{x}$.

Definition 3.2. Let $\mathbf{x}, \mathbf{y} \in \mathcal{S}(t_0, \mathbf{x}_0)$.

(1) We say that $\mathbf{x}$ is *smaller* than $\mathbf{y}$ (and we denote $\mathbf{x} \prec \mathbf{y}$), if and only if

   (i) $I_\mathbf{x} \subset I_\mathbf{y}$;

   (ii) $\sup I_\mathbf{y} > \sup I_\mathbf{x}$;

   (iii) $\mathbf{y}|_{I_\mathbf{x}} = \mathbf{x}$.

   In this case, we say that $\mathbf{x}$ is *extendible to the right* and $\mathbf{y}$ is *a prolongation to the right* of $\mathbf{x}$.

(2) We write that $\mathbf{x} \preceq \mathbf{y} \iff \mathbf{x} \prec \mathbf{y}$ or $\mathbf{x} = \mathbf{y}$.

Remark 3.3. It is worth mentioning that given a solution $\mathbf{x} : [t_0, t_1) \to \mathbb{R}^n$ of (3.1) such that

$$\mathbf{x}(t_1^-) := \lim_{t \to t_1^-} \mathbf{x}(t) \in O \quad \text{and} \quad t_1 \in D_g,$$

to not increase the notation, we will replace $\mathbf{x}$ by the function $\mathbf{x} : [t_0, t_1] \to \mathbb{R}^n$ defined by

$$\mathbf{x} = \begin{cases} \mathbf{x}(t) & \text{if } t \in [t_0, t_1), \\ \mathbf{x}(t_1^-) & \text{if } t = t_1. \end{cases}$$

The next result asserts that two prolongations to the right of a solution are equal on the common interval of existence, the reader is referred to [17, Theorem 4.2.3] for the proof.

Theorem 3.4. *Assume that* $(\mathrm{H}_{\mathbf{x}_0,t_0})$ *and* $(\mathrm{H}_{\mathbf{f},t_0})$ *hold. Let* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{S}(t_0, \mathbf{x}_0)$ *be such that* $\mathbf{y}$ *and* $\mathbf{z}$ *are two prolongations of* $\mathbf{x}$, *then* $\mathbf{y} = \mathbf{z}$ *on* $I_\mathbf{y} \cap I_\mathbf{z}$.

In [17, Theorem 4.2.4], extendible solutions to the right were characterized as follows.

Theorem 3.5. *Assume that* $(\mathrm{H}_{\mathbf{x}_0,t_0})$ *and* $(\mathrm{H}_{\mathbf{f},t_0})$ *hold. Let* $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$. *The following assumptions are equivalent:*

(1) $\mathbf{x}$ *is extendible to the right;*

(2) (i) *Graph*$(\mathbf{x}) := \{(t, \mathbf{x}(t)) : t \in I_{\mathbf{x}}\}$ *is bounded;*

(ii) $A \cup A^+ \subset \Omega$ *where*

$$A = \{(\bar{t}_{\mathbf{x}}, \mathbf{u}) \in [t_0, \infty) \times \mathbb{R}^n : \exists \{t_n\}_n \subset I_{\mathbf{x}},\ t_n \nearrow \bar{t}_{\mathbf{x}} \text{ and } \mathbf{u} = \lim_{n\to\infty} \mathbf{x}(t_n)\},$$
$$A^+ = \{(\bar{t}_{\mathbf{x}}, \mathbf{u}^+_{\mathbf{f},\bar{t}_{\mathbf{x}}}) : (\bar{t}_{\mathbf{x}}, \mathbf{u}) \in A\}.$$

Definition 3.6. Let $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$. We say that $\mathbf{x}$ is a *maximal solution* of (3.1) defined on an interval $I_{\mathbf{x}}$ if, for every $\mathbf{y} \in \mathcal{S}(t_0, \mathbf{x}_0)$ satisfying $\mathbf{x} \preceq \mathbf{y}$, we have $\mathbf{x} = \mathbf{y}$. $I_{\mathbf{x}}$ is referred to as the *maximal interval of existence.*

As shown in [17, Theorem 4.2.5], the existence of the maximal solution holds. In addition, Theorem 3.4 guarantees the uniqueness of the maximal solution, and we obtain the following theorem.

Theorem 3.7. *Assume that* $(\mathrm{H}_{\mathbf{x}_0,t_0})$, $(\mathrm{H}_{\mathbf{f},t_0})$ *and* $(\mathrm{H}_{\Omega,t_0})$ *hold. Then, there exists a unique maximal solution* $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$ *such that* $\omega(t_0, \mathbf{x}_0) := \sup I_{\mathbf{x}} \leq \infty$.

The next theorem highlights three alternative cases that occur, the reader is referred to [17, Theorem 4.2.6] for the proof.

Theorem 3.8. *Assume that* $(\mathrm{H}_{\mathbf{x}_0,t_0})$, $(\mathrm{H}_{\mathbf{f},t_0})$ *and* $(\mathrm{H}_{\Omega,t_0})$ *hold. Let* $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$ *be the maximal solution of* (3.1)*, then one of the alternatives holds:*

(A1) $\omega(t_0, \mathbf{x}_0) = \infty$;

(A2) $\omega(t_0, \mathbf{x}_0) < \infty$*, and for every* $\{t_n\}_n \subset I_{\mathbf{x}}$ *such that* $t_n \nearrow \omega(t_0, \mathbf{x}_0)$, $\{\mathbf{x}(t_n)\}_n$ *is not bounded;*

(A3) $\omega(t_0, \mathbf{x}_0) < \infty$*, and there exists* $\{t_n\}_n \subset I_{\mathbf{x}}$ *satisfying* $t_n \nearrow \omega(t_0, \mathbf{x}_0)$ *and* $\{\mathbf{x}(t_n)\}_n$ *is a bounded sequence such that, for every subsequence* $\{t_{n_k}\}_k$ *verifying* $\mathbf{x}(t_{n_k}) \to \mathbf{u}$*, we have that*

$$\{(\omega(t_0, \mathbf{x}_0), \mathbf{u}), (\omega(t_0, \mathbf{x}_0), \mathbf{u}^+_{\mathbf{f},\omega(t_0,\mathbf{x}_0)})\} \not\subset \Omega.$$

Remark 3.9. Notice that, if Alternative (A1) holds, then $I_{\mathbf{x}} = [t_0, \infty)$, while if Alternative (A2) or (A3) holds, then $I_{\mathbf{x}} = [t_0, \omega(t_0, \mathbf{x}_0))$ or $I_{\mathbf{x}} = [t_0, \omega(t_0, \mathbf{x}_0)]$. In [5], the assumptions made on the Kurzweil integral of $\mathbf{f}$ do not allow the existence of a maximal solution on an interval of the form $[t_0, \omega(t_0, \mathbf{x}_0)]$.

Here is a corollary of Theorem 3.8 which ensures the global existence of the maximal solution over the whole interval $[t_0, \infty)$.

Corollary 3.10. *Let* I *be an interval containing* $[t_0, \infty)$*. Assume that* $(\mathrm{H}_{\mathbf{x}_0,t_0})$, $(\mathrm{H}_{\mathbf{f},t_0})$ *and* $(\mathrm{H}_{\Omega,t_0})$ *hold. Let* $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$ *be the maximal solution of* (3.1)*. If*

$$\Omega^- := \left\{(t, \mathbf{u}) \in \Omega : \left(t, \mathbf{u}^+_{\mathbf{f},t}\right) \in \Omega\right\} = \Omega,$$

and there exists a compact set $D \subset O$ *such that* $\mathbf{x}(t) \in D$ *for every* $t \in I_{\mathbf{x}}$*, then* $\omega(t_0, \mathbf{x}_0) = \infty$.

Proof. Let us assume that $\omega(t_0, \mathbf{x}_0) < \infty$. Since D is compact, for all $\{t_n\}_n \subset I_{\mathbf{x}}$ such that $t_n \nearrow \omega(t_0, \mathbf{x}_0)$, $\{\mathbf{x}(t_n)\}_n$ is bounded. Thus, it results from Theorem 3.8 that Alternative (A3) holds. Therefore, there exists $\{t_n\}_n \subset I_{\mathbf{x}}$ with $t_n \nearrow \omega(t_0, \mathbf{x}_0)$ and $\{(t_n, \mathbf{x}(t_n))\}_n$ is bounded and such that, for a fixed subsequence $\{t_{n_k}\}_k$ verifying $\mathbf{x}(t_{n_k}) \to \mathbf{u}$, we have that

$$\left\{ (\omega(t_0, \mathbf{x}_0), \mathbf{u}), \left(\omega(t_0, \mathbf{x}_0), \mathbf{u}^+_{\mathbf{f}, \omega(t_0, \mathbf{x}_0)}\right) \right\} \not\subset \Omega. \tag{3.2}$$

On the other hand, D being a compact set yields that

$$(t_{n_k}, \mathbf{x}(t_{n_k})) \to (\omega(t_0, \mathbf{x}_0), \mathbf{u}) \in [t_0, \omega(t_0, \mathbf{x}_0)] \times D \subset \Omega.$$

Now, since $\Omega^- = \Omega$, we deduce that $\left(\omega(t_0, \mathbf{x}_0), \mathbf{u}^+_{\mathbf{f}, \omega(t_0, \mathbf{x}_0)}\right) \in \Omega$, which contradicts (3.2). Hence, $\omega(t_0, \mathbf{x}_0) = \infty$. □

Now, based on Theorems 3.5 and 3.8, we provide a characterization of the maximal solution of the problem (3.1).

Theorem 3.11. *Assume that* $(\mathrm{H}_{\mathbf{x}_0, t_0})$, $(\mathrm{H}_{\mathbf{f}, t_0})$ *and* $(\mathrm{H}_{\Omega, t_0})$ *hold. Let* $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$. *The following assumptions are equivalent:*

(1) $\mathbf{x}$ *is maximal;*

(2) *for every compact set* $K \subset \Omega$, *there exists* $t_K \in I_{\mathbf{x}}$ *such that*

$$\left\{ (t, \mathbf{x}(t)), \left(t, (\mathbf{x}(t))^+_{\mathbf{f}, t}\right) \right\} \not\subset K,$$

for all $t \geq t_K$, $t \in I_{\mathbf{x}}$.

Proof. Assume that $\mathbf{x}$ is maximal. By Theorem 3.8, two cases may occur:

Case 1: If $\omega(t_0, \mathbf{x}_0) = \sup I_{\mathbf{x}} = \infty$, by contradiction, assume that there exist a compact set $K \subset \Omega$ and a sequence $\{t_n\}_n \subset I_{\mathbf{x}}$ such that

$$t_n \nearrow \omega(t_0, \mathbf{x}_0) \quad \text{and} \quad \left\{ (t_n, \mathbf{x}(t_n)), \left(t_n, (\mathbf{x}(t_n))^+_{\mathbf{f}, t_n}\right) \right\} \subset K \quad \text{for all } n \in \mathbb{N}.$$

This implies that $\{(t_n, \mathbf{x}(t_n))\}$ is bounded. Therefore, there exists a convergent subsequence $\{(t_{n_k}, \mathbf{x}(t_{n_k}))\}$ such that

$$(t_{n_k}, \mathbf{x}(t_{n_k})) \to (\tau, \mathbf{u}) \in K \subset [t_0, \infty) \times O.$$

This contradicts that $t_n \nearrow \omega(t_0, \mathbf{x}_0) = \infty$.

Case 2: If $\omega(t_0, \mathbf{x}_0) < \infty$, then, by Theorem 3.8, there are two subcases:

Subcase 1: if Alternative (A2) holds, then

$$I_{\mathbf{x}} = [t_0, \omega(t_0, \mathbf{x}_0)) \quad \text{and} \quad \|\mathbf{x}(t)\| \to \infty \;\text{ as } t \nearrow \omega(t_0, \mathbf{x}_0).$$

Thus, for every $M > 0$, there exists $t^* \in I_{\mathbf{x}}$ such that $\|\mathbf{x}(t)\| \geq M$ for all $t \geq t^*$ with $t \in I_{\mathbf{x}}$. Hence, for every compact $K \subset \Omega$, there exists $t_K \in I_{\mathbf{x}}$ such that for all $t \geq t_K$ with $t \in I_{\mathbf{x}}$, we have $(t, \mathbf{x}(t)) \notin K$. In particular,

$$\left\{ (t, \mathbf{x}(t)), \left(t, (\mathbf{x}(t))^+_{\mathbf{f}, t}\right) \right\} \not\subset K.$$

Subcase 2: If Alternative (A3) holds, then we distinguish two cases.

