

Configurations of quadratic systems possessing three distinct infinite singularities and one or more invariant parabolas

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Received 13 December 2024, appeared 20 October 2025

Communicated by Gabriele Villari

Abstract. Denote by **QS** the class of all non-degenerate planar quadratic differential systems and by **QSP** the subclass of **QS** of all systems possessing at least one invariant parabola. In this paper we consider the subfamily of **QSP** defined by the condition $\eta \neq 0$, which means the presence of three distinct infinite singularities real or complex. We denote this subfamily by $\mathbf{QSP}_{(\eta \neq 0)}$. We investigate all possible configurations of invariant parabolas and invariant straight lines which systems in $\mathbf{QSP}_{(\eta \neq 0)}$ could possess and their geometric properties encoded in such configurations. The classification presented here is taken modulo the action of the group of real affine transformations and time rescaling and it is given in terms of affine invariant polynomials. It yields a total of 144 distinct configurations. The obtained classification is an algorithm which makes it possible for any given real quadratic differential system in $\mathbf{QSP}_{(\eta \neq 0)}$ to specify its configuration of invariant parabolas and straight lines. This work will prove helpful in studying the integrability of the systems in $\mathbf{QSP}_{(\eta \neq 0)}$.

Keywords: quadratic differential systems, algebraic solution, configuration of algebraic solutions, invariant parabolas, affine invariant polynomials, group action.

2020 Mathematics Subject Classification: 34C23, 34A34.

1 Introduction and statement of main results

For every planar differential system of the form

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y), \quad (1.1)$$

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where $P, Q \in \mathbb{R}[x, y]$, that is, P and Q are polynomials in x and y with real coefficients, we associate the vector field

$$X = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}.$$

The *degree* of such a system is defined as the integer $m = \max(\deg P, \deg Q)$. In particular, system (1.1) is called a *quadratic differential system* when $m = 2$; here, **QS** denotes the entire class of real quadratic differential systems. From now on, we assume that P and Q are coprime polynomials. Otherwise, by an appropriate rescaling of time, system (1.1) can be reduced to a linear or constant system. Quadratic differential systems satisfying this coprimality condition are referred to as *non-degenerate quadratic systems*.

Quadratic systems appear in various research fields including models of population dynamics [6], fluid dynamics [9], control systems [11] and even quantum dynamics [2]. They are also of theoretical interest because we have open problems on these systems stated more than a century ago; see for example [1] for a bibliographical survey.

Given $f \in \mathbb{C}[x, y]$, we say that the algebraic curve $f(x, y) = 0$ is an *invariant algebraic curve* of systems (1.1) if there exists $K \in \mathbb{C}[x, y]$ (it is called cofactor of the invariant curve $f = 0$) such that

$$P \frac{\partial f}{\partial x} + Q \frac{\partial f}{\partial y} = Kf.$$

Quadratic systems with invariant algebraic curves have been studied extensively by many authors. For example, Druzhkova [10] (1968) presented necessary and sufficient conditions on the coefficients of a quadratic system, as well as on the coefficients of a conic, for the conic to be an invariant curve of the system. Christopher [7] (1989) provided a normal form for quadratic systems possessing invariant parabolas. Qin Yuan-xun [18] (1996) investigated quadratic systems having an ellipse as a limit cycle. Cairó and Llibre [5] (2002) studied quadratic systems with invariant algebraic conics in the context of Darboux integrability. Schlomiuk and Vulpe [19, 21] (2004, 2008) classified quadratic systems with invariant straight lines of total multiplicity at least four, according to their geometric configurations. Many other works have also contributed to this topic.

The primary objective of this research is to study non-degenerate quadratic systems that possess invariant conics. Irreducible affine conics over the real field $\mathbb{R}$ include hyperbolas, ellipses, and parabolas. These conics can be distinguished by analyzing their behavior at infinity. Specifically, a hyperbola is a real irreducible affine conic with two distinct real points at infinity. A parabola, in contrast, has a single real point at infinity, where the conic meets the line at infinity with multiplicity two. An ellipse, on the other hand, has two complex conjugate points at infinity.

The classifications of **QS** with invariant hyperbolas [14, 15] and with invariant ellipses [13, 16] were obtained in previous works. In this study, we focus on the class **QSP** of non-degenerate quadratic differential systems that possess an invariant parabola. The novel contribution of this work lies in adopting a global approach, employing global tools such as the theory of invariant polynomials for differential systems.

The group of real affine transformations combined with time rescaling acts on the class **QS**. Consequently, modulo this group action, quadratic systems in this class depend on five essential parameters. The same group also acts on the subclass **QSP**, and, modulo the group action, systems in **QSP** depend on at most three parameters. To ensure our study is intrinsic and independent of any particular normal form representation, we employ invariant polynomials and geometric invariants to carry out the desired classification.

In the paper [24], the necessary and sufficient conditions for a non-degenerate quadratic system in **QS** to have invariant parabolas are established. Moreover, [24] presents invariant criteria that determine the number, position, and multiplicity of such parabolas.

The present paper is a continuation of [24]. More precisely, using the conditions from that work, we classify all possible configurations of invariant parabolas and invariant lines that a system in **QSP**_($\eta \neq 0$) may possess. The investigation of configurations within the family **QSP**_($\eta=0$) is currently in progress.

A key concept in this study is the notion of a *configuration of algebraic invariant curves* for a polynomial differential system. This concept was first introduced in [22], with an earlier version focusing solely on invariant lines presented in [19]. Following Darboux's definition, an *algebraic solution* of a polynomial differential system is an algebraic invariant curve defined by an irreducible polynomial over $\mathbb{C}$.

Definition 1.1. A *configuration of invariant algebraic curves* of a real polynomial differential system is defined as a finite set of algebraic invariant curves of the system, each endowed with its multiplicity, together with the real singularities, whether finite or infinite, located on these curves, each also endowed with its multiplicity.

It is worth noting that [8] introduces various notions of multiplicity for an algebraic invariant curve, including infinitesimal, integrable, algebraic, geometric, and holonomic multiplicities. In this work, we adopt the concept of *geometric multiplicity*, defined via perturbations within the family **QS** as follows.

Definition 1.2. An invariant conic

$$\Phi(x, y) = p + qx + ry + sx^2 + 2vxy + uy^2 = 0,$$

with $(s, v, u) \neq (0, 0, 0)$ and parameters $(p, q, r, s, v, u) \in \mathbb{C}^6$, for a quadratic vector field X , is said to have *multiplicity* m if there exists a sequence of real quadratic vector fields $\{X_k\}$ converging to X (under the topology induced by their coefficients on the sphere S^{11}) such that each X_k admits m distinct (complex) invariant conics

$$\Phi_k^1 = 0, \dots, \Phi_k^m = 0,$$

all converging to $\Phi = 0$ as $k \rightarrow \infty$ (under the topology induced by their coefficients on the sphere S^5). Moreover, this property does not hold for $m + 1$. When an invariant conic $\Phi(x, y) = 0$ has multiplicity one, it is called *simple*.

We note that two non-equivalent systems, modulo the group action, may have the “same configuration” of invariant parabolas and straight lines. Therefore, it is necessary to define when two configurations are considered “the same” or equivalent.

Definition 1.3. Suppose we have two systems (S_1) and (S_2) in **QSP**, each with a finite number of singularities (finite or infinite), a finite set of invariant parabolas

$$\mathcal{P}_i : g_i(x, y) = 0, \quad i = 1, \dots, k,$$

of (S_1) (respectively

$$\mathcal{P}'_i : g'_i(x, y) = 0, \quad i = 1, \dots, k,$$

of (S_2)), and a finite set (possibly empty) of invariant straight lines

$$\mathcal{L}_j : f_j(x, y) = 0, \quad j = 1, \dots, k',$$

of (S_1) (respectively

$$\mathcal{L}'_j : f'_j(x, y) = 0, \quad j = 1, \dots, k',$$

of (S_2)). We say that the configurations $\mathcal{C}_1$ and $\mathcal{C}_2$ of parabolas and lines of these systems are *equivalent* if there exist one-to-one correspondences

$$\phi_p : \{\mathcal{P}_i\} \rightarrow \{\mathcal{P}'_i\} \quad \text{and} \quad \phi_l : \{\mathcal{L}_j\} \rightarrow \{\mathcal{L}'_j\}$$

such that:

- (i) ϕ_p and ϕ_l preserve multiplicities of parabolas and lines, and map real invariant curves to real invariant curves, and complex invariant curves to complex invariant curves;
- (ii) for each parabola $\mathcal{P} : g(x, y) = 0$ in $\mathcal{C}_1$ (respectively each line $\mathcal{L} : f(x, y) = 0$), there is a one-to-one correspondence between the real singularities on $\mathcal{P}$ (respectively on $\mathcal{L}$) and the real singularities on $\phi_p(\mathcal{P})$ (respectively on $\phi_l(\mathcal{L})$), preserving both their multiplicities and locations;
- (iii) furthermore, consider the total curves

$$\mathcal{F} : \prod_i G_i(X, Y, Z) \cdot \prod_j F_j(X, Y, Z) \cdot Z = 0,$$

and

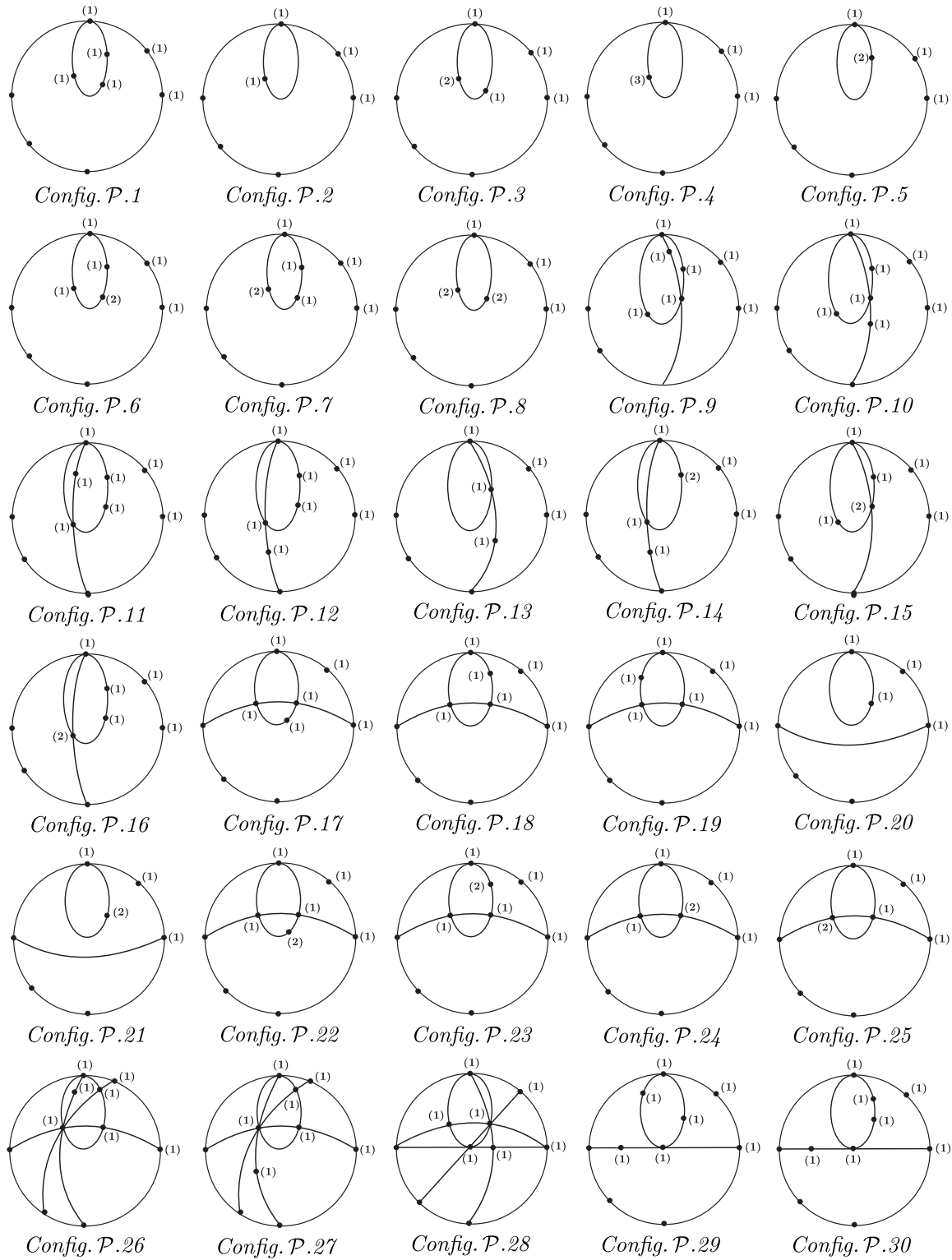
$$\mathcal{F}' : \prod_i G'_i(X, Y, Z) \cdot \prod_j F'_j(X, Y, Z) \cdot Z = 0,$$

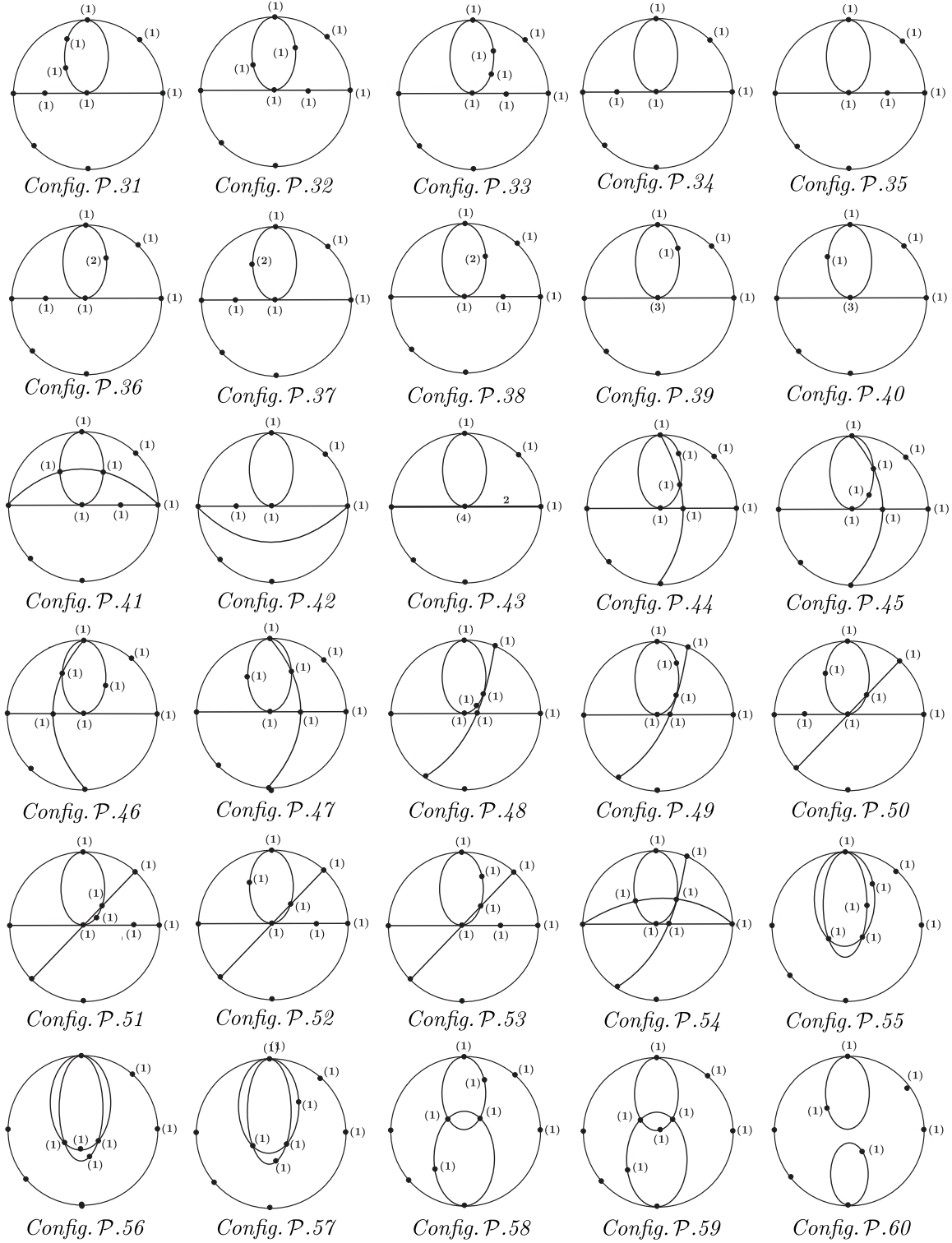
where $G_i(X, Y, Z) = 0$ and $F_j(X, Y, Z) = 0$ (respectively $G'_i(X, Y, Z) = 0$ and $F'_j(X, Y, Z) = 0$) are the projective completions of $\mathcal{P}_i$ and $\mathcal{L}_j$ (respectively $\mathcal{P}'_i$ and $\mathcal{L}'_j$). Then there is a correspondence ψ between the singularities of $\mathcal{F}$ and $\mathcal{F}'$, preserving their multiplicities as singularities of the total curves.

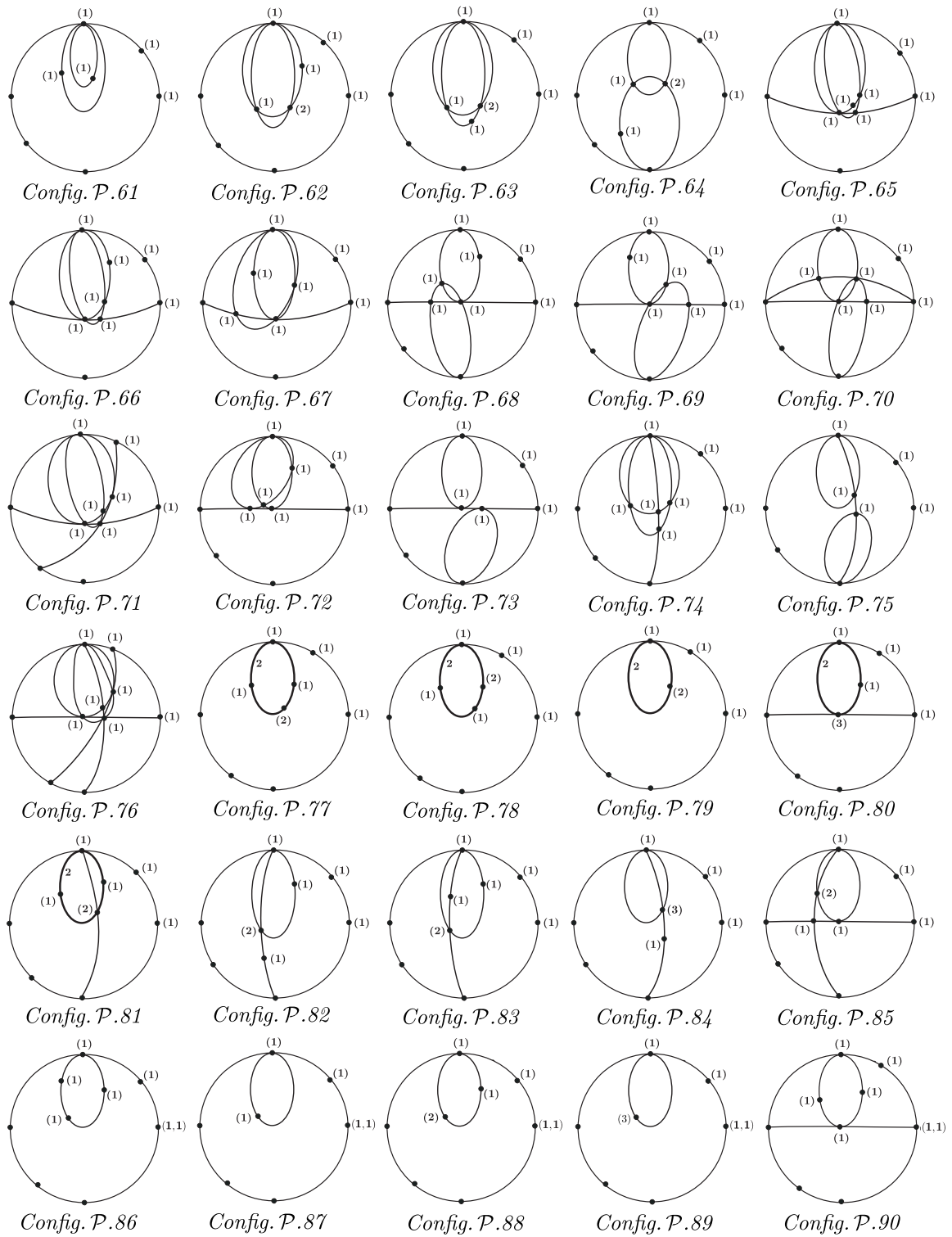
Our main results are summarized in the following theorem.

Main Theorem. Consider the class $\mathbf{QSP}_{(\eta \neq 0)}$ of all non-degenerate quadratic differential systems (1.1) possessing an invariant parabola and three distinct infinite singularities (real or complex).

- (A) This family is classified according to the configurations of invariant parabolas and invariant straight lines, resulting in 144 distinct configurations. Among these, 112 configurations belong to the subclass $\mathbf{QSP}_{(\eta > 0)}$ and 32 to the subclass $\mathbf{QSP}_{(\eta < 0)}$. This geometric classification is illustrated in Figures 1.1 and 1.2, and is characterized by necessary and sufficient invariant conditions presented in Diagrams 1.1, 1.2, and 1.3.
- (B) Using 70 invariant polynomials, we derive the bifurcation diagram in the space $\mathbb{R}^{12}$ of the coefficients of systems in $\mathbf{QSP}_{(\eta \neq 0)}$. These diagrams, shown in Diagrams 1.1, 1.2, and 1.3, classify the systems according to their configurations of invariant parabolas and straight lines. Furthermore, the diagrams provide an algorithmic procedure to compute the configuration of any quadratic differential system with an invariant parabola, irrespective of its chosen normal form.
- (C) Figures 1.1 and 1.2 present all possible configurations for systems in the subclasses $\mathbf{QSP}_{(\eta > 0)}$ and $\mathbf{QSP}_{(\eta < 0)}$, respectively. We prove that all 144 configurations are realizable within $\mathbf{QSP}_{(\eta \neq 0)}$, and that these configurations are topologically distinct. This proof, based on geometric invariants, is provided in Subsection 3.4 and is illustrated in Diagrams 3.1 and 3.2.


 Figure 1.1: Configurations of systems in $\mathbf{QSP}$ in the case $\eta > 0$.

Figure 1.1 (continued): Configurations of systems in QSP in the case $\eta > 0$.


 Figure 1.1 (continued): Configurations of systems in QSP in the case $\eta > 0$.

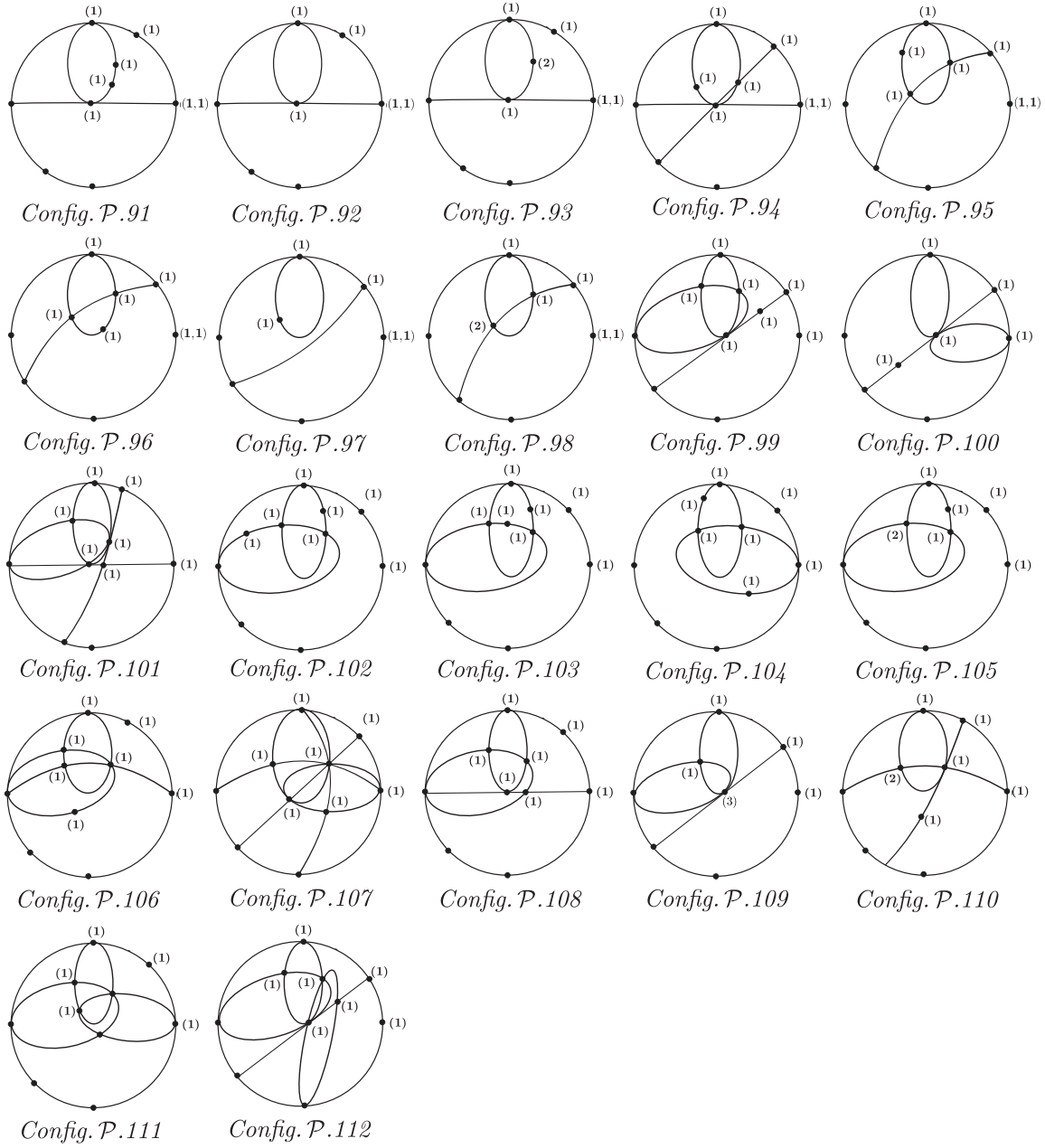
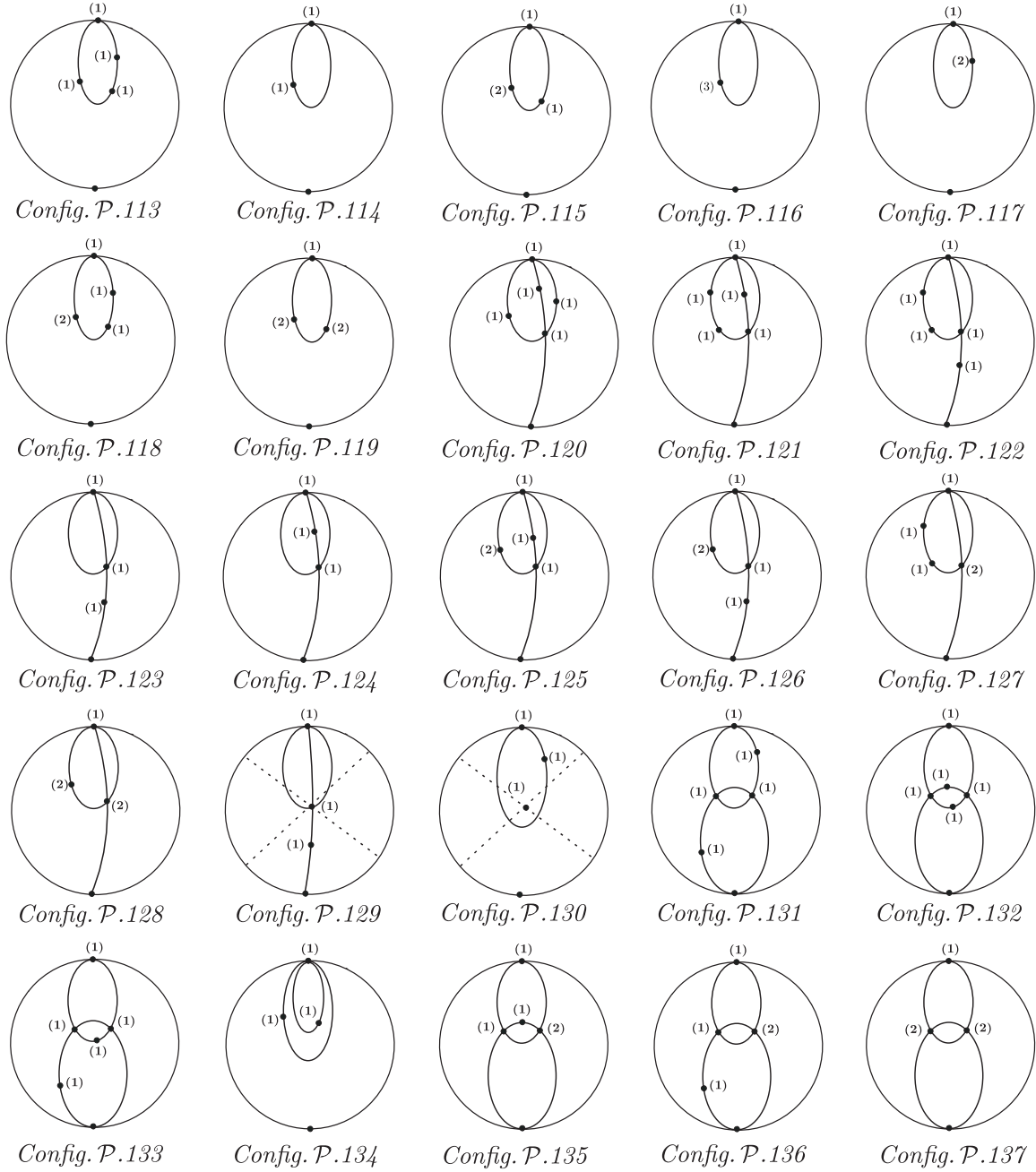


Figure 1.1 (continued): Configurations of systems in QSP in the case $\eta > 0$.


 Figure 1.2: Configurations of systems in **QSP** in the case $\eta < 0$.

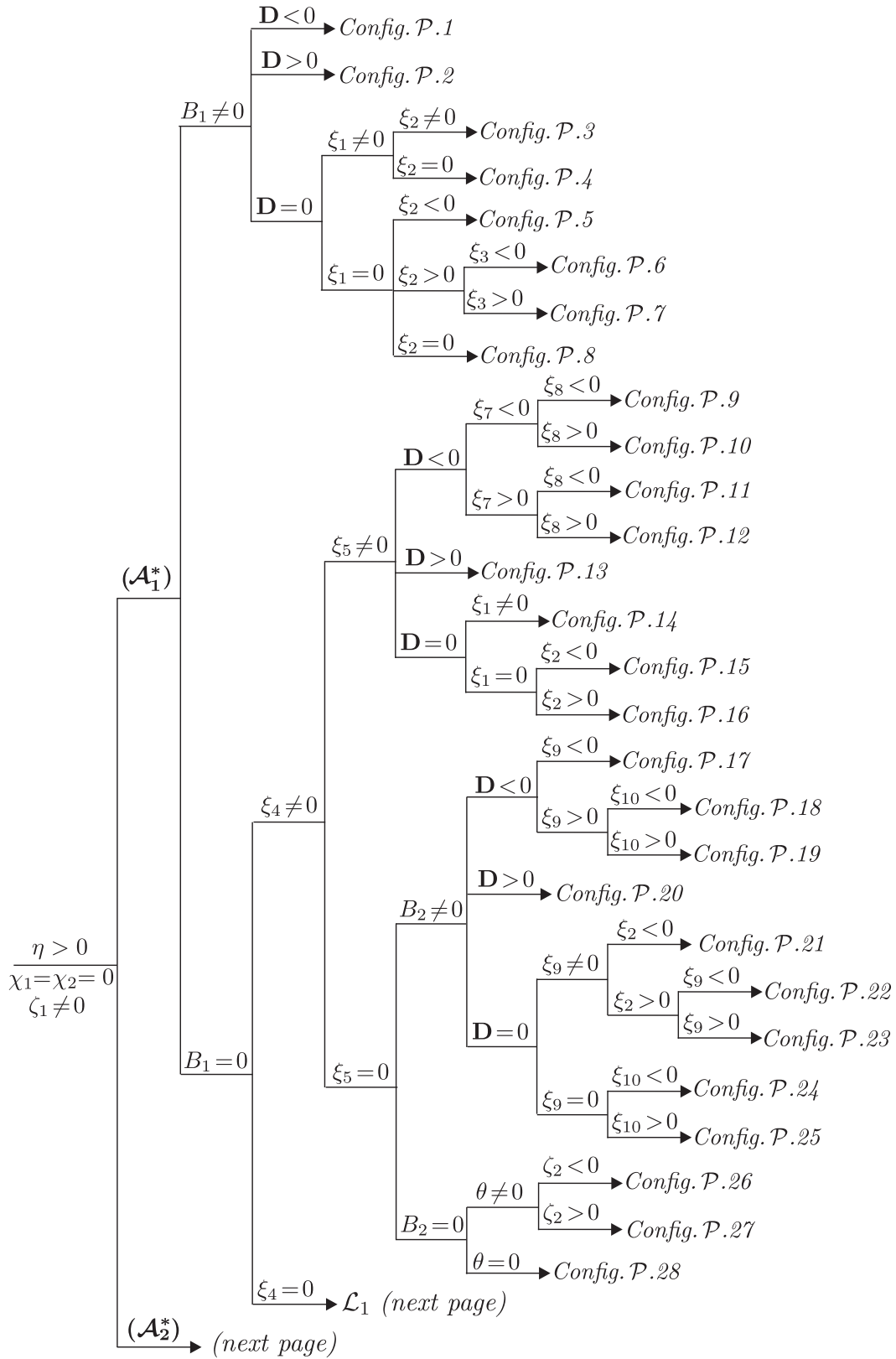


Diagram 1.1: Conditions for the configurations of systems in **QSP** in the case $\eta > 0, \zeta_1 \neq 0$.

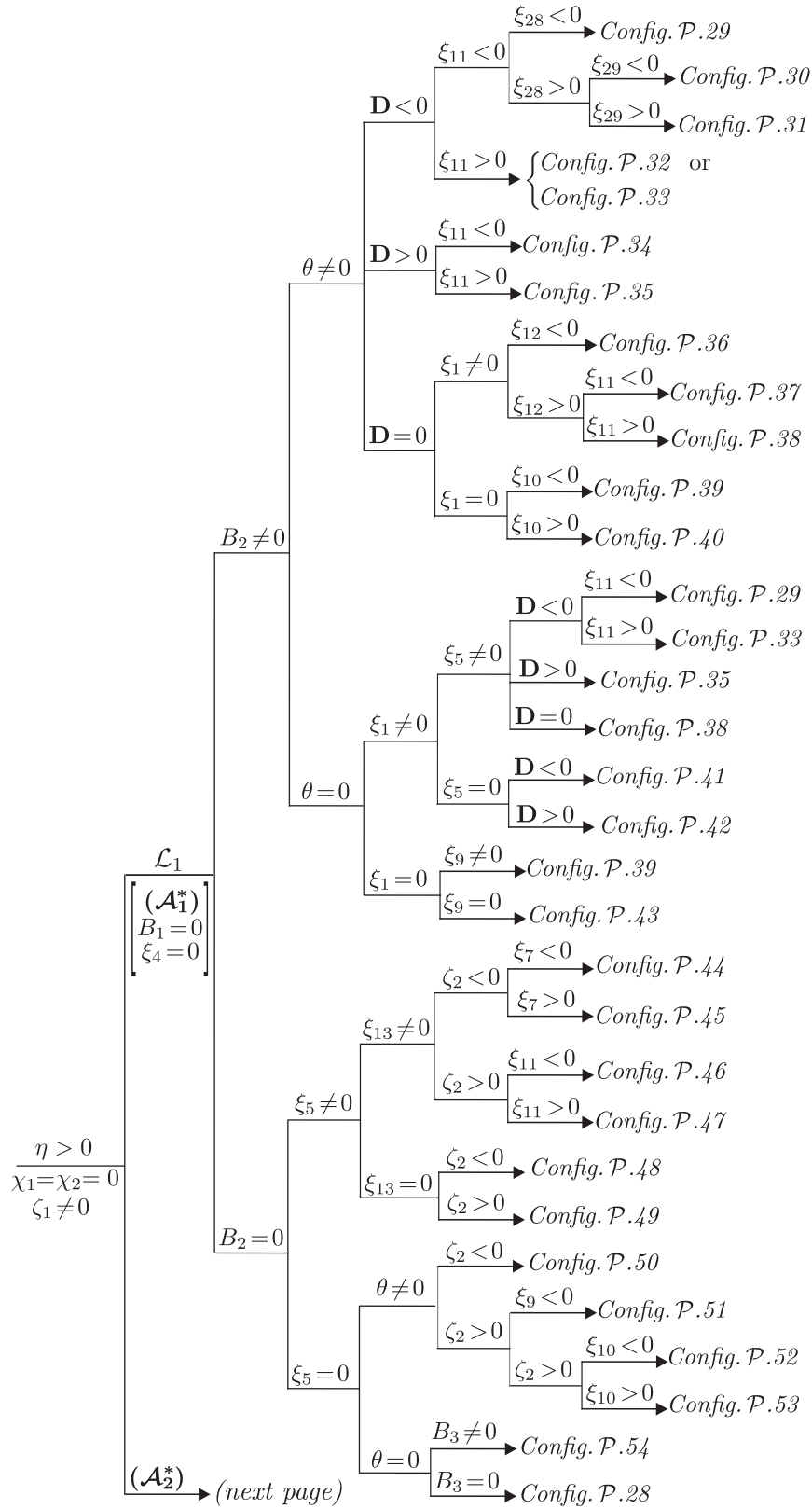


Diagram 1.1 (continued): Conditions for the configurations of systems in QSP in the case $\eta > 0$, $\zeta_1 \neq 0$.