If $I_{\mathbf{x}} = [t_0, \omega(t_0, \mathbf{x}_0))$, then the distance between Graph$(\mathbf{x})$ and the boundary of Ω is zero, so

$$\big(\omega(t_0, \mathbf{x}_0), \mathbf{x}(\omega(t_0, \mathbf{x}_0)^-)\big) \notin \Omega.$$

Thus, for every compact $K \subset \Omega$, there exists $t_K \in [t_0, \omega(t_0, \mathbf{x}_0))$ such that, for all $t \geq t_K$ with $t \in I_{\mathbf{x}}$, $(t, \mathbf{x}(t)) \notin K$, which yields

$$\big\{(t, \mathbf{x}(t)), \big(t, (\mathbf{x}(t))^+_{\mathbf{f},t}\big)\big\} \not\subset K.$$

Now, if $I_{\mathbf{x}} = [t_0, \omega(t_0, \mathbf{x}_0)]$, then $\omega(t_0, \mathbf{x}_0) \in D_g$, $(\omega(t_0, \mathbf{x}_0), \mathbf{x}(\omega(t_0, \mathbf{x}_0))) \in \Omega$ and

$$\Big(\omega(t_0, \mathbf{x}_0), \big(\mathbf{x}(\omega(t_0, \mathbf{x}_0))\big)^+_{\mathbf{f},\omega(t_0,\mathbf{x}_0)}\Big) \notin \Omega.$$

Thus, for every compact $K \subset \Omega$, and for $t_K = \omega(t_0, \mathbf{x}_0)$,

$$\big(t_K, (\mathbf{x}(t_K))^+_{\mathbf{f},t_K}\big) \notin K, \quad \text{hence} \quad \big\{(t_K, \mathbf{x}(t_K)), \big(t_K, (\mathbf{x}(t_K))^+_{\mathbf{f},t_K}\big)\big\} \not\subset K.$$

Conversely, by contradiction, let us assume that $\mathbf{x} : I_{\mathbf{x}} \to \mathbb{R}^n$ is not maximal. Thus, $\bar{t}_{\mathbf{x}} = \sup I_{\mathbf{x}} < \infty$ and $\mathbf{x}$ is extendible to the right. From Theorem 3.5, it follows that Graph$(\mathbf{x})$ is bounded and $A \cup A^+ \subset \Omega$. Thus, $[t_0, \bar{t}_{\mathbf{x}}] \times \overline{\text{Graph}(\mathbf{x})}$ is compact,

$$\mathbf{x}(\bar{t}_{\mathbf{x}}^-) = \mathbf{x}(\bar{t}_{\mathbf{x}}) \in O \quad \text{and} \quad \Big(\mathbf{x}(\bar{t}_{\mathbf{x}})\Big)^+_{\mathbf{f},\bar{t}_{\mathbf{x}}} \in O.$$

Therefore, for the compact

$$K = \overline{\text{Graph}(\mathbf{x})} \cup \Big\{\Big(\bar{t}_{\mathbf{x}}, \big(\mathbf{x}(\bar{t}_{\mathbf{x}})\big)^+_{\mathbf{f},\bar{t}_{\mathbf{x}}}\Big)\Big\} \subset \Omega,$$

we have that

$$\big\{(t, \mathbf{x}(t)), \big(t, (\mathbf{x}(t))^+_{\mathbf{f},t}\big)\big\} \subset K \quad \text{for all } t \in I_{\mathbf{x}},$$

which yields a contradiction. Hence, $\mathbf{x}$ is maximal. □

The negation of Theorem 3.11 provides also an interesting characterization of extendible solutions.

Corollary 3.12. *Assume that* $(\mathrm{H}_{\mathbf{x}_0,t_0})$, $(\mathrm{H}_{\mathbf{f},t_0})$ *and* $(\mathrm{H}_{\Omega,t_0})$ *hold. Let* $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$. *The following statements are equivalent:*

(1) $\mathbf{x}$ *extendible to the right;*

(2) *there exist a compact set* $K \subset \Omega$ *and a sequence* $\{t_n\} \subset I_{\mathbf{x}}$ *with* $t_n \nearrow \omega(t_0, \mathbf{x}_0)$ *such that*

$$\Big\{(t_n, \mathbf{x}(t_n)), \Big(t_n, \big(\mathbf{x}(t_n)\big)^+_{\mathbf{f},t_n}\Big)\Big\} \subset K \quad \textit{for all } n \in \mathbb{N}.$$

Given the generalization of the Grönwall lemma, Lemma 2.12, we can state the next theorem which provides the global existence of the solution over $[t_0, \infty)$ and a bound of the solution.

Theorem 3.13. *Let* $\Omega \supset [t_0, \infty) \times \mathbb{R}^n$. *Assume that* $(\mathrm{H}_{\mathbf{x}_0,t_0})$, $(\mathrm{H}_{\mathbf{f},t_0})$ *and* $(\mathrm{H}_{\Omega,t_0})$ *hold. In addition, assume that*

(H_{LG}) *there exist $k \in \mathcal{L}^1_{g,\mathrm{loc}}([t_0,\infty),\mathbb{R})$, and $p \in \mathcal{L}^1_{g,\mathrm{loc}}([t_0,\infty),[0,\infty))$ such that*

$$\|\mathbf{f}(t,\mathbf{u})\| \le k(t) + p(t)\|\mathbf{u}\| \quad \text{for } g\text{-almost all } t \in [t_0,\infty) \text{ and all } \mathbf{u} \in \mathbb{R}^n.$$

If $\mathbf{x} \in \mathcal{S}(t_0,\mathbf{x}_0)$ is the maximal solution of (3.1), then

$$\|\mathbf{x}(t)\| \le e_p(t,t_0)\left(\int_{[t_0,t)} \frac{e_p^{-1}(s,t_0)k(s)}{1+p(s)\mu_g(\{s\})}\,d\mu_g(s) + \|\mathbf{x}_0\|\right), \quad \text{for all } t \in [t_0,T] \text{ with } T \in I_{\mathbf{x}}. \tag{3.3}$$

Moreover, we have $I_{\mathbf{x}} = [t_0,\infty)$.

Proof. Let $\mathbf{x} \in \mathcal{S}(t_0,\mathbf{x}_0)$ be the maximal solution of (3.1) defined on the maximal interval $I_{\mathbf{x}}$. By (H_{LG}), for $T > t_0$ with $T \in I_{\mathbf{x}}$, we have that

$$(\|\mathbf{x}\|)'_g(t) \le \|\mathbf{x}'_g(t)\| = \|\mathbf{f}(t,\mathbf{x}(t))\| \le k(t) + p(t)\|\mathbf{x}(t)\| \quad \text{for } g\text{-almost all } t \in [t_0,T).$$

Observe that $\|\cdot\|$ is Lipschitz continuous on $\mathbb{R}^n$. Thus, by Lemma 2.8, $\|\mathbf{x}(\cdot)\| \in \mathcal{AC}_g([t_0,T],\mathbb{R})$. Using the generalized version of the Grönwall Lemma, Lemma 2.12, we obtain

$$\|\mathbf{x}(t)\| \le e_p(t,t_0)\left(\int_{[t_0,t)} \frac{e_p^{-1}(s,t_0)k(s)}{1+p(s)\mu_g(\{s\})}\,d\mu_g(s) + \|\mathbf{x}_0\|\right) \quad \text{for all } t \in [t_0,T].$$

Assume by contradiction that $\omega(t_0,\mathbf{x}_0) < \infty$. Since $p \in \mathcal{L}^1_{g,\mathrm{loc}}([t_0,\infty),[0,\infty))$, then

$$e_p(t,t_0)(1+p(t)\mu_g(\{t\})) \ge 1 \quad \text{for all } t \in [t_0,\infty).$$

Consequently,

$$\begin{aligned}\|\mathbf{x}(t)\| &\le \sup_{t\in[t_0,T]} e_p(t,t_0)\left(\int_{[t_0,t)} |k(s)|\,d\mu_g(s) + \|\mathbf{x}_0\|\right)\\ &\le e_p(\omega(t_0,\mathbf{x}_0),t_0)\left(\|k\|_{\mathcal{L}^1_g([t_0,\omega(t_0,\mathbf{x}_0)),\mathbb{R})} + \|\mathbf{x}_0\|\right)\\ &= M_1.\end{aligned}$$

As $T \nearrow \omega(t_0,\mathbf{x}_0) < \infty$, $\mathbf{x}(t) \in \overline{B_{\mathbb{R}^n}(\mathbf{0},M_1)}$ for all $t \in I_{\mathbf{x}}$. As $\mathbf{x}(\omega(t_0,\mathbf{x}_0)^-) = \mathbf{u} \in \mathbb{R}^n$, then

$$\mathbf{u}^+_{\mathbf{f},\omega(t_0,\mathbf{x}_0)} \in \mathbb{R}^n.$$

Therefore, for the compact set

$$K = [t_0,\omega(t_0,\mathbf{x}_0)] \times \overline{B_{\mathbb{R}^n}(\mathbf{0},M)} \quad \text{with} \quad M = \max\left\{M_1, \left\|\mathbf{u}^+_{\mathbf{f},\omega(t_0,\mathbf{x}_0)}\right\|\right\},$$

we have that

$$\left\{(t,\mathbf{x}(t)),(t,(\mathbf{x}(t))^+_{\mathbf{f},t})\right\} \subset K \quad \text{for all } t \in I_{\mathbf{x}}.$$

By Corollary 3.12, we obtain that $\mathbf{x}$ is extendible to the right which is a contradiction. Hence, $\omega(t_0,\mathbf{x}_0) = \infty$. □

4 Lyapunov-like stability results

In this section, we assume that $[0,\infty) \subset I$ and $\Omega = I \times B_{\mathbb{R}^n}(\mathbf{0}, r)$. We present Lyapunov-type results in the context of Stieltjes dynamical systems, based on the classical Lyapunov's second method, considering Stieltjes dynamical systems of the form:

$$\mathbf{x}'_g(t) = \mathbf{f}(t, \mathbf{x}(t)) \quad \text{for } g\text{-almost all } t \geq \theta \geq 0,\, t \in I, \tag{4.1$_\theta$}$$

where $\mathbf{f} : \Omega \to \mathbb{R}^n$ satisfies $\mathbf{f}(t, \mathbf{0}) = \mathbf{0}$ for all $t \geq 0$. This study permits to draw conclusions about the behavior of solutions of the dynamical system (4.1$_\theta$) around the *equilibrium* $\mathbf{x} = \mathbf{0}$, which is also called the *trivial solution*.

Notice that, under hypotheses ($\mathrm{H}_{\Omega,0}$) and ($\mathrm{H}_{\mathbf{f},0}$) and for $(t_0, \mathbf{x}_0) \in [0,\infty) \times B_{\mathbb{R}^n}(\mathbf{0}, r)$, if ($\mathrm{H}_{\mathbf{x}_0,t_0}$) is satisfied, it follows from Theorem 3.7 that there exists a unique maximal solution $\mathbf{x} = \mathbf{x}(\cdot, t_0, \mathbf{x}_0) \in \mathcal{S}(t_0, \mathbf{x}_0) \cap \mathcal{AC}_{g,\mathrm{loc}}(I_{t_0,\mathbf{x}_0}, \mathbb{R}^n)$ of (4.1$_\theta$) with $\theta = t_0$ and satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$ which is defined on a maximal interval of existence $I_{t_0,\mathbf{x}_0}$. As before, we denote $\omega(t_0, \mathbf{x}_0) = \sup I_{t_0,\mathbf{x}_0} \leq \infty$.

4.1 Lyapunov stability notions

In this subsection, we present stability concepts within the framework of Stieltjes' differentiation. Through illustrative examples, we highlight the influence of the sets C_g and D_g on the change of stability properties.

Definition 4.1. The trivial solution $\mathbf{x} = \mathbf{0}$ of the system (4.1$_\theta$) with $\theta = 0$ is said to be

- *stable* if, for all $\varepsilon > 0$ and $t_0 \in [0,\infty)$, there exists $\delta = \delta(\varepsilon, t_0) \in (0, r)$ such that

$$\|\mathbf{x}_0\| < \delta \quad \text{implies that} \quad \|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \varepsilon \quad \text{for all } t \in I_{t_0,\mathbf{x}_0};$$

- *uniformly stable* if, for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) \in (0, r)$ such that for all $t_0 \in [0,\infty)$,

$$\|\mathbf{x}_0\| < \delta \quad \text{implies that} \quad \|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \varepsilon \quad \text{for all } t \in I_{t_0,\mathbf{x}_0}.$$

Remark 4.2. In the case where the trivial solution $\mathbf{x} = \mathbf{0}$ of the system (4.1$_\theta$) with $\theta = 0$ is stable, observe that, for $\varepsilon = r$ and $t_0 \geq 0$, there exists $\delta = \delta(\varepsilon, t_0) \in (0, r)$ such that, for all $\mathbf{x}_0 \in B_{\mathbb{R}^n}(\mathbf{0}, \delta)$,

$$\|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \varepsilon \text{ for all } t \in I_{t_0,\mathbf{x}_0}.$$

Using Theorem 3.8 under ($\mathrm{H}_{\Omega,0}$), ($\mathrm{H}_{\mathbf{f},0}$) and ($\mathrm{H}_{\mathbf{x}_0,t_0}$), we deduce that $I_{t_0,\mathbf{x}_0} = [t_0, \infty)$. Furthermore, observe that if the stability of the trivial solution $\mathbf{x} = \mathbf{0}$ is uniform, then δ do not depend on t_0.

The following definition is a notion of *asymptotic stability*. This concerns the behavior of solutions as $t \to \infty$.