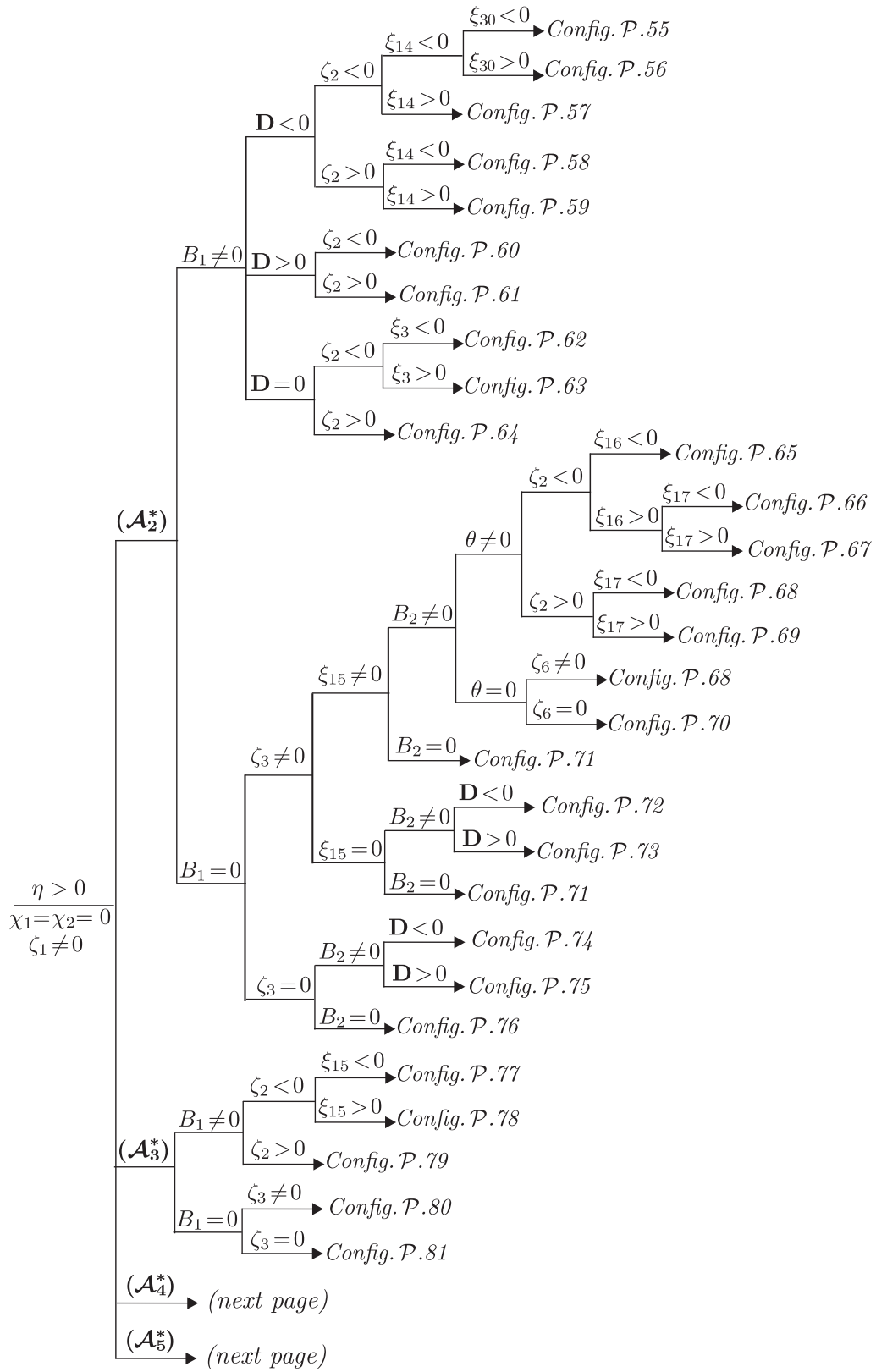
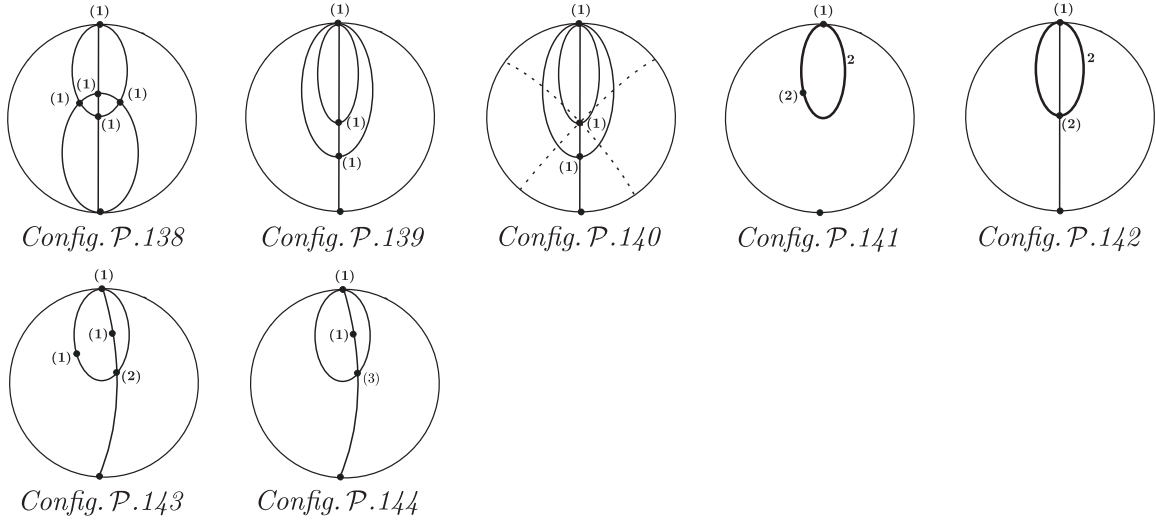
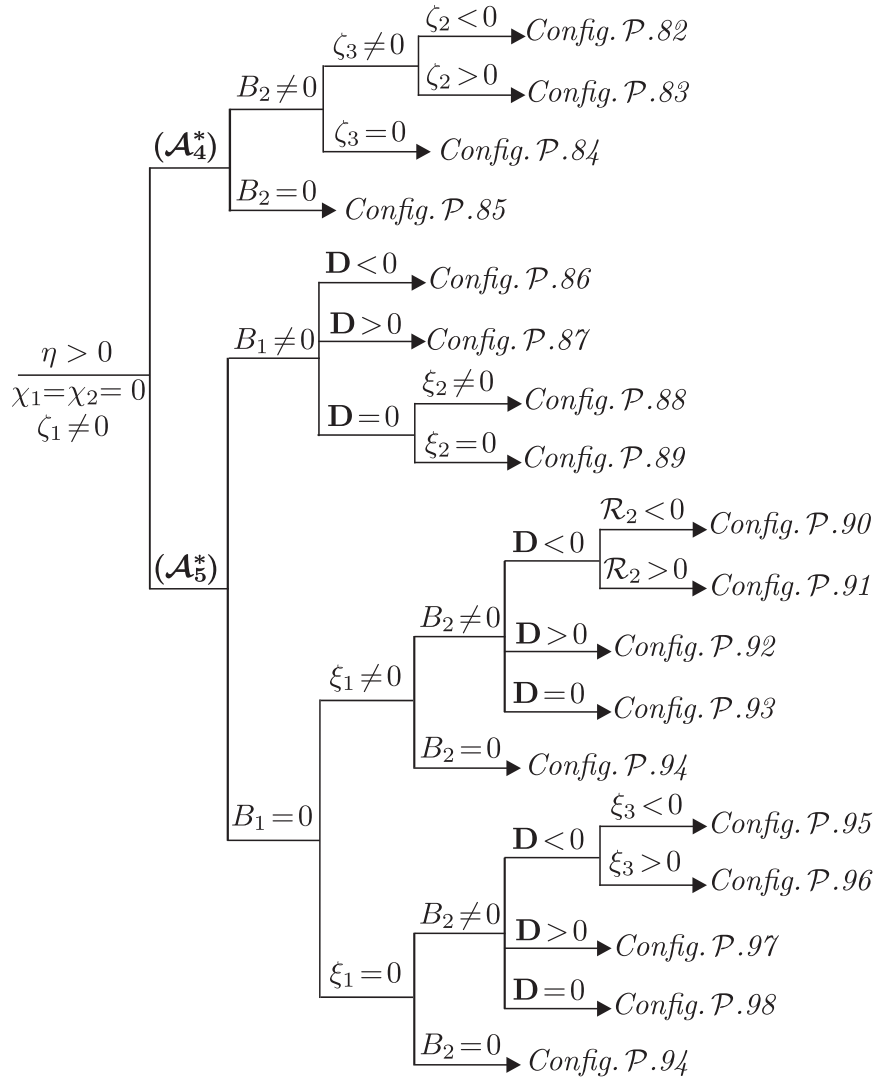


Diagram 1.1 (continued): Conditions for the configurations of systems in QSP in the case $\eta > 0, \zeta_1 \neq 0$.


 Figure 1.2 (continued): Configurations of systems in QSP in the case $\eta < 0$.

 Diagram 1.1 (continued): Conditions for the configurations of systems in QSP in the case $\eta > 0$, $\zeta_1 \neq 0$.

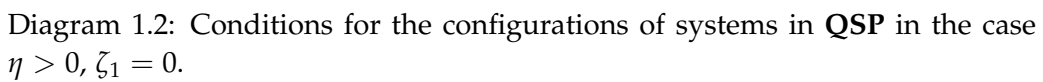


Diagram 1.2: Conditions for the configurations of systems in **QSP** in the case $\eta > 0, \zeta_1 = 0$.

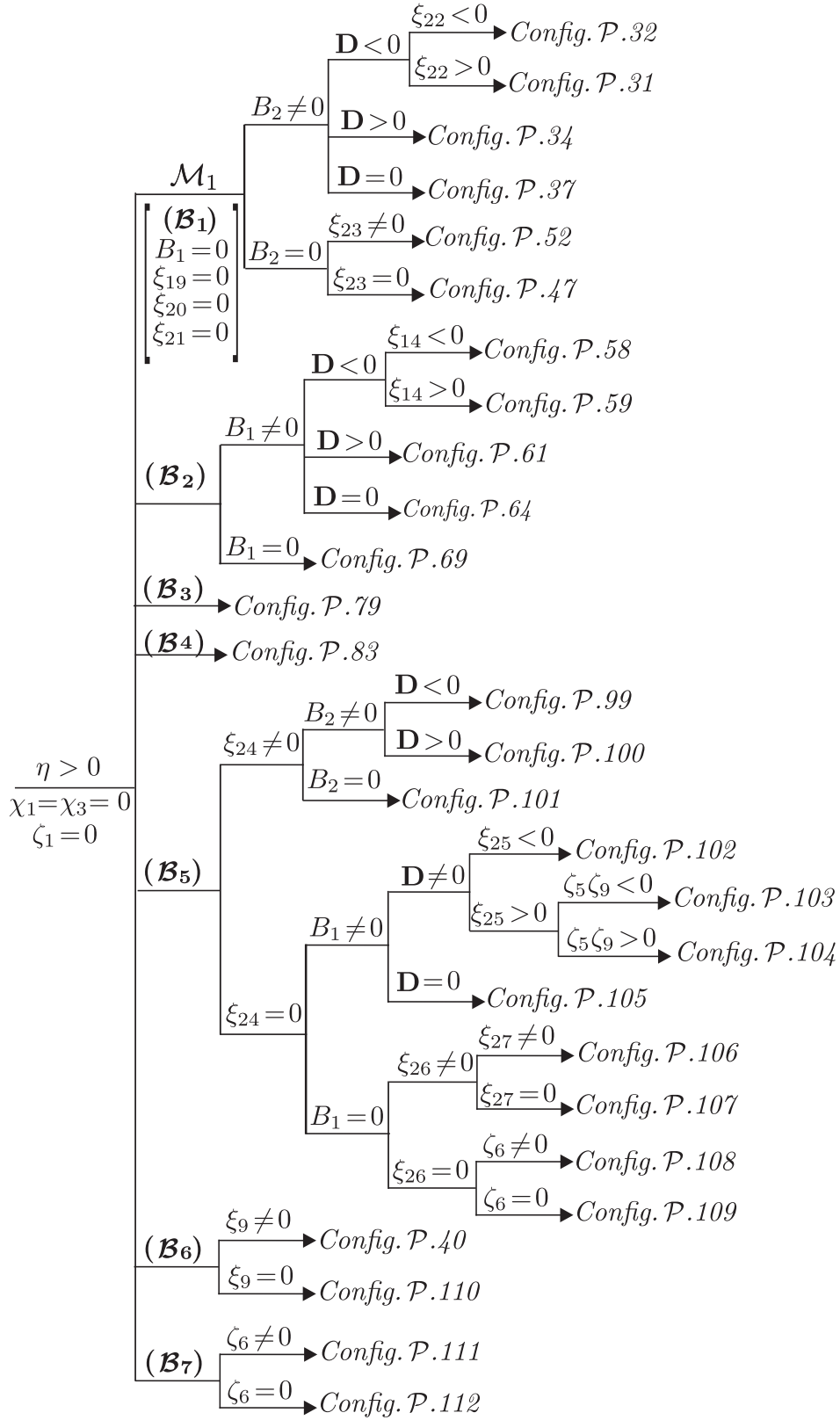


Diagram 1.2 (continued): Conditions for the configurations of systems in QSP in the case $\eta > 0$, $\zeta_1 = 0$.

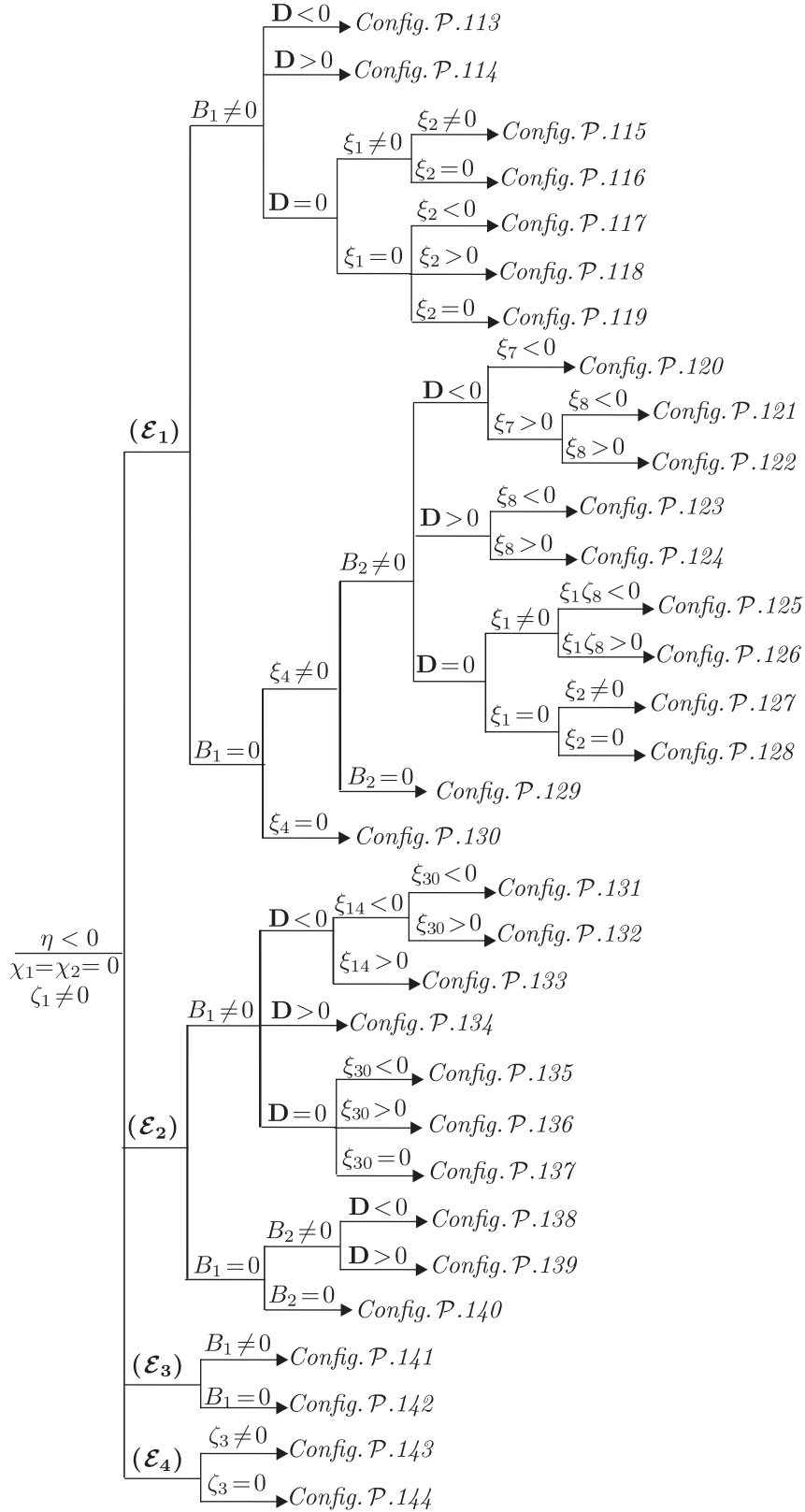


Diagram 1.3: Conditions for the configurations of systems in QSP in the case $\eta < 0$.

Remark 1.4. Every branch of the graphs in Diagrams 1.1, 1.2, and 1.3 terminates at a unique configuration *Config. P.1* through *Config. P.144*, with a single exception. Specifically, in Diagram 1.1 (page 11), there is a branch that leads to an indeterminacy between two configurations: either *Config. P.32* or *Config. P.33*. We are convinced that there exists an invariant polynomial capable of distinguishing between these two configurations, although we have not yet identified it. The determination of such an invariant remains an open problem.

2 Preliminaries

Consider real quadratic systems of the form:

$$\begin{aligned}\frac{dx}{dt} &= p_0 + p_1(x, y) + p_2(x, y) \equiv P(x, y), \\ \frac{dy}{dt} &= q_0 + q_1(x, y) + q_2(x, y) \equiv Q(x, y),\end{aligned}\tag{2.1}$$

where p_i and q_i ($i = 0, 1, 2$) are homogeneous polynomials of degree i in x and y :

$$\begin{aligned}p_0 &= a_{00}, & p_1(x, y) &= a_{10}x + a_{01}y, & p_2(x, y) &= a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ q_0 &= b_{00}, & q_1(x, y) &= b_{10}x + b_{01}y, & q_2(x, y) &= b_{20}x^2 + 2b_{11}xy + b_{02}y^2.\end{aligned}$$

Such a system (2.1) can be identified with a point in $\mathbb{R}^{12}$. Let

$$\tilde{a} = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}),$$

and consider the polynomial ring $\mathbb{R}[a_{00}, \dots, a_{02}, b_{00}, \dots, b_{02}, x, y]$, which we denote by $\mathbb{R}[\tilde{a}, x, y]$.

It is known that the group $\text{Aff}(2, \mathbb{R})$ of affine transformations of the plane acts on the set **QS** of all quadratic differential systems (2.1) (cf. [20]). For every subgroup $G \subseteq \text{Aff}(2, \mathbb{R})$, there is an induced action of G on **QS**.

We can identify the set **QS** with a subset of $\mathbb{R}^{12}$ via the map $\text{QS} \rightarrow \mathbb{R}^{12}$, which associates to each system (2.1) the 12-tuple $\tilde{a} = (a_{00}, \dots, b_{02})$ of its coefficients.

To study this group action, we associate to the systems certain polynomials in x, y and the system parameters that transform in a controlled way under the action. These are known as *GL*-comitants, *T*-comitants, and *CT*-comitants. For detailed definitions and constructions of these comitants, we refer the reader to [20] (see also [1]).

2.1 The main invariant polynomials associated to invariant parabolas

We single out the following five polynomials, which serve as basic ingredients in constructing invariant polynomials for systems (2.1):

$$\begin{aligned}C_i(\tilde{a}, x, y) &= yp_i(x, y) - xq_i(x, y), \quad (i = 0, 1, 2), \\ D_i(\tilde{a}, x, y) &= \frac{\partial p_i}{\partial x} + \frac{\partial q_i}{\partial y}, \quad (i = 1, 2).\end{aligned}\tag{2.2}$$

As shown in [23], these polynomials, which are linear in the coefficients of systems (2.1), are *GL*-comitants of these systems.

For $f, g \in \mathbb{R}[\tilde{a}, x, y]$, we define the *transvectant of index k* of (f, g) by

$$(f, g)^{(k)} = \sum_{h=0}^k (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h} \cdot \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

The resulting polynomial $(f, g)^{(k)}$ belongs to $\mathbb{R}[\tilde{a}, x, y]$ (cf. [12, 17]).

Proposition 2.1 (see [25]). *Any GL -comitant of systems (2.1) can be constructed from the elements (2.2) by using the operations $+$, $-$, $\times$, and by applying the differential operation $(*, *)^{(k)}$.*

Remark 2.2. We point out that the elements (2.2) generate the entire set of GL -comitants, and consequently also the full set of affine comitants and T -comitants, since any affine comitant and any T -comitant can be constructed from GL -comitants using the same operations: $+$, $-$, $\times$, and $(*, *)^{(k)}$.

We construct the following GL -comitants of second degree with respect to the coefficients of the initial systems:

$$\begin{aligned} T_1 &= (C_0, C_1)^{(1)}, & T_2 &= (C_0, C_2)^{(1)}, & T_3 &= (C_0, D_2)^{(1)}, \\ T_4 &= (C_1, C_1)^{(2)}, & T_5 &= (C_1, C_2)^{(1)}, & T_6 &= (C_1, C_2)^{(2)}, \\ T_7 &= (C_1, D_2)^{(1)}, & T_8 &= (C_2, C_2)^{(2)}, & T_9 &= (C_2, D_2)^{(1)}. \end{aligned} \quad (2.3)$$

Using these GL -comitants, together with the polynomials defined in (2.2), we construct additional invariant polynomials. To enable the direct computation of the required invariant polynomials for each canonical system, we now define a family of T -comitants expressed in terms of the polynomials C_i ($i = 0, 1, 2$) and D_j ($j = 1, 2$):

$$\begin{aligned} \hat{A} &= (C_1, T_8 - 2T_9 + D_2^2)^{(2)} / 144, \\ \hat{D} &= [2C_0(T_8 - 8T_9 - 2D_2^2) + C_1(6T_7 - T_6 - (C_1, T_5)^{(1)} + 6D_1(C_1D_2 - T_5) - 9D_1^2C_2)] / 36, \\ \hat{E} &= [D_1(2T_9 - T_8) - 3(C_1, T_9)^{(1)} - D_2(3T_7 + D_1D_2)] / 72, \\ \hat{F} &= [6D_1^2(D_2^2 - 4T_9) + 4D_1D_2(T_6 + 6T_7) + 48C_0(D_2, T_9)^{(1)} - 9D_2^2T_4 + 288D_1\hat{E} \\ &\quad - 24(C_2, \hat{D})^{(2)} + 120(D_2, \hat{D})^{(1)} - 36C_1(D_2, T_7)^{(1)} + 8D_1(D_2, T_5)^{(1)}] / 144, \\ \hat{K} &= (T_8 + 4T_9 + 4D_2^2) / 72, \\ \hat{H} &= (8T_9 - T_8 + 2D_2^2) / 72, \\ \hat{B} &= \left\{ 16D_1(D_2, T_8)^{(1)}(3C_1D_1 - 2C_0D_2 + 4T_2) + 32C_0(D_2, T_9)^{(1)}(3D_1D_2 - 5T_6 + 9T_7) \right. \\ &\quad + 2(D_2, T_9)^{(1)}(27C_1T_4 - 18C_1D_1^2 - 32D_1T_2 + 32(C_0, T_5)^{(1)}) \\ &\quad + 6(D_2, T_7)^{(1)}[8C_0(T_8 - 12T_9) - 12C_1(D_1D_2 + T_7) \\ &\quad + D_1(26C_2D_1 + 32T_5) + C_2(9T_4 + 96T_3)] \\ &\quad + 6(D_2, T_6)^{(1)}[32C_0T_9 - C_1(12T_7 + 52D_1D_2) - 32C_2D_1^2] + 48D_2(D_2, T_1)^{(1)}(2D_2^2 - T_8) \\ &\quad - 32D_1T_8(D_2, T_2)^{(1)} + 9D_2^2T_4(T_6 - 2T_7) - 16D_1(C_2, T_8)^{(1)}(D_1^2 + 4T_3) \\ &\quad + 12D_1(C_1, T_8)^{(2)}(C_1D_2 - 2C_2D_1) + 6D_1D_2T_4(T_8 - 7D_2^2 - 42T_9) \\ &\quad + 12D_1(C_1, T_8)^{(1)}(T_7 + 2D_1D_2) + 96D_2^2[D_1(C_1, T_6)^{(1)} + D_2(C_0, T_6)^{(1)}] - \\ &\quad \left. - 16D_1D_2T_3(2D_2^2 + 3T_8) - 4D_1^3D_2(D_2^2 + 3T_8 + 6T_9) + 6D_1^2D_2^2(7T_6 + 2T_7) \right. \\ &\quad \left. - 252D_1D_2T_4T_9 \right\} / (2^8 3^3), \end{aligned}$$

These polynomials, together with those defined in (2.2) and (2.3), will serve as fundamental building blocks for constructing affine invariant polynomials for systems (2.1).

The following 42 affine invariants, labeled $A_1, \dots, A_{42}$, constitute a minimal polynomial basis of affine invariants up to degree 12. This result was established in [4], where the invariants $A_1, \dots, A_{42}$ were explicitly constructed using the aforementioned building blocks:

$$\begin{aligned}
 A_1 &= \hat{A}, \\
 A_2 &= (C_2, \hat{D})^{(3)} / 12, \\
 A_3 &= [C_2, D_2]^{(1)}, D_2]^{(1)} / 48, \\
 A_4 &= (\hat{H}, \hat{H})^{(2)}, \\
 A_5 &= (\hat{H}, \hat{K})^{(2)} / 2, \\
 A_6 &= (\hat{E}, \hat{H})^{(2)} / 2, \\
 A_7 &= [C_2, \hat{E}]^{(2)}, D_2]^{(1)} / 8, \\
 A_8 &= [\hat{D}, \hat{H}]^{(2)}, D_2]^{(1)} / 8, \\
 A_9 &= [\hat{D}, D_2]^{(1)}, D_2]^{(1)} / 48, \\
 A_{10} &= [\hat{D}, \hat{K}]^{(2)}, D_2]^{(1)} / 8, \\
 A_{11} &= (\hat{F}, \hat{K})^{(2)} / 4, \\
 A_{12} &= (\hat{F}, \hat{H})^{(2)} / 4, \\
 A_{13} &= [C_2, \hat{H}]^{(1)}, \hat{H}]^{(2)}, D_2]^{(1)} / 24, \\
 A_{14} &= (\hat{B}, C_2)^{(3)} / 36, \\
 A_{15} &= (\hat{E}, \hat{F})^{(2)} / 4, \\
 A_{16} &= [\hat{E}, D_2]^{(1)}, C_2]^{(1)}, \hat{K}]^{(2)} / 16, \\
 A_{17} &= [\hat{D}, \hat{D}]^{(2)}, D_2]^{(1)}, D_2]^{(1)} / 64, \\
 A_{18} &= [\hat{D}, \hat{F}]^{(2)}, D_2]^{(1)} / 16, \\
 A_{19} &= [\hat{D}, \hat{D}]^{(2)}, \hat{H}]^{(2)} / 16, \\
 A_{20} &= [C_2, \hat{D}]^{(2)}, \hat{F}]^{(2)} / 16, \\
 A_{21} &= [\hat{D}, \hat{D}]^{(2)}, \hat{K}]^{(2)} / 16, \\
 A_{22} &= [C_2, \hat{D}]^{(1)}, D_2]^{(1)}, D_2]^{(1)}, D_2]^{(1)} / 1152, \\
 A_{23} &= [\hat{F}, \hat{H}]^{(1)}, \hat{K}]^{(2)} / 8, \\
 A_{24} &= [C_2, \hat{D}]^{(2)}, \hat{K}]^{(1)}, \hat{H}]^{(2)} / 32, \\
 A_{25} &= [\hat{D}, \hat{D}]^{(2)}, \hat{E}]^{(2)} / 16, \\
 A_{26} &= (\hat{B}, \hat{D})^{(3)} / 36, \\
 A_{27} &= [\hat{B}, D_2]^{(1)}, \hat{H}]^{(2)} / 24, \\
 A_{28} &= [C_2, \hat{K}]^{(2)}, \hat{D}]^{(1)}, \hat{E}]^{(2)} / 16, \\
 A_{29} &= [\hat{D}, \hat{F}]^{(1)}, \hat{D}]^{(3)} / 96, \\
 A_{30} &= [C_2, \hat{D}]^{(2)}, \hat{D}]^{(1)}, \hat{D}]^{(3)} / 288, \\
 A_{31} &= [\hat{D}, \hat{D}]^{(2)}, \hat{K}]^{(1)}, \hat{H}]^{(2)} / 64,
 \end{aligned}$$

$$\begin{aligned}
A_{32} &= [\widehat{D}, \widehat{D})^{(2)}, D_2)^{(1)}, \widehat{H})^{(1)}, D_2)^{(1)} / 64, \\
A_{33} &= [\widehat{D}, D_2)^{(1)}, \widehat{F})^{(1)}, D_2)^{(1)}, D_2)^{(1)} / 128, \\
A_{34} &= [\widehat{D}, \widehat{D})^{(2)}, D_2)^{(1)}, \widehat{K})^{(1)}, D_2)^{(1)} / 64, \\
A_{35} &= [\widehat{D}, \widehat{D})^{(2)}, \widehat{E})^{(1)}, D_2)^{(1)}, D_2)^{(1)} / 128, \\
A_{36} &= [\widehat{D}, \widehat{E})^{(2)}, \widehat{D})^{(1)}, \widehat{H})^{(2)} / 16, \\
A_{37} &= [\widehat{D}, \widehat{D})^{(2)}, \widehat{D})^{(1)}, \widehat{D})^{(3)} / 576, \\
A_{38} &= [C_2, \widehat{D})^{(2)}, \widehat{D})^{(2)}, \widehat{D})^{(1)}, \widehat{H})^{(2)} / 64, \\
A_{39} &= [\widehat{D}, \widehat{D})^{(2)}, \widehat{F})^{(1)}, \widehat{H})^{(2)} / 64, \\
A_{40} &= [\widehat{D}, \widehat{D})^{(2)}, \widehat{F})^{(1)}, \widehat{K})^{(2)} / 64, \\
A_{41} &= [C_2, \widehat{D})^{(2)}, \widehat{D})^{(2)}, \widehat{F})^{(1)}, D_2)^{(1)} / 64, \\
A_{42} &= [\widehat{D}, \widehat{F})^{(2)}, \widehat{F})^{(1)}, D_2)^{(1)} / 16.
\end{aligned}$$

In the above list, the bracket “[” is used as a typographical device to avoid writing up to five consecutive parentheses “(” in some of the expressions.

Using the elements of the minimal polynomial basis listed above, we construct two groups of affine invariant polynomials. The first group contains invariant polynomials associated with the existence of an invariant parabola for a quadratic system, and they are:

$$\begin{aligned}
\chi_1 &= 32A_3 + 45A_4 - 160A_5; \\
\chi_2 &= 32A_8(14A_8 - 48A_9 + 37A_{10} + 24A_{11}) + 16A_5(76A_{17} + 74A_{18} + 313A_{19} - 80A_{20} \\
&\quad - 167A_{21}) + A_4(160A_2^2 + 368A_{18} - 3363A_{19} + 736A_{20} + 2109A_{21}) + 32(17A_{10}^2 \\
&\quad + 27A_{10}A_{11} + 24A_{11}^2 - 48A_9A_{12} + 51A_{10}A_{12} + 24A_{11}A_{12} + 288A_6A_{14} - 96A_7A_{14}); \\
\chi_3 &= 6520480A_{20}(407A_{18} - 2253A_{21}) + 24A_{18}(1057715458A_{19} + 5944853225A_{21}) \\
&\quad + 28800A_{14}(1872476A_{25} - 122259A_{26}) + 144A_{12}(3620283092A_{29} - 1554910481A_{30}) \\
&\quad + 1440A_{15}(107225339A_{25} - 19561440A_{26}) - 72A_{11}(8198511476A_{29} - 2965514443A_{30}) \\
&\quad + 652048(4544A_{18}^2 + 125A_{20}^2 - 8955A_2A_{42}) - 9(264364688A_{19}^2 + 39417454842A_{19}A_{21} \\
&\quad - 54474141921A_{21}^2) + 3448898760A_{19}A_{20}; \\
\chi_4 &= 62713A_{10}^2 + 45787A_{10}A_{11} - 157928A_{11}^2 + 81202A_{10}A_{12} + A_{19}353474A_{11}A_{12} - 145848A_{12}^2 \\
&\quad + 64320A_7A_{15} + 28600A_5A_{17}; \\
\zeta_1 &= 13A_4 - 24A_5; \\
\zeta_2 &= -A_4; \\
\zeta_3 &= 16A_5 - 17A_4; \\
\zeta_4 &= 9A_1A_4 - 7A_1A_5 - 2A_{16}; \\
\zeta_5 &= 166A_8 + 384A_9 - 1120A_{10} - 512A_{11} - 62A_{12}; \\
\zeta_6 &= A_6; \\
\zeta_7 &= 40(71436A_7A_{20} - 640883A_7A_{21} + 1008622A_1A_{32}) + 12A_{12}(3585035A_{14} + 14919259A_{15}) \\
&\quad - 5(8092193A_{10} + 15970731A_{11})A_{14} - (129780821A_{10} + 269944167A_{11})A_{15}; \\
\zeta_8 &= A_2;
\end{aligned}$$

$$\begin{aligned}
 \zeta_9 &= 1040(2256A_7A_{15} + 143A_3A_{21}) - 264(162941A_{10} + 315202A_{11})A_{12} \\
 &\quad + 3A_{11}(25887132A_{10} + 24385177A_{11}) + 20603609A_{10}^2 + 24896016A_{12}^2; \\
 \zeta_{10} &= 250A_1^2 + 34A_{11} - 41A_{12}; \\
 \mathcal{R}_1 &= 531A_2A_4 - 1472A_2A_5 - 8352A_1A_6 + 320A_{22} - 3216A_{23} + 1488A_{24}; \\
 \mathcal{R}_2 &= 15A_{10} - 10A_8 - 6A_9; \\
 \mathcal{R}_3 &= 4800(6650951968A_{14}A_{15} - 2382132830A_{14}^2 - 9860550485A_{15}^2) + 1600(4765089473A_{11} \\
 &\quad - 7838161089A_{12})A_{20} + 640(15664652914A_{11} - 50944340271A_{12})A_{18} \\
 &\quad - 6(20392663986679A_{10} + 34357804389813A_{11} - 739275727012A_{12})A_{21} \\
 &\quad + 3(46944212550227A_{10} + 83455057317969A_{11} - 22899810934956A_{12})A_{19}; \\
 \mathcal{R}_4 &= 251A_1^2 + 25A_{12}; \\
 \mathcal{R}_7 &= 62250A_1^2 + 8956A_9 - 46223A_{10} - 50129A_{11} + 14766A_{12}.
 \end{aligned}$$