Definition 4.3. The trivial solution $\mathbf{x} = \mathbf{0}$ of the system (4.1$_\theta$) with $\theta = 0$ is said to be

- *asymptotically stable* if it is *stable* and, for every $t_0 \in [0,\infty)$, there exists $\delta = \delta(t_0) \in (0, r)$ such that, for all $\mathbf{x}_0 \in B_{\mathbb{R}^n}(\mathbf{0}, \delta)$ and $\varepsilon > 0$, there exists $\sigma = \sigma(t_0, \mathbf{x}_0, \varepsilon) > 0$ such that

$$\|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \varepsilon \quad \text{for all } t \in [t_0 + \sigma, \infty) \cap I_{t_0,\mathbf{x}_0};$$

- *uniformly asymptotically stable* if it is *uniformly stable* and there exists $\delta \in (0, r)$ such that, for every $\varepsilon > 0$, there exists $\sigma = \sigma(\varepsilon) > 0$ such that, for all $t_0 \in [0, \infty)$ and $\mathbf{x}_0 \in B_{\mathbb{R}^n}(\mathbf{0}, \delta)$,

$$\|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \varepsilon \text{ for all } t \in [t_0 + \sigma, \infty) \cap I_{t_0, \mathbf{x}_0}.$$

In the following example, we compare the stability properties of the trivial solution of a linear Stieltjes dynamical system to the ones in the classical case, observing the change of the stability properties depending on the sets C_g and D_g. The resolution of linear Stieltjes differential equations has been studied in the literature, see for instance [7, 9, 17, 26, 27].

Example 4.4. Let us consider the linear Stieltjes dynamical system

$$x_g'(t) = cx(t) \quad \text{for } g\text{-almost all } t \geq \theta \geq 0, \tag{4.2$_\theta$}$$

for $c \in \mathbb{R}$. In the classical case of derivation where $g \equiv \mathrm{id}_{\mathbb{R}}$, for $c > 0$, the equilibrium $x = 0$ is not stable given that the solutions of (4.2$_\theta$) with $\theta = t_0$ and satisfying $x(t_0) = x_0$ are $x(t, t_0, x_0) = x_0 e^{c(t-t_0)}$ for every $x_0 \in \mathbb{R}$ and $t_0 \geq 0$, and they are not bounded on $[t_0, \infty)$ for $x_0 \neq 0$.

However, for $c < 0$, the equilibrium $x = 0$ is asymptotically stable since stability holds and $x(t, t_0, x_0) = x_0 e^{c(t-t_0)} \xrightarrow{t \to \infty} 0$ for all $x_0 \in \mathbb{R}$. In the case where $c = 0$, every constant $z \in \mathbb{R}$ is a uniformly stable equilibrium.

Now, let us reconsider the dynamical system (4.2$_\theta$), where $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(t) = t$ for $t \leq 1$ and $g(t) = 1$ for $t \geq 1$. Thus, the solution of the problem (4.2$_\theta$) with $\theta = t_0$ and satisfying $x(t_0) = x_0$ is $x(t, t_0, x_0) = x_0 e^{c(g(t) - g(t_0))}$, for every $x_0 \in \mathbb{R}$ and $t_0 \geq 0$. Now, we show that the stability properties of the trivial solution $x = 0$ of the system (4.2$_\theta$) with $\theta = 0$ differ from the classical case, for $c \neq 0$. Indeed, for $c > 0$, we deduce the stability of the trivial solution $x = 0$ since, for all $\varepsilon > 0$ and $t_0 \geq 0$, one can take $\delta = \varepsilon / \left(e^{c(g(1) - g(t_0))}\right)$ which is such that

$$|x_0| < \delta \quad \text{implies that} \quad |x_0 e^{c(g(t) - g(t_0))}| < \varepsilon \quad \text{for all } t \geq t_0.$$

Whereas, for $c < 0$ the equilibrium $x = 0$ is uniformly stable since, for all $\varepsilon > 0$, with $\delta = \varepsilon > 0$, one has that

$$|x_0| < \delta \quad \text{implies that} \quad |x_0 e^{c(g(t) - g(t_0))}| < \varepsilon \quad \text{for all } t \geq t_0.$$

Observe that asymptotic stability does not hold for any $c \in \mathbb{R}$, since

$$x_0 e^{c(g(t) - g(t_0))} \to x_0 e^{c(g(1) - g(t_0))} \nrightarrow 0,$$

as $t \to \infty$ for all $t_0 \geq 0$ and $x_0 \neq 0$.

Next, we define the derivator g_1 by $g_1(t) = t + \sum_{n \in \mathbb{N}} \chi_{[n, \infty)}(t)$ for all $t \in \mathbb{R}$. We consider the dynamical system with g_1:

$$x_{g_1}'(t) = \begin{cases} cx(t) & \text{if } t \geq \theta \geq 0, \text{ with } t \notin D_{g_1} = \mathbb{N}, \\ \nu x(t), & \text{if } t \geq \theta \geq 0, \text{ with } t \in D_{g_1}, \end{cases} \tag{4.3$_\theta$}$$

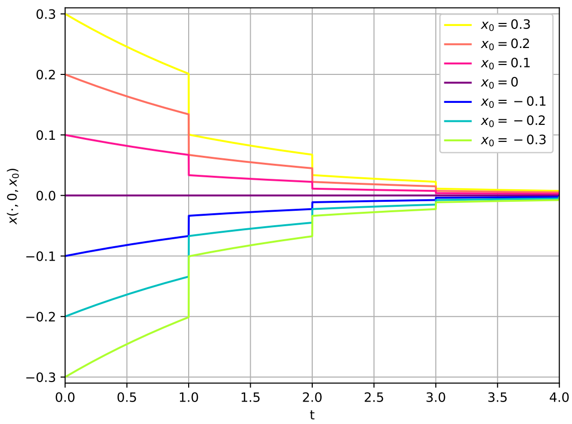
where $c, \nu \in \mathbb{R}$ with $\nu \in (-1, \infty) \setminus \{0\}$. Again the solution of (4.3$_\theta$) with $\theta = t_0$ and satisfying $x(t_0) = x_0$ has the form

$$x(t, t_0, x_0) = x_0 e_{c_*}(t, t_0) = x_0 e^{\int_{[t_0, t)} c_*(s) \, d\mu_{g_1}(s)},$$

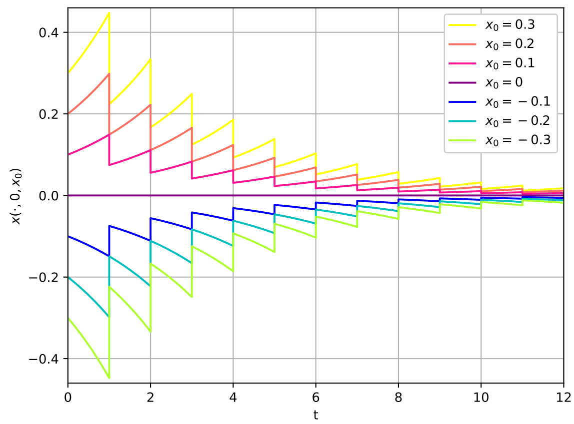
where

$$c_*(t) = \begin{cases} c & \text{if } t \in [t_0, \infty) \setminus D_{g_1}, \\ \log(1+\nu) & \text{if } t \in [t_0, \infty) \cap D_{g_1}. \end{cases}$$

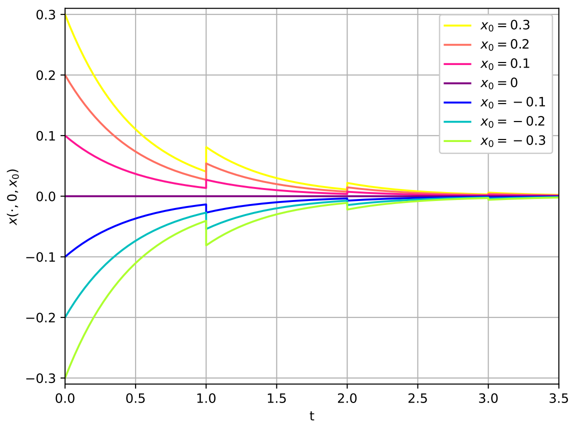
In Figure 4.1, we can observe different patterns depending on the values of c and ν. This implies that the presence of discontinuities can destabilize or restore the stability properties of a dynamical system.



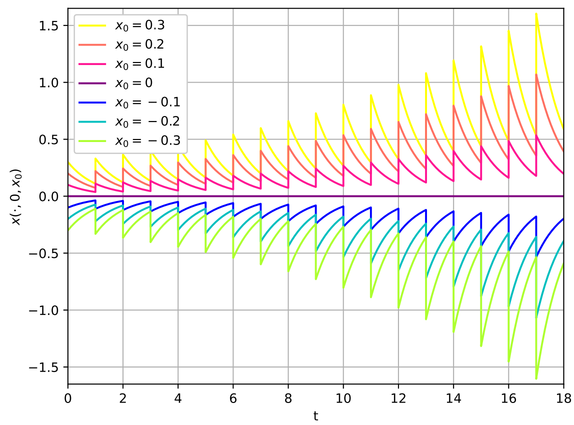
(a) The case of $c = -0.4$, and $\nu = -0.5$.



(b) The case of $c = 0.4$, and $\nu = -0.5$.



(c) The case of $c = -2$, and $\nu = 1$.



(d) The case of $c = -1$, and $\nu = 2$.

Figure 4.1: Behaviour of solutions of (4.3_θ) with $\theta = 0$ in a neighborhood of $x = 0$.

4.2 Stability results based on Lyapunov's function

In order to establish sufficient conditions for different types of stability of the trivial solution $\mathbf{x} = \mathbf{0}$ of the system (4.1_θ) with $\theta = 0$, we introduce specific sets of functions.

Definition 4.5. A function $V : [0, \infty) \times B_{\mathbb{R}^n}(\mathbf{0}, r_0) \to \mathbb{R}$ is said to belong to class $\mathcal{V}_1^g$ if it satisfies the following conditions:

1. $V(t, \cdot)$ is continuous for all $t \geq 0$;

2. $V(\cdot, \mathbf{x}(\cdot)) \in \mathcal{AC}_{g,\mathrm{loc}}(I_{t_0,\mathbf{x}_0}, \mathbb{R})$ for every function $\mathbf{x} : I_{t_0,\mathbf{x}_0} \to B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ of $\mathcal{AC}_{g,\mathrm{loc}}(I_{t_0,\mathbf{x}_0}, \mathbb{R}^n)$ maximal solution of the system (4.1_θ) with $\theta = 0$;
3. $V(t, \mathbf{0}) = 0$ for all $t \geq 0$.

Definition 4.6. A function $\varphi : [0, \infty) \to [0, \infty)$ belongs to the class $\mathcal{K}$ if it fulfills the following assumptions:

1. φ is continuous;
2. $\varphi(0) = 0$;
3. φ is increasing.

In the following, we simply write $\varphi \in \mathcal{K}$ to mean that φ belongs to the class $\mathcal{K}$. Now, we state the first stability result.

Theorem 4.7. *Assume that Conditions* $(\mathrm{H}_{\Omega,0})$ *and* $(\mathrm{H}_{\mathbf{f},0})$ *hold. If there exist functions* $V \in \mathcal{V}_1^g$ *and* $a \in \mathcal{K}$ *such that*

(a) $a(\|\mathbf{u}\|) \leq V(t, \mathbf{u})$ *for all* $(t, \mathbf{u}) \in [0, \infty) \times B_{\mathbb{R}^n}(\mathbf{0}, r_0)$;

(b) *there exists* $r \in (0, r_0]$ *such that, for every* $(t_0, \mathbf{x}_0) \in [0, \infty) \times B_{\mathbb{R}^n}(\mathbf{0}, r)$, $(\mathrm{H}_{\mathbf{x}_0,t_0})$ *holds and, for* $\mathbf{x} : I_{t_0,\mathbf{x}_0} \to B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ *the maximal solution of the system* (4.1_θ) *with* $\theta = t_0$ *and satisfying* $\mathbf{x}(t_0) = \mathbf{x}_0$, *one has that* $V_g'(t, \mathbf{x}(t)) \leq 0$ *for* g*-almost all* $t \in I_{t_0,\mathbf{x}_0}$.

Then, the trivial solution of the system (4.1_θ) *with* $\theta = 0$ *is*

(1) *stable,*

(2) *uniformly stable if there exists* $b \in \mathcal{K}$ *such that*

$$V(t, \mathbf{u}) \leq b(\|\mathbf{u}\|) \quad \text{for all } (t, \mathbf{u}) \in [0, \infty) \times B_{\mathbb{R}^n}(\mathbf{0}, r_0). \tag{4.4}$$

Proof. (1) Since $V \in \mathcal{V}_1^g$, then, for all $\varepsilon > 0$ and $t_0 \in [0, \infty)$, there exists $\delta = \delta(\varepsilon, t_0) \in (0, r)$ such that

$$\sup_{\|\mathbf{u}\| < \delta} V(t_0, \mathbf{u}) < a(\varepsilon).$$

For $\mathbf{x}_0 \in B_{\mathbb{R}^n}(\mathbf{0}, \delta)$, let $\mathbf{x} : I_{t_0,\mathbf{x}_0} \to B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ be the maximal solution of the system (4.1_θ) with $\theta = t_0$ and $\mathbf{x}(t_0) = \mathbf{x}_0$. It follows from Conditions (a) and (b) that

$$a(\|\mathbf{x}(t)\|) \leq V(t, \mathbf{x}(t)) \leq V(t_0, \mathbf{x}_0) < a(\varepsilon).$$

Thus, for all $t \in I_{t_0,\mathbf{x}_0}$, we have that

$$\|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \varepsilon.$$

Therefore, the trivial solution $\mathbf{x} = \mathbf{0}$ is stable.