The invariant polynomials from the second group are responsible for the classification of the configurations of invariant parabolas and lines. They are:

$$\begin{aligned}
 \zeta_1 &= 342A_1^2A_2 + A_2(35A_{10} - 15A_8 - 16A_9 + 97A_{11} - 83A_{12}) - 48A_1(4A_{14} + 3A_{15}) \\
 &\quad + 16(2A_{32} + A_{33} - 3A_{34}) + 90A_{31}; \\
 \zeta_2 &= -A_{19}; \\
 \zeta_3 &= 12(49836514A_8^2 - 40804544A_8A_9 - 63384469A_8A_{10} - 4515985A_{10}^2 + 93824435A_8A_{11} \\
 &\quad - 23552547A_{10}A_{11} + 51595312A_{11}^2 + 202411827A_1^2A_{12}) - 763176315A_4A_{21} \\
 &\quad - 16(30603408A_9A_{12} + 10917387A_7A_{14} + 14011860A_7A_{15} - 75865539A_5A_{17} \\
 &\quad - 115398446A_5A_{18} - 54568383A_5A_{21}) - 4(86656770A_6A_{14} + 404823654A_6A_{15} \\
 &\quad - 68396637A_5A_{19} + 25391678A_5A_{20}) - 6A_{12}(154041735A_8 + 47473233A_{10} \\
 &\quad - 170661233A_{11} + 202411827A_{12}); \\
 \zeta_4 &= 800(175A_2A_5A_7 - 336A_1A_3A_8 - 16500A_{13}A_{14} - 9300A_{13}A_{15} - 47001A_6A_{22} \\
 &\quad + 39861A_7A_{23} - 3150A_6A_{24} - 10242A_7A_{24} + 168792A_5A_{28}) + 240(173478A_8A_{16} \\
 &\quad + 128774A_{10}A_{16} + 151602A_{11}A_{16} + 134102A_{12}A_{16} + 8799A_4A_{27} - 134102A_5A_{27}) \\
 &\quad - 1879552(3A_9A_{16} - A_7A_{22}) + 75(50400A_6A_{23} - 646151A_4A_{28}); \\
 \zeta_5 &= 2000(802A_{13}A_{14} + 315A_6A_{23} - 210A_6A_{24}) + 320(28A_1A_3A_{11} - 13757A_8A_{16} \\
 &\quad - 11282A_{12}A_{16} + 3336A_7A_{24} + 11282A_5A_{27}) + 80(16038A_{13}A_{15} - 30398A_{10}A_{16} \\
 &\quad - 36154A_{11}A_{16} + 46738A_6A_{22} - 45142A_7A_{23} - 162339A_5A_{28}) + 151552(3A_9A_{16} \\
 &\quad - A_7A_{22}) - 15A_4(28392A_{27} - 313721A_{28}); \\
 \zeta_6 &= 1536(16671538A_7A_{14} - 5655800A_{11}^2 - 5655800A_{11}A_{12} - 134975925A_6A_{15} \\
 &\quad + 14236220A_7A_{15}) + 128(42330182A_8A_9 + 279065017A_8A_{11} - 857954A_8A_{12} \\
 &\quad + 138313062A_9A_{12} - 633595086A_6A_{14} - 35417298A_5A_{20}) + 64(171565045A_8^2 \\
 &\quad + 343921603A_5A_{17}) - 32(1111806317A_8A_{10} + 256225409A_{10}^2 + 874265715A_{10}A_{11} \\
 &\quad + 2536914399A_{10}A_{12} - 936841383A_5A_{18}) - 16A_5(2168875001A_{19} + 1048355233A_{21}) \\
 &\quad + A_4(26458433203A_{19} - 4734012269A_{21}); \\
 \zeta_7 &= -A_4[3200A_{12}(14657A_8 - 1615148A_{10} + 318175A_{11}) - 640(388968A_9^2 - 7748782A_{10}^2 \\
 &\quad - 592379A_9A_{12}) - 160(13079737A_8A_{10} - 27509045A_8A_{11} - 63353923A_9A_{11} \\
 &\quad - 16215395A_{10}A_{11} - 36662125A_{11}^2) + 4121433952A_9A_{10}];
 \end{aligned}$$

$$\begin{aligned}
\zeta_8 &= -A_4[512A_9(1275434A_{10} + 2193137A_{11} - 170333A_{12}) - 1280(30087A_9^2 + 424036A_{10}^2 \\
&\quad + 1052798A_{10}A_{11} + 48550A_{11}^2 + 61603A_8A_{12}) - 640(608587A_8A_{10} + 248041A_8A_{11} \\
&\quad + 430261A_{10}A_{12} + 525475A_{11}A_{12})]; \\
\zeta_9 &= -A_4[48(675908847A_8A_9 + 1141726617A_9A_{12} + 7216376855A_{10}A_{12} \\
&\quad - 4015621128A_6A_{14} + 3915909450A_7A_{15}) - 12(16745223889A_8^2 + 5997051735A_8A_{11} \\
&\quad - 26372062499A_{10}A_{11} + 2601951027A_8A_{12} - 7916516650A_7A_{14} - 30105649725A_6A_{15} \\
&\quad + 20512413539A_5A_{17} - 1497206278A_4A_{19} - 4791714129A_4A_{21}) + 2(220220676003A_8A_{10} \\
&\quad + 58687175103A_{10}^2 + 14685562719A_{11}^2 + 9716839839A_{11}A_{12} - 219193688911A_5A_{18} \\
&\quad - 4467110471A_5A_{20}) + 3A_5(36033875127A_{19} - 37652431103A_{21})]; \\
\zeta_{10} &= A_4[48(568199091031A_8A_9 - 248186616391A_9A_{10} + 314207594667A_9A_{11} \\
&\quad + 5804879973A_9A_{12} - 3905825755777A_{10}A_{12} - 2095407390920A_6A_{14} \\
&\quad - 546799764750A_7A_{15}) + 12(6550908482493A_8^2 - 3402501855145A_8A_{11} \\
&\quad - 3448022811579A_{10}A_{11} + 2284925158471A_8A_{12} + 2482932379806A_7A_{14} \\
&\quad - 11017448610465A_6A_{15} + 5894909506479A_5A_{17}) - 2(131290745988327A_8A_{10} \\
&\quad - 17334476527245A_{10}^2 - 11980168965A_{11}^2 + 21428060568795A_{11}A_{12} \\
&\quad - 62352140313275A_5A_{18} + 3924064256285A_5A_{20}) - 3(2258722903315A_5A_{19} \\
&\quad + 9533558573843A_4A_{21} - 10218122423819A_5A_{21})]; \\
\zeta_{11} &= \zeta_1\zeta_2\zeta_6; \\
\zeta_{12} &= 1288A_1^2 + 117A_{10} + 351A_{11} - 352A_{12}; \\
\zeta_{13} &= 61A_2^2 - 20A_{17} - 8A_{18} + 24A_{19} - 28A_{20} + 12A_{21}; \\
\zeta_{14} &= 9854A_{11} - 3005A_8 - 3296A_9 + 13578A_{10} - 991A_{12}; \\
\zeta_{15} &= 8A_5 - 9A_4; \\
\zeta_{16} &= (525A_8 - 4448A_9 + 10554A_{10} - 1378A_{11} + 8087A_{12}); \\
\zeta_{17} &= 10005A_8 + 9856A_9 - 38348A_{10} - 27404A_{11} + 8371A_{12}; \\
\zeta_{18} &= 2240(15452233775A_{14}^2 + 742923092360A_{14}A_{15} - 145263086200A_{15}^2 \\
&\quad + 10151798384A_{11}A_{18} - 68919094926A_{12}A_{18} - 14663220305A_{11}A_{20} \\
&\quad + 7194838365A_{12}A_{20}) + 16A_{19}(88266907919051A_8 + 12824946044853A_{11} \\
&\quad + 119819326860153A_{12}) - 7A_{21}(138073671324637A_{10} + 258358507987439A_{11} \\
&\quad - 32813284182036A_{12}); \\
\zeta_{19} &= 429A_9(629A_{10} + 1275A_{11} - 900A_{12}) + 100(2145A_8A_{11} - 1595A_5A_{17} - 2970A_5A_{18} \\
&\quad + 2886A_2A_{23} - 559A_2A_{24}); \\
\zeta_{20} &= 4A_2(47A_2^2 - 468A_{18} + 3478A_{19} + 9A_{20}) - 9189A_2A_{21} + 12(-682A_1A_{25} + 2592A_1A_{26} \\
&\quad + 395A_{38} + 35A_{39}); \\
\zeta_{21} &= 24(675906A_{40} - 672409A_{39} + 6578A_{41} + 110106A_{42}) - 73404A_2(74A_{18} + A_{20}) \\
&\quad + 4(99911A_2^3 - 2048846A_{38}) - 15133791A_2A_{21}; \\
\zeta_{22} &= 84A_{12} - 68A_{10} - 141A_{11}; \\
\zeta_{23} &= 5A_8 - 3A_9; \\
\zeta_{24} &= 625A_2(12A_2A_3 - 775A_1A_6) - 62(13500A_1^2 + 275A_8 - 276A_9)A_9 + 10A_3(2561A_{17} \\
&\quad + 3240A_{18} + 2550A_{19});
\end{aligned}$$

$$\xi_{25} = -(46A_{18} + 537A_{19} + 134A_{20});$$

$$\xi_{26} = 41A_1A_2 + 16A_{14} - 18A_{15};$$

$$\xi_{27} = A_1;$$

$$\begin{aligned} \xi_{28} = & 64(72137434664A_8^2 + 3322490880A_9^2 - 58216412276A_{10}^2 - 217656099219A_{10}A_{11} \\ & - 63098236389A_{11}^2 - 250756327503A_{10}A_{12} - 71858710389A_{11}A_{12} + 96A_9(449920640A_{11} \\ & + 1009660963A_{12}) + 6A_8(21795888048A_9 - 66020231422A_{10} - 21118997424A_{11} \\ & + 2573485725A_{12})) - 384(62739943233A_6A_{14} - 27065693406A_7A_{14} + 7592410800A_6A_{15} \\ & - 10442342780A_7A_{15}) + A_4(2998959134256A_{17} + 4635359414448A_{18} + 1132776129074A_{19} \\ & - 1187818900002A_{20} - 5542617623395A_{21}) + 32A_3(19078937382A_{20} + 81853956367A_{21}); \end{aligned}$$

$$\begin{aligned} \xi_{29} = & 497213324620A_8^2 - 1001736600522A_{10}^2 - 870653569536A_9A_{11} + 337754949134A_{11}^2 \\ & + A_8(2170429037822A_{10} - 1858453397512A_9 + 2112595332132A_{11} - 304022217484A_{12}) \\ & - 987799827976A_9A_{12} + 949933240214A_{11}A_{12} + A_{10}(-648979472052A_{11} \\ & + 956487534504A_{12}) - 4(125652578829A_6A_{14} + 240347919318A_7A_{14} \\ & - 775425835368A_6A_{15} + 405563103412A_7A_{15}) - A_4(197626785161A_{20} \\ & + 1540932760870A_{21}) + A_5(1910970964424A_{17} + 2668708281714A_{18} + 182967974851A_{19} \\ & + 280452031438A_{20} + 2136843181298A_{21}); \end{aligned}$$

$$\xi_{30} = 3512A_{10} - 1695A_8 - 544A_9 + 4576A_{11} - 3329A_{12}.$$

2.2 Results involving the use of polynomial invariants

A few more definitions and results, which play an important role in the proof of part (A) of the Main Theorem, are needed. We do not prove these results here but indicate where they can be found.

Consider the differential operator $\mathcal{L} = x \cdot L_2 - y \cdot L_1$, introduced in [3], acting on $\mathbb{R}[\tilde{a}, x, y]$, where the operators L_1 and L_2 are given by

$$\begin{aligned} L_1 &= 2a_{00}\frac{\partial}{\partial a_{10}} + a_{10}\frac{\partial}{\partial a_{20}} + \frac{1}{2}a_{01}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{10}} + b_{10}\frac{\partial}{\partial b_{20}} + \frac{1}{2}b_{01}\frac{\partial}{\partial b_{11}}, \\ L_2 &= 2a_{00}\frac{\partial}{\partial a_{01}} + a_{01}\frac{\partial}{\partial a_{02}} + \frac{1}{2}a_{10}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{01}} + b_{01}\frac{\partial}{\partial b_{02}} + \frac{1}{2}b_{10}\frac{\partial}{\partial b_{11}}. \end{aligned}$$

Using the differential operator $\mathcal{L}$ introduced above and the affine invariant

$$\mu_0 = \frac{\text{Res}_x(p_2(\tilde{a}, x, y), q_2(\tilde{a}, x, y))}{y^4},$$

we construct the following family of polynomials:

$$\mu_i(\tilde{a}, x, y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0), \quad i = 1, \dots, 4,$$

where the iterated operator $\mathcal{L}^{(i)}$ is defined recursively by

$$\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0)), \quad \text{with } \mathcal{L}^{(0)}(\mu_0) = \mu_0.$$

These polynomials μ_i are $GL(2, \mathbb{R})$ -comitants of the quadratic systems (2.1), as established in [3]. Their geometric interpretation is detailed in Lemma 5.2 of [1].

From these invariant polynomials, one constructs the affine invariant polynomials $\mathbf{D}$ and $\mathbf{R}$, which characterize the existence of multiple finite singularities in quadratic differential systems:

$$\mathbf{D} = \frac{1}{48} \left[3((\mu_3, \mu_3)^{(2)}, \mu_2)^{(2)} - (6\mu_0\mu_4 - 3\mu_1\mu_3 + \mu_2^2, \mu_4)^{(4)} \right], \quad \mathbf{R} = 3\mu_1^2 - 8\mu_0\mu_2,$$

where $(*, *)^{(k)}$ denotes the bilinear operation transvectant.

Next, we construct the following T -comitants (for the definition of T -comitants, see [20]), which play a fundamental role in characterizing the existence of invariant straight lines for systems (2.1):

$$\begin{aligned} B_3(\tilde{a}, x, y) &= (C_2, \widehat{D})^{(1)} = \text{Jacob}(C_2, \widehat{D}), \\ B_2(\tilde{a}, x, y) &= (B_3, B_3)^{(2)} - 6B_3(C_2, \widehat{D})^{(3)}, \\ B_1(\tilde{a}) &= \frac{\text{Res}_x(C_2, \widehat{D})}{y^9} = -2^{-9}3^{-8} (B_2, B_3)^{(4)}. \end{aligned}$$

The following result, whose proof can be found in [19], provides a necessary condition for the existence of invariant straight lines in quadratic differential systems.

Lemma 2.3 (see [19]). *For system (2.1) to possess invariant straight lines in one, two, or three distinct directions in the affine plane, it is necessary that the following conditions hold, respectively:*

$$B_1 = 0, \quad B_2 = 0, \quad B_3 = 0.$$

In order to detect the presence of parallel invariant straight lines, we require the following invariant polynomials:

$$\begin{aligned} N(\tilde{a}, x, y) &= D_2^2 + T_8 - 2T_9 = 9\widehat{N}, \\ \theta(\tilde{a}) &= 2A_5 - A_4 \quad (\equiv \text{Discriminant}(N(\tilde{a}, x, y)) / 1296). \end{aligned}$$

With these definitions, the following necessary condition holds.

Lemma 2.4 (see [19]). *A necessary condition for the existence of one couple (respectively two couples) of parallel invariant straight lines in system (2.1), corresponding to a parameter vector $\tilde{a} \in \mathbb{R}^{12}$, is that $\theta(\tilde{a}) = 0$ (respectively $N(\tilde{a}, x, y) = 0$).*

Next, we introduce some important GL -comitants relevant to the study of invariant conics. Let us consider

$$C_2(\tilde{a}, x, y) = y p_2(\tilde{a}, x, y) - x q_2(\tilde{a}, x, y),$$

which defines a cubic binary form in x and y . Using this form, we define the following polynomials:

$$\eta = \text{Discrim}[C_2], \quad M = \text{Hessian}[C_2].$$

It is worth noting (see [23]) that the invariant polynomials C_2 , η , and M are responsible for controlling the number of infinite singularities (real or complex) of system (2.1).

Remark 2.5. In order to describe the various kinds of multiplicities for infinite singularities, we use the concepts and notations introduced in [19]. Thus, we denote by (a, b) the maximum number a (respectively, b) of infinite (respectively, finite) singularities that can be obtained by perturbation of a multiple infinite singularity. In this case, we say that an infinite singular point *has multiplicity* (a, b) .

In this paper, we consider only the case $\eta \neq 0$, that is, $\eta > 0$ or $\eta < 0$. In the first case, according to [23], a quadratic system possesses at infinity three real distinct singularities, whereas in the second case, it possesses one real and two complex singularities.

In [24], necessary and sufficient conditions for a system to belong to the family **QSP** of quadratic systems possessing at least one invariant parabola, in terms of invariant polynomials, are determined.

We extract from [24] only the information related to the case $\eta \neq 0$, and for this, we need some notations.

Definition 2.6. By the *direction of an invariant parabola* of a quadratic system (S) , we mean the direction of its axis of symmetry, which intersects the invariant line $Z = 0$ at an infinite singular point of (S) .

In order to distinguish the invariant parabolas that a quadratic system could have, we use the following notations:

- $\cup$ for a simple invariant parabola;
- $\mathbb{U}$ for two simple invariant parabolas in the same direction (they could intersect);
- $\cup\subset$ for two simple invariant parabolas in different directions;
- $\mathbf{U}^2$ for one double invariant parabola;
- $\mathbb{U}\subset$ for three simple invariant parabolas: two in one direction and one in another direction.

The proof of the next proposition can be found in [24].

Proposition 2.7. Assume that for a non-degenerate arbitrary quadratic system, the conditions $\eta > 0$ and $\zeta_1 \neq 0$ are satisfied. Then, this system could possess invariant parabolas only in one direction. More exactly, it could only possess one of the following sets of invariant parabolas: $\cup$, $\mathbb{U}$ and $\mathbf{U}^2$. Moreover, this system has one of the above sets of parabolas if and only if $\chi_1 = \chi_2 = 0$ and one of the following sets of conditions are satisfied, correspondingly:

- | | | |
|-------------------|---|-----------------------------|
| $(\mathcal{A}_1)$ | $\zeta_2 \neq 0, \zeta_3 \neq 0, \zeta_4 \neq 0, \mathcal{R}_1 \neq 0$ | $\Rightarrow \cup;$ |
| $(\mathcal{A}_2)$ | $\zeta_2 \neq 0, \zeta_3 \neq 0, \zeta_4 = 0, \mathcal{R}_2 \neq 0, \zeta_5 \neq 0$ | $\Rightarrow \mathbb{U};$ |
| $(\mathcal{A}_3)$ | $\zeta_2 \neq 0, \zeta_3 \neq 0, \zeta_4 = 0, \mathcal{R}_2 \neq 0, \zeta_5 = 0$ | $\Rightarrow \mathbf{U}^2;$ |
| $(\mathcal{A}_4)$ | $\zeta_2 \neq 0, \zeta_3 \neq 0, \zeta_4 = 0, \mathcal{R}_2 = 0, \zeta_5 \neq 0$ | $\Rightarrow \cup;$ |
| $(\mathcal{A}_5)$ | $\zeta_2 \neq 0, \zeta_3 = 0, \zeta_4 \neq 0, \mathcal{R}_1 \neq 0$ | $\Rightarrow \cup;$ |
| $(\mathcal{A}_6)$ | $\zeta_2 \neq 0, \zeta_3 = 0, \zeta_4 = 0, \mathcal{R}_2 \neq 0, \zeta_5 \neq 0$ | $\Rightarrow \mathbb{U};$ |
| $(\mathcal{A}_7)$ | $\zeta_2 \neq 0, \zeta_3 = 0, \zeta_4 = 0, \mathcal{R}_2 \neq 0, \zeta_5 = 0$ | $\Rightarrow \mathbf{U}^2;$ |
| $(\mathcal{A}_8)$ | $\zeta_2 \neq 0, \zeta_3 = 0, \zeta_4 = 0, \mathcal{R}_2 = 0, \zeta_5 \neq 0$ | $\Rightarrow \cup;$ |
| $(\mathcal{A}_9)$ | $\zeta_2 = 0, \zeta_6 \neq 0, \mathcal{R}_1 = 0, \mathcal{R}_2 \neq 0$ | $\Rightarrow \cup.$ |

Furthermore, in the case of the existence of an invariant parabola, a system with $\eta > 0$ and $\zeta_1 \neq 0$ could be brought via an affine transformation and time rescaling to the following canonical form:

$$\dot{x} = m + nx - \frac{1}{2}(1+g)y + gx^2 + xy, \quad \dot{y} = 2mx + 2ny + (g-1)xy + 2y^2 \quad (2.4)$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y = 0$.

However, examining the conditions $(\mathcal{A}_1)$ – $(\mathcal{A}_9)$, we detect that some of the sets of these conditions could be joined. More exactly, we observe that $(\mathcal{A}_1)$ and $(\mathcal{A}_5)$ contain the same conditions except for the condition involving ζ_3 . Therefore, by eliminating the conditions involving only ζ_3 , we obtain a new set of conditions which we denote by $(\mathcal{A}_1^*)$: $\zeta_2 \neq 0$, $\zeta_4 \neq 0$, $\mathcal{R}_1 \neq 0$.

We perform the analogous operation on $(\mathcal{A}_2)$ and $(\mathcal{A}_6)$ (respectively $(\mathcal{A}_3)$ and $(\mathcal{A}_7)$; $(\mathcal{A}_4)$ and $(\mathcal{A}_8)$), resulting in the new conditions $(\mathcal{A}_2^*)$ (respectively $(\mathcal{A}_3^*)$; $(\mathcal{A}_4^*)$).

Thus, we can replace the first part of Proposition 2.7 obtaining the following one:

Proposition 2.7*. Assume that for a non-degenerate arbitrary quadratic system, the conditions $\eta > 0$ and $\zeta_1 \neq 0$ are satisfied. Then, this system could possess invariant parabolas only in one direction. More exactly, it could only possess one of the following sets of invariant parabolas: $\cup$, $\mathbb{U}$ and $\mathbf{U}^2$. Moreover, this system has one of the above sets of parabolas if and only if $\chi_1 = \chi_2 = 0$ and one of the following sets of conditions are satisfied, correspondingly:

$$\begin{aligned} (\mathcal{A}_1^*) \quad & \zeta_2 \neq 0, \zeta_4 \neq 0, \mathcal{R}_1 \neq 0 & \Rightarrow \cup; \\ (\mathcal{A}_2^*) \quad & \zeta_2 \neq 0, \zeta_4 = 0, \mathcal{R}_2 \neq 0, \zeta_5 \neq 0 & \Rightarrow \mathbb{U}; \\ (\mathcal{A}_3^*) \quad & \zeta_2 \neq 0, \zeta_4 = 0, \mathcal{R}_2 \neq 0, \zeta_5 = 0 & \Rightarrow \mathbf{U}^2; \\ (\mathcal{A}_4^*) \quad & \zeta_2 \neq 0, \zeta_4 = 0, \mathcal{R}_2 = 0, \zeta_5 \neq 0 & \Rightarrow \cup; \\ (\mathcal{A}_5^*) \quad & \zeta_2 = 0, \zeta_6 \neq 0, \mathcal{R}_1 = 0, \mathcal{R}_2 \neq 0 & \Rightarrow \cup. \end{aligned}$$

The proof of the next two propositions could also be found in [24].

Proposition 2.8. Assume that for a non-degenerate arbitrary quadratic system, the conditions $\eta > 0$ and $\zeta_1 = 0$ are satisfied. Then, this system could possess invariant parabolas in one or two directions. More exactly, it could only possess one of the following sets of invariant parabolas: $\cup$, $\mathbb{U}$, $\mathbf{U}^2$, $\cup\subset$ and $\mathbb{U}\subset$. Moreover, this system has one of the above sets of invariant parabolas if and only if $\chi_1 = \chi_3 = 0$ and one of the following sets of conditions are satisfied, correspondingly:

$$\begin{aligned} (\mathcal{B}_1) \quad & \chi_4 \neq 0, \zeta_7 \neq 0, \mathcal{R}_3 \neq 0 & \Rightarrow \cup; \\ (\mathcal{B}_2) \quad & \chi_4 \neq 0, \zeta_7 = 0, \mathcal{R}_4 \neq 0, \zeta_8 \neq 0 & \Rightarrow \mathbb{U}; \\ (\mathcal{B}_3) \quad & \chi_4 \neq 0, \zeta_7 = 0, \mathcal{R}_4 \neq 0, \zeta_8 = 0 & \Rightarrow \mathbf{U}^2; \\ (\mathcal{B}_4) \quad & \chi_4 \neq 0, \zeta_7 = 0, \mathcal{R}_4 = 0 & \Rightarrow \cup; \\ (\mathcal{B}_5) \quad & \chi_4 = 0, \zeta_5 \neq 0, \zeta_9 \neq 0 & \Rightarrow \cup\subset; \\ (\mathcal{B}_6) \quad & \chi_4 = 0, \zeta_5 \neq 0, \zeta_9 = 0, \zeta_{10} \neq 0 & \Rightarrow \cup; \\ (\mathcal{B}_7) \quad & \chi_4 = 0, \zeta_5 = 0, \zeta_6 \neq 0 & \Rightarrow \mathbb{U}\subset. \end{aligned}$$

Furthermore, in the case of the existence of an invariant parabola, a system with $\eta > 0$ and $\zeta_1 = 0$ could be brought via an affine transformation and time rescaling to the systems (2.4) with $g = 2$.

Proposition 2.9. Assume that for a non-degenerate arbitrary quadratic system, the conditions $\eta < 0$ is satisfied. Then, this system could only possess one of the following sets of invariant parabolas: $\cup$, $\mathbb{U}$ and $\mathbf{U}^2$. Moreover, this system has one of the above sets of invariant parabolas if and only if $\chi_1 = \chi_2 = 0$ and $\zeta_1 \neq 0$ and one of the following sets of conditions are satisfied, correspondingly:

$$\begin{aligned} (\mathcal{E}_1) \quad & \zeta_4 \neq 0, \mathcal{R}_1 \neq 0 & \Rightarrow \cup; \\ (\mathcal{E}_2) \quad & \zeta_4 = 0, \mathcal{R}_7 \neq 0, \zeta_5 \neq 0 & \Rightarrow \mathbb{U}; \\ (\mathcal{E}_3) \quad & \zeta_4 = 0, \mathcal{R}_7 \neq 0, \zeta_5 = 0 & \Rightarrow \mathbf{U}^2; \\ (\mathcal{E}_4) \quad & \zeta_4 = 0, \mathcal{R}_7 = 0, \zeta_5 \neq 0 & \Rightarrow \cup. \end{aligned}$$

Furthermore, in the case of the existence of an invariant parabola, a system with $\eta < 0$ could be brought via an affine transformation and time rescaling to the following canonical form:

$$\dot{x} = m + (2n - 1)x/2 + gx^2 - gy/2 - xy, \quad \dot{y} = 2mx - x^2 + 2ny + gxy - 2y^2, \quad (2.5)$$

with $C_2 = x(x^2 + y^2)$, possessing the invariant parabola $\Phi(x, y) = x^2 - y = 0$.

3 The proof of the Main Theorem

The statement **(B)** follows directly from the form of the conditions given in Diagrams 1.1, 1.2, and 1.3. These conditions can be evaluated for any point $a \in \mathbb{R}^{12}$ corresponding to a quadratic system satisfying $\eta \neq 0$.

In order to prove the statement **(A)** of the Main Theorem, we must examine the sets of conditions provided by each one of Propositions 2.7*, 2.8, and 2.9.

3.1 Systems in $\mathbf{QSP}_{(\eta>0)}$ with the condition $\zeta_1 \neq 0$

In what follows, we examine the configurations of the systems in $\mathbf{QSP}_{(\eta>0)}$ in each one of the cases provided by Proposition 2.7*. According to Proposition 2.7, we consider the canonical form (2.4), i.e., the systems:

$$\dot{x} = m + nx - \frac{1}{2}(1+g)y + gx^2 + xy, \quad \dot{y} = 2mx + 2ny + (g-1)xy + 2y^2, \quad (3.1)$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y = 0$.

3.1.1 The statement $(\mathcal{A}_1^*)$

For systems (3.1), we calculate:

$$\begin{aligned} \zeta_1 &= 2(g-2)(3+g), \quad \zeta_2 = 4g(1+g), \\ \zeta_4 &= (g-2)(3+g)(1+7g+15g^2+9g^3-4m+2n+6gn)/16, \\ \mathcal{R}_1 &= -15g(1+g)(g-2)(3+g)(1+7g+15g^2+9g^3-4m+2n+6gn)/2, \\ B_1 &= m(g+8m+4n)(gn-2m-n)(1+2g+g^2-4m+2n+2gn) \\ &\quad \times (g+2g^2+g^3+4m+2n+2gn)/4. \end{aligned} \quad (3.2)$$

We consider two cases: $B_1 \neq 0$ and $B_1 = 0$.

The case $B_1 \neq 0$. Then, according to Lemma 2.3, systems (3.1) could not possess any invariant line.

We examine the finite singularities of these systems. Following [1, Proposition 5.1], we calculate the invariant polynomial $\mathbf{D} = 48F_1^2F_2$, where

$$\begin{aligned} F_1 &= -4m^2 + 2(g+1)m(g^2-2n) - (g+1)^2n^2, \\ F_2 &= 108m^2 + 2(g-1)m(1-2g+g^2-18n) + n^2(16n-1+2g-g^2). \end{aligned} \quad (3.3)$$

Thus, we discuss two subcases: $\mathbf{D} \neq 0$ and $\mathbf{D} = 0$.

The subcase $\mathbf{D} \neq 0$. We determine that systems (3.1) possess four finite singularities $M_i(x_i, y_i)$ ($i = 1, 2, 3, 4$) with the coordinates

$$\begin{aligned}
 x_1 &= -\frac{2m+n+gn}{g(1+g)}, \quad y_1 = \frac{2m}{1+g}; \quad x_2 = \frac{1}{6\mathcal{Z}^{1/3}}[\mathcal{Y} + (1-g)\mathcal{Z}^{1/3} + \mathcal{Z}^{2/3}], \\
 y_2 &= \frac{1}{36\mathcal{Z}}[3(\mathcal{Y}+4n)\mathcal{Z} + \mathcal{Y}^2\mathcal{Z}^{1/3} - 2(g-1)\mathcal{Y}\mathcal{Z}^{2/3} - 2(g-1)\mathcal{Z}^{4/3} + \mathcal{Z}^{5/3}; \\
 x_3 &= \frac{1}{12\mathcal{Z}^{1/3}}[-(1+i\sqrt{3})\mathcal{Y} + 2(1-g)\mathcal{Z}^{1/3} - (1-i\sqrt{3})\mathcal{Z}^{2/3}], \\
 y_3 &= -\frac{1}{72\mathcal{Z}}[-6(\mathcal{Y}+4n)\mathcal{Z} + (1-i\sqrt{3})\mathcal{Y}^2\mathcal{Z}^{1/3} - 2(1+i\sqrt{3})(g-1)\mathcal{Y}\mathcal{Z}^{2/3} \\
 &\quad - 2(1-i\sqrt{3})(g-1)\mathcal{Z}^{4/3} + (1+i\sqrt{3})\mathcal{Z}^{5/3}]; \\
 x_4 &= \frac{1}{12\mathcal{Z}^{1/3}}[(-1+i\sqrt{3})\mathcal{Y} + 2(1-g)\mathcal{Z}^{1/3} - (1+i\sqrt{3})\mathcal{Z}^{2/3}], \\
 y_4 &= -\frac{1}{72\mathcal{Z}}[-6(\mathcal{Y}+4n)\mathcal{Z} + (1+i\sqrt{3})\mathcal{Y}^2\mathcal{Z}^{1/3} - 2(1-i\sqrt{3})(g-1)\mathcal{Y}\mathcal{Z}^{2/3} \\
 &\quad - 2(1+i\sqrt{3})(g-1)\mathcal{Z}^{4/3} + (1-i\sqrt{3})\mathcal{Z}^{5/3}],
 \end{aligned} \tag{3.4}$$

where

$$\mathcal{Z} = 1 - 3g + 3g^2 - g^3 - 108m - 18n + 18gn + 6\sqrt{3}\sqrt{F_2}, \quad \mathcal{Y} = (1-g)^2 - 12n.$$

Calculations yield:

$$\Phi(x_2, y_2) = \Phi(x_3, y_3) = \Phi(x_4, y_4) = 0, \quad \Phi(x_1, y_1) = -\frac{F_1}{g^2(1+g)^2},$$

and therefore we deduce that three singularities M_2, M_3 and M_4 of systems (3.1) are located on the invariant parabola. Moreover, M_1 is located outside the parabola and could belong to it if and only if the condition $F_1 = 0$ holds, where F_1 is given in (3.3). However, we have $\mathbf{D} = 48F_1^2F_2 \neq 0$, and hence on the parabola we always have exactly three distinct singularities.

On the other hand, according to [1, Proposition 5.1], if $\mathbf{D} > 0$, systems (3.1) possess two real and two complex finite singularities. For $\mathbf{D} < 0$, we could either have four real or four complex finite singularities. However, since M_1 is a real singular point for these systems, we conclude that in the case $\mathbf{D} < 0$ we have four real finite distinct singularities.

Thus, since the real singularity M_1 is outside the invariant parabola and all other three finite singularities on the parabola (real or complex) are distinct and furthermore we could not have any invariant line, we arrive at the configuration *Config. P.1* if $\mathbf{D} < 0$ and *Config. P.2* if $\mathbf{D} > 0$.

The subcase $\mathbf{D} = 0$. This implies $F_1F_2 = 0$, and for systems (3.1), we calculate:

$$\xi_1 = -6\zeta_4F_1 \Rightarrow F_1 = 0 \Leftrightarrow \xi_1 = 0.$$

So we examine two possibilities: $\xi_1 \neq 0$ and $\xi_1 = 0$.

1: The possibility $\xi_1 \neq 0$. Then $F_1 \neq 0$, and therefore the condition $\mathbf{D} = 0$ implies $F_2 = 0$.

We observe that the polynomial F_2 is quadratic with respect to the parameter m , and we calculate

$$\text{Discrim} [F_2, m] = 4(1 - 2g + g^2 - 12n)^3.$$

Therefore, since the parameters m , n , and g of systems (3.1) must be real, we conclude that the condition $1 - 2g + g^2 - 12n \geq 0$ has to be fulfilled. Setting a new parameter v by $1 - 2g + g^2 - 12n = v^2 \geq 0$, we get $n = [(g - 1)^2 - v^2] / 12$, and then we calculate

$$F_2 = \frac{1}{432} [216m - (1 - g + v)^2(g - 1 + 2v)] [216m - (1 - g - v)^2(g - 1 - 2v)] = 0$$

and due to the change $v \mapsto -v$, we may force the first factor to vanish. Then we obtain

$$m = \frac{(1 - g + v)^2(g - 1 + 2v)}{216},$$

and considering the expressions for the parameters m and n , we arrive at the two-parameter family of systems

$$\begin{aligned} \dot{x} &= \frac{(1 - g + v)^2(g - 1 + 2v)}{216} + \frac{(g - 1)^2 - v^2}{12} x - \frac{1}{2}(1 + g)y + gx^2 + xy, \\ \dot{y} &= \frac{(1 - g + v)^2(g - 1 + 2v)}{108} x + \frac{(g - 1)^2 - v^2}{6} y + (g - 1)xy + 2y^2, \end{aligned} \quad (3.5)$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y = 0$. We observe that for the above systems, we have the following conditions on the parameters g and v :

$$\begin{aligned} \zeta_1 \zeta_2 \zeta_4 \mathcal{R}_1 \neq 0 &\Leftrightarrow g(g - 2)(1 + g)(3 + g)(2 + 4g - v)(4 + 8g + v) \neq 0; \\ \zeta_1 \neq 0 &\Leftrightarrow (g - 2)(3 + g)(g - 1 - v)(2 + g - v)(2 + 4g - v)(4 + 8g + v)^2 \\ &\quad \times (4 - 2g - 2g^2 - 4v - 8gv + v^2) \neq 0; \\ B_1 \neq 0 &\Leftrightarrow (g - 1 - v)(2 + g - v)(2 + 4g - v)(2g - 2 + v)(1 + 2g + v)(4 + 2g + v) \\ &\quad \times (g - 1 + 2v)(2 + g + 2v) \neq 0. \end{aligned} \quad (3.6)$$

We determine that systems (3.5) possess three finite singularities $M_i(x_i, y_i)$ ($i = 1, 2, 3$) with the coordinates

$$\begin{aligned} x_1 &= \frac{1 - g + v}{6}, \quad y_1 = \frac{(1 - g + v)^2}{36}; \quad x_2 = \frac{1 - g - 2v}{6}, \quad y_2 = \frac{(1 - g - 2v)^2}{36}; \\ x_3 &= \frac{(1 - g + v)(5g^2 - 4 - g + 4v + 5gv - v^2)}{54g(1 + g)}, \quad y_3 = \frac{(1 - g + v)^2(g - 1 + 2v)}{108(1 + g)}. \end{aligned} \quad (3.7)$$

We calculate

$$\begin{aligned} \Phi(x_1, y_1) &= \Phi(x_2, y_2) = 0, \\ \Phi(x_3, y_3) &= \frac{(g - 1 - v)^2(2 + g - v)^2(4 - 2g - 2g^2 - 4v - 8gv + v^2)}{2916g^2(1 + g)^2}, \end{aligned}$$

and we conclude that the singular points M_1 and M_2 are located on the invariant parabola.

On the other hand, considering the conditions (3.6), we obtain that M_3 will be located on $\Phi(x, y) = 0$ if and only if

$$\alpha = 4 - 2g(1 + g) - 4v - 8gv + v^2 = 0.$$

However, considering (3.6), we conclude that $\alpha \neq 0$ (due to $\zeta_1 \neq 0$), and hence the singularity M_3 is not located on the invariant parabola in the considered case.

We claim that M_1 is a multiple singularity of systems (3.5). Indeed, applying the corresponding translation, we could place M_1 at the origin of coordinates and arrive at the systems

$$\begin{aligned}\dot{x} &= -\frac{(g-v-1)(4g-v+2)}{18}x - \frac{4g-v+2}{6}y + gx^2 + xy, \\ \dot{y} &= \frac{(g-v-1)^2(2g+v-2)}{54}x + \frac{(g-v-1)(2g+v-2)}{18}y + (g-1)xy + 2y^2,\end{aligned}$$

where $M_0(0,0)$ is a singularity of the above systems corresponding to the singularity M_2 .

Considering [1, Lemma 5.2], we calculate the following invariant polynomials: $\mu_4 = \mu_3 = 0$, and

$$\mu_2 = -\frac{1}{324}v(g-v-1)(g-v+2)[(2g+v-2)x+6y][g(g-v-1)x+(2+4g-v)y].$$

Therefore, by [1, Lemma 5.2, statement (ii)], the point M_0 is of multiplicity at least 2. We observe that due to the condition $\zeta_1 \neq 0$, we have $\mu_2 = 0$ if and only if $v = 0$. In this case, we calculate

$$\mu_2 = 0, \quad \mu_1 = \frac{1}{27}(g-1)(g+2)[g(g-1)x+2(2g+1)y] \neq 0,$$

due to $\zeta_1 \neq 0$. According to [1, Lemma 5.2, statement (ii)], $M_0(0,0)$ is a double point if $v \neq 0$, and it is a triple one if $v = 0$.

On the other hand, for systems (3.5), we have

$$\zeta_2 = \frac{1}{209952}(g-1-v)^2(2+g-v)^2v^2\alpha^2,$$

and due to the conditions (3.6), we conclude that the condition $v = 0$ is equivalent to $\zeta_2 = 0$.

Thus, for systems (3.5), we have the configuration *Config. P.3* if $\zeta_2 \neq 0$ and *Config. P.4* if $\zeta_2 = 0$.