(2) Arguing as in (1), we can choose a $\delta = \delta(\varepsilon) \in (0, r)$ independent of t_0 such that

$$b(\delta) < a(\varepsilon).$$

Thus, using (4.4), we obtain for all $t \in I_{t_0,\mathbf{x}_0}$,

$$a(\|\mathbf{x}(t)\|) \leq V(t, \mathbf{x}(t)) \leq V(t_0, \mathbf{x}_0) \leq b(\|\mathbf{x}_0\|) < b(\delta) < a(\varepsilon).$$

This yields that the trivial solution $\mathbf{x} = \mathbf{0}$ is uniformly stable. □

In the next theorem, we impose additional assumptions which will permit to ensure the asymptotic stability of the trivial solution $\mathbf{x} = \mathbf{0}$ to the system (4.1_θ) with $\theta = 0$.

Theorem 4.8. *Assume that Conditions* ($\mathrm{H}_{\Omega,0}$) *and* ($\mathrm{H}_{\mathbf{f},0}$) *hold. Let* $V \in \mathcal{V}_1^g$, $\phi : [0,\infty) \to [0,\infty)$ *continuous,* $a, b \in \mathcal{K}$, *and a g-measurable function* $v : [0,\infty) \to [0,\infty)$ *be such that*

(a) $a(\|\mathbf{u}\|) \leq V(t,\mathbf{u})$ *for every* $(t,\mathbf{u}) \in [0,\infty) \times B_{\mathbb{R}^n}(\mathbf{0}, r_0)$;

(b) $\phi(s) = 0$ *if and only if* $s = 0$;

(c) *there exists* $r \in (0, r_0]$ *such that, for every* $(t_0, \mathbf{x}_0) \in [0,\infty) \times B_{\mathbb{R}^n}(\mathbf{0}, r)$, ($\mathrm{H}_{\mathbf{x}_0,t_0}$) *holds and, for* $\mathbf{x} : I_{t_0,\mathbf{x}_0} \to B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ *the maximal solution of the system* (4.1_θ) *with* $\theta = t_0$ *and satisfying* $\mathbf{x}(t_0) = \mathbf{x}_0$, *one has*

$$V_g'(t,\mathbf{x}(t)) \leq -v(t)\phi(\|\mathbf{x}(t)\|) \quad \textit{for g-almost all } t \in I_{t_0,\mathbf{x}_0};$$

(d) $\inf_{t_0\in[0,\infty)} \lim_{t\to+\infty} \int_{[t_0,t_0+t)} v(s)\, d\mu_g(s) = \infty$.

If the trivial solution $\mathbf{x} = \mathbf{0}$ *of the system* (4.1_θ) *with* $\theta = 0$ *is uniformly stable, then* $\mathbf{x} = \mathbf{0}$ *is asymptotically stable.*

Proof. The stability of the trivial solution $\mathbf{x} = \mathbf{0}$ holds from uniform stability. Let us choose $\delta_0 \in (0, r)$ associated to an $\varepsilon_0 \leq r$ given by the uniform stability. Now, for a fixed $t_0 \geq 0$, let $\varepsilon > 0$. Again, by the uniform stability, there exists $\delta \in (0, \delta_0)$ such that, for all $\hat{t} \in [0,\infty)$ and every $\hat{\mathbf{x}}_0$ satisfying $\|\hat{\mathbf{x}}_0\| < \delta$, one has

$$\|\hat{\mathbf{x}}(t, \hat{t}, \hat{\mathbf{x}}_0)\| < \varepsilon \quad \text{for all } t \in [\hat{t}, \infty) \cap I_{\hat{t},\hat{\mathbf{x}}_0}.$$

We denote

$$M = \inf_{s\in[\delta,r_0)} |\phi(s)|. \tag{4.5}$$

By Condition (b), observe that $M > 0$.

Let $\mathbf{x}_0 \in B_{\mathbb{R}^n}(\mathbf{0}, \delta_0)$. Since

$$\lim_{t\to+\infty} \int_{[t_0,t_0+t)} v(s)\, d\mu_g(s) = \infty,$$

we can choose $\sigma > 0$ such that

$$\int_{[t_0,t_0+\sigma)} v(s)\, d\mu_g(s) > \frac{V(t_0, \mathbf{x}_0)}{M}.$$

Let $\mathbf{x} : I_{t_0,\mathbf{x}_0} \to B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ be the maximal solution of (4.1_θ) with $\theta = t_0$ and satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$. Using Remark 4.2, notice that $I_{t_0,\mathbf{x}_0} = [t_0, \infty)$. Now, if there exists $\hat{t} \in [t_0, t_0 + \sigma]$ such that $\|\mathbf{x}(\hat{t})\| < \delta$, then, by the uniform stability

$$\|\hat{\mathbf{x}}(t)\| < \varepsilon \quad \text{for all } t \in [\hat{t}, \infty) \cap I_{\hat{t},\mathbf{x}(\hat{t})},$$

where $\hat{\mathbf{x}} : I_{\hat{t},\mathbf{x}(\hat{t})} \to B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ is the maximal solution of (4.1_θ) with $\theta = \hat{t}$ satisfying the initial condition $\hat{\mathbf{x}}(\hat{t}) = \mathbf{x}(\hat{t})$. By the uniqueness of the maximal solution, one has

$$\omega(t_0, \mathbf{x}_0) = \omega(\hat{t}, \mathbf{x}(\hat{t})) = \infty \quad \text{and} \quad \mathbf{x}(t) = \hat{\mathbf{x}}(t) \quad \text{for all } t \in [\hat{t}, \infty).$$

Hence,

$$\|\mathbf{x}(t)\| < \varepsilon \quad \text{for all } t \in [t_0 + \sigma, \infty) \cap I_{t_0, \mathbf{x}_0}.$$

On the other hand, if $\|\mathbf{x}(t)\| \geq \delta$ for all $t \in [t_0, t_0 + \sigma]$, then using Conditions (a), (c), Theorem 2.7, and (4.5), we obtain

$$\begin{aligned} a(\|\mathbf{x}(t_0+\sigma)\|) &\leq V(t_0+\sigma, \mathbf{x}(t_0+\sigma)) \\ &= V(t_0, \mathbf{x}(t_0, t_0, \mathbf{x}_0)) + \int_{[t_0, t_0+\sigma)} V_g'(s, \mathbf{x}(s))\, d\mu_g(s) \\ &\leq V(t_0, \mathbf{x}_0) - \int_{[t_0, t_0+\sigma)} v(s)\phi(\|\mathbf{x}(s)\|)\, d\mu_g(s) \\ &\leq V(t_0, \mathbf{x}_0) - M \int_{[t_0, t_0+\sigma)} v(s)\, d\mu_g(s) \\ &< 0. \end{aligned}$$

This is a contradiction. Therefore, $\mathbf{x} = \mathbf{0}$ is asymptotically stable. □

In the example below, we present an application of Theorem 4.8.

Example 4.9. Let us consider the Stieltjes dynamical system

$$x_g'(t) = f(t, x(t)) \quad \text{for } g\text{-almost all } t \geq \theta \geq 0, \tag{4.6$_\theta$}$$

with $g : \mathbb{R} \to \mathbb{R}$ defined by $g(t) = t$ for all $t \leq 1$, and $g(t) = t + 1$ for $t > 1$, and where $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a function defined by

$$f(t, x) = \begin{cases} -\dfrac{xt}{1+t^2} & \text{if } t \in [0, \infty) \setminus D_g, \\ \nu x & \text{if } t \in [0, \infty) \cap D_g, \end{cases}$$

for some $\nu \in \mathbb{R} \setminus \{-1, 0\}$. The function f satisfies conditions of Theorem 3.13, thus, for every $(t_0, x_0) \in [0, \infty) \times \mathbb{R}$, the problem (4.6$_\theta$) with $\theta = t_0$ has a maximal solution $x : [t_0, +\infty) \to \mathbb{R}$ satisfying $x(t_0) = x_0$. Observe that $x = 0$ is an equilibrium of the dynamical system (4.6$_\theta$) with $\theta = 0$.

Let us define the function $V : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ for every $(t, x) \in [0, \infty) \times \mathbb{R}$ by

$$V(t, x) = \begin{cases} x^2 & \text{if } t \in [0, 1], \\ \dfrac{x^2}{(1+\nu)^2} & \text{if } t > 1. \end{cases}$$

Clearly $V \in \mathcal{V}_g^1$, and $a(|u|) \leq V(t, u) \leq b(|u|)$ for all $(t, u) \in [0, \infty) \times \mathbb{R}$, where $a, b \in \mathcal{K}$ are given by

$$a(s) = \min\left\{s^2, \frac{s^2}{(1+\nu)^2}\right\} \quad \text{and} \quad b(s) = \max\left\{s^2, \frac{s^2}{(1+\nu)^2}\right\} \quad \text{for all } s \in [0, \infty).$$

In addition, for all $(t, x) \in [0, \infty) \times \mathbb{R}$:

$$\begin{aligned} \frac{\partial V}{\partial_g t}(t, x) &= \begin{cases} 0 & \text{if } t \in [0, 1) \cup (1, \infty), \\ \dfrac{\frac{x^2}{(1+\nu)^2} - x^2}{g(1^+) - g(1)} & \text{if } t = 1, \end{cases} \\ &= \begin{cases} 0 & \text{if } t \in [0, 1) \cup (1, \infty), \\ \dfrac{1-(1+\nu)^2}{(1+\nu)^2} x^2 & \text{if } t = 1. \end{cases} \end{aligned}$$

Thus, by means of Proposition 2.10, for $t \in [t_0, \infty) \setminus D_g$, we obtain

$$
\begin{aligned}
V'_g(t,x(t)) &= \frac{\partial V}{\partial_g t}(t,x(t)) + \frac{\partial V}{\partial x}(t,x(t))f(t,x(t)) \\
&= \begin{cases} -\dfrac{2t}{1+t^2}x(t)^2 & \text{if } t \in [t_0,\infty) \cap [0,1), \\ -\dfrac{2t}{(1+t^2)(1+\nu)^2}x(t)^2 & \text{if } t \in [t_0,\infty) \cap (1,\infty). \end{cases}
\end{aligned}
$$

For $t \in [t_0, \infty) \cap D_g$, if $t_0 \leq 1$, then $t = 1$ and we have that

$$
\begin{aligned}
V'_g(1,x(1)) &= \frac{V(1^+,x(1^+)) - V(1,x(1))}{g(1^+) - g(1)} \\
&= \frac{V(1^+,x(1) + \mu_g(\{1\})f(1,x(1))) - V(1,x(1))}{g(1^+) - g(1)} \\
&= \frac{\frac{(1+\nu)^2 x(1)^2}{(1+\nu)^2} - x(1)^2}{g(1^+) - g(1)} \\
&= 0.
\end{aligned}
$$

This implies that

$$
V'_g(t,x(t)) \leq -v(t)\phi(|x(t)|), \text{ for every } t \geq t_0,
$$

with $v : [0,\infty) \to [0,\infty)$, defined by

$$
v(t) = \begin{cases} \dfrac{2t}{1+t^2} & \text{if } t \in [0,1), \\ 0 & \text{if } t = 1, \\ \dfrac{2t}{(1+t^2)(1+\nu)^2} & \text{if } t \in (1,\infty), \end{cases}
$$

and $\phi \in \mathcal{K}$, defined by $\phi(y) = y^2$ for all $y \in [0,\infty)$. Moreover, for every $t_0 \in [0,\infty)$, we have

$$
\begin{aligned}
\lim_{t\to\infty} \int_{[t_0,t_0+t)} v(s)\, d\mu_g(s) &= \lim_{t\to\infty} \int_{[t_0,t_0+t)\cap([0,1)\cup\{1\}\cup(1,\infty))} v(s)\, d\mu_g(s) \\
&\geq \lim_{t\to\infty} \frac{1}{(1+\nu)^2} \int_{[t_0,t_0+t)\cap(1,\infty)} \frac{2s}{1+s^2}\, ds \\
&= \lim_{t\to\infty} \frac{1}{(1+\nu)^2} \big(\log(1+(t_0+t)^2) - \sup\{\log(2), \log(1+t_0^2)\} \big) \\
&= \infty.
\end{aligned}
$$

Hence,

$$
\inf_{t_0 \in [0,\infty)} \lim_{t\to+\infty} \int_{[t_0,t_0+t)} v(s)\, d\mu_g(s) = \infty.
$$

Thus, Condition (d) holds. Therefore, by means of Theorem 4.8, we deduce that $x = 0$ is an asymptotically stable equilibrium. Figure 4.2 illustrates the asymptotic behavior of solutions of the dynamical system (4.6_θ) with $\theta = 0$.

In the following example, we consider the case where g has an infinite number of discontinuities.

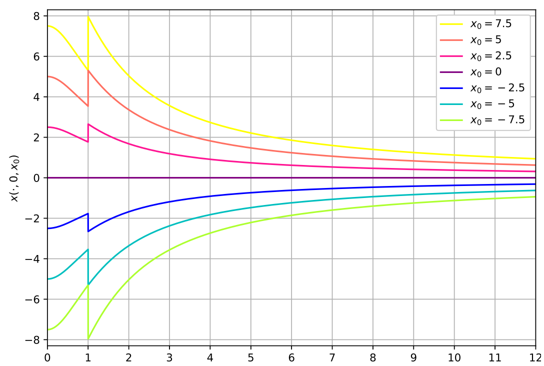


Figure 4.2: Asymptotic behavior of solutions of the dynamical system (4.6_θ) with $\theta = 0$ obtained with a time-discretization step-size of 10^{-3}, with $\nu = 0.5$.

Example 4.10. Let us consider the dynamical system

$$x_g'(t) = f(t, x(t)) \quad \text{for } g\text{-almost all } t \geq \theta \geq 0, \tag{4.7_θ}$$

where $g : \mathbb{R} \to \mathbb{R}$ is the derivator defined by

$$g(t) = t + \sum_{k \in \mathbb{N}} \chi_{[t_k, +\infty)}(t) \quad \text{for all } t \in \mathbb{R}, \tag{4.8}$$

where $\{t_k\}$ is an unbounded, increasing sequence in $(0, \infty)$, and where the function $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f(t, x) = \begin{cases} -\dfrac{xt}{1 + t^2} & \text{if } t \in [0, \infty) \setminus D_g, \\ \nu x & \text{if } t \in [0, \infty) \cap D_g, \end{cases}$$

for some $\nu \in [-2, -1)$. The function f satisfies conditions of Theorem 3.13. Thus, for every $(t_0, x_0) \in [0, \infty) \times \mathbb{R}$, the problem ($4.6_\theta$) with $\theta = t_0$ has a maximal solution $x : [t_0, +\infty) \to \mathbb{R}$ satisfying $x(t_0) = x_0$. Observe that $x = 0$ is an equilibrium of the dynamical system (4.7_θ) with $\theta = 0$.

Let us define the function $V : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ by $V(t, x) = x^2$. Clearly, $V \in \mathcal{V}_g^1$ and

$$a(|u|) \leq V(t, u) \leq b(|u|) \quad \text{for all } (t, u) \in [0, \infty) \times \mathbb{R},$$

where $a, b \in \mathcal{K}$ are given by

$$a(s) = s^2 \quad \text{and} \quad b(s) = \frac{s^2}{(1 + \nu)^2} \quad \text{for all } s \in [0, \infty).$$

In addition,

$$\frac{\partial V}{\partial_g t}(t, x) = 0 \quad \text{for all } (t, x) \in [0, \infty) \times \mathbb{R}.$$

Thus, by means of Proposition 2.10, for g-almost every $t \in [t_0, \infty) \setminus D_g$, we obtain

$$\begin{aligned} V_g'(t, x(t)) &= \frac{\partial V}{\partial_g t}(t, x(t)) + \frac{\partial V}{\partial x}(t, x(t)) f(t, x(t)) \\ &= -\frac{2t}{1 + t^2} x(t)^2. \end{aligned}$$

For $t_k \in [t_0, \infty) \cap D_g$, we have that

$$\begin{aligned} V_g'(t_k, x(t_k)) &= \frac{V(t_k^+, x(t_k^+)) - V(t_k, x(t_k))}{g(t_k^+) - g(t_k)} \\ &= \frac{V(t_k^+, x(t_k) + \mu_g(\{t_k\}) f(t_k, x(t_k))) - V(t_k, x(t_k))}{g(t_k^+) - g(t_k)} \\ &= \frac{((1+\nu)x(t_k))^2 - x(t_k)^2}{g(t_k^+) - g(t_k)} \\ &= \nu(2+\nu)x(t_k)^2. \end{aligned}$$

This implies that

$$V_g'(t, x(t)) \le -v(t)\phi(|x(t)|) \quad \text{for } g\text{-almost every } t \ge t_0,$$

with $v : [0, \infty) \to [0, \infty)$ defined by

$$v(t) = \begin{cases} \dfrac{2t}{1+t^2} & \text{if } t \in [0, t_1) \cup (t_k, t_{k+1}),\ k \in \mathbb{N}, \\ -\nu(2+\nu) & \text{if } t \in \{t_k\}_{k\in\mathbb{N}}, \end{cases}$$

and $\phi \in \mathcal{K}$ defined by $\phi(y) = y^2$ for all $y \in [0, \infty)$. For $t_0, t \in [0, \infty)$, observe that v satisfies:

$$\begin{aligned} \int_{[t_0, t_0+t)} v(s)\, d\mu_g(s) &= \sum_{s \in [t_0, t_0+t) \cap D_g} -\nu(2+\nu)\mu_g(\{s\}) + \int_{t_0}^{t_0+t} \frac{2s}{1+s^2}\, ds \\ &= \sum_{s \in [t_0, t_0+t) \cap D_g} -\nu(2+\nu)\mu_g(\{s\}) + \log\left(\frac{1+(t_0+t)^2}{1+t_0^2}\right). \end{aligned}$$

Thus,

$$\inf_{t_0 \in [0,\infty)} \lim_{t \to +\infty} \int_{[t_0, t_0+t)} v(s)\, d\mu_g(s) = +\infty.$$

Consequently, Condition (d) holds. Therefore, by means of Theorem 4.8, we deduce that $x = 0$ is an asymptotically stable equilibrium. Figure 4.3 illustrates the asymptotic behavior of solutions of the dynamical system (4.7_θ) with $\theta = 0$.

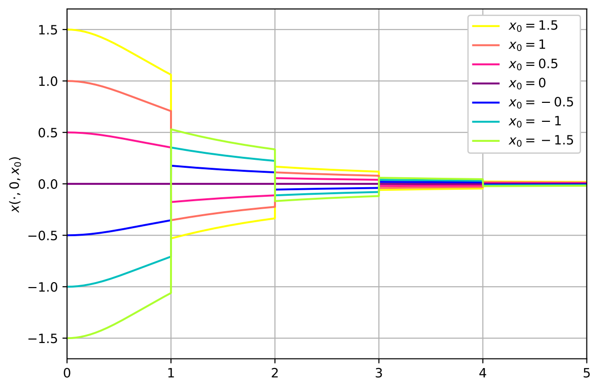


Figure 4.3: Asymptotic behavior of solutions of the dynamical system (4.7_θ) with $\theta = 0$ obtained with a time-discretization step-size of 10^{-3}, with $\nu = -3/2$.

The following result gives conditions to ensure the uniform asymptotic stability of the trivial solution of (4.1_θ) with $\theta = 0$.

Theorem 4.11. *Assume that Conditions* $(\mathrm{H}_{\Omega,0})$, $(\mathrm{H}_{\mathbf{f},0})$ *hold. If there exist* $V \in \mathcal{V}_1^g$, $a, b \in \mathcal{K}$, $\phi : [0,\infty) \to [0,\infty)$ *continuous, and a g-measurable function* $v : [0,\infty) \to [0,\infty)$ *such that*

(a) $a(\|\mathbf{u}\|) \leq V(t,\mathbf{u}) \leq b(\|\mathbf{u}\|)$ *for every* $(t,\mathbf{u}) \in [0,\infty) \times B_{\mathbb{R}^n}(\mathbf{0}, r_0)$*;*

(b) $\phi(s) = 0$ *if and only if* $s = 0$*;*

(c) *there exists* $r \in (0, r_0]$ *such that, for every* $(t_0, \mathbf{x}_0) \in [0,\infty) \times B_{\mathbb{R}^n}(\mathbf{0}, r)$, $(\mathrm{H}_{\mathbf{x}_0,t_0})$ *holds and, for* $\mathbf{x} : I_{t_0,\mathbf{x}_0} \to B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ *the maximal solution of the system* (4.1_θ) *with* $\theta = t_0$ *and satisfying* $\mathbf{x}(t_0) = \mathbf{x}_0$*, one has that*

$$V_g'(t, \mathbf{x}(t)) \leq -v(t)\phi(\|\mathbf{x}(t)\|) \quad \textit{for g-almost all } t \in I_{t_0,\mathbf{x}_0}.$$

(d) $\lim_{t\to+\infty} \inf_{t_0\in[0,\infty)} \int_{[t_0,t_0+t)} v(s)\, d\mu_g(s) = +\infty$.

Then, the trivial solution $\mathbf{x} = \mathbf{0}$ *of the system* (4.1_θ) *with* $\theta = 0$ *is uniformly asymptotically stable.*

Proof. By (2) of Theorem 4.7, the trivial solution $\mathbf{x} = \mathbf{0}$ is uniformly stable. Thus, let us choose $\delta_0 \in (0, r)$ associated to an $\varepsilon_0 \leq r$. Let $\varepsilon > 0$. Again, by uniform stability, there exists $\delta \in (0, r)$ such that, for all $\hat{t} \in [0,\infty)$ and every $\hat{\mathbf{x}}_0$ such that $\|\hat{\mathbf{x}}_0\| < \delta$, one has

$$\|\hat{\mathbf{x}}(t, \hat{t}, \hat{\mathbf{x}}_0)\| < \varepsilon \quad \text{for all } t \in [\hat{t}, \infty) \cap I_{\hat{t},\hat{\mathbf{x}}_0}.$$

Let M be as defined in (4.5). Since

$$\lim_{t\to+\infty} \inf_{t_0\in[0,\infty)} \int_{[t_0,t_0+t)} v(s)\, d\mu_g(s) = \infty,$$

we can choose $\sigma > 0$ such that

$$\int_{[t_0,t_0+\sigma)} v(s)\, d\mu_g(s) > \frac{b(\delta_0)}{M} \quad \text{for all } t_0 \in [0,\infty).$$

Let $(t_0, \mathbf{x}_0) \in [0,\infty) \times B_{\mathbb{R}^n}(\mathbf{0}, \delta_0)$ and $\mathbf{x} : I_{t_0,\mathbf{x}_0} \to B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ a maximal solution of (4.1_θ) with $\theta = t_0$ and satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$.

If there exists $\hat{t} \in [t_0, t_0 + \sigma] \cap I_{t_0,\mathbf{x}_0}$ such that $\|\mathbf{x}(\hat{t})\| < \delta$, then

$$\|\hat{\mathbf{x}}(t)\| < \varepsilon \quad \text{for all } t \in [\hat{t}, \infty) \cap I_{\hat{t},\mathbf{x}(\hat{t})},$$

where $\hat{\mathbf{x}} : I_{\hat{t},\mathbf{x}(\hat{t})} \to B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ is the maximal solution of (4.1_θ) with $\theta = \hat{t}$ satisfying the initial condition $\hat{\mathbf{x}}(\hat{t}) = \mathbf{x}(\hat{t})$. By the uniqueness of the maximal solution, one has

$$\omega(t_0, \mathbf{x}_0) = \omega(\hat{t}, \mathbf{x}(\hat{t})) \quad \text{and} \quad \mathbf{x}(t) = \hat{\mathbf{x}}(t) \quad \text{for all } t \in [\hat{t}, \infty) \cap I_{t_0,\mathbf{x}_0}.$$

Hence,

$$\|\mathbf{x}(t)\| < \varepsilon \quad \text{for all } t \in [t_0 + \sigma, \infty) \cap I_{t_0,\mathbf{x}_0}.$$

On the other hand, if $\|\mathbf{x}(t)\| \geq \delta$ for all $t \in [t_0, t_0+\sigma]$, then, using Conditions (a), (c), Theorem 2.7, and (4.5), we obtain

$$\begin{aligned} a(\|\mathbf{x}(t_0+\sigma)\|) &\leq V(t_0+\sigma, \mathbf{x}(t_0+\sigma)) \\ &= V(t_0, \mathbf{x}(t_0, t_0, \mathbf{x}_0)) + \int_{[t_0, t_0+\sigma)} V'_g(s, \mathbf{x}(s))\, d\mu_g(s) \\ &\leq V(t_0, \mathbf{x}_0) - \int_{[t_0, t_0+\sigma)} v(s)\phi(\|\mathbf{x}(s)\|)\, d\mu_g(s) \\ &\leq V(t_0, \mathbf{x}_0) - M \int_{[t_0, t_0+\sigma)} v(s)\, d\mu_g(s) \\ &< b(\|\mathbf{x}_0\|) - b(\|\mathbf{x}_0\|) \\ &= 0. \end{aligned}$$

This is a contradiction. Hence, we conclude that $\mathbf{x} = \mathbf{0}$ is uniformly asymptotically stable. □

In the next example, we provide an application of Theorem 4.11 for a system subject to a derivator having an infinite number of discontinuities and for which the trivial solution $\mathbf{x} = \mathbf{0}$ is uniformly asymptotically stable.