2: The possibility $\zeta_1 = 0$. This implies $F_1 = 0$, and since the polynomial F_1 is quadratic with respect to the parameter m , we calculate

$$\text{Discrim}[F_1, m] = 4g^2(1+g)^2(g^2-4n).$$

Since $g(g+1) \neq 0$ (due to $\zeta_2 \neq 0$), we must have $g^2-4n \geq 0$. So we set a new parameter u as follows: $g^2-4n = u^2 \geq 0$, and we get $n = (g^2-u^2)/4$. Then, calculation yields

$$F_1 = -\frac{1}{16}[8m - (1+g)(g+u)^2][8m - (1+g)(g-u)^2] = 0,$$

and due to the change $u \mapsto -u$, we may force the second factor to vanish. In this case, we obtain

$$m = \frac{(1+g)(g-u)^2}{8},$$

and considering the expression for the parameters m and n , we arrive at the two-parameter family of systems

$$\begin{aligned}\dot{x} &= \frac{(1+g)(g-u)^2}{8} + \frac{g^2-u^2}{4}x - \frac{1}{2}(1+g)y + gx^2 + xy, \\ \dot{y} &= \frac{(1+g)(g-u)^2}{4}x + \frac{g^2-u^2}{2}y + (g-1)xy + 2y^2,\end{aligned}\tag{3.8}$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y = 0$. We observe that for the above systems, we have the following condition on the parameters g and u :

$$\begin{aligned} \zeta_1 \zeta_2 \zeta_4 \mathcal{R}_1 \neq 0 &\Leftrightarrow (g-2)g(1+g)(3+g)(1+2g+u)(1+5g+5g^2-u-2gu) \neq 0; \\ B_1 \neq 0 &\Leftrightarrow g(1+g)(g-u)(1+g-u)(1+2g-u)(-1+u)(1+u) \neq 0. \end{aligned} \quad (3.9)$$

We determine that systems (3.8) possess three finite singularities $M_i(x_i, y_i)$ ($i = 1, 2, 3$) with the coordinates

$$\begin{aligned} x_1 &= \frac{u-g}{2}, \quad y_1 = \frac{(u-g)^2}{4}; \quad x_{2,3} = \frac{1}{4}(1-u \pm \sqrt{Z_1}), \\ y_{2,3} &= \frac{1}{8}[1-2g-2g^2+2gu+u^2 \mp (u-1)\sqrt{Z_1}], \quad Z_1 = -4g^2+4g(-1+u)+(1+u)^2. \end{aligned} \quad (3.10)$$

We calculate

$$\Phi(x_1, y_1) = \Phi(x_2, y_2) = \Phi(x_3, y_3) = 0,$$

and therefore all three singularities are located on the invariant parabola.

We point out that M_1 is a multiple singularity of systems (3.8). Indeed, applying the corresponding translation, we could place M_1 at the origin of coordinates and arrive at the systems

$$\begin{aligned} \dot{x} &= -\frac{1}{2}g(g-u)x + \frac{1}{2}(u-2g-1)y + gx^2 + xy, \\ \dot{y} &= \frac{1}{2}g(g-u)^2x + \frac{1}{2}(2g-u+1)(g-u)y + (g-1)xy + 2y^2, \end{aligned}$$

where $M_0(0,0)$ is a singularity of the above systems corresponding to the singularity M_1 .

Considering [1], we calculate the invariant polynomials μ_4 , μ_3 , μ_2 , and we obtain $\mu_4 = \mu_3 = 0$, and

$$\mu_2 = \frac{1}{2}g(g+1)(g-u)(g-u+1)[g(g-u)x^2 + (2g-1-u)xy + 2y^2] \neq 0,$$

due to the conditions (3.9). By [1, Lemma 5.2, statement (ii)], the point M_0 is of multiplicity exactly 2.

On the other hand, it is clear that the singularities M_2 and M_3 could be complex (respectively real; coinciding) if $Z_1 < 0$ (respectively $Z_1 > 0$; $Z_1 = 0$). We observe that for systems (3.8), we have:

$$\zeta_2 = g^2(1+g)^2(g-u)^2(1+g-u)^2Z_1,$$

and due to the conditions (3.9), we conclude that the sign of Z_1 is governed by the invariant polynomial ζ_2 . So we discuss three cases: $\zeta_2 < 0$, $\zeta_2 > 0$, and $\zeta_2 = 0$.

2.1: The case $\zeta_2 < 0$. This implies $Z_1 < 0$, and then systems (3.5) possess only one real singular point M_1 (which is double), and evidently we get the configuration *Config. P.5*.

2.2: The case $\zeta_2 > 0$. Then $Z_1 > 0$, and this implies the existence of three real singularities, and we have to determine the position of the double point with respect to the simple ones. So we calculate

$$(x_3 - x_1)(x_2 - x_1) = \frac{(g-u)(1+g-u)}{2} \equiv \frac{\alpha_1}{2}, \quad \text{sign}((x_3 - x_1)(x_2 - x_1)) = \text{sign}(\alpha_1), \quad (3.11)$$

where $\alpha_1 \neq 0$ due to $B_1 \neq 0$. This means that the singularity M_1 could not coalesce with one of the singularities M_2 or M_3 .

On the other hand, for systems (3.5), calculations yield:

$$\xi_3 = \frac{27249129}{2} g^2 (1+g)^2 \alpha_1^3 Z_1.$$

So, due to the conditions (3.9), we deduce that $\text{sign}(\xi_3) = \text{sign}(\alpha_1 Z_1)$.

Therefore, in the case $\xi_3 < 0$, the double singular point M_1 is located on the parabola between M_2 and M_3 , and we arrive at the configuration *Config. P.6*. If $\xi_3 > 0$, we evidently get the configuration *Config. P.7*.

2.3: The case $\xi_2 = 0$. Then $Z_1 = 0$, which implies the coalescence of the singularities M_2 and M_3 . Therefore, systems (3.5) possess two double singularities located on the invariant parabola. So we obtain the configuration *Config. P.8*.

It remains to mention that the case $u = 0$ (i.e., when the discriminant of F_1 vanishes) is included in the previous examination because the condition $u \neq 0$ was not necessary. So, in this case, we obtain the same configurations for the corresponding conditions, respectively.

The case $B_1 = 0$. Considering (3.2), we observe that the condition $B_1 = 0$ splits into five conditions at the coefficient level. However, due to an affine transformation, we can reduce this number. More precisely, we have the following lemma.

Lemma 3.1. *The condition $(g + 8m + 4n)(1 + 2g + g^2 - 4m + 2n + 2gn) = 0$ for systems (3.1) can be transformed into the condition $m(gn - 2m - n) = 0$ via an affine transformation.*

Proof. Applying to systems (3.1) the transformation

$$x_1 = -x + \frac{1}{2}, \quad y_1 = -x + y + \frac{1}{4},$$

we obtain the systems

$$\begin{aligned} \dot{x}_1 &= -\frac{1}{8}(g + 8m + 4n) + \frac{1}{4}(1 + 2g + 4n)x_1 + \frac{g}{2}y_1 - (1 + g)x_1^2 + x_1y_1, \\ \dot{y}_1 &= -\frac{1}{4}(g + 8m + 4n)x_1 + \frac{1}{2}(1 + 2g + 4n)y_1 - (g + 2)x_1y_1 + 2y_1^2. \end{aligned}$$

So, setting the new parameters

$$\begin{aligned} m_1 &= -\frac{1}{8}(g + 8m + 4n), & n_1 &= \frac{1}{4}(1 + 2g + 4n), & g_1 &= -(1 + g) \Rightarrow \\ m &= -\frac{1}{8}(g_1 + 8m_1 + 4n_1), & n &= \frac{1}{4}(1 + 2g_1 + 4n_1), & g &= -(1 + g_1), \end{aligned} \quad (3.12)$$

we obtain the family of systems

$$\dot{x}_1 = m_1 + n_1x_1 - \frac{1 + g_1}{2}y_1 + g_1x_1^2 + x_1y_1, \quad \dot{y}_1 = 2m_1x_1 + 2n_1y_1 + (g_1 - 1)x_1y_1 + 2y_1^2.$$

which has the same form as (3.1).

Then, considering (3.12), calculations yield:

$$g + 8m + 4n = -8m_1, \quad 1 + 2g + g^2 - 4m + 2n + 2gn = 2(2m_1 + n_1 - g_1n_1),$$

and this completes the proof of the lemma. $\square$

Thus, by Lemma 3.1, in order to examine the condition $B_1 = 0$, it is sufficient to consider the condition

$$m(gn - 2m - n)(g + 2g^2 + g^3 + 4m + 2n + 2gn) = 0.$$

In order to determine the invariant conditions that distinguish the three possibilities provided by the above equality, for systems (3.1) we calculate:

$$\begin{aligned}\zeta_4 &= 21 \cdot 2^6 5^4 m(g + 8m + 4n)\zeta_4, \\ \zeta_5 &= -14 \cdot 5^5 (gn - 2m - n)(1 + 2g + g^2 - 4m + 2n + 2gn)\zeta_4.\end{aligned}\tag{3.13}$$

Hence, due to $\zeta_4 \neq 0$, the condition $\zeta_4 = 0$ is equivalent to $m(g + 8m + 4n) = 0$ (which implies $B_1 = 0$), whereas the condition $\zeta_5 = 0$ is equivalent to $(gn - 2m - n)(1 + 2g + g^2 - 4m + 2n + 2gn) = 0$ (which also implies $B_1 = 0$).

The subcase $\zeta_4 \neq 0$. Then $m(g + 8m + 4n) \neq 0$, and we consider two possibilities: $\zeta_5 \neq 0$ and $\zeta_5 = 0$.

1: The possibility $\zeta_5 \neq 0$. In this case, we have $(gn - 2m - n)(1 + 2g + g^2 - 4m + 2n + 2gn) \neq 0$, and therefore the condition $B_1 = 0$ implies $g + 2g^2 + g^3 + 4m + 2n + 2gn = 0$. This yields $m = -(1 + g)(g + g^2 + 2n)/4$, and we get the family of systems

$$\begin{aligned}\dot{x} &= -\frac{1}{4}(1 + g - 2x)(g + g^2 + 2n + 2gx + 2y), \\ \dot{y} &= -\frac{(1 + g)(g + g^2 + 2n)}{2}x + 2ny + (g - 1)xy + 2y^2\end{aligned}\tag{3.14}$$

possessing the invariant line $x = (g + 1)/2$. For these systems, we calculate

$$\begin{aligned}B_2 &= -81g^2(1 + g)^2(g + g^2 + 2n)(1 + 4g + 2g^2 + 4n)(1 + 2g + g^2 + 4n)^2x^4, \\ \zeta_4 &= 13125g(1 + g)(g - 2)(3 + g)(1 + 2g)(g + g^2 + 2n)(1 + 4g + 2g^2 + 4n)(1 + 6g + 5g^2 + 4n), \\ \zeta_5 &= -21875(g - 2)g(1 + g)(3 + g)(1 + 2g)(1 + 2g + g^2 + 4n)^2(1 + 6g + 5g^2 + 4n)/16,\end{aligned}$$

and we observe that the condition $\zeta_4\zeta_5 \neq 0$ implies $B_2 \neq 0$.

Then, by Lemma 2.3, besides the invariant line $x = (g + 1)/2$, systems (3.14) could not possess invariant lines in other directions. However, they could have a parallel invariant line, and by Lemma 2.4, for this to occur it is necessary that $\theta = 0$. This condition implies $(g - 1)(g + 2) = 0$. A straightforward calculation shows that neither of the conditions $g = 1$ nor $g = -2$ could imply the appearance of an additional parallel invariant line.

Next, we determine that systems (3.14) possess four finite singularities $M_i(x_i, y_i)$ ($i = 1, 2, 3, 4$) with the coordinates

$$\begin{aligned}x_1 &= \frac{1 + g}{2}, y_1 = \frac{(1 + g)^2}{4}; \quad x_2 = \frac{1 + g}{2}, y_2 = -\frac{g + g^2 + 2n}{2}; \quad x_{3,4} = \frac{1}{2}(-g \pm \sqrt{Z_2}), \\ y_{3,4} &= \frac{1}{2}(-g - 2n \mp g\sqrt{Z_2}), \quad Z_2 = -(2g + g^2 + 4n).\end{aligned}\tag{3.15}$$

We determine that the singularities M_1, M_3 , and M_4 are located on the invariant parabola. At the same time, M_1 and M_2 are located on the invariant line $x = (g + 1)/2$, and M_1 is the point of intersection of this invariant line with the parabola.

In order to determine the relative position of the singularities M_1 and M_2 on the vertical invariant line, we calculate

$$y_2 - y_1 = -\frac{1 + 4g + 3g^2 + 4n}{4} \equiv -\frac{\alpha_2}{4} \Rightarrow \text{sign}(y_2 - y_1) = -\text{sign}(\alpha_2). \quad (3.16)$$

Since the singularities M_3 and M_4 are either complex, real, or coinciding depending on the value of Z_2 , we need to distinguish these conditions using affine invariant polynomials. For systems (3.14), we calculate:

$$\begin{aligned} \zeta_4 &= \frac{1}{16}(g-2)(3+g)(1+2g)(1+6g+5g^2+4n) \equiv \frac{1}{16}(g-2)(3+g)(1+2g)\beta_2, \\ \mathbf{D} &= -\frac{3}{4}g^4(1+g)^4\beta_2^2\alpha_2^2Z_2, \quad \zeta_2 = 4g(1+g), \end{aligned} \quad (3.17)$$

and due to $\zeta_2\zeta_4 \neq 0$, we conclude that $\mathbf{D} = 0$ is equivalent to $\alpha_2Z_2 = 0$. Moreover, if $\mathbf{D} \neq 0$, then $\text{sign}(\mathbf{D}) = -\text{sign}(Z_2)$. Thus, we discuss three cases: $\mathbf{D} < 0$, $\mathbf{D} > 0$, and $\mathbf{D} = 0$.

1.1: The case $\mathbf{D} < 0$. This implies $Z_2 > 0$, and systems (3.14) possess four real singularities. Clearly, it is necessary to know the position of the real singularities $M_{3,4}$ with respect to M_1 , all located on the invariant parabola. We calculate:

$$\begin{aligned} (x_3 - x_1)(x_4 - x_1) &= \frac{\beta_2}{4}, \quad (x_3 - x_1) + (x_4 - x_1) = -(1 + 2g), \\ \text{sign}((x_3 - x_1)(x_4 - x_1)) &= \text{sign}(\beta_2), \quad \text{sign}((x_1 - x_3) + (x_1 - x_4)) = -\text{sign}(1 + 2g). \end{aligned}$$

We observe that $\beta_2 \neq 0$ due to the condition $\zeta_4 \neq 0$, and moreover $\alpha_2 \neq 0$ due to $\mathbf{D} \neq 0$.

On the other hand, we need the invariant polynomials that govern the signs of β_2 and α_2 . Thus, for systems (3.14), we calculate:

$$\zeta_7 = 1174627500g^2(1+g)^2(1+2g)^2\alpha_2^2\beta_2Z_2, \quad \zeta_8 = 61822500g^2(1+g)^2(1+2g)^2\alpha_2\beta_2^2Z_2.$$

Due to the conditions $\zeta_4 \neq 0$ and $\mathbf{D} < 0$, which imply $g(1+g)(1+2g)\alpha_2\beta_2 \neq 0$ and $Z_2 > 0$ (also ensuring $\zeta_7\zeta_8 \neq 0$), we have the following relations:

$$\text{sign}(\beta_2) = \text{sign}(\zeta_7), \quad \text{sign}(\alpha_2) = \text{sign}(\zeta_8).$$

Thus, considering the above relations, in the case $\mathbf{D} < 0$ we detect the following configurations:

$$\begin{aligned} \zeta_7 < 0, \zeta_8 < 0 &\Rightarrow (x_3 - x_1)(x_4 - x_1) < 0, y_2 > y_1 \Rightarrow \text{Config. } \mathcal{P}.9; \\ \zeta_7 < 0, \zeta_8 > 0 &\Rightarrow (x_3 - x_1)(x_4 - x_1) < 0, y_2 < y_1 \Rightarrow \text{Config. } \mathcal{P}.10; \\ \zeta_7 > 0, \zeta_8 < 0 &\Rightarrow (x_3 - x_1)(x_4 - x_1) > 0, y_2 > y_1 \Rightarrow \text{Config. } \mathcal{P}.11; \\ \zeta_7 > 0, \zeta_8 > 0 &\Rightarrow (x_3 - x_1)(x_4 - x_1) > 0, y_2 < y_1 \Rightarrow \text{Config. } \mathcal{P}.12. \end{aligned}$$

1.2: The case $\mathbf{D} > 0$. Then $Z_2 < 0$, and we claim that this condition implies $\alpha_2 > 0$. Indeed, supposing the contrary (i.e., $\alpha_2 < 0$), we must have $Z_2 + \alpha_2 < 0$. However, calculations yield:

$$Z_2 + \alpha_2 = -(2g + g^2 + 4n) + (1 + 4g + 3g^2 + 4n) = (1 + g)^2 + g^2 > 0. \quad (3.18)$$

The contradiction obtained proves our claim.

Therefore, since M_3 and M_4 are complex, we arrive at the configuration *Config. P.13*.

1.3: The case $\mathbf{D} = 0$. Considering (3.17), we deduce that, due to $\zeta_2\zeta_4 \neq 0$, the condition $\mathbf{D} = 0$ implies $\alpha_2Z_2 = 0$.

On the other hand, for systems (3.14), we calculate:

$$\xi_1 = \frac{3}{32}g^2(1+g)^2(g-2)(3+g)(1+2g)\alpha_2\beta_2.$$

Thus, due to $\zeta_2\zeta_4 \neq 0$ (i.e., $g(1+g)(g-2)(3+g)(1+2g)\beta_2 \neq 0$), we obtain that the condition $\alpha_2 = 0$ is equivalent to $\xi_1 = 0$. Therefore, we discuss two subcases: $\xi_1 \neq 0$ and $\xi_1 = 0$.

1.3.1: *The subcase $\xi_1 \neq 0$.* In this case, we have $\alpha_2 \neq 0$, and the condition $\mathbf{D} = 0$ implies $Z_2 = 0$. Then M_3 and M_4 coalesce, producing a double point located on the invariant parabola. Considering (3.18), we deduce that the condition $Z_2 = 0$ implies $\alpha_2 > 0$.

Thus, it is not difficult to determine that, in this case, we arrive at the configuration *Config. P.14*.

1.3.2: *The subcase $\xi_1 = 0$.* This implies $\alpha_2 = 0$, and, as we have mentioned earlier (see formulas (3.16)), in this case, we get $y_2 = y_1$, and hence the intersection point M_1 of the invariant line $x = (g+1)/2$ with the parabola becomes a double singularity of systems (3.14). Moreover, the position of the real singularities M_3 and M_4 with respect to M_1 depends on the sign of β_2 .

Thus, the condition $\alpha_2 = 0$ implies $n = -(1+g)(1+3g)/4$, and then we obtain

$$\beta_2 = 2g(1+g), \quad \zeta_2 = 4g(1+g) \Rightarrow \text{sign}(\beta_2) = \text{sign}(\zeta_2).$$

Therefore, in the case $\alpha_2 = 0$ (i.e., $\xi_1 = 0$), we obtain the following two configurations:

$$\begin{aligned} \zeta_2 < 0 &\Rightarrow (x_3 - x_1)(x_4 - x_1) < 0, y_2 = y_1 \Rightarrow \text{Config. P.15}; \\ \zeta_2 > 0 &\Rightarrow (x_3 - x_1)(x_4 - x_1) > 0, y_2 = y_1 \Rightarrow \text{Config. P.16}. \end{aligned}$$

2: The possibility $\xi_5 = 0$. Considering (3.13) and the condition $\zeta_4 \neq 0$, we obtain that the condition $\xi_5 = 0$ implies

$$(gn - 2m - n)(1 + 2g + g^2 - 4m + 2n + 2gn) = 0.$$

On the other hand, according to Lemma 3.1, it is sufficient to examine the condition given by the first factor, because the condition defined by the second factor can be brought to the first one via an affine transformation.

Thus, in what follows, we assume that for systems (3.1), the condition $gn - 2m - n = 0$ holds. Then $m = n(g-1)/2$, and we arrive at the family of systems

$$\dot{x} = \frac{n(g-1)}{2} + nx - \frac{1}{2}(1+g)y + gx^2 + xy, \quad \dot{y} = (n+y)(gx - x + 2y), \quad (3.19)$$

which possess the invariant line $y = -n$ and the invariant parabola $\Phi(x, y) = x^2 - y = 0$. For these systems, we calculate

$$\begin{aligned} B_2 &= -81g^2(1+4n)[(1+g)^2 + 4n]^2y^4/2, \\ \zeta_4 &= 26250(g-1)g(1+g)(g-2)(3+g)n(1+4n)(1+6g+9g^2+4n), \end{aligned} \quad (3.20)$$

and we consider two cases: $B_2 \neq 0$ and $B_2 = 0$.

2.1: *The case $B_2 \neq 0$.* In this case, by Lemma 2.3, systems (3.19) cannot possess invariant lines in other directions than the invariant line $y = -n$. However, by Lemma 2.4, these systems could possess an invariant line parallel to the existing one if $\theta = -8(g-1)(2+g) = 0$. Thus,

due to $\xi_4 \neq 0$, this condition implies $g = -2$. However, in this case, systems (3.19) do not have any invariant line parallel to $y = -n$.

Next, we determine that systems (3.19) possess the finite singularities $M_i(x_i, y_i)$ ($i = 1, 2, 3, 4$) with coordinates

$$\begin{aligned} x_1 = \sqrt{-n}, y_1 = -n; \quad x_2 = -\sqrt{-n}, y_2 = -n; \quad x_3 = \frac{1-g}{2}, y_3 = \frac{(1-g)^2}{4}; \\ x_4 = -\frac{2n}{1+g}, y_4 = \frac{n(g-1)}{1+g}. \end{aligned} \quad (3.21)$$

We observe that the singular points M_1, M_2 , and M_3 lie on the invariant parabola $\Phi(x, y) = x^2 - y = 0$. Moreover, M_1 and M_2 are the points of intersection between the parabola and the invariant line $y = -n$, and since $n \neq 0$ (due to $\xi_4 \neq 0$), they are either complex (for $n > 0$) or real (for $n < 0$).

On the other hand, for systems (3.19), calculations yield:

$$\mathbf{D} = 48g^4n^3(1 - g^2 + 4n)^2(1 - 2g + g^2 + 4n)^2 \equiv 48g^4n^3\alpha_3^2\beta_3^2,$$

and it is clear that, in the case $\mathbf{D} \neq 0$, we have $\text{sign}(\mathbf{D}) = \text{sign}(n)$.

To determine the position of the singular point M_4 , we calculate

$$\Phi(x_4, y_4) = \frac{n\alpha_3}{(1+g)^2},$$

and since $n \neq 0$ (due to $\xi_4 \neq 0$), we deduce that the singular point M_4 lies on the invariant parabola if and only if $\alpha_3 = 0$.

To examine the configurations of the systems, we consider three subcases: $\mathbf{D} < 0$, $\mathbf{D} > 0$, and $\mathbf{D} = 0$.

2.1.1: *The subcase $\mathbf{D} < 0$.* Then $n < 0$, and the singular points M_1 and M_2 are real. In order to determine the position of the singularity M_3 with respect to the real singularities M_1 and M_2 , we calculate

$$\begin{aligned} (x_3 - x_1)(x_3 - x_2) &= \frac{1}{4}[(1-g)^2 + 4n] \equiv \beta_3, \quad (x_3 - x_1) + (x_3 - x_2) = 1 - g; \\ \text{sign}((x_3 - x_1)(x_3 - x_2)) &= \text{sign}(\beta_3), \quad \text{sign}((x_3 - x_1) + (x_3 - x_2)) = \text{sign}(1 - g). \end{aligned}$$

We observe that $\alpha_3\beta_3 \neq 0$ due to $\mathbf{D} \neq 0$, and we need to determine the invariant polynomials responsible for the signs of β_3 and $g - 1$. Calculations yield:

$$\begin{aligned} \xi_9 &= 22359252960 g^6(1+g)^2(1+2g+g^2+4n)^2\beta_3, \\ \xi_{10} &= 24814861965 (g-1)g^2(1+g)^2(1+2g+g^2+4n)^2(1+6g+9g^2+4n)^2/2. \end{aligned} \quad (3.22)$$

Taking into account the condition $\xi_4 B_2 \neq 0$ and from (3.20), we deduce that $\text{sign}(\beta_3) = \text{sign}(\xi_9)$ and $\text{sign}(g - 1) = \text{sign}(\xi_{10})$.

Thus, considering the above relations, in the case $\mathbf{D} < 0$, we arrive at the following configurations:

$$\begin{aligned} \xi_9 < 0 &\Rightarrow (x_3 - x_1)(x_3 - x_2) < 0 &\Rightarrow \text{Config. } \mathcal{P}.17; \\ \xi_9 > 0, \xi_{10} < 0 &\Rightarrow (x_3 - x_1) > 0, (x_3 - x_2) > 0 &\Rightarrow \text{Config. } \mathcal{P}.18; \\ \xi_9 > 0, \xi_{10} > 0 &\Rightarrow (x_3 - x_1) < 0, (x_3 - x_2) < 0 &\Rightarrow \text{Config. } \mathcal{P}.19. \end{aligned}$$

2.1.2: *The subcase $\mathbf{D} > 0$.* Then $n > 0$, and the singular points M_1 and M_2 are complex. Therefore, due to the condition $\alpha_3 \neq 0$, we arrive at the configuration *Config. $\mathcal{P}.20$* .

2.1.3: The subcase $\mathbf{D} = 0$. Since $n \neq 0$ (due to $\zeta_4 \neq 0$), this implies $\alpha_3 \beta_3 = 0$, and we have to distinguish two cases: $\beta_3 \neq 0$ and $\beta_3 = 0$. From (3.22) we observe that, due to $\zeta_4 B_2 \neq 0$, the condition $\zeta_9 = 0$ is equivalent to $\beta_3 = 0$.

2.1.3.1: The possibility $\zeta_9 \neq 0$. Then $\beta_3 \neq 0$, and the condition $\mathbf{D} = 0$ implies $\alpha_3 = 0$, yielding $n = (g^2 - 1)/4$. Considering (3.21), we observe that in this case the singular point M_4 coalesces with M_3 , producing a double singular point on the invariant parabola. Thus, the finite singularities of systems (3.19) have the following coordinates:

$$\begin{aligned} x_1 &= \frac{\sqrt{1-g^2}}{2}, & y_1 &= \frac{1-g^2}{4}; & x_2 &= -\frac{\sqrt{1-g^2}}{2}, & y_2 &= \frac{1-g^2}{4}; \\ x_3 &= x_4 = \frac{1-g}{2}, & y_3 &= y_4 = \frac{(1-g)^2}{4}. \end{aligned}$$

We note that in this case $\beta_3 = 2g(g-1)$, and it is necessary to determine, in an invariant way, the signs of the expressions $1-g^2$ and $g(g-1)$. For systems (3.19) with $n = (g^2 - 1)/4$, we calculate:

$$\zeta_2 = \frac{1}{2}(1-g^2)^3 g^4, \quad \zeta_9 = 44718505920(g-1)g^9(1+g)^4.$$

We observe that $\text{sign}(\zeta_2) = \text{sign}(1-g^2)$ and $\text{sign}(\zeta_9) = \text{sign}(g(g-1))$.

Thus, in the case $\alpha_3 = 0$ which implies $\mathbf{D} = 0$ (and the existence of a double real singularity on the invariant parabola), we obtain the following configurations:

$$\begin{aligned} \zeta_2 < 0 &\Rightarrow M_1 \text{ and } M_2 \text{ are complex} &\Rightarrow \text{Config. } \mathcal{P}.21; \\ \zeta_2 > 0, \zeta_9 < 0 &\Rightarrow (x_3 - x_1)(x_3 - x_2) < 0 &\Rightarrow \text{Config. } \mathcal{P}.22; \\ \zeta_2 > 0, \zeta_9 > 0 &\Rightarrow (x_3 - x_1) > 0, (x_3 - x_2) > 0 &\Rightarrow \text{Config. } \mathcal{P}.23. \end{aligned}$$

2.1.3.2: The possibility $\zeta_9 = 0$. This implies $\beta_3 = 0$, and hence we get $n = -(g-1)^2/4$. We observe that in this case, considering (3.21), we obtain:

$$\begin{aligned} x_1 &= \frac{1}{2}\sqrt{(g-1)^2}, & y_1 &= \frac{(g-1)^2}{4}; & x_2 &= -\frac{1}{2}\sqrt{(g-1)^2}, & y_2 &= \frac{(g-1)^2}{4}; \\ x_3 &= \frac{1-g}{2}, & y_3 &= \frac{(g-1)^2}{4}; & x_4 &= \frac{(g-1)^2}{2(1+g)}, & y_4 &= \frac{(g-1)^3}{4(1+g)}. \end{aligned}$$

We observe that the singular point M_3 either coincides with M_1 or with M_2 . Since x_1 is positive and x_2 is negative, we conclude that M_3 coalesces with M_1 if $1-g > 0$, and with M_2 if $1-g < 0$.

On the other hand, for systems (3.19) with $n = -(g-1)^2/4$, we have:

$$\zeta_{10} = 12705209326080(g-1)g^6(1+g)^4,$$

and hence we obtain $\text{sign}(\zeta_{10}) = \text{sign}(g-1)$. Therefore, it is not difficult to determine that we obtain the configuration *Config. P.24* if $\zeta_{10} < 0$, and *Config. P.25* if $\zeta_{10} > 0$.

2.2: The case $B_2 = 0$. Since $\zeta_4 \neq 0$ (i.e., $g(1+4n) \neq 0$), considering (3.20), this condition implies $(1+g)^2 + 4n = 0$.

Then we get $n = -(1+g)^2/4$, which leads to the family of systems:

$$\dot{x} = -\frac{1}{8}(1+g-2x)(-1+g^2+4gx+4y), \quad \dot{y} = -\frac{1}{4}(1+2g+g^2-4y)(-x+gx+2y), \quad (3.23)$$

which possess the following three invariant affine lines:

$$1+g-2x=0, \quad 1+2g+g^2-4y=0, \quad 1-g^2-4x+4y=0.$$

For these systems, we have $B_2 = B_3 = 0$, and we see that they possess invariant lines in three distinct directions. However, parallel invariant lines may appear, and by Lemma 2.4, for this to occur, it is necessary that $\theta = 0$. Thus, we discuss two subcases: $\theta \neq 0$ and $\theta = 0$.

2.2.1: *The subcase $\theta \neq 0$.* For the above systems, we calculate the coordinates of the finite singularities $M_i(x_i, y_i)$ ($i = 1, 2, 3, 4$):

$$\begin{aligned} x_1 &= \frac{1+g}{2}, & y_1 &= \frac{(1+g)^2}{4}; & x_2 &= -\frac{1+g}{2}, & y_2 &= \frac{(1+g)^2}{4}; \\ x_3 &= \frac{1-g}{2}, & y_3 &= \frac{(1-g)^2}{4}; & x_4 &= \frac{1+g}{2}, & y_4 &= \frac{1-g^2}{4}. \end{aligned}$$

We observe that the singular point M_1 is the intersection point of all three invariant lines, as well as lying on the invariant parabola. Since this point, together with M_4 , lies on the vertical invariant line $1+g-2x=0$, the relative positions of these two points are crucial for determining the configurations of systems (3.23). Thus, we calculate:

$$y_4 - y_1 = -\frac{g(g+1)}{2} \Rightarrow \text{sign}(y_4 - y_1) = -\text{sign}(g(g+1)).$$

We also point out that the position of the vertical invariant line $x = (g+1)/2$ is important, and we must consider $\text{sign}(g+1)$.

On the other hand, for systems (3.23), we calculate: $\zeta_2 = 4g(1+g)$, and then we determine the following configurations:

$$\begin{aligned} \zeta_2 < 0 \text{ (i.e., } -1 < g < 0) &\Rightarrow x_1 > 0, y_4 > y_1 \Rightarrow \text{Config. } \mathcal{P}.26; \\ \zeta_2 > 0 \text{ and } g < -1 &\Rightarrow x_1 < 0, y_4 < y_1 \Rightarrow \text{Config. } \mathcal{P}.27; \\ \zeta_2 > 0 \text{ and } g > 0 &\Rightarrow x_1 > 0, y_4 < y_1 \Rightarrow \simeq \text{Config. } \mathcal{P}.27. \end{aligned}$$

2.2.2: *The subcase $\theta = 0$.* This condition implies $(g-1)(g+2) = 0$.

If $g = 1$, we arrive at the system:

$$\dot{x} = (x-1)(x+y), \quad \dot{y} = 2(y-1)y, \quad (3.24)$$

which possesses four invariant affine lines: $x = 1$, $y = 0$, $y = 1$, and $y = x$. Therefore, it is easy to determine that this system corresponds to the configuration *Config. P.28*.

Assuming $g = -2$, we arrive at the system:

$$\dot{x} = \frac{1}{8}(1+2x)(3-8x+4y), \quad \dot{y} = -\frac{1}{4}(4y-1)(3x-2y),$$

which, via the transformation $x_1 = -x + 1/2$, $y_1 = -x + y + 1/4$, can be brought to the system (3.24), thus also corresponding to configuration *Config. P.28*.

The subcase $\zeta_4 = 0$. Considering (3.13) and the condition $\zeta_4 \neq 0$, we obtain that the condition $\zeta_4 = 0$ implies:

$$m(g+8m+4n) = 0.$$

On the other hand, according to Lemma 3.1, it is sufficient to examine the condition $m = 0$ because the condition $g+8m+4n = 0$ can be brought to $m = 0$ via an affine transformation.

Thus, setting $m = 0$, we arrive at the family of systems:

$$\dot{x} = nx - \frac{1}{2}(1+g)y + gx^2 + xy, \quad \dot{y} = y(2n - x + gx + 2y), \quad (3.25)$$

which possess the invariant line $y = 0$ and the invariant parabola $\Phi(x, y) = x^2 - y = 0$. It is clear that the invariant line $y = 0$ is tangent to the invariant parabola at the origin.

We determine that for the above systems, the condition $\zeta_1\zeta_2\zeta_4\mathcal{R}_1 \neq 0$ implies:

$$g(1+g)(g-2)(3+g)(1+3g)(1+4g+3g^2+2n) \neq 0. \quad (3.26)$$

For systems (3.25), we calculate:

$$\begin{aligned} B_2 &= -\frac{81}{2}(1+g)^2(1+g+2n)(g+g^2+2n)(g+4n)y^4, \\ \theta &= -8(g-1)(2+g). \end{aligned} \quad (3.27)$$

We now consider two possibilities: $B_2 \neq 0$ and $B_2 = 0$.

1: *The possibility $B_2 \neq 0$.* Then, besides the invariant line $y = 0$, systems (3.25) cannot possess invariant lines in other directions. However, there could exist an invariant line parallel to $y = 0$, and by Lemma 2.4, a necessary condition for this is $\theta = 0$. Therefore, we discuss two cases: $\theta \neq 0$ and $\theta = 0$.

1.1: *The case $\theta \neq 0$.* The systems (3.25) possess four finite singularities $M_i(x_i, y_i)$ ($i = 1, 2, 3, 4$) with coordinates:

$$\begin{aligned} x_1 &= 0, \quad y_1 = 0; \\ x_2 &= -\frac{n}{g}, \quad y_2 = 0; \\ x_{3,4} &= \frac{1}{4}(1-g \pm \sqrt{Z_3}), \quad y_{3,4} = \frac{1}{8}[(1-g)^2 - 8n \pm (1-g)\sqrt{Z_3}], \\ Z_3 &= (1-g)^2 - 16n. \end{aligned} \quad (3.28)$$

We observe that $\Phi(x_3, y_3) = \Phi(x_4, y_4) = 0$, meaning that the singular points M_3 and M_4 lie on the invariant parabola. Moreover, the singularity M_2 lies on the invariant line $y = 0$ and coalesces with M_1 if and only if $n = 0$.

The singularities M_3 and M_4 are complex (respectively, real) if $Z_3 < 0$ (respectively, $Z_3 > 0$), and they coincide (producing a multiple singular point) when $Z_3 = 0$.

On the other hand, for systems (3.25), we have:

$$\mathbf{D} = 48(1+g)^4n^6(-1+2g-g^2+16n) = -48(1+g)^4n^6Z_3,$$

and we proceed to discuss three subcases: $\mathbf{D} < 0$, $\mathbf{D} > 0$, and $\mathbf{D} = 0$.

1.1.1: *The subcase $\mathbf{D} < 0$.* Then $Z_3 > 0$, and therefore the finite singularities M_3 and M_4 are real and distinct [1, Proposition 5.1]. To determine their positions on the parabola with respect to the singularity M_1 , we calculate:

$$\begin{aligned} (x_3 - x_1)(x_4 - x_1) &= n, \quad (x_3 - x_1) + (x_4 - x_1) = (1-g)/2; \\ \text{sign}((x_3 - x_1)(x_4 - x_1)) &= \text{sign}(n), \quad \text{sign}((x_3 - x_1) + (x_4 - x_1)) = \text{sign}(1-g). \end{aligned} \quad (3.29)$$

We point out that $g - 1 \neq 0$ (due to $\theta \neq 0$), and the sign of $1 - g$ is only relevant when $n > 0$ (i.e., when $(x_3 - x_1)(x_4 - x_1) > 0$).

Furthermore:

$$x_2 - x_1 = -n/g \quad \Rightarrow \quad \text{sign}(x_2 - x_1) = -\text{sign}(gn).$$

For systems (3.25), calculations yield:

$$\begin{aligned}\xi_{11} &= -95982880gn(g-2)^2(1+g)^2(3+g)^2(1+3g)^2(1+4g+3g^2+2n)^2, \\ \xi_{28} &= 3244620(1+g)^2(3+g)^2(1+3g)^2(1+4g+3g^2+2n)^2(g+4n), \\ \xi_{29} &= 1700495253(g-1)g(1+g)^2(3+g)(1+3g)(1+4g+3g^2+2n)^2(g+4n)/16.\end{aligned}$$

Remark 3.2. We observe that due to the condition (3.26), we have $\xi_{11} \neq 0$, with $\text{sign}(\xi_{11}) = -\text{sign}(gn)$. If $\xi_{11} < 0$ (i.e., $gn > 0$), then $\text{sign}(\xi_{28}) = \text{sign}(g+4n)$. Moreover, in the case $g > 0$ and $n > 0$, we also have $\text{sign}(\xi_{29}) = \text{sign}(g-1)$.

Thus, considering the above relations in the case $\mathbf{D} < 0$, we obtain the following configurations:

$$\begin{aligned}\xi_{11} < 0, n < 0 \text{ (then } g < 0) &\Rightarrow (x_3 - x_1)(x_4 - x_1) < 0, x_2 < x_1 \Rightarrow \text{Config. } \mathcal{P}.29; \\ \xi_{11} < 0, n > 0 \text{ (then } g > 0), g < 1 &\Rightarrow x_3 > x_1, x_4 > x_1, x_2 < x_1 \Rightarrow \text{Config. } \mathcal{P}.30; \\ \xi_{11} < 0, n > 0 \text{ (then } g > 0), g > 1 &\Rightarrow x_3 < x_1, x_4 < x_1, x_2 < x_1 \Rightarrow \text{Config. } \mathcal{P}.31; \\ \xi_{11} > 0, n < 0 \text{ (then } g > 0) &\Rightarrow (x_3 - x_1)(x_4 - x_1) < 0, x_2 > x_1 \Rightarrow \text{Config. } \mathcal{P}.32; \\ \xi_{11} > 0, n > 0 \text{ (then } g < 0) &\Rightarrow x_3 > x_1, x_4 > x_1, x_2 > x_1 \Rightarrow \text{Config. } \mathcal{P}.33.\end{aligned}$$

Taking into account Remark 3.2, we obtain the following invariant conditions:

$$\begin{aligned}\xi_{11} < 0, \xi_{28} < 0 &\Rightarrow \text{Config. } \mathcal{P}.29; \\ \xi_{11} < 0, \xi_{28} > 0, \xi_{29} < 0 &\Rightarrow \text{Config. } \mathcal{P}.30; \\ \xi_{11} < 0, \xi_{28} > 0, \xi_{29} > 0 &\Rightarrow \text{Config. } \mathcal{P}.31; \\ \xi_{11} > 0 &\Rightarrow \begin{cases} \text{Config. } \mathcal{P}.32, & \text{or} \\ \text{Config. } \mathcal{P}.33. \end{cases}\end{aligned}$$

1.1.2: The subcase $\mathbf{D} > 0$. Then $Z_3 < 0$, and hence the finite singularities M_3 and M_4 are complex. This condition also implies $n > 0$, meaning that the singular point $M_2(-n/g, 0)$ cannot coalesce with $M_1(0, 0)$. Moreover, its position relative to M_1 depends on the sign of the parameter g .

Since $n > 0$, we have:

$$\text{sign}(\xi_{11}) = -\text{sign}(gn) = -\text{sign}(g).$$

Therefore, we obtain Config. $\mathcal{P}.34$ if $\xi_{11} < 0$, and Config. $\mathcal{P}.35$ if $\xi_{11} > 0$.

1.1.3: The subcase $\mathbf{D} = 0$. Considering (3.26), we deduce that $\mathbf{D} = 0$ implies $nZ_3 = 0$. For systems (3.25), we calculate:

$$\xi_1 = 3(g-2)(1+g)^2(3+g)(1+3g)n^2(1+4g+3g^2+2n)/8.$$

By (3.26), the condition $n = 0$ is equivalent to $\xi_1 = 0$. Thus, we examine two possibilities: $\xi_1 \neq 0$ and $\xi_1 = 0$.

1.1.3.1: The possibility $\xi_1 \neq 0$. Then $\mathbf{D} = 0$ implies $Z_3 = 0$. From (3.28), we get $n = (1-g)^2/16 \neq 0$, and thus:

$$\begin{aligned}x_1 &= 0, \quad y_1 = 0; \quad x_2 = -\frac{(1-g)^2}{16g}, \quad y_2 = 0; \\ x_3 &= x_4 = \frac{1-g}{4}, \quad y_3 = y_4 = \frac{(1-g)^2}{16}; \\ \text{sign}(x_2 - x_1) &= -\text{sign}(g), \quad \text{sign}(x_3 - x_1) = \text{sign}(1-g).\end{aligned}$$

Therefore, there is a double singular point on the invariant parabola. For the parameter g , the bifurcation values are $g \in \{0, 1\}$.

For systems (3.25) with $n = (1 - g)^2/16$ (i.e., $Z_3 = 0$), we compute:

$$\begin{aligned}\zeta_{11} &= -\frac{2999465}{32}(g-2)^2(g-1)^2g(1+g)^2(3+g)^2(1+3g)^2(3+5g)^4, \quad \theta = -8(g-1)(2+g), \\ \zeta_2 &= 4g(1+g), \quad \zeta_4 = (g-2)(3+g)(1+3g)(3+5g)^2/128, \quad \zeta_{12} = g(g-1)^3\psi_1(g)/256,\end{aligned}$$

where $\psi_1(g) = 1105 + 1774g + 961g^2$. Observing that $\text{Discrim}[\psi_1(g), g] = -1100544 < 0$, we note that $\psi_1(g)$ does not vanish for real g .

Considering the condition $\zeta_4\theta \neq 0$, we conclude:

$$\text{sign}(\zeta_{11}) = -\text{sign}(g), \quad \text{sign}(\zeta_{12}) = \text{sign}(g(g-1)).$$

Thus, we determine the following configurations:

$$\begin{aligned}\zeta_{12} < 0 &\Rightarrow x_2 < x_1, x_3 > x_1 \Rightarrow \text{Config. } \mathcal{P}.36; \\ \zeta_{12} > 0, \zeta_{11} < 0 &\Rightarrow x_2 < x_1, x_3 > x_1 \Rightarrow \text{Config. } \mathcal{P}.37; \\ \zeta_{12} > 0, \zeta_{11} > 0 &\Rightarrow x_2 > x_1, x_3 > x_1 \Rightarrow \text{Config. } \mathcal{P}.38.\end{aligned}$$

1.1.3.2: The possibility $\zeta_1 = 0$. In this case, $n = 0$, and the singular point $M_2(-n/g, 0)$ coalesces with $M_1(0, 0)$. Moreover, one of the singularities M_3 or M_4 also coalesces with M_1 , resulting in a triple finite singularity at $M_1(0, 0)$. There could still remain a distinct simple singularity (M_3 or M_4), whose position depends on $\text{sign}(1 - g)$ (see (3.29)).

For $n = 0$, systems (3.25) satisfy:

$$\zeta_{10} = 24814861965(g-1)g^2(1+g)^6(1+3g)^4/2,$$

so that: $\text{sign}(\zeta_{10}) = \text{sign}(g-1)$. Therefore:

$$\begin{aligned}\zeta_{10} < 0 &\Rightarrow \text{Config. } \mathcal{P}.39; \\ \zeta_{10} > 0 &\Rightarrow \text{Config. } \mathcal{P}.40.\end{aligned}$$

1.2: The case $\theta = 0$. This condition implies $(g-1)(g+2) = 0$. For systems (3.25), we calculate:

$$\begin{aligned}\zeta_1 &= 3(g-2)(1+g)^2(3+g)(1+3g)n^2(1+4g+3g^2+2n)/8, \\ \zeta_5 &= -21875(g-1)(1+g)(g-2)(3+g)(1+3g)n(1+g+2n)(1+4g+3g^2+2n)/8.\end{aligned}\tag{3.30}$$

We now discuss two subcases: $\zeta_1 \neq 0$ and $\zeta_1 = 0$.

1.2.1: The subcase $\zeta_1 \neq 0$. This implies $n \neq 0$, and considering (3.26) and $B_2 \neq 0$ (i.e., $1+g+2n \neq 0$), we conclude that the condition $g = 1$ is equivalent to $\zeta_5 = 0$. Thus, we analyze two possibilities: $\zeta_5 \neq 0$ and $\zeta_5 = 0$.

1.2.1.1: The possibility $\zeta_5 \neq 0$. Then $g-1 \neq 0$, and the condition $\theta = 0$ implies $g = -2$. Straightforward calculations show that for $g = -2$, the systems (3.25) do not possess any invariant line parallel to $y = 0$.

For $g = -2$, the systems (3.25) yield:

$$\begin{aligned}\mathbf{D} &= 48n^6(16n-9), \\ \zeta_{11} &= 76786304000n(5+2n)^2, \\ \zeta_4 &= 5(5+2n)/4.\end{aligned}$$

Thus, $\text{sign}(\xi_{11}) = \text{sign}(n)$, and $\text{sign}(\mathbf{D}) = \text{sign}(16n - 9)$. Moreover, since $n \neq 0$, we conclude that $\mathbf{D} = 0$ corresponds to $16n - 9 = 0$.

Therefore, considering that $g = -2 < 0$ and applying the previous analysis for systems (3.25), we determine the following configurations:

$$\begin{aligned} \mathbf{D} < 0, \xi_{11} < 0 &\Rightarrow \text{Config. } \mathcal{P}.29; \\ \mathbf{D} < 0, \xi_{11} > 0 &\Rightarrow \text{Config. } \mathcal{P}.33; \\ \mathbf{D} > 0 &\Rightarrow \text{Config. } \mathcal{P}.35; \\ \mathbf{D} = 0 &\Rightarrow \text{Config. } \mathcal{P}.38. \end{aligned}$$

1.2.1.2: The possibility $\xi_5 = 0$. Then $g = 1$, leading to the family of systems:

$$\dot{x} = nx - y + x^2 + xy, \quad \dot{y} = 2y(n + y), \quad (3.31)$$

which possess the additional invariant line $y + n = 0$.

From (3.28), we find:

$$\begin{aligned} x_1 &= 0, \quad y_1 = 0; \\ x_2 &= -n, \quad y_2 = 0; \\ x_{3,4} &= \pm\sqrt{-n}, \quad y_3 = y_4 = -n. \end{aligned}$$

We observe that the invariant line $y = -n$ intersects the invariant parabola $\Phi(x, y) = x^2 - y = 0$ at two points $M_{3,4}(\pm\sqrt{-n}, -n)$, which are distinct due to $\xi_1 \neq 0$ (i.e., $n \neq 0$). These points are real if $n < 0$ and complex if $n > 0$.

Calculating the invariant $\mathbf{D} = 12288n^7$, we get $\text{sign}(\mathbf{D}) = \text{sign}(n)$. Therefore:

$$\begin{aligned} \mathbf{D} < 0 &\Rightarrow \text{Config. } \mathcal{P}.41; \\ \mathbf{D} > 0 &\Rightarrow \text{Config. } \mathcal{P}.42. \end{aligned}$$

1.2.2: The subcase $\xi_1 = 0$. This implies $n = 0$, and for systems (3.25) we obtain:

$$B_2 = -81g^2(1+g)^4y^4/2, \quad \xi_9 = 5589813240(g-1)^2g^6(1+g)^6.$$

Therefore, since $B_2 \neq 0$, we conclude that the condition $\xi_9 = 0$ is equivalent to $g - 1 = 0$. Thus, we discuss two cases: $\xi_9 \neq 0$ and $\xi_9 = 0$.

1.2.2.1: The case $\xi_9 \neq 0$. Then, $\theta = 0$ implies $g = -2$, and we obtain the system:

$$\dot{x} = \frac{y}{2} - 2x^2 + xy, \quad \dot{y} = -y(3y - 2x).$$

This system belongs to the family (3.25) with $n = 0$ (i.e., $\xi_1 = 0$) and $g = -2$. As previously shown (see point 1.1.3.2 on page 41), in the case $n = 0$, the singular point $M_2(-n/g, 0)$ coalesces with $M_1(0, 0)$. Moreover, one of the singular points M_3 or M_4 also coalesces with $M_1(0, 0)$, yielding a triple finite singularity at $M_1(0, 0)$.

On the other hand, since $\text{sign}(\xi_{10}) = \text{sign}(g - 1)$, for $g = -2$ we get $\xi_{10} < 0$. Therefore, the system has the configuration *Config. P.39*.

1.2.2.2: The case $\xi_9 = 0$. Then $g = 1$, and we obtain the systems (3.31). For $n = 0$, the invariant line $y = -n$ coalesces with $y = 0$, producing a double invariant line. Moreover, all finite singular points coalesce, generating a quadruple singularity at $M_1(0, 0)$. As a result, the system has the configuration *Config. P.43*.

2: The possibility $B_2 = 0$. We first make a remark.

Remark 3.3. The condition $B_2 = 0$ implies $n \neq 0$ for systems (3.25). Indeed, in the case $n = 0$, for systems (3.25), we obtain:

$$B_2 = -81g^2(1+g)^4y^4/2 \neq 0$$

due to the condition $\zeta_2 \neq 0$ (i.e., $g(g+1) \neq 0$).

Thus, $n \neq 0$, and since $g+1 \neq 0$, considering (3.27), we get the condition:

$$(1+g+2n)(g+g^2+2n)(g+4n) = 0.$$

Considering (3.30), we examine two cases: $\zeta_5 \neq 0$ and $\zeta_5 = 0$.

2.1: The case $\zeta_5 \neq 0$. Then, by (3.30), we get $1+g+2n \neq 0$, and hence the condition $B_2 = 0$ implies: $(g+g^2+2n)(g+4n) = 0$.

On the other hand, for systems (3.25), we calculate:

$$\zeta_{13} = 27(1+g)^2(3+g)(1+3g)n^2(g+4n)/4,$$

and considering Remark 3.3 and the condition (3.26), we conclude that $\zeta_{13} = 0$ is equivalent to $g+4n = 0$.

2.1.1: The subcase $\zeta_{13} \neq 0$. Then $g+4n \neq 0$, and therefore $B_2 = 0$ implies $g+g^2+2n = 0$. In this case, we get $n = -g(g+1)/2$, and we arrive at the following family of systems:

$$\dot{x} = (2x-1-g)(gx+y)/2, \quad \dot{y} = -y(g+g^2+x-gx-2y), \quad (3.32)$$

which possess the invariant lines $y = 0$ and $x = (g+1)/2$.

Considering Lemma 2.3, for these systems we calculate:

$$\begin{aligned} B_3 &= -3(g-1)g(1+g)^2(1+2g)x^2y^2/4, \quad \theta = -8(g-1)(g+2), \\ \zeta_5 &= -21875(g-2)(g-1)^2g(1+g)^4(3+g)(1+2g)(1+3g)/16. \end{aligned}$$

We observe that $B_3 \neq 0$ due to $\zeta_5 \neq 0$, and hence by Lemma 2.3, the above systems cannot have any invariant line in the third direction.

However, according to Lemma 2.4, we could have parallel invariant lines if $\theta = 0$. Due to $B_3 \neq 0$ (i.e., $g-1 \neq 0$), the condition $\theta = 0$ is equivalent to $g+2 = 0$. It is straightforward to check that for $g = -2$, systems (3.32) do not have any invariant line parallel to $y = 0$ or to $x = (g+1)/2$.

The systems (3.32) possess the following finite singularities $M_i(x_i, y_i)$, ($i = 1, 2, 3, 4$), with coordinates:

$$x_1 = 0, \quad y_1 = 0; \quad x_2 = \frac{1+g}{2}, \quad y_2 = 0; \quad x_3 = -g, \quad y_3 = g^2; \quad x_4 = \frac{1+g}{2}, \quad y_4 = \frac{(1+g)^2}{4}.$$

We observe that the invariant line $x = (g+1)/2$ intersects the invariant parabola at point M_4 and the invariant line $y = 0$ at singularity M_2 . Furthermore, the invariant line $y = 0$ is tangent to the parabola at M_1 .

To determine the relative positions of the line $x = (g+1)/2$ and the singularities, we calculate:

$$x_2 - x_1 = \frac{1+g}{2}, \quad x_3 - x_1 = -g, \quad x_3 - x_4 = -\frac{1+3g}{2},$$

$$\text{sign}(x_2 - x_1) = \text{sign}(1+g), \quad \text{sign}(x_3 - x_1) = -\text{sign}(g), \quad \text{sign}(x_3 - x_4) = -\text{sign}(1+3g).$$

Thus, for the parameter g , we have the following possible bifurcation values: $g \in \{-1, -1/3, 0\}$.

On the other hand, for systems (3.32), we calculate:

$$\begin{aligned}\zeta_2 &= 4g(g+1), \quad \zeta_7 = 1174627500g^4(1+g)^7(1+2g)^2(1+3g), \\ \zeta_{11} &= 47991440(g-2)^2g^2(1+g)^5(3+g)^2(1+2g)^2(1+3g)^2\end{aligned}$$

and we observe that:

$$\text{sign}(\zeta_2) = \text{sign}(g(g+1)), \quad \text{sign}(\zeta_7) = \text{sign}((g+1)(1+3g)), \quad \text{sign}(\zeta_{11}) = \text{sign}(g+1).$$

Moreover, in the case $\zeta_2 < 0$, we have $-1 < g < 0$ (i.e., $g+1 > 0$), and then:

$$\text{sign}(\zeta_7) = \text{sign}(1+3g).$$

Thus, considering the above relations, we obtain the following configurations:

$$\begin{aligned}\zeta_2 < 0, \zeta_7 < 0 \text{ (i.e., } -1 < g < -1/3) &\Rightarrow x_2 > x_1, x_3 > x_1, x_3 > x_4 \Rightarrow \text{Config. } \mathcal{P}.44; \\ \zeta_2 < 0, \zeta_7 > 0 \text{ (i.e., } -1/3 < g < 0) &\Rightarrow x_2 > x_1, x_1 < x_3 < x_4 \Rightarrow \text{Config. } \mathcal{P}.45; \\ \zeta_2 > 0, \zeta_{11} < 0 \text{ (i.e., } g < -1) &\Rightarrow x_2 < x_1, x_3 > x_1 \Rightarrow \text{Config. } \mathcal{P}.46; \\ \zeta_2 > 0, \zeta_{11} > 0 \text{ (i.e., } g > 0) &\Rightarrow x_2 > x_1, x_3 < x_1 \Rightarrow \text{Config. } \mathcal{P}.47.\end{aligned}$$

2.1.2: The subcase $\zeta_{13} = 0$. This implies $g + 4n = 0$ (i.e., $n = -g/4$) and we arrive at the family of systems

$$\dot{x} = -g(1+g)x/2 - (1+g)y/2 + gx^2 + xy, \quad \dot{y} = -y(g+g^2+x-gx-2y), \quad (3.33)$$

possessing the invariant lines $y = 0$ and $y = x - 1/4$. For these systems we have

$$\begin{aligned}\zeta_5 &= 21875(g-2)g(1+g)(g-1)(2+g)(3+g)(1+2g)(1+3g)(2+3g)/128, \\ B_3 &= 3g(1+g)(1+2g)(x-y)^2y^2/8, \quad \theta = -8(g-1)(2+g),\end{aligned}$$

and since $\zeta_5 \neq 0$, we obtain $B_3\theta \neq 0$. So, by Lemmas 2.3 and 2.4, we conclude that the above systems could not have a third invariant line.

The systems (3.33) possess the finite singularities $M_i(x_i, y_i)$ ($i = 1, 2, 3, 4$) with coordinates

$$\begin{aligned}x_1 &= 0, y_1 = 0; \quad x_2 = 1/4, y_2 = 0; \quad x_3 = 1/2, y_3 = 1/4; \quad x_4 = -g/2, y_4 = g^2/4; \\ \text{sign}(x_4 - x_1) &= -\text{sign}(g), \quad x_4 - x_3 = -(g+1)/2 \Rightarrow \text{sign}(x_4 - x_3) = -\text{sign}(g+1).\end{aligned}$$

It could be checked directly that the invariant line $y = x - 1/4$ is tangent to the invariant parabola at the singular point $M_3(1/2, 1/4)$. Therefore, considering the above relations, we obtain the following configurations:

$$\begin{aligned}\zeta_2 < 0 \text{ (i.e., } -1 < g < 0) &\Rightarrow x_4 > x_1, x_4 < x_3 \Rightarrow \text{Config. } \mathcal{P}.48; \\ \zeta_2 > 0 \text{ and } g < -1 &\Rightarrow x_4 > x_1, x_4 > x_3 \Rightarrow \text{Config. } \mathcal{P}.49; \\ \zeta_2 > 0 \text{ and } g > 0 &\Rightarrow x_4 < x_1, x_4 < x_3 \Rightarrow \simeq \text{Config. } \mathcal{P}.49.\end{aligned}$$

2.2: The case $\zeta_5 = 0$. Considering (3.30), the conditions (3.26) and Remark 3.3 imply $(g-1)(1+g+2n) = 0$, and we examine two subcases: $\theta \neq 0$ and $\theta = 0$.

2.2.1: The subcase $\theta \neq 0$. Then $g-1 \neq 0$, and we get $1+g+2n = 0$. Therefore, $n = -(1+g)/2 \neq 0$, and we arrive at the family of systems

$$\dot{x} = -(1+g)(x+y)/2 + gx^2 + xy, \quad \dot{y} = -y(1+g+x-gx-2y), \quad (3.34)$$

possessing the invariant lines $y = 0$ and $y = x$, and the finite singularities $M_i(x_i, y_i)$ ($i = 1, 2, 3, 4$) with the coordinates:

$$x_1 = 0, \quad y_1 = 0; \quad x_2 = \frac{1+g}{2g}, \quad y_2 = 0; \quad x_3 = 1, \quad y_3 = 1; \quad x_4 = -\frac{1+g}{2}, \quad y_4 = \frac{(1+g)^2}{4}.$$

On the other hand, considering Lemma 2.3, we calculate B_3 and we have:

$$B_3 = 3(g-1)(1+g)^2(x-y)^2y^2/4 \neq 0,$$

due to the conditions (3.26) and $\theta \neq 0$. Then, by Lemma 2.3, we could not have any invariant line in the third direction. Moreover, by Lemma 2.4, we could not have parallel invariant lines due to $\theta \neq 0$.

Next, considering the coordinates of the finite singularities of these systems, it follows immediately:

$$\begin{aligned} \text{sign}(x_4 - x_1) &= -\text{sign}(1+g), \quad \text{sign}(x_2 - x_1) = \text{sign}(g(1+g)), \\ x_4 - x_3 &= -(g+3)/2 \Rightarrow \text{sign}(x_4 - x_3) = -\text{sign}(g+3). \end{aligned}$$

We remark that $g(g+1)(g+3) \neq 0$ due to the condition (3.26), and hence for the parameter g we have the following possible bifurcation values: $g \in \{-3, -1, 0\}$.

On the other hand, for systems (3.34), we calculate:

$$\begin{aligned} \zeta_2 &= 4g(1+g), \quad \zeta_9 = 5589813240(g-1)^2g^2(1+g)^9(3+g), \\ \zeta_{10} &= -223333757685(g-1)^2g^2(1+g)^6(2+g)(1+3g)^2/2, \end{aligned}$$

and hence we have

$$\text{sign}(\zeta_2) = \text{sign}(g(1+g)), \quad \text{sign}(\zeta_9) = \text{sign}((1+g)(3+g)), \quad \text{sign}(\zeta_{10}) = -\text{sign}(2+g).$$

Remark 3.4. We observe that the conditions $\zeta_2 > 0$ and $\zeta_9 > 0$ imply either $g > 0$ or $g < -3$. In order to distinguish these two possibilities, we use the invariant ζ_{10} even though this invariant does not vanish in the bifurcation values of g .

Considering the above remark, we arrive at the following configurations:

$$\begin{aligned} \zeta_2 < 0 \text{ (i.e., } -1 < g < 0) &\Rightarrow x_2 < x_1, x_4 < x_1 &\Rightarrow \text{Config. } \mathcal{P}.50; \\ \zeta_2 > 0, \zeta_9 < 0 \text{ (i.e., } -3 < g < -1) &\Rightarrow x_2 > x_1, x_1 < x_4 < x_3 &\Rightarrow \text{Config. } \mathcal{P}.51; \\ \zeta_2 > 0, \zeta_9 > 0, \zeta_{10} < 0 \text{ (i.e., } g > 0) &\Rightarrow x_2 > x_1, x_3 < x_1 &\Rightarrow \text{Config. } \mathcal{P}.52; \\ \zeta_2 > 0, \zeta_9 > 0, \zeta_{10} > 0 \text{ (i.e., } g < -3) &\Rightarrow x_2 > x_1, x_4 > x_3 &\Rightarrow \text{Config. } \mathcal{P}.53. \end{aligned}$$

2.2.2: The subcase $\theta = 0$. This implies $(g-1)(g+2) = 0$, and we discuss two possibilities: $B_3 \neq 0$ and $B_3 = 0$.

2.2.2.1: The possibility $B_3 \neq 0$. We claim that in this case we get the same configuration either if $g = 1$ or if $g = -2$.

Indeed, assume first $g = -2$. Then, calculations yield

$$\begin{aligned} \zeta_5 &= -328125n(2n-1)(5+2n)/2, \quad B_2 = -162(1+n)(2n-1)^2y^4, \\ B_3 &= 3y^2[n(4n-5)x^2 + 2(1+n)xy - (1+n)y^2]/2, \end{aligned} \tag{3.35}$$

and the condition $\zeta_5 = B_2 = 0$ gives us $n = 1/2$. This leads to the system

$$\dot{x} = (x+y)/2 - 2x^2 + xy, \quad \dot{y} = y(1-3x+2y), \tag{3.36}$$

possessing three invariant affine lines: $y = 0$, $y = x$ and $y = x - 1/4$. Then, this system has the configuration equivalent to *Config. P.54*.

Suppose now $g = 1$. Then, we have

$$\zeta_5 = 0, \quad B_2 = -648(1+n)^2(1+4n)y^4, \quad B_3 = -3(1+n)y^2(4nx^2 + 2xy - y^2),$$

and due to $B_3 \neq 0$, the condition $B_2 = 0$ implies $n = -1/4$. In this case, we arrive at the system

$$\dot{x} = -x/4 - y + x^2 + xy, \quad \dot{y} = y(4y - 1)/2,$$

which via the affine transformation $x_1 = -x + 1/2$, $y_1 = -x + y + 1/4$ could be brought to system (3.36). Thus, our claim is proved and we get the configuration *Config. P.54*.

2.2.2.2: The possibility $B_3 = 0$. Considering (3.35), we conclude that the condition $g = -2$ implies $B_3 \neq 0$, and hence the condition $\theta = 0$ gives us $g = 1$. In this case, we arrive at the system

$$\dot{x} = (x - 1)(x + y), \quad \dot{y} = 2(y - 1)y,$$

possessing four invariant affine lines: $x = 1$, $y = 0$, $y = 1$ and $y = x$. Therefore, it is easy to determine that this system possesses the configuration equivalent to *Config. P.28*.

3.1.2 The statement ($\mathcal{A}_2^*$)

According to the statement ($\mathcal{A}_2^*$) of Proposition 2.7*, for systems (3.1) the condition $\zeta_4 = 0$ must hold. Considering (3.2), we obtain

$$(g - 2)(3 + g)(1 + 7g + 15g^2 + 9g^3 - 4m + 2n + 6gn) = 0,$$

and since $(g - 2)(3 + g) \neq 0$ (due to $\zeta_1 \neq 0$), we get

$$m = \frac{1}{4}(1 + 3g)(1 + 4g + 3g^2 + 2n).$$

Then, we arrive at the two-parameter family of systems

$$\begin{aligned} \dot{x} &= \frac{1}{4}(1 + 3g)(1 + 4g + 3g^2 + 2n) + nx - \frac{1}{2}(1 + g)y + gx^2 + xy, \\ \dot{y} &= \frac{1}{2}(1 + 3g)(1 + 4g + 3g^2 + 2n)x + 2ny + (g - 1)xy + 2y^2 \end{aligned} \quad (3.37)$$

possessing the following two invariant parabolas: $\Phi_1(x, y) = x^2 - y = 0$ and

$$\begin{aligned} \Phi_2(x, y) &= -(1 + 4g + 3g^2 + 2n)(1 + 4g + 3g^2 + 4n) + 2(1 + g)(1 + 4g + 3g^2 + 4n)x \\ &\quad + 4g(1 + g)x^2 - 2(1 + 6g + 5g^2 + 4n)y = 0. \end{aligned} \quad (3.38)$$

Following the statement ($\mathcal{A}_2^*$), for the above systems we calculate

$$\begin{aligned} \zeta_1 &= 2(g - 2)(3 + g), \quad \zeta_2 = 4g(1 + g), \quad \zeta_4 = 0, \\ \zeta_5 &= 19(g - 2)(3 + g)(1 + 4g + 3g^2 + 4n)^2/4, \\ \mathcal{R}_2 &= -(g - 2)(3 + g)(8 + 27g + 27g^2)(1 + 6g + 5g^2 + 4n)/16, \\ B_1 &= g(1 + g)(1 + 2g)(1 + 3g)(2 + 3g)(1 + 4g + 3g^2 + 2n)(1 + 6g + 5g^2 + 4n) \\ &\quad \times (1 + 6g + 6g^2 + 4n)(1 + 6g + 9g^2 + 4n)(5 + 14g + 9g^2 + 4n)/32. \end{aligned} \quad (3.39)$$

According to Lemma 2.3, for the existence of an invariant line of systems (3.37), the condition $B_1 = 0$ is necessary.

The case $B_1 \neq 0$. Then we could not have any invariant line. The systems (3.37) possess four finite singularities $M_i(x_i, y_i)$ ($i = 1, 2, 3, 4$) with the coordinates

$$\begin{aligned} x_1 &= -\frac{1+3g}{2}, \quad y_1 = \frac{(1+3g)^2}{4}; \quad x_2 = -\frac{(1+g)(1+3g)^2 + 4(1+2g)n}{2g(1+g)}, \\ y_2 &= \frac{(1+3g)(1+4g+3g^2+2n)}{2(1+g)}; \quad x_{3,4} = \frac{1}{2}(1+g \pm \sqrt{Z_4}), \\ y_{3,4} &= -\frac{1}{2}(2g+2g^2+2n \mp (g+1)\sqrt{Z_4}), \quad Z_4 = -(1+6g+5g^2+4n). \end{aligned} \quad (3.40)$$

In order to determine the position of the finite singularities with respect to the parabolas $\Phi_1(x, y) = 0$ and $\Phi_2(x, y) = 0$, we calculate

$$\Phi_1(x_1, y_1) = \Phi_1(x_3, y_3) = \Phi_1(x_4, y_4) = 0; \quad \Phi_2(x_2, y_2) = \Phi_2(x_3, y_3) = \Phi_2(x_4, y_4) = 0. \quad (3.41)$$

Therefore, the singularities M_3 and M_4 are the finite intersection points of these two invariant parabolas. We observe that the points of intersection of the invariant parabolas are complex if $Z_4 < 0$ and they are real if $Z_4 > 0$.

On the other hand, for systems (3.37) we calculate:

$$\mathbf{D} = -3Z_4(1+4g+3g^2+4n)^2 \alpha_4^2 \beta_4^2 / 4, \quad (3.42)$$

where

$$\alpha_4 = 5 + 22g + 21g^2 + 4n, \quad \beta_4 = (1+g)(1+3g)(1+6g+7g^2) + 4(1+2g)^2 n. \quad (3.43)$$

So if $\mathbf{D} \neq 0$, then $\text{sign}(\mathbf{D}) = -\text{sign}(Z_4)$ and we discuss three possibilities: $\mathbf{D} < 0$, $\mathbf{D} > 0$ and $\mathbf{D} = 0$.

1: The possibility $\mathbf{D} < 0$. Then $Z_4 > 0$ and systems (3.37) possess four real singularities and it is necessary to know the positions of the singularities $M_{3,4}$ with respect to M_1 and M_2 . We calculate

$$\begin{aligned} (x_1 - x_3)(x_1 - x_4) &= \frac{\alpha_4}{4}, \quad (x_1 - x_3) + (x_1 - x_4) = -2(1+2g), \\ (x_2 - x_3)(x_2 - x_4) &= -\frac{Z_4 \beta_4}{4g^2(1+g)^2}, \quad (x_2 - x_3) + (x_2 - x_4) = -\frac{(1+2g)Z_4}{g(1+g)}. \end{aligned}$$

Therefore, considering the condition $Z_4 > 0$, we obtain

$$\begin{aligned} \text{sign}((x_1 - x_3)(x_1 - x_4)) &= \text{sign}(\alpha_4), \quad \text{sign}((x_1 - x_3) + (x_1 - x_4)) = -\text{sign}(1+2g) \quad \text{if } \alpha_4 > 0; \\ \text{sign}((x_2 - x_3)(x_2 - x_4)) &= -\text{sign}(\beta_4), \\ \text{sign}((x_2 - x_3) + (x_2 - x_4)) &= -\text{sign}(g(1+g)(1+2g)) \quad \text{if } \beta_4 < 0, \end{aligned}$$

where $1+2g \neq 0$ due to $B_1 \neq 0$.

We need invariant polynomials governing the signs of α_4 and β_4 . For systems (3.37), we calculate:

$$\zeta_{14} = 1235\alpha_4\beta_4/2, \quad \zeta_{30} = 1235[Z_4\beta_4 - g^2(1+g)^2\alpha_4]/4, \quad \zeta_2 = 4g(1+g),$$

and we have

$$\text{sign}(\zeta_{14}) = \text{sign}(\alpha_4\beta_4), \quad \text{sign}(\zeta_2) = \text{sign}(g(1+g)).$$

Moreover, in the case $\zeta_{14} < 0$ (i.e. $\alpha_4\beta_4 < 0$) and $\mathbf{D} < 0$ (i.e. $Z_4 > 0$), we obtain

$$\text{sign}(\zeta_{30}) = \text{sign}(Z_4\beta_4 - g^2(1+g)^2\alpha_4) = \text{sign}(\beta_4).$$

On the other hand, considering the form of the invariant parabola $\Phi_2(x, y) = 0$, we have

$$y = -\frac{1}{Z_4} \left[(1+g)(1+4g+3g^2+4n)x - \frac{1}{2}(1+4g+3g^2+2n)(1+4g+3g^2+4n) \right] - \frac{2g(1+g)}{Z_4} x^2. \quad (3.44)$$

Therefore, since $Z_4 > 0$, we deduce that the invariant parabolas $\Phi_1(x, y) = 0$ and $\Phi_2(x, y) = 0$ have the same point at infinity on the Poincaré disk if $\zeta_2 < 0$, and opposite points at infinity if $\zeta_2 > 0$. So we consider these two cases separately.

1.1: The case $\zeta_2 < 0$. Then $g(g+1) < 0$ and considering the above relations, in this case we obtain the following configurations:

$$\begin{aligned} \zeta_{14} < 0, \beta_4 < 0, 2g+1 < 0 \text{ (i.e. } \beta_4 < 0, \alpha_4 > 0, -1 < g < -1/2) &\Rightarrow \\ x_2 - x_3 > 0, x_2 - x_4 > 0, x_1 - x_3 > 0, x_1 - x_4 > 0 &\Rightarrow \text{Config. } \mathcal{P}.55; \\ \zeta_{14} < 0, \beta_4 < 0, 2g+1 > 0 \text{ (i.e. } \beta_4 < 0, \alpha_4 > 0, -1/2 < g < 0) &\Rightarrow \\ x_2 - x_3 < 0, x_2 - x_4 < 0, x_1 - x_3 < 0, x_1 - x_4 < 0 &\Rightarrow \simeq \text{Config. } \mathcal{P}.55; \\ \zeta_{14} < 0, \beta_4 > 0 \text{ (i.e. } \beta_4 > 0, \alpha_4 < 0, -1 < g < 0) &\Rightarrow \\ (x_2 - x_3)(x_2 - x_4) < 0, (x_1 - x_3)(x_1 - x_4) < 0 &\Rightarrow \text{Config. } \mathcal{P}.56; \\ \zeta_{14} > 0, \beta_4 < 0, 2g+1 < 0 \text{ (i.e. } \beta_4 < 0, \alpha_4 < 0, -1 < g < -1/2) &\Rightarrow \\ x_2 - x_3 > 0, x_2 - x_4 > 0, (x_1 - x_3)(x_1 - x_4) < 0 &\Rightarrow \text{Config. } \mathcal{P}.57; \\ \zeta_{14} > 0, \beta_4 < 0, 2g+1 > 0 \text{ (i.e. } \beta_4 < 0, \alpha_4 < 0, -1/2 < g < 0) &\Rightarrow \\ x_2 - x_3 < 0, x_2 - x_4 < 0, (x_1 - x_3)(x_1 - x_4) < 0 &\Rightarrow \simeq \text{Config. } \mathcal{P}.57; \\ \zeta_{14} > 0, \beta_4 > 0, 2g+1 < 0 \text{ (i.e. } \beta_4 > 0, \alpha_4 > 0, -1 < g < -1/2) &\Rightarrow \\ (x_2 - x_3)(x_2 - x_4) < 0, x_1 - x_3 > 0, x_1 - x_4 > 0 &\Rightarrow \simeq \text{Config. } \mathcal{P}.57; \\ \zeta_{14} > 0, \beta_4 > 0, 2g+1 > 0 \text{ (i.e. } \beta_4 > 0, \alpha_4 > 0, -1/2 < g < 0) &\Rightarrow \\ (x_2 - x_3)(x_2 - x_4) < 0, x_1 - x_3 < 0, x_1 - x_4 < 0 &\Rightarrow \simeq \text{Config. } \mathcal{P}.57. \end{aligned}$$

1.2: The case $\zeta_2 > 0$. Then $g(g+1) > 0$ and we obtain the following configurations:

$$\begin{aligned} \zeta_{14} < 0, \beta_4 < 0, 2g+1 < 0 \text{ (i.e. } \beta_4 < 0, \alpha_4 > 0, g < -1) &\Rightarrow \\ x_2 - x_3 < 0, x_2 - x_4 < 0, x_1 - x_3 > 0, x_1 - x_4 > 0 &\Rightarrow \text{Config. } \mathcal{P}.58; \\ \zeta_{14} < 0, \beta_4 < 0, 2g+1 > 0 \text{ (i.e. } \beta_4 < 0, \alpha_4 > 0, g > 0) &\Rightarrow \\ x_2 - x_3 > 0, x_2 - x_4 > 0, x_1 - x_3 < 0, x_1 - x_4 < 0 &\Rightarrow \simeq \text{Config. } \mathcal{P}.58; \\ \zeta_{14} > 0, \beta_4 < 0, 2g+1 < 0 \text{ (i.e. } \beta_4 < 0, \alpha_4 < 0, g < -1) &\Rightarrow \\ x_2 - x_3 < 0, x_2 - x_4 < 0, (x_1 - x_3)(x_1 - x_4) < 0 &\Rightarrow \text{Config. } \mathcal{P}.59; \\ \zeta_{14} > 0, \beta_4 < 0, 2g+1 > 0 \text{ (i.e. } \beta_4 < 0, \alpha_4 < 0, g > 0) &\Rightarrow \\ x_2 - x_3 > 0, x_2 - x_4 > 0, (x_1 - x_3)(x_1 - x_4) < 0 &\Rightarrow \simeq \text{Config. } \mathcal{P}.59; \\ \zeta_{14} > 0, \beta_4 > 0, 2g+1 < 0 \text{ (i.e. } \beta_4 > 0, \alpha_4 > 0, g < -1) &\Rightarrow \\ (x_2 - x_3)(x_2 - x_4) < 0, x_1 - x_3 > 0, x_1 - x_4 > 0 &\Rightarrow \simeq \text{Config. } \mathcal{P}.59; \\ \zeta_{14} > 0, \beta_4 > 0, 2g+1 > 0 \text{ (i.e. } \beta_4 > 0, \alpha_4 > 0, g < -1) &\Rightarrow \\ (x_2 - x_3)(x_2 - x_4) < 0, x_1 - x_3 < 0, x_1 - x_4 < 0 &\Rightarrow \simeq \text{Config. } \mathcal{P}.59. \end{aligned}$$

Applying the Mathematica function “FindInstance” (or “Reduce”) we detect that the conditions $\mathbf{D} < 0$, $\zeta_2 > 0$, $\zeta_{14} < 0$, and $\beta_4 > 0$ (i.e. $Z_4 > 0$, $g(g+1) > 0$, $\alpha_4 < 0$, and $\beta_4 > 0$) are incompatible.