Example 4.12. Let us consider the dynamical system

$$x'_g(t) = f(t, x(t)) \quad \text{for } g\text{-almost all } t \geq \theta \geq 0, \tag{4.9$_\theta$}$$

where $g : \mathbb{R} \to \mathbb{R}$ is given in (4.8), and $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is the function defined by

$$f(t,x) = \begin{cases} -t\arctan(x) & \text{if } t \in [0,\infty) \setminus D_g, \\ \nu_k x & \text{if } t = t_k,\ k \in \mathbb{N}, \end{cases}$$

where $\{\nu_k\}_k \subset \mathbb{R}^*_+$ is a sequence satisfying

$$\lim_{k\to\infty} \frac{1}{\prod_{i=1}^k (1+\nu_i)^2} = a_0 > 0.$$

The map f satisfies conditions of Theorem 3.13. Thus, for every $(t_0, x_0) \in [0, \infty) \times \mathbb{R}$, the problem (4.9$_\theta$) with $\theta = t_0$ has a maximal solution $x : [t_0, +\infty) \to \mathbb{R}$ satisfying $x(t_0) = x_0$. Observe that $x = 0$ is an equilibrium of the Stieltjes dynamical system (4.9$_\theta$) with $\theta = 0$.

Let us define the function $V : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ by

$$V(t,x) = \begin{cases} x^2 & \text{if } t \in [0, t_1], \\ \dfrac{x^2}{\prod_{i=1}^k (1+\nu_i)^2} & \text{if } t \in (t_k, t_{k+1}],\ k \in \mathbb{N}. \end{cases}$$

Clearly $V \in \mathcal{V}^1_g$, and $a(|x|) \leq V(t,x) \leq b(|x|)$ for all $(t,x) \in [0,\infty) \times \mathbb{R}$, where $a, b \in \mathcal{K}$ are functions defined by $a(s) = a_0 s^2$ and $b(s) = s^2$ for all $s \in [0, \infty)$. In addition, for all $(t,x) \in [0,\infty) \times \mathbb{R}$:

$$\begin{aligned} \frac{\partial V}{\partial_g t}(t,x) &= \begin{cases} 0 & \text{if } t \in [0,t_1] \cup (t_k, t_{k+1}),\ k \in \mathbb{N}, \\ \dfrac{\frac{x^2}{\prod_{i=1}^k (1+\nu_i)^2} - \frac{x^2}{\prod_{i=1}^{k-1} (1+\nu_i)^2}}{g(t_k^+) - g(t_k)} & \text{if } t = t_k,\ k \in \mathbb{N}, \end{cases} \\ &= \begin{cases} 0 & \text{if } t \in [0,t_1) \cup (t_k, t_{k+1}),\ k \in \mathbb{N}, \\ \dfrac{1-(1+\nu_k)^2}{\prod_{i=1}^k (1+\nu_i)^2} x^2 & \text{if } t = t_k,\ k \in \mathbb{N}. \end{cases} \end{aligned}$$

For g-almost every $t \in [t_0, \infty) \setminus D_g$, we have that $t \in [0, t_1)$ or there exists $k \in \mathbb{N}$ such that $t \in (t_k, t_{k+1})$. Thus, by means of Proposition 2.10, we obtain if $t \in [0, t_1)$,

$$\begin{aligned} V'_g(t, x(t)) &= \frac{\partial V}{\partial_g t}(t, x(t)) + \frac{\partial V}{\partial x}(t, x(t)) f(t, x(t)) \\ &= -2x(t)t \arctan(x(t)) \\ &\leq -2t \frac{x(t)^2}{1 + x(t)^2}, \end{aligned}$$

where the last inequality follows from the Mean Value Theorem. While if there exists $k \in \mathbb{N}$ such that $t \in (t_k, t_{k+1})$, then

$$\begin{aligned} V'_g(t, x(t)) &= \frac{\partial V}{\partial_g t}(t, x(t)) + \frac{\partial V}{\partial x}(t, x(t)) f(t, x(t)) \\ &= \frac{-2x(t)}{\prod_{i=1}^k (1 + \nu_i)^2} t \arctan(x(t)) \\ &\leq \frac{-2t x(t)^2}{(1 + x(t)^2) \left(\prod_{i=1}^k (1 + \nu_i)^2 \right)}. \end{aligned}$$

For $t_k \in [t_0, \infty) \cap D_g$, we have that

$$\begin{aligned} V'_g(t_k, x(t_k)) &= \frac{V(t_k^+, x(t_k^+)) - V(t_k, x(t_k))}{g(t_k^+) - g(t_k)} \\ &= \frac{V(t_k^+, x(t_k) + \mu_g(\{t_k\}) f(t_k, x(t_k))) - V(t_k, x(t_k))}{g(t_k^+) - g(t_k)} \\ &= \frac{\frac{(1+\nu_k)^2 x(t_k)^2}{\prod_{i=1}^k (1+\nu_i)^2} - \frac{x(t_k)^2}{\prod_{i=1}^{k-1} (1+\nu_i)^2}}{g(t_k^+) - g(t_k)} \\ &= 0. \end{aligned}$$

Therefore, we conclude that

$$V'_g(t, x(t)) \leq -v(t) \phi(|x(t)|) \quad \text{for } g\text{-almost all } t \geq t_0,$$

where $v : [0, \infty) \to [0, \infty)$ is the function defined for every $t \in [0, \infty)$ by

$$v(t) = \begin{cases} 2t & \text{if } t \in [0, t_1), \\ 0 & \text{if } t = t_k,\ k \in \mathbb{N}, \\ \dfrac{2t}{\prod_{i=1}^k (1 + \nu_i)^2} & \text{if } t \in (t_k, t_{k+1}),\ k \in \mathbb{N}, \end{cases}$$

and $\phi \in \mathcal{K}$ the function given by $\phi(y) = y^2/(1 + y^2)$ for all $y \in [0, \infty)$.

Observe that the function v satisfies

$$\begin{aligned} \lim_{t \to +\infty} \inf_{t_0 \in [0,\infty)} \int_{[t_0, t_0 + t)} v(s)\, d\mu_g(s) &\geq \lim_{t \to +\infty} \inf_{t_0 \in [0,\infty)} \int_{t_0}^{t_0 + t} a_0 2s\, ds \\ &= a_0 \lim_{t \to +\infty} \inf_{t_0 \in [0,\infty)} 2t t_0 + t^2 \\ &= \infty. \end{aligned}$$

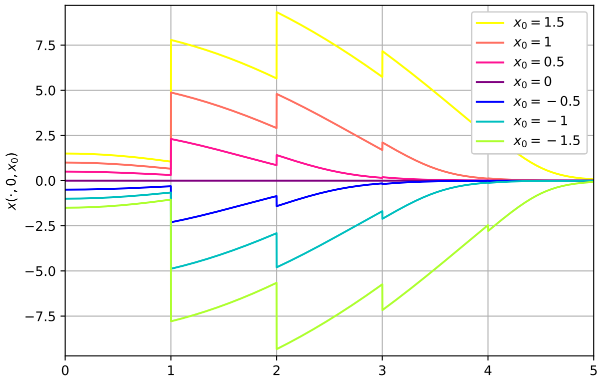


Figure 4.4: Asymptotic behavior of solutions of the dynamical system (4.9_θ) with $\theta = 0$ and $D_g = \mathbb{N}$, obtained with a time-discretization step-size of 10^{-3}, with $\nu_k = e^{\frac{2}{k^2}} - 1$ for all $k \in \mathbb{N}$.

By means of Theorem 4.11, we deduce that $x = 0$ is uniformly asymptotically stable equilibrium. Figure 4.4 illustrates the asymptotic behavior of solutions of the dynamical system (4.9_θ) with $\theta = 0$ and $D_g = \mathbb{N}$.

Remark 4.13. Observe that, in Example 4.9, the function v satisfies

$$\lim_{t \to +\infty} \inf_{t_0 \in [0,\infty)} \int_{[t_0, t_0+t)} v(s)\, d\mu_g(s) = 0 \neq \infty.$$

Thus, Condition (d) of Theorem 4.11 does not hold. Consequently, uniform asymptotic stability cannot be deduced.

In the classical case where $g \equiv \mathrm{id}_{\mathbb{R}}$, corollary results [13, Theorem 4.2] and [23] are well-known when Conditions (b) of Theorem 4.8 is replaced by $V_g'(t, \mathbf{x}(t))$ being negative definite along each maximal solution $\mathbf{x}$ for every $t \geq t_0$. However, to present an analogous statement, we require an additional assumption to avoid the case when $\lim_{t\to\infty} g(t) = l < \infty$, and in particular, when there exists $T \geq 0$ such that $(T, \infty) \subset C_g$. Example 4.4 provides an interesting illustration of attractivity lack for asymptotic stability.

Corollary 4.14. *Assume that g is not bounded from above and Conditions ($\mathrm{H}_{\Omega,0}$) and ($\mathrm{H}_{\mathbf{f},0}$) hold. If there exist $V : [0, \infty) \times B_{\mathbb{R}^n}(\mathbf{0}, r_0) \to \mathbb{R}$; $V \in \mathcal{V}_1^g$ and $a, b, \phi \in \mathcal{K}$ such that*

(a) *$a(\|\mathbf{u}\|) \leq V(t, \mathbf{u}) \leq b(\|\mathbf{u}\|)$, for every $(t, \mathbf{u}) \in [0, \infty) \times B_{\mathbb{R}^n}(\mathbf{0}, r_0)$;*

(b) *there exists $r \in (0, r_0]$ such that, for every $(t_0, \mathbf{x}_0) \in [0, \infty) \times B_{\mathbb{R}^n}(\mathbf{0}, r)$, ($\mathrm{H}_{\mathbf{x}_0, t_0}$) holds and, for $\mathbf{x} : I_{t_0, \mathbf{x}_0} \to B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ the maximal solution of the system (4.1_θ) with $\theta = t_0$ and satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$, one has that*

$$V_g'(t, \mathbf{x}(t)) \leq -\phi(\|\mathbf{x}(t)\|), \quad \text{for } g\text{-almost all } t \geq t_0.$$

Then, the trivial solution $\mathbf{x} = \mathbf{0}$ of the system (4.1_θ) with $\theta = 0$ is asymptotically stable. Furthermore, if

$$\lim_{t \to +\infty} \inf_{t_0 \in [0,\infty)} g(t + t_0) - g(t_0) = \infty,$$

then, the trivial solution $\mathbf{x} = \mathbf{0}$ of the system (4.1_θ) with $\theta = 0$ is uniformly asymptotically stable.

Proof. Observe that Conditions of Theorem 4.8 hold for $v \equiv 1$ as

$$\inf_{t_0\in[0,\infty)} \lim_{t\to+\infty} \int_{[t_0,t+t_0)} \omega(s)\,d\mu_g(s) = \inf_{t_0\in[0,\infty)} \lim_{t\to+\infty} g(t+t_0) - g(t_0) = \infty.$$

Hence, the trivial solution is $\mathbf{x} = \mathbf{0}$ of the system (4.1_θ) with $\theta = 0$ is asymptotically stable. Moreover, if

$$\lim_{t\to+\infty} \inf_{t_0\in[0,\infty)} g(t+t_0) - g(t_0) = \infty, \tag{4.10}$$

then, Theorem 4.11 ensures that $\mathbf{x} = \mathbf{0}$ is uniformly asymptotically stable. □

Remark 4.15. In Corollary 4.14, it is worth mentioning that the assumption that the derivator g is not bounded from above does not necessarily imply that (4.10) holds. Indeed, let us consider the increasing sequence $\{t_n\}_{n\in\mathbb{N}}$ defined by

$$\begin{cases} t_n = 2t_{n-1} - t_{n-2} + 1, & n \in \mathbb{N},\, n \geq 3, \\ t_2 = 3,\ t_1 = 1. \end{cases}$$

Now, consider the derivator $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(t) = \begin{cases} t & \text{if } t \leq 0, \\ 0 & \text{if } t \in [0, t_1], \\ n & \text{if } t \in (t_n, t_{n+1}],\ n \in \mathbb{N}. \end{cases}$$

Observe that

$$D_g = \{t_n\}_{n\in\mathbb{N}} = \{1, 3, 6, 10, 15, \dots\} \quad \text{and} \quad C_g = (0, t_1) \cup \bigcup_{n\in\mathbb{N}} (t_n, t_{n+1}).$$

Notice that for every $t > 0$, there exists $t_0 \in [0, \infty)$ such that $[t_0, t + t_0] \subset C_g$. Thus,

$$\lim_{t\to+\infty} \inf_{t_0\in[0,\infty)} g(t+t_0) - g(t_0) = 0 \neq \infty.$$

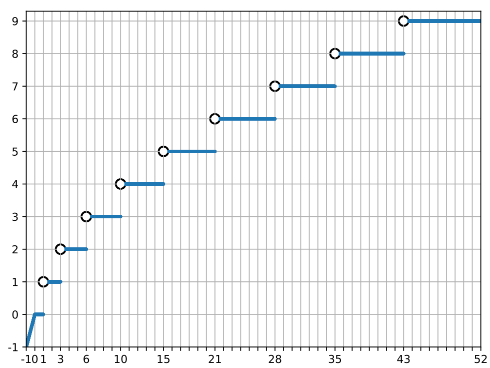


Figure 4.5: Graph of the derivator g in Remark 4.15.