We observe that in both cases (i.e. $\zeta_2 < 0$ and $\zeta_2 > 0$) the configurations do not depend on the sign $(1+2g)$. As a result we obtain the following lemma.

Lemma 3.5. Assume that for the systems (3.37) the condition $\mathbf{D} < 0$ holds. Then, these systems exhibit the following configurations if and only if the respective conditions are satisfied:

$$\begin{aligned} \zeta_2 < 0, \tilde{\zeta}_{14} < 0, \tilde{\zeta}_{30} < 0 &\Leftrightarrow \text{Config. } \mathcal{P}.55; \\ \zeta_2 < 0, \tilde{\zeta}_{14} < 0, \tilde{\zeta}_{30} > 0 &\Leftrightarrow \text{Config. } \mathcal{P}.56; \\ \zeta_2 < 0, \tilde{\zeta}_{14} > 0 &\Leftrightarrow \text{Config. } \mathcal{P}.57; \\ \zeta_2 > 0, \tilde{\zeta}_{14} < 0 &\Leftrightarrow \text{Config. } \mathcal{P}.58; \\ \zeta_2 > 0, \tilde{\zeta}_{14} > 0 &\Leftrightarrow \text{Config. } \mathcal{P}.59. \end{aligned}$$

2: The possibility $\mathbf{D} > 0$. Then $Z_4 < 0$ and the systems (3.37) possess only two real singularities: M_1 (located on the parabola $\Phi_1(x, y) = 0$) and M_2 (located on the parabola $\Phi_2(x, y) = 0$). As mentioned earlier, the direction of the second invariant parabola depends on the sign of $g(1 + g)$ (see (3.44)).

Thus, considering the condition $\mathbf{D} > 0$ (i.e., $Z_4 < 0$), we have the configuration *Config. P.60* if $\zeta_2 < 0$ (i.e., $g(g + 1) < 0$) and *Config. P.61* if $\zeta_2 > 0$ (i.e., $g(g + 1) > 0$).

3: The possibility $\mathbf{D} = 0$. From (3.42), (3.39), and the condition $\zeta_5 \mathcal{R}_2 \neq 0$ (i.e., $Z_4(1 + 4g + 3g^2 + 4n) \neq 0$), it follows that $\mathbf{D} = 0$ implies $\alpha_4 \beta_4 = 0$. This yields the next lemma.

Lemma 3.6. For systems (3.37), the condition $\beta_4 = 0$ can be transformed by an affine change of variables into the condition $\alpha_4 = 0$.

Proof. Apply to the systems (3.37) the transformation

$$\begin{aligned} x_1 &= \delta x - \frac{(1 + g)(1 + 4g + 3g^2 + 4n)}{2Z_4}, \quad y_1 = \delta y - \frac{(1 + g)(1 + 3g)(1 + 4g + 3g^2 + 4n)}{4Z_4}, \\ t_1 &= \frac{1}{\delta}, \quad \delta = -\frac{2g(1 + g)}{Z_4}. \end{aligned} \tag{3.45}$$

Define

$$\begin{aligned} n_1 &= -\frac{(1 + g)(1 + 11g + 31g^2 + 21g^3 + 4n + 20gn)}{4(1 + 6g + 5g^2 + 4n)} \Rightarrow \\ n &= -\frac{(1 + g)(1 + 11g + 31g^2 + 21g^3 + 4n_1 + 20gn_1)}{4(1 + 6g + 5g^2 + 4n_1)}. \end{aligned} \tag{3.46}$$

Then we arrive at the family

$$\begin{aligned} \dot{x}_1 &= \frac{1}{4}(1 + 3g)(1 + 4g + 3g^2 + 2n_1) + n_1 x_1 - \frac{1}{2}(1 + g)y_1 + gx_1^2 + x_1 y_1, \\ \dot{y}_1 &= \frac{1}{2}(1 + 3g)(1 + 4g + 3g^2 + 2n_1)x_1 + 2n_1 y_1 + (g - 1)x_1 y_1 + 2y_1^2. \end{aligned} \tag{3.47}$$

Observe that this family coincides with (3.37) up to notation of variables and parameters. Calculating for the above system, we get:

$$\begin{aligned} \beta_4(g, n_1) &= (1 + g)(1 + 3g)(1 + 6g + 7g^2) + 4(1 + 2g)^2 n_1 \\ &= \frac{g^2(1 + g)^2(5 + 22g + 21g^2 + 4n)}{1 + 6g + 5g^2 + 4n} = -\frac{g^2(1 + g)^2 \alpha_4(g, n)}{Z_4}, \\ \alpha_4(g, n_1) &= 5 + 22g + 21g^2 + 4n_1 \\ &= \frac{4(1 + 10g + 34g^2 + 46g^3 + 21g^4 + 4n + 16gn + 16g^2 n)}{1 + 6g + 5g^2 + 4n} = -\frac{4\beta_4(g, n)}{Z_4}. \end{aligned}$$

Since $g(g+1)Z_4 \neq 0$, the condition $\beta_4(g, n_1) = 0$ (respectively $\alpha_4(g, n_1) = 0$) for systems (3.47) implies $\alpha_4(g, n) = 0$ (respectively $\beta_4(g, n) = 0$) for the original systems (3.37). This completes the proof. $\square$

Thus, in what follows, we assume that the condition

$$\alpha_4 = 5 + 22g + 21g^2 + 4n = 0$$

holds, which implies

$$n = -\frac{(1+3g)(5+7g)}{4}.$$

We then obtain the following family of systems:

$$\begin{aligned} \dot{x} &= -\frac{1}{8}(1+3g)^2(3+5g) - \frac{1}{4}(1+3g)(5+7g)x - \frac{1}{2}(1+g)y + gx^2 + xy, \\ \dot{y} &= -\frac{1}{4}(1+3g)^2(3+5g)x - \frac{1}{2}(1+3g)(5+7g)y + (g-1)xy + 2y^2, \end{aligned} \quad (3.48)$$

possessing the two invariant parabolas $\Phi_1(x, y) = x^2 - y = 0$ and

$$\begin{aligned} \Phi_2(x, y) &= -(1+3g)^2(2+3g)(3+5g) - 4(1+g)(1+3g)(2+3g)x \\ &\quad + 4g(1+g)x^2 + 8(1+2g)^2y = 0. \end{aligned}$$

Considering the coordinates of the finite singularities (3.40), we observe that for $\alpha_4 = 0$, the singular point M_4 coalesces with M_1 , producing a double finite singularity. Thus, systems (3.48) possess three finite singularities $M_i(x_i, y_i)$, $i = 1, 2, 3$ (with M_1 being double), with coordinates

$$\begin{aligned} x_1 &= -\frac{1+3g}{2}, & y_1 &= \frac{(1+3g)^2}{4}; & x_2 &= \frac{(1+3g)(4+13g+11g^2)}{2g(1+g)}, \\ y_2 &= -\frac{(1+3g)^2(3+5g)}{4(1+g)}; & x_3 &= \frac{3+5g}{2}, & y_3 &= \frac{(3+5g)^2}{4}. \end{aligned}$$

Using (3.41), we see that the singular points $M_1(\equiv M_4)$ and M_3 are the intersection points of the invariant parabolas, while M_2 lies on the parabola $\Phi_2(x, y) = 0$. We calculate

$$\begin{aligned} (x_2 - x_1)(x_2 - x_3) &= \frac{(1+2g)^2(1+3g)(2+3g)(2+7g+7g^2)}{g^2(1+g)^2}, \\ (x_2 - x_1) + (x_2 - x_3) &= \frac{4(1+2g)^3}{g(1+g)} \neq 0 \quad \text{if } (x_2 - x_1)(x_2 - x_3) > 0. \end{aligned}$$

Since the discriminant $\text{Discrim}[2+7g+7g^2, g] = -7 < 0$, it follows that

$$\begin{aligned} \text{sign}((x_2 - x_1)(x_2 - x_3)) &= \text{sign}((1+3g)(2+3g)), \\ \text{sign}((x_2 - x_1) + (x_2 - x_3)) &= \text{sign}(g(1+g)(1+2g)). \end{aligned}$$

Note that $\text{sign}(1+2g)$ is only relevant if $(1+3g)(2+3g) > 0$.

On the other hand, for systems (3.48) we have

$$\begin{aligned} \zeta_2 &= 4g(1+g), & \zeta_5 &= 19(g-2)(3+g)(1+3g)^2(2+3g)^2, \\ \zeta_3 &= 217993032g(1+g)(1+3g)^3(2+3g)^3(2+7g+7g^2)^2, \end{aligned}$$

and since $\zeta_2\zeta_5 \neq 0$, it follows that

$$\text{sign}(\zeta_3) = \text{sign}(g(1+g)(1+3g)(2+3g)), \quad \text{sign}(\zeta_2) = \text{sign}(g(1+g)).$$

We claim that $\zeta_2 > 0$ implies $\zeta_3 > 0$. Indeed, suppose $\zeta_2 > 0$ but $\zeta_3 < 0$. This implies $(1+3g)(2+3g) < 0$, i.e., $-2/3 < g < -1/3$. Hence, $-1 < g < 0$, which contradicts $\zeta_2 > 0$. This proves the claim.

Therefore, considering the above relations, for systems (3.48) we obtain the following configurations:

$$\begin{aligned} \zeta_2 < 0, \zeta_3 < 0, 2g+1 < 0 \text{ (i.e. } -1 < g < -2/3) &\Rightarrow x_2 - x_1 > 0, x_2 - x_3 > 0 \Rightarrow \text{Config. } \mathcal{P}.62; \\ \zeta_2 < 0, \zeta_3 < 0, 2g+1 > 0 \text{ (i.e. } -1/3 < g < 0) &\Rightarrow x_2 - x_1 < 0, x_2 - x_3 < 0 \Rightarrow \simeq \text{Config. } \mathcal{P}.62; \\ \zeta_2 < 0, \zeta_3 > 0 \text{ (i.e. } -2/3 < g < -1/3) &\Rightarrow (x_2 - x_1)(x_2 - x_3) < 0 \Rightarrow \text{Config. } \mathcal{P}.63. \\ \zeta_2 > 0, 2g+1 < 0 \text{ (i.e. } g < -1) &\Rightarrow x_2 - x_1 < 0, x_2 - x_3 < 0 \Rightarrow \text{Config. } \mathcal{P}.64; \\ \zeta_2 > 0, 2g+1 > 0 \text{ (i.e. } g > 0) &\Rightarrow x_2 - x_1 > 0, x_2 - x_3 > 0 \Rightarrow \simeq \text{Config. } \mathcal{P}.64. \end{aligned}$$

We observe that these configurations do not depend on the sign of $2g+1$. Hence, we arrive at the following lemma.

Lemma 3.7. *Assume that for systems (3.37) the condition $\mathbf{D} = 0$ holds. Then these systems possess the following configurations if and only if the corresponding conditions are satisfied:*

$$\begin{aligned} \zeta_2 < 0, \zeta_3 < 0 &\Leftrightarrow \text{Config. } \mathcal{P}.62; \\ \zeta_2 < 0, \zeta_3 > 0 &\Leftrightarrow \text{Config. } \mathcal{P}.63; \\ \zeta_2 > 0 &\Leftrightarrow \text{Config. } \mathcal{P}.64. \end{aligned}$$

The case $B_1 = 0$. Considering (3.39) and the condition $\zeta_2\mathcal{R}_2 \neq 0$ (i.e., $g(g+1)(1+6g+5g^2+4n) \neq 0$), we conclude that the condition $B_1 = 0$ is equivalent to

$$\begin{aligned} (1+2g)(1+3g)(2+3g)(1+4g+3g^2+2n)(1+6g+6g^2+4n) \\ \times (1+6g+9g^2+4n)(5+14g+9g^2+4n) = 0. \end{aligned} \quad (3.49)$$

However, through suitable transformations, we can reduce the number of cases arising from the condition $B_1 = 0$. This is stated in the following lemma.

Lemma 3.8. *The condition (3.49) can be transformed, via affine transformations and time rescaling, into the simpler condition*

$$(1+2g)(1+3g)(1+4g+3g^2+2n) = 0. \quad (3.50)$$

Proof. We prove this lemma in two steps: (i) by applying a transformation that replaces the line $y = 0$ with $y = x$ while preserving the invariant parabola $\Phi_1(x, y) = x^2 - y = 0$; and (ii) by applying a transformation that maps the invariant parabola $\Phi_2(x, y) = 0$ (see (3.38)) to the parabola $\Phi_1(x, y) = x^2 - y = 0$.

Step (i): Applying to systems (3.37) the change of variables

$$x_1 = -x + 1/2, \quad y_1 = -x + y + 1/4,$$

we obtain the transformed system

$$\begin{aligned} \dot{x}_1 &= -\frac{1}{8}(2+3g)(1+6g+6g^2+4n) + \frac{1}{4}(1+2g+4n)x_1 + \frac{g}{2}y_1 - (1+g)x_1^2 + x_1y_1, \\ \dot{y}_1 &= -\frac{1}{4}(2+3g)(1+6g+6g^2+4n)x_1 + \frac{1}{2}(1+2g+4n)y_1 - (g+2)x_1y_1 + 2y_1^2. \end{aligned}$$

Setting new parameters

$$\begin{aligned} n_1 &= \frac{1}{4}(1 + 2g + 4n), \quad g_1 = -(1 + g), \quad \Rightarrow \\ n &= \frac{1}{4}(1 + 2g_1 + 4n_1), \quad g = -(1 + g_1), \end{aligned} \quad (3.51)$$

the system becomes

$$\begin{aligned} \dot{x}_1 &= \frac{1}{4}(1 + 3g_1)(1 + 4g_1 + 3g_1^2 + 2n_1) + n_1x_1 - \frac{1 + g_1}{2}y_1 + g_1x_1^2 + x_1y_1, \\ \dot{y}_1 &= \frac{1}{2}(1 + 3g_1)(1 + 4g_1 + 3g_1^2 + 2n_1)x_1 + 2n_1y_1 + (g_1 - 1)x_1y_1 + 2y_1^2, \end{aligned}$$

which has the same form as (3.37).

Moreover, calculations show that

$$\begin{aligned} 2 + 3g &= -(1 + 3g_1), \quad 1 + 6g + 6g^2 + 4n = 2(1 + 4g_1 + 3g_1^2 + 2n_1), \\ 5 + 14g + 9g^2 + 4n &= 1 + 6g_1 + 9g_1^2 + 4n_1, \end{aligned}$$

thus reducing the condition (3.49) to

$$(1 + 2g)(1 + 3g)(1 + 4g + 3g^2 + 2n)(1 + 6g + 9g^2 + 4n) = 0.$$

Step (ii): As shown in the proof of Lemma 3.6, via the transformation (3.45), systems (3.37) can be brought to the canonical form (3.47) but with a new parameter n_1 given by (3.46). Calculations give

$$1 + 6g + 9g^2 + 4n_1 = \frac{8g^2(1 + 4g + 3g^2 + 2n)}{4(1 + 6g + 5g^2 + 4n)}.$$

Since $g \neq 0$, the condition $1 + 6g + 9g^2 + 4n = 0$ can be transformed into $1 + 4g + 3g^2 + 2n = 0$. Consequently, we arrive at the condition (3.50), completing the proof of Lemma 3.8. $\square$

For systems (3.37), we compute

$$\zeta_3 = 8(1 + 2g)^2, \quad \xi_{15} = 2(1 + 3g)(2 + 3g),$$

and distinguish two subcases: $\zeta_3 \neq 0$ and $\zeta_3 = 0$.

1: *The subcase $\zeta_3 \neq 0$.* Then $1 + 2g \neq 0$, and considering (3.50) together with Lemma 3.8, to satisfy the condition $B_1 = 0$ it is sufficient to consider $(1 + 3g)(1 + 4g + 3g^2 + 2n) = 0$. Clearly, we must distinguish two possibilities: $\xi_{15} \neq 0$ and $\xi_{15} = 0$.

1.1: *The possibility $\xi_{15} \neq 0$.* Then $1 + 3g \neq 0$, and in this case we obtain $1 + 4g + 3g^2 + 2n = 0$, i.e., $n = -(1 + g)(1 + 3g)/2$. This leads to the following one-parameter family of systems:

$$\begin{aligned} \dot{x} &= -\frac{1}{2}(1 + g)(1 + 3g)x - \frac{1}{2}(1 + g)y + gx^2 + xy, \\ \dot{y} &= -y(1 + 4g + 3g^2 + x - gx - 2y), \end{aligned} \quad (3.52)$$

which, besides the invariant parabolas

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = (1 + g)(1 + 3g)x - 2gx^2 - (1 + g)y = 0,$$

also possess the invariant line $y = 0$.

Using Lemmas 2.3 and 2.4, we compute:

$$\theta = -8(g-1)(2+g), \quad B_2 = \frac{243}{2}g(1+g)^4(1+2g)^2(2+3g)y^4. \quad (3.53)$$

Thus, we need to consider the cases $B_2 \neq 0$ and $B_2 = 0$.

1.1.1: The case $B_2 \neq 0$. Then, by Lemma 2.3, the system cannot have invariant lines in directions other than $y = 0$. However, by Lemma 2.4, parallel invariant lines could exist if $\theta = 0$.

1.1.1.1: The subcase $\theta \neq 0$. The systems (3.52) possess four finite singularities $M_i(x_i, y_i)$ for $i = 1, 2, 3, 4$, with coordinates:

$$\begin{aligned} x_1 = 0, \quad y_1 = 0; \quad x_2 = \frac{(1+g)(1+3g)}{2g}, \quad y_2 = 0; \\ x_3 = 1+g, \quad y_3 = (1+g)^2; \quad x_4 = -\frac{1+3g}{2}, \quad y_4 = \frac{(1+3g)^2}{4}. \end{aligned}$$

From the conditions stated in the statement $(\mathcal{A}_2^*)$, for systems (3.52) we have:

$$\zeta_1 \zeta_2 \zeta_5 \mathcal{R}_2 \neq 0 \Leftrightarrow g(1+g)(g-2)(3+g)(1+3g)(8+27g+27g^2) \neq 0. \quad (3.54)$$

We observe that the invariant parabolas intersect at two points: M_1 and M_3 . Moreover, the invariant line $y = 0$ has a contact point at M_1 with the parabola $\Phi_1(x, y) = 0$ and two intersection points, M_1 and M_2 , with the parabola $\Phi_2(x, y) = 0$.

Thus, three finite singularities are fixed by the intersections of the invariant curves, with their positions determined by the value of the parameter g .

On the other hand, the singular point M_4 lies on the invariant parabola $\Phi_1(x, y) = 0$ and is a floating singularity. Therefore, we must determine its position relative to the other two singularities on the same invariant curve. To this end, we calculate:

$$(x_4 - x_1)(x_4 - x_3) = \frac{1}{4}(1+3g)(3+5g), \quad (x_4 - x_1) + (x_4 - x_3) = -2(1+2g).$$

Therefore, we obtain:

$$\begin{aligned} \text{sign}((x_4 - x_1)(x_4 - x_3)) &= \text{sign}((1+3g)(3+5g)), \\ \text{sign}((x_4 - x_1) + (x_4 - x_3)) &= -\text{sign}(1+2g). \end{aligned}$$

We observe that the direction of the second invariant parabola depends on $\text{sign}(g(g+1))$.

Furthermore, for systems (3.52), calculations yield:

$$\begin{aligned} \zeta_2 = 4g(g+1), \quad \zeta_{16} = \frac{3705}{2}g^2(1+g)^2(1+3g)(3+5g), \\ \zeta_{17} = \frac{3705}{4}(1+g)^2(1+2g)(1+3g)(3+5g). \end{aligned}$$

Thus, we have:

$$\begin{aligned} \text{sign}(\zeta_2) &= \text{sign}(g(g+1)), \quad \text{sign}(\zeta_{16}) = \text{sign}((1+3g)(3+5g)), \\ \text{sign}(\zeta_{17}) &= \text{sign}((1+2g)(1+3g)(3+5g)). \end{aligned}$$

Therefore, we establish the following configurations:

$$\begin{aligned}
\zeta_2 < 0, \xi_{16} < 0 \text{ (i.e. } -3/5 < g < -1/3) &\Rightarrow (x_4 - x_1)(x_4 - x_3) < 0 \Rightarrow \text{Config. } \mathcal{P}.65; \\
\zeta_2 < 0, \xi_{16} > 0, \xi_{17} < 0 \text{ (i.e. } -1 < g < -3/5) &\Rightarrow x_4 > x_1, x_4 > x_3 \Rightarrow \text{Config. } \mathcal{P}.66; \\
\zeta_2 < 0, \xi_{16} > 0, \xi_{17} > 0 \text{ (i.e. } -1/3 < g < 0) &\Rightarrow x_4 < x_1, x_4 < x_3 \Rightarrow \text{Config. } \mathcal{P}.67; \\
\zeta_2 > 0, \xi_{17} < 0 \text{ (i.e. } g < -1) &\Rightarrow x_4 > x_1, x_4 > x_3 \Rightarrow \text{Config. } \mathcal{P}.68; \\
\zeta_2 > 0, \xi_{17} > 0 \text{ (i.e. } g > 0) &\Rightarrow x_4 < x_1, x_4 < x_3 \Rightarrow \text{Config. } \mathcal{P}.69.
\end{aligned}$$

1.1.1.2: The subcase $\theta = 0$. This condition implies $(g - 1)(g + 2) = 0$, and since for systems (3.52) we have

$$\zeta_6 = \frac{(g - 1)(1 + g)(2 + 5g + 5g^2)}{8},$$

we consider two possibilities: $\zeta_6 \neq 0$ and $\zeta_6 = 0$.

1.1.1.2.1: The possibility $\zeta_6 \neq 0$. In this case, the condition $\theta = 0$ implies $g = -2$. We then obtain the system:

$$\dot{x} = -\frac{5x}{2} + \frac{y}{2} - 2x^2 + xy, \quad \dot{y} = -y(5 + 3x - 2y),$$

which, besides the invariant parabolas:

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = 5x + 4x^2 + y = 0,$$

possesses only one invariant line: $y = 0$. This means that the condition $g = -2$ does not imply the appearance of any additional parallel invariant line.

Since $g = -2 < -1$, we conclude that this case corresponds to configuration *Config. P.68* (as detected earlier).

1.1.1.2.2: The possibility $\zeta_6 = 0$. Then $\theta = 0$ implies $g = 1$, and we arrive at the system:

$$\dot{x} = -4x - y + x^2 + xy, \quad \dot{y} = 2y(y - 4),$$

which possesses the invariant lines $y = 0$ and $y = 4$, as well as the invariant parabolas:

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = -4x + x^2 + y = 0.$$

In this case, we obtain configuration *Config. P.70*.

1.1.2: The case $B_2 = 0$. Considering (3.53) and (3.54), the condition $B_2 = 0$ implies $g = -2/3$, and we arrive at the system:

$$\dot{x} = \frac{x}{6} - \frac{y}{6} - \frac{2x^2}{3} + xy, \quad \dot{y} = \frac{1}{3}y(1 - 5x + 6y), \quad (3.55)$$

which possesses the invariant lines $y = 0$ and $y = x - 1/4$, as well as the invariant parabolas:

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = x - 4x^2 + y = 0.$$

We observe that the invariant line $y = x - 1/4$ is tangent to the invariant parabola $\Phi_1(x, y) = 0$ at the point $M_4(1/2, 1/4)$ and also tangent to the parabola $\Phi_2(x, y) = 0$ at the point $M_2(1/4, 0)$. Therefore, in this case, we obtain configuration *Config. P.71*.

Thus, we have proved the following lemma.

Lemma 3.9. Assume that for systems (3.37) the conditions $B_1 = 0$ and $\zeta_3\zeta_{15} \neq 0$ hold. Then these systems possess the following configurations, provided the corresponding conditions are satisfied:

$$\begin{aligned}
 B_2 \neq 0, \theta \neq 0, \zeta_2 < 0, \xi_{16} < 0 &\Rightarrow \text{Config. } \mathcal{P}.65; \\
 B_2 \neq 0, \theta \neq 0, \zeta_2 < 0, \xi_{16} > 0, \xi_{17} < 0 &\Rightarrow \text{Config. } \mathcal{P}.66; \\
 B_2 \neq 0, \theta \neq 0, \zeta_2 < 0, \xi_{16} > 0, \xi_{17} > 0 &\Rightarrow \text{Config. } \mathcal{P}.67; \\
 B_2 \neq 0, \theta \neq 0, \zeta_2 > 0, \xi_{17} < 0 &\Rightarrow \text{Config. } \mathcal{P}.68; \\
 B_2 \neq 0, \theta \neq 0, \zeta_2 > 0, \xi_{17} > 0 &\Rightarrow \text{Config. } \mathcal{P}.69; \\
 B_2 \neq 0, \theta = 0, \zeta_6 \neq 0 &\Rightarrow \text{Config. } \mathcal{P}.68; \\
 B_2 \neq 0, \theta = 0, \zeta_6 = 0 &\Rightarrow \text{Config. } \mathcal{P}.70; \\
 B_2 = 0 &\Rightarrow \text{Config. } \mathcal{P}.71.
 \end{aligned}$$

1.2: The possibility $\zeta_{15} = 0$. Then, considering the proof of Lemma 3.8, we may assume $1 + 3g = 0$. This yields $g = -1/3$, leading to the following one-parameter family of systems:

$$\dot{x} = nx - \frac{y}{3} - \frac{x^2}{3} + xy, \quad \dot{y} = \frac{2}{3}y(3n - 2x + 3y), \quad (3.56)$$

which, besides the invariant parabolas:

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = 9n^2 - 6nx + x^2 + (9n - 1)y = 0,$$

also possesses the invariant line $y = 0$.

Considering Lemmas 2.3 and 2.4, we compute:

$$\begin{aligned}
 B_2 &= -8(1 + 3n)(9n - 1)(12n - 1)y^4/9, \quad \mathbf{D} = 4096n^6(9n - 1)/243, \\
 \theta &= 160/9 \neq 0, \quad \zeta_5 = -4256n^2/9, \quad \mathcal{R}_2 = 28(9n - 1)/81.
 \end{aligned}$$

Thus, we discuss the cases $B_2 \neq 0$ and $B_2 = 0$.

1.2.1: The case $B_2 \neq 0$. By Lemmas 2.3 and 2.4, no additional invariant line can exist.

Moreover, considering (3.40), the systems (3.56) possess four finite singularities $M_i(x_i, y_i)$ for $i = 1, 2, 3, 4$, with coordinates:

$$x_1 = 0, y_1 = 0; \quad x_2 = 3n, y_2 = 0; \quad x_{3,4} = \frac{1}{3} \left(1 \pm \sqrt{1 - 9n} \right), \quad y_{3,4} = \frac{1}{9} \left(2 - 9n \pm 2\sqrt{1 - 9n} \right).$$

We observe that in this case $n(1 - 9n) \neq 0$ due to $\zeta_5\mathcal{R}_2 \neq 0$, ensuring all finite singularities are distinct. Moreover, the invariant line $y = 0$ is tangent to the parabola $\Phi_1(x, y) = 0$ at the singular point $M_1(0, 0)$ and tangent to $\Phi_2(x, y) = 0$ at $M_2(3n, 0)$.

Since $\mathbf{D} \neq 0$ and $\text{sign}(1 - 9n) = -\text{sign}(\mathbf{D})$, we examine two subcases: $\mathbf{D} < 0$ and $\mathbf{D} > 0$.

1.2.1.1: The subcase $\mathbf{D} < 0$. Then $1 - 9n > 0$ (i.e., $n < 1/9$), and we arrive at configuration *Config. P.72*, independent of the relative position of the singularity $M_2(3n, 0)$ with respect to $M_1(0, 0)$.

1.2.1.2: The subcase $\mathbf{D} > 0$. This implies $1 - 9n < 0$, so the singularities M_3 and M_4 become complex. In this case, we arrive at configuration *Config. P.73*.

1.2.2: The case $B_2 = 0$. Since $9n - 1 \neq 0$ due to $\mathcal{R}_2 \neq 0$, the condition $B_2 = 0$ implies:

$$(1 + 3n)(12n - 1) = 0.$$

1.2.2.1: The subcase $1 + 3n = 0$. Then $n = -1/3$, leading to the system:

$$\dot{x} = -\frac{1}{3}x - \frac{1}{3}y - \frac{1}{3}x^2 + xy, \quad \dot{y} = \frac{2}{3}y(-1 - 2x + 3y).$$

Applying the transformation $x_1 = (1 - x)/4$, $y_1 = (y - x)/4$, and $t_1 = 4t$, we recover system (3.55), corresponding to configuration *Config. P.71*.

1.2.2.2: The subcase $12n - 1 = 0$. This implies $n = 1/12$, and we arrive at the system:

$$\dot{x} = \frac{1}{12}x - \frac{1}{3}y - \frac{1}{3}x^2 + xy, \quad \dot{y} = \frac{1}{6}y(1 - 8x + 12y).$$

Applying the affine transformation $x_1 = -x + 1/2$, $y_1 = y - x + 1/4$, the system is brought to the form (3.55), again corresponding to configuration *Config. P.71*.

2: The subcase $\zeta_3 = 0$. Then $g = -1/2$, which implies $B_1 = 0$. This leads to the following one-parameter family of systems:

$$\dot{x} = (4x - 1)(8n - 1 - 4x + 8y)/32, \quad \dot{y} = (x - 8nx + 32ny - 24xy + 32y^2)/16, \quad (3.57)$$

which possess the following two invariant parabolas: $\Phi_1(x, y) = x^2 - y = 0$, and

$$\Phi_2(x, y) = (8n - 1)(16n - 1) - 4(16n - 1)x + 16x^2 + 8(16n - 3)y = 0,$$

as well as the invariant line $x = 1/4$.

For the above systems, we calculate:

$$\zeta_5 = -475(16n - 1)^2/256, \quad \mathcal{R}_2 = 125(16n - 3)/1024,$$

and according to the conditions given by statement $(\mathcal{A}_2^*)$, we require $\zeta_5 \mathcal{R}_2 \neq 0$. This implies $(16n - 1)(16n - 3) \neq 0$, and by Proposition 2.7*, the invariant parabolas $\Phi_1(x, y) = 0$ and $\Phi_2(x, y) = 0$ are distinct.

For these systems, we compute:

$$B_1 = 0, \quad B_2 = -\frac{81}{2048}(8n - 1)^2(1 + 16n)^2x^4, \quad \theta = 18 \neq 0, \quad (3.58)$$

and since $\theta \neq 0$, by Lemma 2.4, these systems cannot possess an invariant line parallel to $x = 1/4$.

On the other hand, according to Lemma 2.3, the existence of an invariant line in another direction would require $B_2 = 0$. Thus, we consider the possibilities $B_2 \neq 0$ and $B_2 = 0$.

2.1: The possibility $B_2 \neq 0$. The systems (3.57) possess four finite singularities $M_i(x_i, y_i)$ ($i = 1, 2, 3, 4$) with coordinates:

$$\begin{aligned} x_1 &= \frac{1}{4}, \quad y_1 = \frac{1}{16}; & x_2 &= \frac{1}{4}, \quad y_2 = \frac{1 - 8n}{8}; \\ x_{3,4} &= \frac{1}{4} \left(1 \pm \sqrt{3 - 16n} \right), & y_{3,4} &= \frac{1}{8} \left(2 - 8n \pm \sqrt{3 - 16n} \right). \end{aligned} \quad (3.59)$$

We observe that the invariant parabolas intersect at two points: M_3 and M_4 . The invariant line $x = 1/4$ intersects the parabola $\Phi_1(x, y) = 0$ at M_1 and the parabola $\Phi_2(x, y) = 0$ at M_2 .

Thus, all four finite singularities are located at the intersections of invariant curves, and their positions are determined by the parameter n . Furthermore, the singularities M_3 and M_4 are real if $3 - 16n > 0$, and complex if $3 - 16n < 0$.

For systems (3.57), we compute:

$$\mathbf{D} = 3(16n - 3)^3(16n - 1)^2/1048576 \neq 0,$$

due to $\zeta_5 \mathcal{R}_2 \neq 0$. Therefore, $\text{sign}(\mathbf{D}) = \text{sign}(16n - 3)$.

Thus, for $B_2 \neq 0$, we conclude: if $\mathbf{D} < 0$, the system corresponds to configuration *Config. P.74*, and if $\mathbf{D} > 0$, it corresponds to configuration *Config. P.75*.

2.2: The possibility $B_2 = 0$. From (3.58), this condition implies $(8n - 1)(1 + 16n) = 0$. We analyze two cases: $(1 + 16n) \neq 0$ and $(1 + 16n) = 0$.

If $8n - 1 = 0$, i.e., $n = 1/8$, we obtain the system:

$$\dot{x} = -(4x - 1)(x - 2y)/8, \quad \dot{y} = y(1 - 6x + 8y)/4, \quad (3.60)$$

which possesses the following five invariant curves (two parabolas and three invariant lines):

$$\Phi_1(x, y) = x^2 - y, \quad \Phi_2(x, y) = -x + 4x^2 - 2y, \quad x = 1/4, \quad y = 0, \quad y = x - 1/4.$$

For this system, we obtain configuration *Config. P.76*.

In the case $1 + 16n = 0$, we get $n = -1/16$, leading to the system:

$$\dot{x} = -(4x - 1)(3 + 8x - 16y)/64, \quad \dot{y} = -(16y - 1)(3x - 4y)/32,$$

which can be brought to system (3.60) via the affine transformation and time rescaling:

$$x_1 = x/2 + 1/8, \quad y_1 = y/2 - 1/32, \quad t_1 = 2t.$$

Thus, in this case, we again obtain configuration *Config. P.76*.

3.1.3 The statement ($\mathcal{A}_3^*$)

In this case, the conditions $\zeta_4 = \zeta_5 = 0$, together with (3.39) and the assumption $\zeta_1 \neq 0$ (i.e., $(g - 2)(g + 3) \neq 0$), yield the condition $1 + 4g + 3g^2 + 4n = 0$, which implies:

$$n = -\frac{1}{4}(1 + g)(1 + 3g).$$

This leads to the following family of systems:

$$\begin{aligned} \dot{x} &= \frac{1}{8}(1 + g)(1 + 3g)^2 - \frac{1}{4}(1 + g)(1 + 3g)x - \frac{1}{2}(1 + g)y + gx^2 + xy, \\ \dot{y} &= \frac{1}{4}(1 + g)(1 + 3g)^2x - \frac{1}{2}(1 + g)(1 + 3g)y + (g - 1)xy + 2y^2, \end{aligned} \quad (3.61)$$

which possess the invariant parabola $\Phi(x, y) = x^2 - y = 0$ with multiplicity 2.

Following statement ($\mathcal{A}_3^*$), for the above systems we calculate:

$$\begin{aligned} \zeta_1 &= 2(g - 2)(g + 3), \quad \zeta_2 = 4g(1 + g), \quad \zeta_4 = \zeta_5 = 0, \\ \mathcal{R}_2 &= -\frac{g(1 + g)(g - 2)(g + 3)(8 + 27g + 27g^2)}{8}, \\ B_1 &= \frac{g^4(1 + g)^4(1 + 2g)(1 + 3g)^3(2 + 3g)^3}{8}. \end{aligned} \quad (3.62)$$

Since the quadratic polynomial $8 + 27g + 27g^2$ has a negative discriminant, for systems (3.61) we have:

$$\zeta_1 \zeta_2 \mathcal{R}_2 \neq 0 \quad \Rightarrow \quad g(1 + g)(g - 2)(g + 3) \neq 0. \quad (3.63)$$

According to Lemma 2.3, the existence of an invariant line for systems (3.61) requires the condition $B_1 = 0$. Thus, we analyze two cases: $B_1 \neq 0$ and $B_1 = 0$.