5 Applications to dynamics of population

5.1 Stable equilibrium of a population subject to train vibrations

In this subsection, by means of a system of Stieltjes differential equations, we study the long-term impact of high-speed train vibrations and noise pollution on a population of animals living near railways. Depending on the species and their sensitivity to vibrations, various implications can be observed, we mention for instance:

- Hearing damage resulting in from the significant noise and vibrations that can potentially harm animals with sensitive hearing such as certain small mammals, birds, and bats which rely heavily on their hearing for communication, navigation, detection of predators, and finding food, thus, prolonged exposure to train vibrations may lead to hearing impairment or damage, disrupting their normal behaviors and increasing their death rate.

- Increased stress levels since some animals may be startled by the vibrations. This can affect their feeding patterns, reduce their reproduction rate, or lead to emigration resulting in a loss of suitable habitat and altering the composition and diversity of the local ecosystem

- Ecological interactions disruption which can affect pollination for instance if the vibrations deter insects that are important pollinators, disturb ground-dwelling organisms (insects, reptiles, and small mammals. . .) which can implicitly impact other species that rely on them as a food source.

In particular, an Allee effect can be observed in this regard, especially when the survival of the population depends on a minimum threshold size $M > 0$. In the follows, we denote by $x(t)$ the number of individuals of a population living in a region near a railway with a carrying capacity $K > M$.

Let us assume that a certain number $m > 0$ of trains pass through the area every day. We refer to $\{\delta_i\}_{i=1}^{i=m}$ as the moments when trains pass in a single day. Once a train pass by the area, its impact is significant for a proportion of individuals living near the railway. They may experience vibrations or direct injuries. In the following analysis, we use a Stieltjes differential equation to model the dynamics of this population affected by train vibrations, and we study the asymptotic behavior of its solutions. In doing so, we require a derivator $g : \mathbb{R} \to \mathbb{R}$ presenting discontinuities for $t \in \{\delta_i\}_{i=1}^{i=m} + 24\mathbb{N}$ such that $\mu_g(\{t\})$ quantifies the rate at which the risk of damage varies. Depending on the specific δ_i, this rate can either increase or decrease, reflecting the varying impact of vibrations during daylight and nighttime hours. For simplicity, we can take for instance:

$$g(t) = t + \sum_{k\in\mathbb{N}} \sum_{i=1}^{m} \chi_{[\delta_i+24k,\infty)}(t) \quad \text{for all } t \in \mathbb{R},$$

with $\mu_g(\{\cdot\}) \equiv 1$ on $D_g = \{\delta_i : i = 1, \dots, m\} + 24\mathbb{N}$.

In the sequel, we suggest to analyse the asymptotic behaviour of the dynamics of this population, through the study of asymptotic stability of the zero equilibrium of the Stieltjes dynamical system:

$$x'_g(t) = f(t, x(t)) \quad \text{for } g\text{-almost every } t \geq \theta \geq 0, \tag{5.1$_\theta$}$$

where $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f(t, x) = \begin{cases} \rho x \left(1 - \dfrac{x}{K}\right)\left(\dfrac{x}{M} - 1\right), & \text{if } t \notin \{\delta_i + 24k\}_{i=1}^{m},\ k = 0, 1, 2 \ldots \\ -dx, & \text{if } t \in \{\delta_i + 24k\}_{i=1}^{m},\ k = 0, 1, 2 \ldots \end{cases} \tag{5.2}$$

The parameters of the model can be understood as:

$K > M$**:** the carrying capacity of the environment;

$\rho > 0$**:** the intrinsic rate of reproduction of the population;

$d \in (0, 1)$**:** a constant related to the impact induced by trains, either a migration rate immediately following the passage of trains or a mortality rate for certain populations that live in close proximity to the railway.

To simplify the analysis, we make the assumption that a train passes every hour over a 24-hour period. Thus, we consider

$$g(t) = t + \sum_{n \in \mathbb{N}} \chi_{[n,\infty)}(t) \quad \text{for all } t \in \mathbb{R},$$

which is such that $D_g = \mathbb{N}$ and $\mu_g(\{t\}) = 1$ for all $t \in D_g$.

Now, we will focus on the zero equilibrium to study its local asymptotic stability within a region $(-r_0, r_0)$ for some $r_0 > 0$ that will be determined to enhance the impact of environmental factors threatening this population.

Since the trivial solution $x = 0$ is an equilibrium of the dynamical system (5.1_θ) with $\theta = 0$, let us consider $r_0 \leq M$. For all $(t_0, x_0) \in ([0, \infty) \cap D_g) \times (-r_0, r_0)$, if x is a solution of (5.1_θ) with $\theta = t_0$ and satisfying $x(t_0) = x_0$, then

$$x_0 + \mu_g(\{t_0\}) f(t_0, x_0) = (1 - d)x_0 \in (-r_0, r_0).$$

Thus, (H_{x_0,t_0}) holds for $r = r_0$. One can also verify that f satisfies ($H_{f,0}$) and ($H_{\Omega,0}$). Hence, Theorem 3.7, yields the existence of a maximal solution $x : I_{t_0,x_0} \to \mathbb{R}$ of (5.1_θ) with $\theta = t_0$ and satisfying $x(t_0) = x_0$. Let us show that $\omega(t_0, x_0) = \infty$. First of all, let $\tau \in (t_0, \omega(t_0, x_0))$ be such that $x : [t_0, \tau] \to (-r_0, r_0)$. Observe that, if $x(\tau) = 0$, by uniqueness of the solution, we deduce that $x \equiv 0$ which lies in $(-r_0, r_0)$. Thus, two other cases occur when $x(\tau) \neq 0$.

Case 1: If $\tau \in D_g$, then $x(\tau^+) = x(\tau) + \mu_g(\{\tau\}) f(\tau, x(\tau)) = (1 - d)x(\tau) \in (-r_0, r_0)$.

Case 2: If $\tau \notin D_g$, and if we assume that $x(\tau) \in (0, r_0)$, then $x'_g(\tau) = f(\tau, x(\tau)) < 0$. Thus, by g-continuity of x at τ, there exists $\tau_1 \in (\tau, \omega(t_0, x_0))$ such that

$$x : [t_0, \tau_1] \to (0, x(\tau)] \subset (0, r_0).$$

Similarly, if $x(\tau) \in (-r_0, 0)$ then $x'_g(\tau) = f(\tau, x(\tau)) > 0$. Thus, there exists $\tau_2 > 0$ such that

$$x : [\tau, \tau_2] \to [x(\tau), 0) \subset (-r_0, 0).$$

Repeating the same argument, we deduce that

$$x(t) \in [-\lambda_{x_0,t_0}, \lambda_{x_0,t_0}] \subset (-r_0, r_0) \quad \text{for all } t \in I_{t_0,x_0}$$

where

$$\lambda_{x_0,t_0} = \sup_{t \in [t_0,\tau]} |x(t, t_0, x_0)|.$$

By Corollary 3.10, we deduce that $\omega(t_0, x_0) = \infty$.

Now, we define the function $V : [0,\infty) \times (-r_0, r_0) \to \mathbb{R}$ by

$$V(t,x) = x^2 \quad \text{for all } (t,x) \in [0,\infty) \times (-r_0, r_0).$$

Clearly, $V \in \mathcal{V}_g^1$. For all $(t_0, x_0) \in [0,\infty) \times (-r_0, r_0)$, let $x : [t_0, \infty) \to (-r_0, r_0)$ be the maximal solution of (5.1_θ) with $\theta = t_0$ and satisfying $x(t_0) = x_0$. Thus, using Proposition 2.10, we obtain, for $t_0 \in [0,\infty)$ and for g-almost all $t \in [t_0,\infty) \setminus D_g$,

$$\begin{aligned} V_g'(t,x(t)) &= \frac{\partial V}{\partial_g t}(t,x(t)) + \frac{\partial V}{\partial x}(t,x(t)) x_g'(t) \\ &= \frac{\partial V}{\partial_g t}(t,x(t)) + \frac{\partial V}{\partial x}(t,x(t)) f(t,x(t)) \\ &= 2\rho x(t)^2 \left(1 - \frac{x(t)}{K}\right)\left(\frac{x(t)}{M} - 1\right). \end{aligned}$$

While for $t \in [t_0,\infty) \cap D_g$, we obtain

$$\begin{aligned} V_g'(t,x(t)) &= \frac{V(t^+, x(t^+)) - V(t,x(t))}{g(t^+) - g(t)} \\ &= \frac{V(t^+, x(t) + \mu_g(\{t\}) f(t,x(t))) - V(t,x(t))}{g(t^+) - g(t)} \\ &= (1-d)^2 x(t)^2 - x(t)^2 \\ &= (-2d + d^2) x(t)^2. \end{aligned}$$

As $(-2d + d^2) < 0$, it follows that $V_g'(t,x(t))$ is negative definite. Since

$$a(|x|) \le V(t,x) \le b(|x|) \quad \text{for all } (t,x) \in [0,\infty) \times (-r_0, r_0),$$

with $a, b \in \mathcal{K}$ defined by $a(s) = b(s) = s^2$ for all $s \in [0,\infty)$, Corollary 4.14 ensures that the trivial solution $x = 0$ is uniformly asymptotically stable.

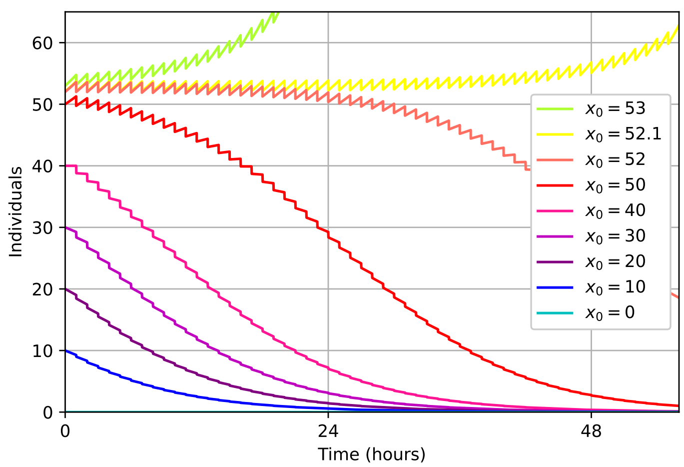


Figure 5.1: Asymptotic behavior of solutions of the dynamical system (5.1_θ) with $\theta = 0$ obtained with a time-discretization step-size of 10^{-3}, with $\rho = 0.1$, $K = 1000$, $M = 40$, and $d = 0.03$.

The asymptotic behavior of solutions is illustrated in Figure 5.1. It illustrates the long-term effect of the high-speed trains vibrations. The outcome raises alarm about the overall stability and the persistence of the whole ecosystem in this region. Specifically, the impact of the vibrations shifts the minimum threshold size M to a new threshold $M' > M$, as shown in Figure 5.1 (where we observe that $M' = 53$ individuals whereas $M = 40$ individuals). This shift indicates a heightened likelihood of population extinction if there is no measure ensuring that the initial population exceeds M' at the onset of trains activity.

5.2 Stable equilibrium of bacteria–ammonia dynamics

Cyanobacteria, similar to plants, participate in oxygenic photosynthesis as they are photosynthetic bacteria. *Photosynthesis* is the biochemical process through which organisms convert light energy into chemical energy in the form of glucose or other organic compounds resulting oxygen release. The energy captured is then used to fuel the synthesis of organic molecules such as glucose, serving as an energy source for the bacteria.

Cyanobacteria are found in diverse habitats, including freshwater, marine environments, and terrestrial ecosystems. In this subsection, we consider a species of cyanobacteria that has also the ability to fix nitrogen such as *Anabaena cyanobacteria*. Through nitrogen fixation, atmospheric nitrogen gas (N_2) is converted into a form that can be used by plants and other organisms by means the enzyme *nitrogenase*. This enzyme catalyzes the conversion of nitrogen gas (N_2) into ammonium ions (NH_4^+) based on a considerable amount of energy obtained from photosynthesis, these ammonium ions are assimilated into amino acids and proteins, which are essential for the growth and survival of Cyanobacteria. As a by product of nitrogen fixation, ammonia gas (NH_3) can be released. Some of the assimilated ammonium ions (NH_4^+) are released back into the environment providing neighboring vegetative cells with a source of nitrogen.

To avoid losing valuable nitrogen nutrients and optimizing nitrogen utilization efficiency, this population has mechanisms allowing ammonium ions (NH_4^+) reabsorption, and ammonia gas (NH_3) assimilation through converting ammonia gas (NH_3) into ammonium ions (NH_4^+). In what follows, the term "ammonia" refers to both the protonated and unprotonated forms, which are denoted as (NH_3) and (NH_4^+) respectively [4]. It is worth mentioning that ammonia is commonly used in cleaning products, fertilizers. Beyond that, ammonia's cooling properties make it an essential refrigerant in air conditioning systems and refrigerators.

To optimize resource utilization and adapt to the varying environmental conditions, the population undergoes a day-night cycling of nitrogen fixation and carbon consumption [32]. This is due to the sensitivity of the nitrogenase enzyme responsible for nitrogen fixation to oxygen. Thus, nitrogen fixation is carried out during daylight hours when the photosynthesis can provide the necessary energy and oxygen levels are relatively low. During the nights, since oxygen levels within the cells would be higher, this population of cyanobacteria reduces their metabolic activity and growth and relies on stored carbon compounds to fulfill their energy needs. Our objective in the sequel is to observe the dynamics of this population which thrives in the presence of ammonia in a culture room without exposure to artificial light, tracking the levels of the ammonia during this process. Here, we assume that the carbon dioxide (CO_2), the nitrogen gas (N_2) and nutrients supply are maintained steady as well as the PH level. Since the population undergoes dormancy phases during the nights, we identify the days with intervals of the form $[2k, 2k+1]$, $k = 0, 1, 2, \ldots$, and the night with intervals $[2k+1, 2k+2]$, $k = 0, 1, 2, \ldots$ In our example, we differentiate with respect to a derivator g whose slopes describe the intensity of light, which is necessary for the photosynthesis process. We require that g presents smaller slopes at the beginning and at the end of the daylight hours, with a maximal slope of 1 representing the peak light intensity at middays where $t = 2k + 1/2$, and remains constant during the dormancy phases in the night $[2k + 1, 2k + 2]$, $k = 0, 1, 2, \dots$ For instance, we consider $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(t) = \begin{cases} \dfrac{\sin(\pi(t - 1/2)) + 1}{\pi} & \text{if } t \in [0, 1] \\ 2/\pi & \text{if } t \in (1, 2], \end{cases}$$

and $g(t) = 2/\pi + g(t - 2)$ for $t \geq 2$.

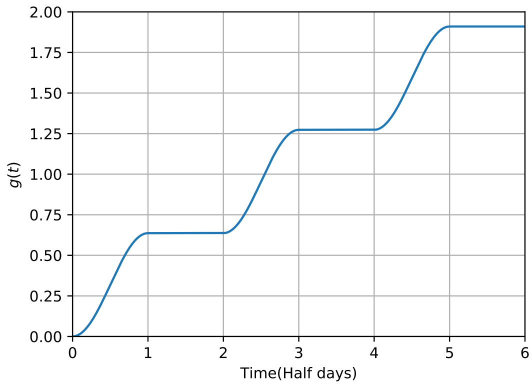


Figure 5.2: Graph of the derivator g.

We denote by $N(t)$ the biomass of cyanobacteria (grams per liter), and $A(t)$ the ammonia concentration in the environment at time $t \geq 0$. Since the population thrives in the presence of ammonia, we can assume that the growth rate is proportional to the level of ammonia. We denote ρ the maximal intrinsic coefficient of reproduction that the population can reach in the presence of one unit of ammonia with maximal sunlight intensity. Thus, the dynamics can be modeled using the autonomous system of Stieltjes differential equations:

$$\mathbf{u}_g'(t) = \mathbf{F}(\mathbf{u}(t)), \quad \text{for } g\text{-almost every } t \geq \theta \geq 0, \tag{5.3$_\theta$}$$

where $\mathbf{u} = (N, A)$ and $\mathbf{F} = (F_1, F_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\mathbf{F}(N, A) = \left(\rho AN \left(1 - \frac{N}{K}\right), (\alpha N - \beta AN)\right), \tag{5.4}$$

and where the parameters of the model can be understood as:

$K > 0$: the carrying capacity of the culture room, which forms a spacial constraint for growth;

$\alpha > 0$: a constant related to the production of ammonia through nitrogen fixation.

$\beta > 0$: a constant related to the proportion of the reabsorption of ammonia by the population depending on the level of ammonia in the environment.

Observe that $\mathbf{u}^* = (K, \alpha/\beta)$ is an equilibrium of the system (5.3$_\theta$) with $\theta = 0$ among other equilibria. Its asymptotic stability would guarantee the persistence of the population with nonzero ammonia production. Therefore, we study local asymptotic stability in a neighborhood $B_{\mathbb{R}^2}(\mathbf{u}^*, r_0)$ for some $r_0 > 0$ of this equilibrium. To this aim, we translate our study in a neighborhood of $\mathbf{x} = \mathbf{0} = (0,0) \in \mathbb{R}^2$ with the change of variables $\mathbf{x} = (x_1, x_2) := \mathbf{u} - \mathbf{u}^*$, we obtain the system

$$\mathbf{x}'_g(t) = \mathbf{f}(\mathbf{x}(t)) \quad \text{for } g\text{-almost every } t \geq \theta \geq 0, \tag{5.5$_\theta$}$$

where $\mathbf{f} = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\mathbf{f}(x_1, x_2) = \left(-\rho \frac{x_1}{K}\left(x_2 + \frac{\alpha}{\beta}\right)(x_1 + K), -\beta x_2(x_1 + K)\right). \tag{5.6}$$

So, $\mathbf{x} = \mathbf{0} = (0,0) \in \mathbb{R}^2$ is an equilibrium of the dynamical system (5.5$_\theta$) with $\theta = 0$. Next, we prove that $\mathbf{x} = \mathbf{0}$ is asymptotically stable, and hence, $\mathbf{u}^* = \big(K, \alpha/\beta\big)$ is an asymptotically stable equilibrium of the system (5.3$_\theta$) with $\theta = 0$.

Let us consider $r_0 = \min\{K, \alpha/\beta\}$, and let $(t_0, \mathbf{x}_0) \in [0,\infty) \times B_{\mathbb{R}^2}(\mathbf{0}, r_0)$. Arguing as in the previous subsection, since the function $\mathbf{f}$ satisfies conditions of Theorem 3.7, there exists, $\mathbf{x} = (x_1, x_2) : I_{t_0,\mathbf{x}_0} \to \mathbb{R}^2$, the maximal solution of the system (5.5$_\theta$) with $\theta = t_0$ and satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$. Let $\tau \in (0, \omega(t_0, \mathbf{x}_0)) \setminus C_g$ be such that $\mathbf{x} = (x_1, x_2) : [t_0, \tau] \to B_{\mathbb{R}^2}(\mathbf{0}, r_0)$. In what follows, we analyze the possible cases:

Case 1: If $x_1(\tau), x_2(\tau) \in (0, r_0)$ (resp. $x_1(\tau), x_2(\tau) \in (-r_0, 0)$), then, for $i = 1,2$, we have $(x_i)'_g(\tau) = f_i(\mathbf{x}(\tau)) < 0$ (resp. $(x_i)'_g(\tau) = f_i(\mathbf{x}(\tau)) > 0$). Therefore, by the g-continuity of $\mathbf{x}$ at τ, there exists $\tau_1 \in (\tau, \omega(t_0, \mathbf{x}_0))$ (τ_1 can be chosen such that $\tau_1 \notin C_g$) such that

$$x_i : [t_0, \tau_1] \to (0, x_i(\tau)] \subset (0, r_0) \quad (\text{resp. } x_i : [t_0, \tau_1] \to [x_i(t_0), 0) \subset (-r_0, 0)).$$

Case 2: If $x_1(\tau) \in (0, r_0)$ and $x_2(\tau) \in (-r_0, 0)$, then

$$(x_1)'_g(\tau) = f_1(\mathbf{x}(\tau)) < 0 \quad \text{and} \quad (x_2)'_g(\tau) = f_2(\mathbf{x}(\tau)) > 0.$$

Therefore, there exists $\tau_2 \in (\tau, \omega(t_0, \mathbf{x}_0)) \setminus C_g$ such that

$$x_1 : [\tau, \tau_2] \to (0, x_1(\tau)] \subset (0, r_0) \quad \text{and} \quad x_2 : [\tau, \tau_2] \to [x_2(\tau), 0) \subset (-r_0, 0).$$

Case 3: If $x_1(\tau) \in (-r_0, 0)$ and $x_2(\tau) \in (0, r_0)$, then, similarly to Case 2, we deduce that there exists $\tau_3 \in (\tau, \omega(t_0, \mathbf{x}_0))$ such that

$$x_1 : [\tau, \tau_3] \to [x_1(\tau), 0) \subset (-r_0, 0) \quad \text{and} \quad x_2 : [\tau, \tau_3] \to (0, x_2(\tau)] \subset (0, r_0).$$

From this argument, we deduce that the solution

$$\mathbf{x}(t) \in [-\lambda_{\mathbf{x}_{0,1},t_0}, \lambda_{\mathbf{x}_{0,1},t_0}] \times [-\lambda_{\mathbf{x}_{0,2},t_0}, \lambda_{\mathbf{x}_{0,2},t_0}] \subset B_{\mathbb{R}^2}(\mathbf{0}, r_0) \quad \text{for all } t \in I_{t_0,\mathbf{x}_0},$$

where

$$\lambda_{\mathbf{x}_{0,i},t_0} = \sup_{t \in [t_0,\tau]} |x_i(t, t_0, \mathbf{x}_0)| \quad \text{for } i = 1,2, \text{ and } \mathbf{x}_0 = (\mathbf{x}_{0,1}, \mathbf{x}_{0,2}).$$

Using Corollary 3.10, we deduce that $\omega(t_0, \mathbf{x}_0) = \infty$.

Now, let us consider the function $V : [0,\infty) \times B_{\mathbb{R}^2}(\mathbf{0}, r_0) \to \mathbb{R}$ defined by $V(t, \mathbf{x}) = x_1^2 + x_2^2$ for all $t \in [0,\infty)$ and $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. It is clear that $V \in \mathcal{V}_1^g$. Let $(t_0, \mathbf{x}_0) \in [0,\infty) \times B_{\mathbb{R}^2}(\mathbf{0}, r_0)$, and $\mathbf{x} : [t_0, \infty) \to B_{\mathbb{R}^2}(\mathbf{0}, r_0)$ the maximal solution of (5.5$_\theta$) with $\theta = t_0$ and satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$. By means of Proposition 2.10, for g-almost all $t \in [t_0, \infty)$, we obtain

$$\begin{aligned} V_g'(t, \mathbf{x}(t)) &= \frac{\partial V}{\partial_g t}(t, \mathbf{x}(t)) + \sum_{i=1}^{2} \frac{\partial V}{\partial x_i}(t, \mathbf{x}(t))(x_i)_g'(t) \\ &= \frac{\partial V}{\partial_g t}(t, \mathbf{x}(t)) + \sum_{i=1}^{2} \frac{\partial V}{\partial x_i}(t, \mathbf{x}(t)) f_i(t, \mathbf{x}(t)) \\ &= -2\rho \frac{x_1(t)^2}{K}(x_2(t) + \alpha/\beta)(x_1(t) + K) - 2\beta x_2(t)^2 (x_1(t) + K). \end{aligned} \tag{5.7}$$

Thus, $V_g'(t, \mathbf{x}(t))$ is negative definite. Since

$$a(\|\mathbf{z}\|) \leq V(t, \mathbf{z}) \leq b(\|\mathbf{z}\|) \quad \text{for all } (t, \mathbf{z}) \in [0, \infty) \times B_{\mathbb{R}^2}(\mathbf{0}, r_0),$$

where $a, b \in \mathcal{K}$ are defined by $a(s) = s^2$ and $b(s) = 2s^2$ for all $s \in [0, \infty)$. It follows from Corollary 4.14 that $\mathbf{x} = \mathbf{0}$ is uniformly asymptotically stable. Hence, $\mathbf{u}^* = (K, \alpha/\beta)$ is a uniformly asymptotically stable equilibrium of the system (5.3$_\theta$). The graph of asymptotic behavior of solutions near the equilibrium $\mathbf{u}^* = (K, \alpha/\beta)$ is given in Figure 5.3 with $\theta = 0$.

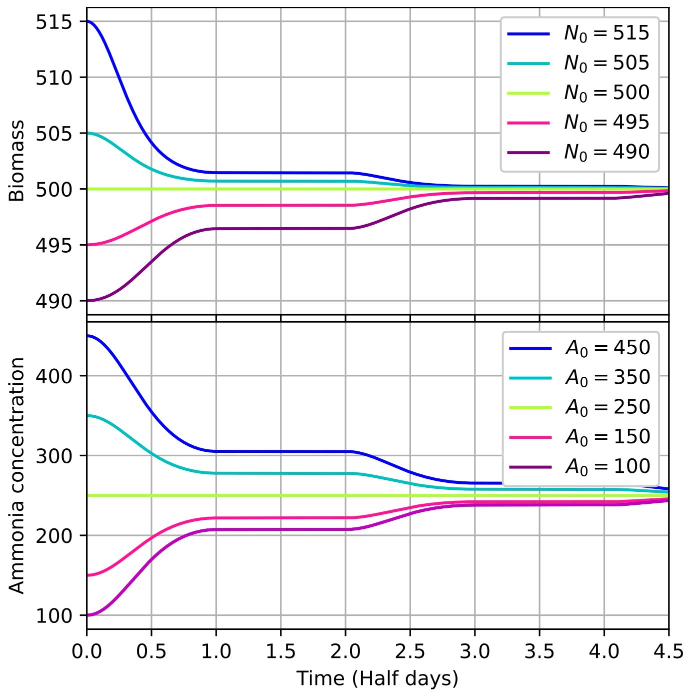


Figure 5.3: Asymptotic behavior of solutions of the dynamical system (5.3$_\theta$) with $\theta = 0$ obtained with a time-discretization step-size of 10^{-3}, with different initial densities N_0 and ammonia concentrations A_0 (g/L), where $\rho = 0.01$, $K = 500$, $\alpha = 1$, and $\beta = 0.004$.

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