The case $B_1 \neq 0$. Then, no invariant line can exist. The systems (3.61) possess three finite singularities $M_i(x_i, y_i)$, for $i = 1, 2, 3$, with coordinates:

$$\begin{aligned} x_1 &= -\frac{1+3g}{2}, \quad y_1 = \frac{(1+3g)^2}{4}; \\ x_{2,3} &= \frac{1}{2}(1+g \pm \sqrt{-2g(1+g)}), \quad y_{2,3} = \frac{g+1}{4}(1-g \pm 2\sqrt{-2g(1+g)}). \end{aligned} \quad (3.64)$$

We point out that M_1 is a multiple singularity of systems (3.61). Indeed, by applying the corresponding translation, M_1 can be placed at the origin, yielding the transformed system:

$$\begin{aligned} \dot{x} &= -\frac{1}{2}g(3g+1)x - (2g+1)y + gx^2 + xy, \\ \dot{y} &= \frac{1}{2}g(3g+1)^2x + (2g+1)(3g+1)y + (g-1)xy + 2y^2, \end{aligned}$$

where $M_0(0,0)$ is the singularity corresponding to M_1 .

Following [1], we compute the corresponding invariant polynomials at M_0 : $\mu_4 = \mu_3 = 0$, and

$$\mu_2 = \frac{1}{2}g(g+1)(3g+1)(3g+2) [g(1+3g)x^2 + 4gxy + 2y^2] \neq 0,$$

due to $B_1 \neq 0$. By [1, Lemma 5.2, statement (ii)], the point M_0 has multiplicity exactly 2.

We also observe that:

$$\Phi(x_1, y_1) = \Phi(x_2, y_2) = \Phi(x_3, y_3) = 0,$$

meaning that all three singularities lie on the invariant parabola.

The nature (real or complex) of M_2 and M_3 depends on the sign of $g(g+1) \neq 0$ (since $\zeta_2 \neq 0$). Given that $\zeta_2 = 4g(g+1)$, we analyze two subcases: $\zeta_2 < 0$ and $\zeta_2 > 0$.

1: The subcase $\zeta_2 < 0$. This implies $g(g+1) < 0$, i.e., $-1 < g < 0$. In this case, all three singularities on the invariant parabola are real. We need to determine the relative position of the double singularity M_1 with respect to the simple singularities M_2 and M_3 .

From (3.64), we calculate:

$$(x_2 - x_1)(x_3 - x_1) = (1+3g)(2+3g)/2 \Rightarrow \text{sign}((x_2 - x_1)(x_3 - x_1)) = \text{sign}((1+3g)(2+3g)).$$

On the other hand, for systems (3.61) we obtain:

$$\xi_{15} = 2(1+3g)(2+3g) \neq 0,$$

due to $B_1 \neq 0$.

Therefore, if $\xi_{15} < 0$, the double point M_1 lies between the singularities M_2 and M_3 , yielding configuration *Config. P.77*, and if $\xi_{15} > 0$, the double point M_1 lies outside the curvilinear interval (M_2, M_3) , yielding configuration *Config. P.78*.

2: The subcase $\zeta_2 > 0$. Then $g(g+1) > 0$, so the singularities M_2 and M_3 are complex. In this case, we get only configuration *Config. P.79*.

The case $B_1 = 0$. Considering (3.62) together with the condition (3.63), we conclude that $B_1 = 0$ is equivalent to the condition $(1 + 2g)(1 + 3g)(2 + 3g) = 0$. For systems (3.61), we also have $\zeta_3 = 8(1 + 2g)^2$, so we consider two subcases: $\zeta_3 \neq 0$ and $\zeta_3 = 0$.

1: The subcase $\zeta_3 \neq 0$. Then $1 + 2g \neq 0$, and the condition $B_1 = 0$ implies $(1 + 3g)(2 + 3g) = 0$. If $g = -1/3$, we arrive at the system:

$$\dot{x} = -(y + x^2 - 3xy)/3, \quad \dot{y} = -2y(2x - 3y)/3, \quad (3.65)$$

which possesses the invariant line $y = 0$, tangent to the double invariant parabola at the singular point $M_1(0, 0)$. In this case, the singular point M_3 has coalesced with M_1 , producing a triple singularity. Therefore, we obtain configuration *Config. P.80*.

Now, consider $g = -2/3$. Then, the system becomes:

$$\dot{x} = (1 + 2x - 4y - 16x^2 + 24xy)/24, \quad \dot{y} = (x + 2y - 20xy + 24y^2)/12,$$

which, via the affine transformation $x_1 = -x + 1/2$, $y_1 = y - x + 1/4$, can be brought to system (3.65), also possessing configuration *Config. P.80*.

2: The subcase $\zeta_3 = 0$. Then $g = -1/2$, and we arrive at the system:

$$\dot{x} = -(4x - 1)(1 + 8x - 16y)/64, \quad \dot{y} = (x + 4y - 48xy + 64y^2)/32,$$

which possesses the invariant line $x = 1/4$ and the double invariant parabola $\Phi_1(x, y) = x^2 - y = 0$. In this case, the singular points M_1 and M_2 have coalesced, and it is straightforward to verify that this yields configuration *Config. P.81*.

Thus, we have proved the following lemma.

Lemma 3.10. Assume that for a quadratic system the conditions $(\mathcal{A}_3^*)$ are satisfied. Then, the system possesses one of the following configurations, if and only if the corresponding conditions below hold:

$$\begin{aligned} B_1 \neq 0, \quad \zeta_2 < 0, \quad \xi_{15} < 0 &\Rightarrow \text{Config. P.77;} \\ B_1 \neq 0, \quad \zeta_2 < 0, \quad \xi_{15} > 0 &\Rightarrow \text{Config. P.78;} \\ B_1 \neq 0, \quad \zeta_2 > 0 &\Rightarrow \text{Config. P.79;} \\ B_1 = 0, \quad \zeta_3 \neq 0 &\Rightarrow \text{Config. P.80;} \\ B_1 = 0, \quad \zeta_3 = 0 &\Rightarrow \text{Config. P.81.} \end{aligned}$$

3.1.4 The statement $(\mathcal{A}_4^*)$

In this case, the conditions $\zeta_4 = \mathcal{R}_2 = 0$ hold. Considering (3.39) and the condition $\zeta_1 \neq 0$ (i.e., $(g - 2)(g + 3) \neq 0$), we get:

$$(8 + 27g + 27g^2)(1 + 6g + 5g^2 + 4n) = 0.$$

However, the discriminant of the quadratic polynomial $8 + 27g + 27g^2$ equals $-135 < 0$, so this factor cannot vanish. Thus, we obtain:

$$1 + 6g + 5g^2 + 4n = 0 \Rightarrow n = -\frac{1}{4}(1 + g)(1 + 5g).$$

This leads to the following family of systems:

$$\begin{aligned} \dot{x} &= \frac{1}{8}(1 + g - 2x)(1 + 4g + 3g^2 - 4gx - 4y), \\ \dot{y} &= \frac{1}{4}(1 + g)^2(1 + 3g)x - \frac{1}{2}(1 + g)(1 + 5g)y + (g - 1)xy + 2y^2, \end{aligned} \quad (3.66)$$

which possess the invariant parabola $\Phi(x, y) = x^2 - y = 0$ and the invariant line $x = (g + 1)/2$.

For the above systems, following the statement $(\mathcal{A}_4^*)$, we calculate:

$$\begin{aligned}\zeta_1 &= 2(g - 2)(3 + g), \quad \zeta_2 = 4g(1 + g), \\ \zeta_4 &= 0 = \mathcal{R}_2, \quad \zeta_5 = 19(g - 2)g^2(1 + g)^2(3 + g), \\ B_1 &= 0, \quad B_2 = -648g^5(1 + g)^5(1 + 3g)(2 + 3g)x^4.\end{aligned}\tag{3.67}$$

Therefore, for systems (3.61), we have:

$$\zeta_1\zeta_2\zeta_5 \neq 0 \Rightarrow g(1 + g)(g - 2)(3 + g) \neq 0.\tag{3.68}$$

We now discuss two possibilities: $B_2 \neq 0$ and $B_2 = 0$.

1: *The case $B_2 \neq 0$.* In this situation, by Lemma 2.3, systems (3.66) cannot possess any invariant line in other direction than $x = (g + 1)/2$.

On the other hand, by Lemma 2.4, these systems could possess an invariant line parallel to the existing one if $\theta = (g - 1)(g + 2) = 0$. However, a straightforward computation shows that neither $g = 1$ nor $g = -2$ leads to the appearance of an additional parallel invariant line.

Systems (3.66) possess three finite singularities $M_i(x_i, y_i)$ ($i = 1, 2, 3$) with coordinates:

$$\begin{aligned}x_1 &= \frac{1 + g}{2}, \quad y_1 = \frac{(1 + g)^2}{4}; \quad x_2 = \frac{1 + g}{2}, \quad y_2 = \frac{1}{4}(1 + g)(1 + 3g); \\ x_3 &= -\frac{1 + 3g}{2}, \quad y_3 = \frac{(1 + 3g)^2}{4}.\end{aligned}$$

We observe that the invariant line $x = (g + 1)/2$ intersects the invariant parabola at the singular point M_1 , and we claim that M_1 is a multiple singularity of systems (3.66). Indeed, applying the corresponding translation, we can place M_1 at the origin of coordinates, resulting in the system:

$$\dot{x} = gx^2 + xy, \quad \dot{y} = g(1 + g)^2x - g(1 + g)y + (g - 1)xy + 2y^2,$$

where $M_0(0, 0)$ is the singularity corresponding to M_1 .

Following [1], we compute the invariant polynomials: $\mu_4 = \mu_3 = 0$, and:

$$\mu_2 = g^2(1 + g)^2(1 + 2g)x(gx + y).$$

Due to condition (3.68), we have that $\mu_2 = 0$ if and only if $1 + 2g = 0$. By [1, Lemma 5.2, statement (ii)], the point M_0 has multiplicity exactly 2 if $1 + 2g \neq 0$, and it is triple if $g = -1/2$.

For systems (3.66), we have $\zeta_3 = 8(1 + 2g)^2$, so we now consider two cases: $\zeta_3 \neq 0$ and $\zeta_3 = 0$.

1.1: *The subcase $\zeta_3 \neq 0$.* Then $1 + 2g \neq 0$, and therefore M_1 is a double singular point of the system, with all three singularities distinct.

Moreover, the singularity M_2 lies on the invariant line, while M_3 lies on the invariant parabola. To determine the relative positions of these points with respect to the double singularity M_1 , we compute:

$$\begin{aligned}y_2 - y_1 &= g(1 + g)/2 \Rightarrow \text{sign}(y_2 - y_1) = \text{sign}(g(g + 1)), \\ x_3 - x_1 &= -(1 + 2g) \Rightarrow \text{sign}(x_3 - x_1) = -\text{sign}(1 + 2g),\end{aligned}$$

and we observe that for systems (3.66), we have $\zeta_2 = 4g(g+1)$, hence $\text{sign}(\zeta_2) = \text{sign}(g(g+1))$.

Thus, we arrive at the following configurations:

$$\begin{aligned} \zeta_2 < 0, 2g+1 < 0 \text{ (i.e., } -1 < g < -1/2) &\Rightarrow y_2 < y_1, x_3 > x_1 \Rightarrow \text{Config. } \mathcal{P}.82; \\ \zeta_2 < 0, 2g+1 > 0 \text{ (i.e., } -1/2 < g < 0) &\Rightarrow y_2 < y_1, x_3 < x_1 \Rightarrow \simeq \text{Config. } \mathcal{P}.82; \\ \zeta_2 > 0, 2g+1 < 0 \text{ (i.e., } g < -1) &\Rightarrow y_2 > y_1, x_3 > x_1 \Rightarrow \text{Config. } \mathcal{P}.83; \\ \zeta_2 > 0, 2g+1 > 0 \text{ (i.e., } g > 0) &\Rightarrow y_2 > y_1, x_3 < x_1 \Rightarrow \simeq \text{Config. } \mathcal{P}.83. \end{aligned}$$

1.2: The subcase $\zeta_3 = 0$. Then $g = -1/2$, leading to the system:

$$\dot{x} = -(4x-1)(-1+8x-16y)/64, \quad \dot{y} = (-x+12y-48xy+64y^2)/32,$$

which possesses the invariant line $x = 1/4$ and the invariant parabola $\Phi_1(x, y) = x^2 - y = 0$. In this case, the singular points M_1 and M_3 coalesce, resulting in the configuration *Config. P.84*.

2: The case $B_2 = 0$. Considering (3.67) and the condition (3.68), we conclude that $B_2 = 0$ is equivalent to $(1+3g)(2+3g) = 0$.

If $g = -1/3$, we arrive at the system:

$$\dot{x} = -(3x-1)(x-3y)/9, \quad \dot{y} = 2y(1-6x+9y)/9, \quad (3.69)$$

which possesses the additional invariant line $y = 0$, tangent to the invariant parabola at the singular point $M_3(0,0)$ and intersecting the invariant line $x = 1/3$ at the singular point $M_2(1/3,0)$. Therefore, we obtain the configuration *Config. P.85*.

Assume now $g = -2/3$. This leads to the system:

$$\dot{x} = -(6x-1)(8x-1-12y)/72, \quad \dot{y} = (-x+14y-60xy+72y^2)/36,$$

which, via the affine transformation $x_1 = -x + 1/2$, $y_1 = -x + y + 1/4$, can be brought to system (3.69), also corresponding to configuration *Config. P.85*.

Thus, we have proved the following lemma.

Lemma 3.11. Assume that for a quadratic system the conditions $(\mathcal{A}_4^*)$ are satisfied. Then this system possesses one of the following configurations if and only if the corresponding conditions are satisfied, respectively:

$$\begin{aligned} B_2 \neq 0, \zeta_3 \neq 0, \zeta_2 < 0 &\Rightarrow \text{Config. } \mathcal{P}.82; \\ B_2 \neq 0, \zeta_3 \neq 0, \zeta_2 > 0 &\Rightarrow \text{Config. } \mathcal{P}.83; \\ B_2 \neq 0, \zeta_3 = 0 &\Rightarrow \text{Config. } \mathcal{P}.84; \\ B_2 = 0 &\Rightarrow \text{Config. } \mathcal{P}.85. \end{aligned}$$

3.1.5 The statement $(\mathcal{A}_5^*)$

According to Proposition 2.7*, the condition $\zeta_2 = 0$ holds. Considering (3.2), we obtain $g(g+1) = 0$.

Following the proof of Lemma 3.1, we conclude that the condition $g+1 = 0$ can be reduced, via an affine transformation (see formulas (3.12)), to the condition $g = 0$.

Therefore, it suffices to study only the case $g = 0$. In this case, we arrive at the following two-parameter family of systems:

$$\dot{x} = m + nx - y/2 + xy, \quad \dot{y} = 2mx + 2ny - xy + 2y^2, \quad (3.70)$$

which possesses the invariant parabola $\Phi(x, y) = x^2 - y = 0$.

Considering the statement $(\mathcal{A}_5^*)$ for these systems, we calculate:

$$\begin{aligned} \zeta_1 &= -12, \quad \zeta_2 = 0, \quad \zeta_6 = (2m + n)/2, \quad \mathcal{R}_1 = 0, \quad \mathcal{R}_2 = 6(2m + n), \\ B_1 &= 2m(4m - 1 - 2n)(2m + n)^3. \end{aligned} \quad (3.71)$$

Remark 3.12. Following [1, Lemma 5.2], for systems (3.70), we compute:

$$\mu_0 = 0, \quad \mu_1 = -2(2m + n)y \neq 0,$$

due to the condition $\zeta_6 \neq 0$. Therefore, according to [1, Lemma 5.2, statement (i)], we conclude that one of the singular points of systems (3.70) has gone to infinity and has coalesced with the infinite singularity $N[1 : 0 : 0]$, producing an infinite singularity of multiplicity $(1, 1)$ (see Remark 2.5).

The case $B_1 \neq 0$. We observe that the family of systems (3.70) is a subfamily of (3.1) defined by the condition $g = 0$. Considering the finite singularities of (3.1) given in (3.4), we remark that in the case $g = 0$, the singular point $M_1(x_1, y_1)$ with coordinates

$$x_1 = -\frac{2m + n + gn}{g(1 + g)}, \quad y_1 = \frac{2m}{1 + g},$$

has moved to infinity. According to Remark 3.12, this singularity coalesced with the infinite singularity $N[1 : 0 : 0]$, producing an infinite singularity of multiplicity $(1, 1)$.

Therefore, systems (3.70) possess three finite singularities $M_i(\tilde{x}_i, \tilde{y}_i)$ for $i = 2, 3, 4$, where, from (3.4), we have

$$\tilde{x}_i = x_i|_{g=0}, \quad \tilde{y}_i = y_i|_{g=0}, \quad i = 2, 3, 4.$$

Taking into consideration [1, Proposition 5.1], for systems (3.70), we compute:

$$\mu_0 = 0, \quad \mathbf{D} = 48(2m + n)^4(108m^2 - 2m + 36mn - n^2 + 16n^3), \quad \mathbf{R} = 12(2m + n)^2y^2.$$

We observe that $\mathbf{R} \neq 0$ due to $\zeta_6 \neq 0$, and by [1, Proposition 5.1], we have three distinct real finite singularities if $\mathbf{D} < 0$, and one real and two complex if $\mathbf{D} > 0$. Considering the point at infinity of multiplicity $(1, 1)$, we arrive at *Config. P.86* if $\mathbf{D} < 0$, and *Config. P.87* if $\mathbf{D} > 0$.

Now assume that for systems (3.70), the condition $\mathbf{D} = 48(F'_1)^2F'_2 = 0$ holds, where, from (3.3), we have:

$$F'_1 = -(2m + n)^2, \quad F'_2 = -2m + 108m^2 + 36mn - n^2 + 16n^3.$$

Since $F'_1 \neq 0$ (due to $\zeta_6 \neq 0$), we conclude that the condition $\mathbf{D} = 0$ is equivalent to $F'_2 = 0$.

Introducing a new parameter v , as in the generic case (see page 29), we obtain $n = (1 - v^2)/12$, $m = (1 + v)^2(2v - 1)/216$, and we arrive at the following one-parameter family of systems:

$$\begin{aligned} \dot{x} &= \frac{(1 + v)^2(2v - 1)}{216} - \frac{v^2 - 1}{12}x - \frac{1}{2}y + xy, \\ \dot{y} &= \frac{(1 + v)^2(2v - 1)}{108}x - \frac{v^2 - 1}{6}y - xy + 2y^2, \end{aligned} \quad (3.72)$$

which is a subfamily of (3.8) defined by $g = 0$.

The systems (3.5) possess three finite singularities given in (3.7), and M_1 is a multiple singularity (of multiplicity at least two). We observe that for $g = 0$, the singular point M_3

has gone to infinity and, according to Remark 3.12, this singularity coalesced with the infinite singularity $N[1 : 0 : 0]$, producing an infinite singularity of multiplicity $(1, 1)$.

Thus, the systems (3.72) possess at most two different finite singularities $M_1(\tilde{x}_1, \tilde{y}_1)$ (multiple) and $M_2(\tilde{x}_2, \tilde{y}_2)$, where, from (3.7), we have:

$$\tilde{x}_1 = \frac{1+v}{6}, \quad \tilde{y}_1 = \frac{(1+v)^2}{36}; \quad \tilde{x}_2 = \frac{1-2v}{6}, \quad \tilde{y}_2 = \frac{(1-2v)^2}{36}.$$

We observe that M_2 coalesces with the double point M_1 if and only if $v = 0$.

Considering that $B_1 \neq 0$ (i.e., systems (3.72) do not possess any invariant line), we conclude that the configuration is *Config. P.88* if $v \neq 0$, and *Config. P.89* if $v = 0$.

Moreover, for systems (3.72), we compute:

$$\xi_2 = \frac{1}{209952}v^2(v-2)^6(1+v)^2, \quad \zeta_6 = \frac{1}{108}(v-2)^2(1+v),$$

and due to $\zeta_6 \neq 0$, we conclude that the condition $v \neq 0$ is equivalent to $\xi_2 \neq 0$. Therefore, we obtain configuration *Config. P.88* if $\xi_2 \neq 0$, and *Config. P.89* if $\xi_2 = 0$.

The case $B_1 = 0$. Considering (3.71) and the condition $\zeta_6 \neq 0$ (i.e., $2m + n \neq 0$), we deduce that the condition $B_1 = 0$ is equivalent to $m(4m - 1 - 2n) = 0$.

On the other hand, for systems (3.70), we calculate:

$$\xi_1 = 9(4m - 1 - 2n)(2m + n)^2/4$$

and, due to $\zeta_6 \neq 0$, we conclude that the condition $4m - 1 - 2n = 0$ is equivalent to $\xi_1 = 0$. Thus, we discuss two subcases: $\xi_1 \neq 0$ and $\xi_1 = 0$.

1: The subcase $\xi_1 \neq 0$. In this case, the condition $B_1 = 0$ yields $m = 0$, which leads to the one-parameter family of systems:

$$\dot{x} = \frac{1}{2}(2nx - y + 2xy), \quad \dot{y} = y(2n - x + 2y), \quad (3.73)$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y = 0$ and the invariant line $y = 0$. Calculations yield:

$$B_2 = -324n^2(1 + 2n)y^4$$

and, according to Lemma 2.3, we discuss two possibilities: $B_2 \neq 0$ and $B_2 = 0$.

1.1: The possibility $B_2 \neq 0$. We determine that systems (3.73) possess three finite singularities $M_i(x_i, y_i)$ ($i = 1, 2, 3$) with coordinates:

$$x_1 = 0, \quad y_1 = 0; \quad x_{2,3} = \frac{1}{4} \left(1 \pm \sqrt{1 - 16n} \right), \quad y_{2,3} = \frac{1}{8} \left(1 - 8n \pm \sqrt{1 - 16n} \right). \quad (3.74)$$

According to Remark 3.12, the fourth finite singularity has coalesced with an infinite one, resulting in a singular point of multiplicity $(1, 1)$.

We observe that M_1 is the tangency point between $y = 0$ and the invariant parabola, and that M_2 and M_3 are either real, complex, or coincident, depending on the value of $1 - 16n$.

To determine the relative position of the real singularities M_2 and M_3 with respect to M_1 , we calculate:

$$(x_2 - x_1)(x_3 - x_1) = n, \quad (x_2 - x_1) + (x_3 - x_1) = 1/2 > 0.$$

On the other hand, for systems (3.73), we calculate:

$$\mathbf{D} = -48n^6(1 - 16n), \quad \mathcal{R}_2 = 6n,$$

and therefore, due to $\mathcal{R}_2 \neq 0$, we have $\text{sign}(\mathcal{R}_2) = \text{sign}(n)$ and $\text{sign}(\mathbf{D}) = -\text{sign}(1 - 16n)$.

Thus, in the case $B_2 \neq 0$, we arrive at the following four configurations:

$$\begin{aligned} \mathbf{D} < 0, \mathcal{R}_2 < 0 &\Rightarrow \text{Config. } \mathcal{P}.90; \\ \mathbf{D} < 0, \mathcal{R}_2 > 0 &\Rightarrow \text{Config. } \mathcal{P}.91; \\ \mathbf{D} > 0 &\Rightarrow \text{Config. } \mathcal{P}.92; \\ \mathbf{D} = 0 &\Rightarrow \text{Config. } \mathcal{P}.93. \end{aligned}$$

1.2: The possibility $B_2 = 0$. This implies that $(1 + 2n)n = 0$, and since $n \neq 0$ (due to $\zeta_6 = n/2 \neq 0$), we get $1 + 2n = 0$. Thus, $n = -1/2$, and we obtain the system:

$$\dot{x} = \frac{1}{2}(-x - y + 2xy), \quad \dot{y} = y(-1 - x + 2y), \quad (3.75)$$

possessing two invariant lines: $y = 0$ and $y = x$. Considering (3.74), we get three real finite singularities, leading to configuration *Config. P.94*.

2: The subcase $\zeta_1 = 0$. This implies $m = (1 + 2n)/4$, and we arrive at the one-parameter family of systems:

$$\dot{x} = \frac{1}{4}(1 + 2n + 4nx - 2y + 4xy), \quad \dot{y} = \frac{1}{2}(x + 2nx + 4ny - 2xy + 4y^2), \quad (3.76)$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y = 0$ and the invariant line $y = x - (2n + 1)/2$. Calculations yield:

$$B_2 = -81(1 + 2n)(1 + 4n)^2(x - y)^4$$

and, considering Lemma 2.3, we examine two possibilities: $B_2 \neq 0$ and $B_2 = 0$.

2.1: The possibility $B_2 \neq 0$. The above systems possess three finite singularities $M_i(x_i, y_i)$ ($i = 1, 2, 3$) with coordinates:

$$x_1 = -\frac{1}{2}, \quad y_1 = \frac{1}{4}; \quad x_{2,3} = \frac{1}{2} \left(1 \pm \sqrt{-(1 + 4n)} \right), \quad y_{2,3} = \frac{1}{2} \left(-2n \pm \sqrt{-(1 + 4n)} \right).$$

The singularities M_2 and M_3 are the intersection points between the invariant line $y = x - (2n + 1)/2$ and the invariant parabola, and they are real (respectively, complex or coincident) if $1 + 4n < 0$ (respectively, $1 + 4n > 0$, $1 + 4n = 0$).

Again, in the case of real singularities, we calculate:

$$(x_2 - x_1)(x_3 - x_1) = (5 + 4n)/4, \quad (x_2 - x_1) + (x_3 - x_1) = 2 > 0.$$

Moreover, for systems (3.76), calculations give:

$$\mathbf{D} = 3(1 + 4n)^5(5 + 4n)^2/4, \quad \zeta_3 = 8164197(1 + 4n)^3(5 + 4n)/2,$$

and therefore, for $\mathbf{D} \neq 0$, we have $\text{sign}(\mathbf{D}) = \text{sign}(1 + 4n)$ and $\text{sign}(\zeta_3) = \text{sign}((1 + 4n)(5 + 4n))$.

Thus, in the case $B_2 \neq 0$ and $\mathbf{D} \neq 0$, we arrive at the following configurations:

$$\begin{aligned} \mathbf{D} < 0, \zeta_3 < 0 &\Rightarrow \text{Config. } \mathcal{P}.95; \\ \mathbf{D} < 0, \zeta_3 > 0 &\Rightarrow \text{Config. } \mathcal{P}.96; \\ \mathbf{D} > 0 &\Rightarrow \text{Config. } \mathcal{P}.97. \end{aligned}$$

Now, assume that $\mathbf{D} = 0$. This implies $(1 + 4n)(5 + 4n) = 0$. Since $\zeta_6 = (1 + 4n)/4 \neq 0$, we get $5 + 4n = 0$, meaning that one of the singularities M_2 or M_3 coalesces with M_1 . Thus, we arrive at configuration *Config. P.98*.

2.2: The possibility $B_2 = 0$. This implies $(1 + 2n)(1 + 4n) = 0$, and since $1 + 4n \neq 0$ (due to $\zeta_6 \neq 0$), we get $1 + 2n = 0$. Thus, $n = -1/2$, and we arrive again at system (3.75), possessing configuration *Config. P.94*.

Thus, we have proved the following lemma.

Lemma 3.13. *Assume that for a quadratic system the conditions $(\mathcal{A}_5^*)$ are satisfied. Then this system possesses one of the following configurations if and only if the corresponding conditions are satisfied, respectively:*

$B_1 \neq 0, \mathbf{D} < 0$	$\Rightarrow$ <i>Config. P.86;</i>
$B_1 \neq 0, \mathbf{D} > 0$	$\Rightarrow$ <i>Config. P.87;</i>
$B_1 \neq 0, \mathbf{D} = 0, \zeta_2 \neq 0$	$\Rightarrow$ <i>Config. P.88;</i>
$B_1 \neq 0, \mathbf{D} = 0, \zeta_2 = 0$	$\Rightarrow$ <i>Config. P.89;</i>
$B_1 = 0, \zeta_1 \neq 0, B_2 \neq 0, \mathbf{D} < 0, \mathcal{R}_2 < 0$	$\Rightarrow$ <i>Config. P.90;</i>
$B_1 = 0, \zeta_1 \neq 0, B_2 \neq 0, \mathbf{D} < 0, \mathcal{R}_2 > 0$	$\Rightarrow$ <i>Config. P.91;</i>
$B_1 = 0, \zeta_1 \neq 0, B_2 \neq 0, \mathbf{D} > 0$	$\Rightarrow$ <i>Config. P.92;</i>
$B_1 = 0, \zeta_1 \neq 0, B_2 \neq 0, \mathbf{D} = 0$	$\Rightarrow$ <i>Config. P.93;</i>
$B_1 = 0, \zeta_1 \neq 0, B_2 = 0$	$\Rightarrow$ <i>Config. P.94;</i>
$B_1 = 0, \zeta_1 = 0, B_2 \neq 0, \mathbf{D} < 0, \zeta_3 < 0$	$\Rightarrow$ <i>Config. P.95;</i>
$B_1 = 0, \zeta_1 = 0, B_2 \neq 0, \mathbf{D} < 0, \zeta_3 > 0$	$\Rightarrow$ <i>Config. P.96;</i>
$B_1 = 0, \zeta_1 = 0, B_2 \neq 0, \mathbf{D} > 0$	$\Rightarrow$ <i>Config. P.97;</i>
$B_1 = 0, \zeta_1 = 0, B_2 \neq 0, \mathbf{D} = 0$	$\Rightarrow$ <i>Config. P.98;</i>
$B_1 = 0, \zeta_1 = 0, B_2 = 0$	$\Rightarrow$ <i>Config. P.94.</i>

Since all the statements provided by Proposition 2.7* have been considered, this proposition is proved. $\square$

3.2 Systems in $\text{QSP}_{(\eta>0)}$ with the condition $\zeta_1 = 0$

In what follows, we examine each of the statements $(\mathcal{B}_1)$ to $(\mathcal{B}_7)$ given by Proposition 2.8.

According to this proposition, a system satisfying the conditions provided by one of the statements $(\mathcal{B}_1)$ to $(\mathcal{B}_7)$ can be brought to the form:

$$\dot{x} = m + nx - \frac{3y}{2} + 2x^2 + xy, \quad \dot{y} = 2mx + 2ny + xy + 2y^2, \quad (3.77)$$

and this system possesses the invariant parabola $\Phi_1(x, y) = x^2 - y = 0$.

3.2.1 The statement $(\mathcal{B}_1)$

According to this statement, for systems (3.77) we calculate $\chi_3 = 0$ and

$$\chi_4 = 61875 \mathcal{U}_1 \mathcal{U}_3, \quad \zeta_7 = -\frac{52875}{2} \mathcal{U}_1 \mathcal{U}_2 \mathcal{U}_3, \quad \mathcal{R}_3 = 3850561006875 \mathcal{U}_1 \mathcal{U}_2 \mathcal{U}_3, \quad (3.78)$$

where

$$\begin{aligned} \mathcal{U}_1 &= 1 + 4m + 2n, & \mathcal{U}_2 &= 4m - 147 - 14n, \\ \mathcal{U}_3 &= 18m + 1372m^2 - 84mn + 27n^2 + 144n^3. \end{aligned}$$

On the other hand, following Lemma 2.3, we calculate:

$$B_1 = m(2m - n)(2m + 3n + 9)(4m - 6n - 9)(1 + 4m + 2n) \quad (3.79)$$

and, considering Lemma 2.3, we discuss two cases: $B_1 \neq 0$ and $B_1 = 0$.

The case $B_1 \neq 0$. We observe that the family of systems (3.77) is a subfamily of (3.1) defined by the condition $g = 2$. Therefore, it is clear that systems (3.77) possess four finite singularities $M_i(\tilde{x}_i, \tilde{y}_i)$ ($i = 1, 2, 3, 4$), where, considering (3.4), we have:

$$\tilde{x}_i = x_i|_{\{g=2\}}, \quad \tilde{y}_i = y_i|_{\{g=2\}}, \quad i = 1, 2, 3, 4.$$

On the other hand, for systems (3.77), we have:

$$\mathbf{D} = 48\tilde{F}_1^2\tilde{F}_2, \quad \tilde{F}_1 = F_1|_{\{g=2\}}, \quad \tilde{F}_2 = F_2|_{\{g=2\}},$$

where F_1 and F_2 are given in (3.3).

As it was proved for the family (3.1), in the case $\mathbf{D} \neq 0$, these systems possess only two distinct configurations: *Config. P.1* if $\mathbf{D} < 0$, and *Config. P.2* if $\mathbf{D} > 0$. We obtain the same two configurations in the particular case $g = 2$, because this value of the parameter g is not a bifurcation value for these two configurations.

Assume now that $\mathbf{D} = 0$. This implies $\tilde{F}_1\tilde{F}_2 = 0$, and we have to distinguish which factor vanishes. We point out that the invariant polynomial ζ_1 , which governed the condition $F_1 = 0$ for systems (3.1) in the generic case (i.e., $g \neq 2$), vanishes for $g = 2$. Therefore, we must use another invariant polynomial, and for systems (3.77) we calculate:

$$\zeta_{18} = 17969284698750 \mathcal{U}_2 \mathcal{U}_3 \tilde{F}_1.$$

Therefore, due to the condition $\zeta_7 \neq 0$, we obtain that the condition $\tilde{F}_1 = 0$ is equivalent to $\zeta_{18} = 0$. Thus, we examine two possibilities: $\zeta_{18} \neq 0$ and $\zeta_{18} = 0$.

1: The possibility $\zeta_{18} \neq 0$. Then $\tilde{F}_1 \neq 0$ and hence the condition $\mathbf{D} = 0$ implies $\tilde{F}_2 = 0$. Following the investigation of the family of systems (3.1) in the particular case $g = 2$, we arrive at the systems (3.5), which for $g = 2$ become:

$$\begin{aligned} \dot{x} &= \frac{1}{216}(2v+1)(v-1)^2 - \frac{1}{12}(v^2-1)x - \frac{3y}{2} + 2x^2 + xy, \\ \dot{y} &= \frac{1}{108}(2v+1)(v-1)^2x - \frac{1}{6}(v^2-1)y + xy + 2y^2. \end{aligned} \quad (3.80)$$

For the above systems, we calculate:

$$\begin{aligned} \zeta_2 &= 2^{-5}3^{-8}(v-4)^2(v-1)^2v^2(v^2-20v-8)^2, \\ \zeta_{18} &= 998293594375(v-10)(v-4)^4(v-1)^4(20+v)^2(4+5v)^2(v^2-20v-8)/3188646, \end{aligned}$$

and we observe that, due to $\zeta_{18} \neq 0$, the condition $\zeta_2 = 0$ is equivalent to $v = 0$.

Therefore, following the examination of the two-parameter family of systems (3.1), we conclude that the one-parameter family of systems (3.80), in the case $B_1\zeta_{18} \neq 0$, possesses the configuration *Config. P.3* if $\zeta_2 \neq 0$ and *Config. P.4* if $\zeta_2 = 0$.

2: The possibility $\xi_{18} = 0$. Then we have $\tilde{F}_1 = 0$, which implies $\mathbf{D} = 0$. Following the investigation of the family of systems (3.1) in the particular case $g = 2$, we arrive at the systems (3.8), which for $g = 2$ become:

$$\begin{aligned}\dot{x} &= \frac{3}{8}(u-2)^2 - \frac{1}{4}(u^2-4)x - \frac{3y}{2} + 2x^2 + xy, \\ \dot{y} &= \frac{3}{4}(u-2)^2x - \frac{1}{2}(u^2-4)y + xy + 2y^2.\end{aligned}\tag{3.81}$$

For the above systems, we calculate:

$$\begin{aligned}\chi_4 &= -556875(u-3)^5(u-2)^2(13+u)/4, \quad \xi_2 = 9(u-3)^2(u-2)^2\tilde{Z}_1/2, \\ \xi_3 &= 490484322\tilde{\alpha}_1^3\tilde{Z}_1, \quad \tilde{Z}_1 = Z_1|_{\{g=2\}}, \quad \tilde{\alpha}_1 = \alpha_1|_{\{g=2\}},\end{aligned}$$

where Z_1 and α_1 are the polynomials defined for systems (3.8) (see (3.10) and (3.11)).

We observe that, due to the condition $\chi_4 \neq 0$, we have:

$$\text{sign}(\xi_2) = \text{sign}(\tilde{Z}_1), \quad \text{sign}(\xi_3) = \text{sign}(\tilde{Z}_1\tilde{\alpha}_1),$$

and, following the examination of the two-parameter family of systems (3.8), we conclude that the one-parameter family of systems (3.81), in the case $B_1 \neq 0$, possesses the following configurations if and only if the corresponding conditions are satisfied:

$$\begin{aligned}\xi_2 < 0 &\Leftrightarrow \text{Config. } \mathcal{P}.5; \\ \xi_2 > 0, \xi_3 < 0 &\Leftrightarrow \text{Config. } \mathcal{P}.6; \\ \xi_2 > 0, \xi_3 > 0 &\Leftrightarrow \text{Config. } \mathcal{P}.7; \\ \xi_2 = 0 &\Leftrightarrow \text{Config. } \mathcal{P}.8.\end{aligned}$$

The case $B_1 = 0$. Considering (3.79) and the condition $\chi_4 \neq 0$ (i.e., $4m + 2n + 1 \neq 0$), we observe that the condition $B_1 = 0$ is equivalent to:

$$m(2m-n)(2m+3n+9)(4m-6n-9) = 0.$$

For systems (3.77), calculations yield:

$$\begin{aligned}\xi_{19} &= -12870000m(2m-n)(2m+3n+9), \\ \xi_{20} &= -540m(2m-n)\mathcal{U}_1\mathcal{U}_3, \quad \xi_{21} = -110106m\mathcal{U}_1\mathcal{U}_2\mathcal{U}_3,\end{aligned}\tag{3.82}$$

and we consider two subcases: $\xi_{19} \neq 0$ and $\xi_{19} = 0$.

The subcase $\xi_{19} \neq 0$. Then $m(2m-n)(2m+3n+9) \neq 0$, and therefore the condition $B_1 = 0$ yields $4m - 6n - 9 = 0$. This implies $m = \frac{3(3+2n)}{4}$, and we arrive at the one-parameter family of systems:

$$\begin{aligned}\dot{x} &= (9 + 6n + 4nx + 8x^2 - 6y + 4xy)/4, \\ \dot{y} &= (9x + 6nx + 4ny + 2xy + 4y^2)/2,\end{aligned}\tag{3.83}$$

which possesses the invariant line $y = x - \frac{3+2n}{2}$ and four finite singularities $M_i(x_i, y_i)$ ($i = 1, 2, 3, 4$) with coordinates:

$$\begin{aligned}x_1 &= -\frac{3}{2}, \quad y_1 = \frac{9}{4}; \quad x_2 = -\frac{3+4n}{4}, \quad y_2 = \frac{3+2n}{4}; \\ x_{3,4} &= \frac{1}{2} \left(1 \pm \sqrt{-(5+4n)} \right), \quad y_{3,4} = \frac{1}{2} \left(-2 - 2n \pm \sqrt{-(5+4n)} \right).\end{aligned}$$

We determine that the singularities M_1 , M_3 , and M_4 lie on the invariant parabola. Moreover, M_3 and M_4 are the points of intersection of the invariant line $y = x - (3 + 2n)/2$ with the invariant parabola $\Phi_1(x, y) = x^2 - y = 0$. We calculate:

$$\Phi_1(x_2, y_2) = \frac{1}{16}(4n - 3)(5 + 4n),$$

and we conclude that M_2 lies on the parabola if and only if $4n - 3 = 0$, because for systems (3.83), we have:

$$\chi_4 \zeta_7 \mathcal{R}_3 \neq 0 \Leftrightarrow (5 + 4n)(9 + 4n)(69 + 4n) \neq 0. \quad (3.84)$$

In order to determine the position of the singularity M_1 with respect to M_3 and M_4 (when they are real), we calculate:

$$\begin{aligned} (x_1 - x_3)(x_1 - x_4) &= (21 + 4n)/4 \Rightarrow \text{sign}((x_1 - x_3)(x_1 - x_4)) = \text{sign}(21 + 4n), \\ (x_1 - x_3) + (x_1 - x_4) &= -4 < 0. \end{aligned}$$

Thus, we observe that for the parameter n , the following possible bifurcation values arise: $n \in \{-21/4, -5/4, 3/4\}$. Moreover, we point out that due to the condition (3.84), the inequality $5 + 4n \neq 0$ must hold (i.e., $n \neq -5/4$), and hence the singularities M_3 and M_4 cannot coincide.

On the other hand, according to Lemma 2.3, for the existence of an invariant line in a direction different from $y = x$, the condition $B_2 = 0$ is necessary. For systems (3.83), we calculate:

$$\begin{aligned} B_2 &= -729(3 + 2n)(9 + 4n)^2(x - y)^4, \quad \mathbf{D} = 243(4n - 3)^2(5 + 4n)^3(21 + 4n)^2/4, \\ \zeta_9 &= 16299895407840(9 + 4n)^2(21 + 4n), \end{aligned}$$

and in the case $\mathbf{D} \neq 0$, we have:

$$\text{sign}(\mathbf{D}) = \text{sign}(5 + 4n), \quad \text{sign}(\zeta_9) = \text{sign}(21 + 4n).$$

Considering Lemma 2.3, we examine two possibilities: $B_2 \neq 0$ and $B_2 = 0$.

1: The possibility $B_2 \neq 0$. We discuss two cases: $\mathbf{D} \neq 0$ and $\mathbf{D} = 0$.

1.1: The case $\mathbf{D} \neq 0$. In this case, all four finite singular points of systems (3.83) are distinct. Considering the bifurcation values of the parameter n mentioned above, for systems (3.83) we obtain the following configurations (depending on the parameter n):

$$\begin{aligned} \mathbf{D} < 0, \zeta_9 < 0 \text{ (i.e., } n < -21/4) &\Rightarrow \simeq \text{Config. } \mathcal{P}.17; \\ \mathbf{D} < 0, \zeta_9 > 0 \text{ (i.e., } -21/4 < n < -5/4) &\Rightarrow \simeq \text{Config. } \mathcal{P}.19; \\ \mathbf{D} > 0 \text{ (i.e., } n > -5/4) &\Rightarrow \simeq \text{Config. } \mathcal{P}.20. \end{aligned}$$

1.2: The case $\mathbf{D} = 0$. Then, due to the condition (3.84), we get $(4n - 3)(21 + 4n) = 0$, and we observe that the condition $21 + 4n = 0$ is governed by the invariant polynomial ζ_9 . Therefore, we arrive at the configuration Config. $\mathcal{P}.21$ if $\zeta_9 \neq 0$ and Config. $\mathcal{P}.25$ if $\zeta_9 = 0$.

2: The possibility $B_2 = 0$. Considering the condition (3.84), we get $n = -3/2$, which leads to the system:

$$\dot{x} = \frac{1}{2}(-3x + 4x^2 - 3y + 2xy), \quad \dot{y} = y(-3 + x + 2y),$$

possessing two invariant lines $y = x$ and $y = 0$, in addition to the invariant parabola. Therefore, in this case, we obtain an equivalent configuration to Config. $\mathcal{P}.52$.

The subcase $\xi_{19} = 0$. Then $m(2m - n)(2m + 3n + 9) = 0$, and we examine two possibilities: $\xi_{20} \neq 0$ and $\xi_{20} = 0$.

1: The possibility $\xi_{20} \neq 0$. In this case, considering (3.82), we obtain $m(2m - n) \neq 0$, and therefore $2m + 3n + 9 = 0$. This implies $m = -3(3 + n)/2$, and we arrive at the one-parameter family of systems:

$$\dot{x} = \frac{1}{2}(2x - 3)(3 + n + 2x + y), \quad \dot{y} = -3(3 + n)x + 2ny + xy + 2y^2, \quad (3.85)$$

which is a subfamily of (3.14) defined by the condition $g = 2$. The family (3.14) was investigated earlier, and considering (3.15), (3.16) and (3.17) for $g = 2$ (i.e., for systems (3.85)), we have:

$$\begin{aligned} Z_2 &= -4(2 + n), \quad \alpha_2 = 21 + 4n, \quad \beta_2 = 33 + 4n, \\ B_2 &= -5832(3 + n)(9 + 4n)^2(17 + 4n)x^4, \quad \mathbf{D} = -972Z_2\alpha_2^2\beta_2^2, \\ B_3 &= -9(9 + 4n)x^2(12x^2 + 4nx^2 - 24xy - 8nxy - 5y^2)/2, \\ \xi_7 &= 1057164750000Z_2\alpha_2^2\beta_2, \quad \xi_8 = 55640250000Z_2\alpha_2\beta_2^2, \\ \xi_{18} &= -7277560302993750(6 + n)(9 + 4n)(57 + 4n)\alpha_2\beta_2. \end{aligned}$$

We observe that for the parameter n , the following possible bifurcation values arise: $n \in \{-33/4, -21/4, -2\}$. Considering (3.78) for systems (3.85), we obtain:

$$\chi_4\xi_7\mathcal{R}_3 \neq 0 \Leftrightarrow (6 + n)(9 + 4n)(17 + 4n)(57 + 4n)\beta_2 \neq 0, \quad (3.86)$$

and therefore $\beta_2B_3 \neq 0$. Moreover, from the above condition we deduce that $\xi_{18} = 0$ if and only if $\alpha_2 = 0$.

Thus, in the case $B_2 \neq 0$, and following the investigation of the family (3.14) for $g = 2$, we get the following configurations (depending on the parameter n):

$$\begin{aligned} \mathbf{D} < 0, \xi_7 < 0 \text{ (i.e., } n < -33/4) &\Rightarrow \text{Config. } \mathcal{P}.9; \\ \mathbf{D} < 0, \xi_7 > 0, \xi_8 < 0 \text{ (i.e., } -33/4 < n < -21/4) &\Rightarrow \text{Config. } \mathcal{P}.11; \\ \mathbf{D} < 0, \xi_7 > 0, \xi_8 > 0 \text{ (i.e., } -21/4 < n < -2) &\Rightarrow \text{Config. } \mathcal{P}.12; \\ \mathbf{D} > 0 \text{ (i.e., } n > -2) &\Rightarrow \text{Config. } \mathcal{P}.13; \\ \mathbf{D} = 0, \xi_{18} \neq 0 \text{ (i.e., } n = -2) &\Rightarrow \text{Config. } \mathcal{P}.14; \\ \mathbf{D} = 0, \xi_{18} = 0 \text{ (i.e., } n = -21/4) &\Rightarrow \text{Config. } \mathcal{P}.16. \end{aligned}$$

Assuming $B_2 = 0$, and considering the condition (3.86), we get $n = -3$ and arrive at the system:

$$\dot{x} = \frac{1}{2}(2x - 3)(2x + y), \quad \dot{y} = y(-6 + x + 2y),$$

which possesses two invariant lines $x = 3/2$ and $y = 0$, in addition to the invariant parabola. Therefore, for this system, we obtain the configuration Config. $\mathcal{P}.47$.

2: The possibility $\xi_{20} = 0$. Then, from (3.82), we obtain $m(2m - n) = 0$, and we discuss two cases: $\xi_{21} \neq 0$ and $\xi_{21} = 0$.

2.1: The case $\xi_{21} \neq 0$. Then $m \neq 0$, and we obtain $m = n/2$. This leads to the following one-parameter family of systems:

$$\dot{x} = \frac{n}{2} + nx - \frac{3y}{2} + 2x^2 + xy, \quad \dot{y} = (n + y)(x + 2y), \quad (3.87)$$

which is a subfamily of (3.19) defined by the condition $g = 2$. The family (3.19) was investigated earlier, and considering (3.21) for $g = 2$ (i.e., for systems (3.87)), we conclude that these systems possess four singularities $M_i(x_i, y_i)$ ($i = 1, 2, 3, 4$) with coordinates:

$$\begin{aligned} x_1 &= \sqrt{-n}, & y_1 &= -n; & x_2 &= -\sqrt{-n}, & y_2 &= -n; \\ x_3 &= -\frac{1}{2}, & y_3 &= \frac{1}{4}; & x_4 &= -\frac{2n}{3}, & y_4 &= \frac{n}{3}. \end{aligned}$$

For these systems, we calculate:

$$\theta = -32 \neq 0, \quad B_2 = -162(1+4n)(9+4n)^2y^4, \quad \chi_4 = 61875n(1+4n)(9+4n)(1+36n),$$

and therefore $B_2 \neq 0$ due to $\chi_4 \neq 0$. Following the examination of the configurations of systems (3.19) for $g = 2$, we obtain:

$$\alpha_3 = 4n - 3, \quad \beta_3 = 1 + 4n, \quad \mathbf{D} = 768n^3\alpha_3^2\beta_3^2, \quad \zeta_9 = 3219732426240(9+4n)^2\beta_3,$$

and due to $\chi_4 \neq 0$, in the case $\mathbf{D} \neq 0$, we have:

$$\text{sign}(\mathbf{D}) = \text{sign}(n), \quad \text{sign}(\zeta_9) = \text{sign}(\beta_3),$$

and the condition $\alpha_3 = 0$ is equivalent to $\mathbf{D} = 0$.

Thus, we arrive at the following configurations (depending on the parameter n):

$$\begin{aligned} \mathbf{D} < 0, \zeta_9 < 0 \text{ (i.e., } n < -1/4) & \Rightarrow \text{Config. } \mathcal{P}.17; \\ \mathbf{D} < 0, \zeta_9 > 0 \text{ (i.e., } -1/4 < n < 0) & \Rightarrow \text{Config. } \mathcal{P}.19; \\ \mathbf{D} > 0 \text{ (i.e., } n > 0, n \neq 3/4) & \Rightarrow \text{Config. } \mathcal{P}.20; \\ \mathbf{D} = 0 \text{ (i.e., } n = 3/4) & \Rightarrow \text{Config. } \mathcal{P}.21. \end{aligned}$$

2.2: The case $\zeta_{21} = 0$. Then $m = 0$, and we obtain the one-parameter family of systems:

$$\dot{x} = nx - \frac{3y}{2} + 2x^2 + xy, \quad \dot{y} = y(2n + x + 2y), \quad (3.88)$$

which is a subfamily of (3.25) defined by the condition $g = 2$. The family (3.25) was investigated earlier, and considering (3.28) for $g = 2$ (i.e., for systems (3.88)), we have:

$$\begin{aligned} Z_3 &= 1 - 16n, \quad B_2 = -1458(3+n)(1+2n)(3+2n)y^4, \quad \theta = -32 \neq 0, \\ B_3 &= -9n(7+4n)x^2y^2/2 - 9(3+n)xy^3 + 9(3+n)y^4/2 \neq 0, \\ \mathbf{D} &= -3888n^6Z_3, \quad \zeta_{22} = 1050n, \quad \chi_4 = 556875n^2(1+2n)(3+16n). \end{aligned}$$

So, due to the condition $\chi_4 \neq 0$, in the case $\mathbf{D} \neq 0$, we have:

$$\text{sign}(\mathbf{D}) = -\text{sign}(Z_3), \quad \text{sign}(\zeta_{22}) = \text{sign}(n).$$

Thus, in the case $B_2 \neq 0$, and following the investigation of the family (3.25) for $g = 2$, we get the following configurations (depending on the parameter n):

$$\begin{aligned} \mathbf{D} < 0, \zeta_{22} < 0 \text{ (i.e., } n < 0) & \Rightarrow \text{Config. } \mathcal{P}.32; \\ \mathbf{D} < 0, \zeta_{22} > 0 \text{ (i.e., } 0 < n < 1/16) & \Rightarrow \text{Config. } \mathcal{P}.31; \\ \mathbf{D} > 0 \text{ (i.e., } n > 1/16) & \Rightarrow \text{Config. } \mathcal{P}.34; \\ \mathbf{D} = 0 \text{ (i.e., } n = 1/16) & \Rightarrow \text{Config. } \mathcal{P}.37. \end{aligned}$$

Assume now $B_2 = 0$. The condition $\chi_4 \neq 0$ implies $1 + 2n \neq 0$, and we get $(3+2n)(3+n) = 0$. Since for systems (3.88) we have $\zeta_{23} = -225(3+n)/4$, we arrive at configuration *Config. P.52* in the case $\zeta_{23} \neq 0$ (i.e., $n = -3/2$), and *Config. P.47* in the case $\zeta_{23} = 0$ (i.e., $n = -3$).

3.2.2 The statements $(\mathcal{B}_2)$, $(\mathcal{B}_3)$, and $(\mathcal{B}_4)$

According to Proposition 2.8, all these three statements share the common condition $\zeta_7 = 0$. Considering (3.78) and the requirement $\chi_4 \neq 0$, we must impose the condition $\mathcal{U}_2 = 4m - 147 - 14n = 0$, leading to $m = 7(21 + 2n)/4$. This gives rise to the following one-parameter family:

$$\begin{aligned} \dot{x} &= (147 + 14n + 4nx + 8x^2 - 6y + 4xy)/4, \\ \dot{y} &= (147x + 14nx + 4ny + 2xy + 4y^2)/2, \end{aligned} \quad (3.89)$$

for which we calculate:

$$\begin{aligned} \chi_4 &= 61875(37 + 4n)(69 + 4n)(357 + 4n)(301 + 36n), \\ \zeta_8 &= 5(21 + 4n)^2/4, \quad \mathcal{R}_4 = 19500(33 + 4n). \end{aligned} \quad (3.90)$$

Thus, following Proposition 2.8, we distinguish three possibilities: $\zeta_8 \mathcal{R}_4 \neq 0$ (statement $(\mathcal{B}_2)$), $\zeta_8 = 0$ (statement $(\mathcal{B}_3)$), and $\mathcal{R}_4 = 0$ (statement $(\mathcal{B}_4)$). We examine each possibility in turn.

The possibility $\zeta_8 \mathcal{R}_4 \neq 0$. Then $33 + 4n \neq 0$, and systems (3.89) possess the following two invariant parabolas:

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = -(21 + 2n)(21 + 4n) + 6(21 + 4n)x + 24x^2 - 2(33 + 4n)y = 0.$$

We observe that systems (3.89) form a subfamily of (3.37) defined by the condition $g = 2$. The family (3.37) was investigated earlier, and considering (3.40) and (3.43) for $g = 2$ (i.e., for systems (3.89)), we have:

$$\begin{aligned} Z_4 &= -(33 + 4n), \quad \alpha_4 = 133 + 4n, \quad \beta_4 = 861 + 100n, \quad \theta = -32 \neq 0, \\ \mathbf{D} &= -3(21 + 4n)^2 Z_4 \alpha_4^2 \beta_4^2 / 4, \quad \zeta_2 = 24 > 0, \quad \zeta_{14} = 1235 \alpha_4 \beta_4 / 2, \\ B_1 &= 105(21 + 2n)(33 + 4n)(37 + 4n)(49 + 4n)(69 + 4n) / 2. \end{aligned}$$

We observe that due to $\zeta_8 \mathcal{R}_4 \neq 0$, we have $Z_4(21 + 4n) \neq 0$, and, in the case $\mathbf{D} \neq 0$, we obtain:

$$\text{sign}(\mathbf{D}) = -\text{sign}(Z_4), \quad \text{sign}(\zeta_{14}) = \text{sign}(\alpha_4 \beta_4).$$

Moreover, the direction of the invariant parabola $\Phi_2(x, y) = 0$ depends on $\text{sign}(33 + 4n)$.

According to Lemma 2.3, the existence of an invariant line in systems (3.89) requires $B_1 = 0$. Thus, we consider two cases: $B_1 \neq 0$ and $B_1 = 0$.

The case $B_1 \neq 0$. Then no invariant line exists. For the parameter n , we have three possible bifurcation values: $n \in \{-133/4, -861/100, -33/4\}$. However, due to $\mathcal{R}_4 \neq 0$, we must have $Z_4 \neq 0$, i.e., $n \neq -33/4$.

Considering these possible bifurcation values, in the case $B_1 \neq 0$, systems (3.89) lead to the following configurations (depending on the parameter n):

$$\begin{aligned} \mathbf{D} < 0, \zeta_{14} < 0 \text{ (i.e., } -133/4 < n < -861/100) & \Rightarrow \text{Config. } \mathcal{P}.58; \\ \mathbf{D} < 0, \zeta_{14} > 0, \beta_4 < 0 \text{ (i.e., } n < -133/4) & \Rightarrow \text{Config. } \mathcal{P}.59; \\ \mathbf{D} < 0, \zeta_{14} > 0, \beta_4 > 0 \text{ (i.e., } -861/100 < n < -33/4) & \Rightarrow \text{Config. } \mathcal{P}.59; \\ \mathbf{D} > 0 \text{ (i.e., } n > -33/4) & \Rightarrow \text{Config. } \mathcal{P}.61; \\ \mathbf{D} = 0, \beta_4 \neq 0 \text{ (i.e., } n = -133/4) & \Rightarrow \text{Config. } \mathcal{P}.64; \\ \mathbf{D} = 0, \beta_4 = 0 \text{ (i.e., } n = -861/100) & \Rightarrow \text{Config. } \mathcal{P}.64. \end{aligned}$$

We may summarize the above as:

$$\begin{aligned} \mathbf{D} < 0, \xi_{14} < 0 &\Rightarrow \text{Config. } \mathcal{P}.58; \\ \mathbf{D} < 0, \xi_{14} > 0 &\Rightarrow \text{Config. } \mathcal{P}.59; \\ \mathbf{D} > 0 &\Rightarrow \text{Config. } \mathcal{P}.61; \\ \mathbf{D} = 0 &\Rightarrow \text{Config. } \mathcal{P}.64. \end{aligned}$$

The case $B_1 = 0$. Considering (3.90) and the condition $\chi_4 \zeta_8 \mathcal{R}_4 \neq 0$, the condition $B_1 = 0$ is equivalent to $49 + 4n = 0$, i.e., $n = -49/4$. The corresponding system (3.89) possesses the invariant line $y = 49/4$, leading to a configuration equivalent to Config. $\mathcal{P}.69$.

The possibility $\zeta_8 = 0$. From (3.90), this condition implies $n = -21/4$, and we arrive at the system:

$$\begin{aligned} \dot{x} &= (147 - 42x - 12y + 16x^2 + 8xy)/8, \\ \dot{y} &= (147x - 42y + 4xy + 8y^2)/4. \end{aligned} \quad (3.91)$$

For $n = -21/4$, we find:

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = 24(x^2 - y) = 0,$$

i.e., the system has a double invariant parabola. Moreover, in this case, there is one real singular point $M_1(x_1, y_1)$ and two complex singularities $M_{2,3}(x_{2,3}, y_{2,3})$, with:

$$x_1 = -\frac{7}{2}, \quad y_1 = \frac{49}{4}; \quad x_{2,3} = \frac{3}{2} \pm i\sqrt{3}, \quad y_{2,3} = -\frac{3}{4} \pm 3i\sqrt{3}.$$

We point out that M_1 is a double singularity of system (3.91), located on the double invariant parabola $\Phi_1(x, y) = x^2 - y = 0$. Therefore, we arrive at configuration Config. $\mathcal{P}.79$.

The possibility $\mathcal{R}_4 = 0$. From (3.90), this condition implies $n = -33/4$, and we obtain the system:

$$\dot{x} = (2x - 3)(-21 + 8x + 4y)/8, \quad \dot{y} = (63x - 66y + 4xy + 8y^2)/4.$$

For $n = -33/4$, the second invariant parabola becomes the reducible conic $\Phi_2(x, y) = 6(2x - 3)^2 = 0$. Thus, the system possesses the invariant line $x = 3/2$ and the invariant parabola $\Phi_1(x, y) = x^2 - y = 0$, yielding a configuration equivalent to Config. $\mathcal{P}.83$.

3.2.3 The statement ($\mathcal{B}_5$)

According to Proposition 2.8, we must have the condition $\chi_4 = 0$ and $\zeta_5 \zeta_9 \neq 0$. Thus, for systems (3.77) we compute:

$$\begin{aligned} \chi_4 &= 61875(1 + 4m + 2n)\mathcal{V}, \quad \zeta_{24} = 140625(4m - 14n - 147)\mathcal{V}/16, \\ \zeta_5 &= 25(196m - 46n - 3)(4m - 14n - 147)/16, \\ \zeta_9 &= -2970000(4m - 14n - 147)\mathcal{W}, \end{aligned} \quad (3.92)$$

where

$$\mathcal{V} = 18m + 1372m^2 - 84mn + 27n^2 + 144n^3, \quad \mathcal{W} = 10m + 196m^2 - 88mn + 15n^2.$$

We consider two cases: $\zeta_{24} \neq 0$ and $\zeta_{24} = 0$.

The case $\xi_{24} \neq 0$. Then $\mathcal{V} \neq 0$ and the condition $1 + 4m + 2n = 0$ implies $m = -(1 + 2n)/4$. This leads to the one-parameter family of systems

$$\dot{x} = -\frac{1}{4}(2n + 1) + nx - \frac{3y}{2} + 2x^2 + xy, \quad \dot{y} = -\frac{1}{2}(2n + 1)x + 2ny + xy + 2y^2, \quad (3.93)$$

which possess the invariant line $y = x - \frac{1}{4}$ and two invariant parabolas:

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = -1 - 2n + 2(1 + 4n)x - 2(-1 + 4n)y - 8y^2 = 0.$$

For these systems, we have

$$\begin{aligned} \zeta_5 &= 25(37 + 4n)(13 + 36n), \quad \xi_{24} = -140625(5 + 4n)^2(37 + 4n)(13 + 36n)/16, \\ \zeta_9 &= 11 \cdot 30^4(1 + 4n)(37 + 4n)(13 + 36n), \quad B_1 = 0, \quad \theta = -32 \neq 0, \\ B_2 &= -81(1 + 2n)(1 + 4n)(17 + 4n)(x - y)^4, \quad \mathbf{D} = 3(1 + 4n)(5 + 4n)^6/4. \end{aligned} \quad (3.94)$$

Following Lemma 2.3, we consider two subcases: $B_2 \neq 0$ and $B_2 = 0$.

1: The subcase $B_2 \neq 0$. Then, by Lemmas 2.3 and 2.4 (since $\theta \neq 0$), we conclude that systems (3.93) can have only one invariant line (namely, $y = x - 1/4$).

Systems (3.93) have four finite singularities $M_i(x_i, y_i)$, $i = 1, \dots, 4$, with coordinates

$$\begin{aligned} x_1 &= \frac{1}{2}, \quad y_1 = \frac{1}{4}; \quad x_2 = \frac{1 - 4n}{12}, \quad y_2 = -\frac{1 + 2n}{6}; \\ x_{3,4} &= -\frac{1}{2} \left(1 \pm \sqrt{-1 - 4n} \right), \quad y_{3,4} = \frac{1}{2} \left(-2n \pm \sqrt{-1 - 4n} \right). \end{aligned}$$

Note that the invariant parabolas intersect at three points: M_1 , M_3 , and M_4 . The singularities M_3 and M_4 may be real or complex depending on $1 + 4n \neq 0$ (due to $\zeta_9 \neq 0$). Moreover, the direction of the parabola $\Phi_2(x, y) = 0$ also depends on $1 + 4n$.

It is easy to verify that the invariant line $y = x - 1/4$ is tangent to both invariant parabolas at M_1 . The singularity M_2 lies on this invariant line, and

$$\Phi_1(x_2, y_2) = \frac{1}{144}(5 + 4n)^2 \neq 0, \quad \Phi_2(x_2, y_2) = -\frac{1}{18}(5 + 4n)^2 \neq 0,$$

due to $\xi_{24} \neq 0$. Considering (3.94), $\mathbf{D} \neq 0$ since $\zeta_9 \xi_{24} \neq 0$ and $\text{sign}(\mathbf{D}) = \text{sign}(5 + 4n)$.

To understand the position of M_2 relative to M_1 , we compute

$$x_2 - x_1 = \frac{1 - 4n}{12} - \frac{1}{2} = -\frac{5 + 4n}{12} \Rightarrow \text{sign}(x_2 - x_1) = -\text{sign}(5 + 4n).$$

Hence, all finite singularities except M_2 are fixed as intersection points of the invariant curves, and their positions depend on n . In the case $B_2 \neq 0$, the possible bifurcation values for n are $-5/4$ and $-1/4$.

Thus, for $B_2 \neq 0$ in systems (3.93) we have the following configurations:

$$\begin{aligned} \mathbf{D} < 0 \text{ and } n < -5/4 &\Rightarrow \text{Config. } \mathcal{P}.99; \\ \mathbf{D} < 0 \text{ and } n > -5/4 &\Rightarrow \simeq \text{Config. } \mathcal{P}.99; \\ \mathbf{D} > 0 \text{ (i.e., } n > -1/4) &\Rightarrow \text{Config. } \mathcal{P}.100. \end{aligned}$$

So, Config. $\mathcal{P}.99$ occurs if $\mathbf{D} < 0$, and Config. $\mathcal{P}.100$ if $\mathbf{D} > 0$.

2: The subcase $B_2 = 0$. From $\zeta_9 \neq 0$, we get $(1 + 2n)(17 + 4n) = 0$.

If $1 + 2n = 0$, then $n = -1/2$, yielding the system

$$\dot{x} = (-x + 4x^2 - 3y + 2xy)/2, \quad \dot{y} = y(-1 + x + 2y),$$

which has four invariant curves (two parabolas and two lines):

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = x - 3y + 4y^2 = 0, \quad y = x - 1/4, \quad y = 0.$$

This corresponds to configuration *Config. P.101*.

If $n = -17/4$, we get

$$\dot{x} = (-3 + 2x)(-5 + 8x + 4y)/8, \quad \dot{y} = (15x - 34y + 4xy + 8y^2)/4,$$

which has invariant lines $y = x - 1/4$ and $x = 3/2$, and invariant parabolas

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = 15 - 64x + 72y - 16y^2 = 0.$$

The line $x = 3/2$ is tangent to $\Phi_2(x, y) = 0$ at $M_4(3/2, 9/4)$. Thus, in this case, we get a configuration equivalent to *Config. P.101*.

The case $\zeta_{24} = 0$. This implies $\mathcal{V} = 0$ and we calculate

$$\text{Discrim} [\mathcal{V}, m] = 36(1 - 28n)(3 + 28n)^2 \equiv \gamma(n).$$

Since $m, n \in \mathbb{R}$, the condition $\gamma(n) \geq 0$ is necessary.

We claim that $\zeta_5 \neq 0$ implies $3 + 28n \neq 0$. Indeed, setting $n = -3/28$ we get

$$\mathcal{V} = \frac{(2744m + 27)^2}{5488} = 0 \quad \Rightarrow \quad m = -\frac{27}{2744},$$

which implies $\zeta_5 = 0$. Thus, the claim is proved.

Therefore, the condition $1 - 28n \geq 0$ is necessary for $\mathcal{V}$ to have real roots. Setting a new parameter u as $1 - 28n = u^2 \geq 0$, we have $n = (1 - u^2)/28$ and

$$\mathcal{V} = \frac{1}{5488}(15 + 2744m + 24u + 3u^2 - 6u^3)(15 + 2744m - 24u + 3u^2 + 6u^3) = 0.$$

By symmetry ($u \mapsto -u$), we may assume the second factor vanishes, yielding

$$m = -3(u - 1)^2(5 + 2u)/2744.$$

This leads to the one-parameter family of systems

$$\begin{aligned} \dot{x} &= -\frac{3(u - 1)^2(5 + 2u)}{2744} - \frac{u^2 - 1}{28}x - \frac{3}{2}y + 2x^2 + xy, \\ \dot{y} &= -\frac{3(u - 1)^2(5 + 2u)}{1372}x - \frac{u^2 - 1}{14}y + xy + 2y^2, \end{aligned} \tag{3.95}$$

which possess two invariant parabolas:

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = 3(u - 1)^4 + 112(u - 1)^3x + 1176(u - 1)^2y - 38416y^2 = 0.$$

For these systems,

$$\begin{aligned}\zeta_5 &= 25(u-50)(u-22)(u^2-4)(3u-10)(3u+46)/38416, \\ \zeta_9 &= 2227500(u-50)(u-22)(u^2-4)(u-1)^3(3u-10)(3u+46)/823543, \\ \zeta_{10} &= -15(354160 + 48u - 1336u^2 + 64u^3 + 27u^4 - 19u^5 + 3u^6)/38416, \\ \mathbf{D} &= -3^5 7^{-18} (u-1)^6 (6+u)^8 (34+u)^2 (2+5u)^2.\end{aligned}\tag{3.96}$$

Also, for (3.95) we have

$$B_1 = -3^3 2^{-5} 7^{-15} (u-22)(u-8)^3(u-1)^3(6+u)^3(13+u)(20+u)(5+2u)(3u-10)(4+3u).\tag{3.97}$$

Following Lemma 2.3, we consider two subcases: $B_1 \neq 0$ and $B_1 = 0$.

1: The subcase $B_1 \neq 0$. Then, by Lemma 2.3, systems (3.95) cannot possess any invariant line.

The systems (3.95) have four finite singularities $M_i(x_i, y_i)$, $i = 1, 2, 3, 4$, with coordinates

$$\begin{aligned}x_1 &= -\frac{5+2u}{14}, \quad y_1 = \frac{(5+2u)^2}{196}; \quad x_2 = \frac{(u-1)(22+26u+u^2)}{1372}, \quad y_2 = -\frac{(u-1)^2(5+2u)}{1372}; \\ x_3 &= \frac{1-u}{14}, \quad y_3 = \frac{(1-u)^2}{196}; \quad x_4 = \frac{3(u-1)}{14}, \quad y_4 = \frac{9(u-1)^2}{196}.\end{aligned}\tag{3.98}$$

The invariant parabolas intersect at points M_3 and M_4 . The singularities M_1 and M_2 lie on $\Phi_1 = 0$ and $\Phi_2 = 0$, respectively. The direction of $\Phi_2 = 0$ depends on $u-1$, and since $\zeta_9 \neq 0$, we have $u \neq 1$.

Because $\mathbf{D} = 0$ indicates a multiple singularity, we consider the possibilities $\mathbf{D} \neq 0$ and $\mathbf{D} = 0$.

1.1: The possibility $\mathbf{D} \neq 0$. Then all singularities are distinct. To determine the relative positions of M_1 and M_2 with respect to M_3 and M_4 (intersection points of the parabolas), we compute:

$$\begin{aligned}x_3 - x_1 &= \frac{6+u}{14}, & x_3 - x_2 &= -\frac{(u-1)(6+u)(20+u)}{1372}, \\ x_4 - x_1 &= \frac{2+5u}{14}, & x_4 - x_2 &= -\frac{(u-8)(u-1)(34+u)}{1372}.\end{aligned}$$

Moreover, the singular point M_2 , which lies on the parabola $\Phi_2(x, y) = 0$, can be located either above or below its axis $y = y_v$, where y_v is the ordinate of the vertex of this parabola. For $\Phi_2(x, y) = 0$, we have $y_v = 3(u-1)^2/196$ and then we calculate

$$y_2 - y_v = -(u-1)^2(13+u)/686 \Rightarrow \text{sign}(y_2 - y_v) = -\text{sign}(13+u).$$

Additionally,

$$\Phi_1(x_2, y_2) = \frac{(u-1)^2(6+u)^3(34+u)}{1882384}, \quad \Phi_2(x_1, y_1) = -(6+u)^3(2+5u).$$

Possible bifurcation values for u are $\{-34, -20, -13, -6, -\frac{2}{5}, 1, 8\}$. Since $\zeta_5 \zeta_9 B_1 \mathbf{D} \neq 0$, we have:

$$(u-8)(u-1)(u+20)(u+13)(u+6)(u+34)(2+5u) \neq 0.$$

Thus, in the case $B_1 \neq 0$ and $\mathbf{D} \neq 0$, systems (3.95) yield the following configurations:

$$\begin{array}{lll}
u < -34 & \Rightarrow x_3 > x_4, x_1 > x_3, x_2 < x_4, y_2 > y_v & \Rightarrow \text{Config. } \mathcal{P}.102; \\
-34 < u < -20 & \Rightarrow x_3 > x_4, x_1 > x_3, x_4 < x_2 < x_3, y_2 > y_v & \Rightarrow \text{Config. } \mathcal{P}.103; \\
-20 < u < -13 & \Rightarrow x_3 > x_4, x_1 > x_3, x_4 < x_3 < x_2, y_2 > y_v & \Rightarrow \simeq \text{Config. } \mathcal{P}.103; \\
-13 < u < -6 & \Rightarrow x_3 > x_4, x_1 > x_3, x_4 < x_3 < x_2, y_2 < y_v & \Rightarrow \simeq \text{Config. } \mathcal{P}.103; \\
-6 < u < -2/5 & \Rightarrow x_3 > x_4, x_4 < x_1 < x_3, x_4 < x_2 < x_3, y_2 < y_v & \Rightarrow \simeq \text{Config. } \mathcal{P}.103; \\
-2/5 < u < 1 & \Rightarrow x_3 > x_4, x_1 < x_3, x_4 < x_2 < x_3, y_2 < y_v & \Rightarrow \simeq \text{Config. } \mathcal{P}.102; \\
1 < u < 8 & \Rightarrow x_3 > x_4, x_1 < x_3, x_4 < x_2 < x_3, y_2 < y_v & \Rightarrow \text{Config. } \mathcal{P}.104; \\
u > 8 & \Rightarrow x_3 > x_4, x_1 < x_3, x_4 < x_2 < x_3, y_2 < y_v & \Rightarrow \simeq \text{Config. } \mathcal{P}.104.
\end{array}$$

We can summarize these as

$$\begin{array}{ll}
\text{Config. } \mathcal{P}.103 & \Leftrightarrow (u + 34)(5u + 2) < 0; \\
\text{Config. } \mathcal{P}.102 & \Leftrightarrow (u + 34)(5u + 2) > 0 \text{ and } u - 1 < 0; \\
\text{Config. } \mathcal{P}.104 & \Leftrightarrow (u + 34)(5u + 2) > 0 \text{ and } u - 1 > 0.
\end{array}$$

On the other hand, for systems (3.95) we have

$$\begin{aligned}
\zeta_{25} &= 2^{-3}7^{-9}5913(u-1)^3(6+u)^4(34+u)(2+5u), \\
\zeta_5\zeta_9 &= 2^{-2}7^{-11}3^45^611(u-1)^3(-50+u)^2(-22+u)^2(-2+u)^2(2+u)^2(-10+3u)^2(46+3u)^2,
\end{aligned}$$

and due to $\mathbf{D}\zeta_5\zeta_9 \neq 0$ we have

$$\zeta_{25} \neq 0, \quad \text{sign}(\zeta_{25}) = \text{sign}((u-1)(34+u)(2+5u)), \quad \text{sign}(\zeta_5\zeta_9) = \text{sign}(u-1).$$

This leads to the following invariant conditions:

$$\begin{array}{ll}
\zeta_{25} < 0 & \Leftrightarrow \text{Config. } \mathcal{P}.102; \\
\zeta_{25} > 0, \zeta_5\zeta_9 < 0 & \Leftrightarrow \text{Config. } \mathcal{P}.103; \\
\zeta_{25} > 0, \zeta_5\zeta_9 > 0 & \Leftrightarrow \text{Config. } \mathcal{P}.104.
\end{array}$$

1.2: The possibility $\mathbf{D} = 0$. Considering the condition $B_1 \neq 0$, this implies $(34+u)(2+5u) = 0$. Taking into consideration the position of the invariant parabolas and the coordinates (3.98) of the singularities of systems (3.95), we obtain:

$$\begin{array}{lll}
u = -34 & \Rightarrow x_3 > x_4, x_1 > x_3, x_2 = x_4, y_2 > y_v & \Rightarrow \text{Config. } \mathcal{P}.105; \\
u = -2/5 & \Rightarrow x_3 > x_4, x_1 = x_4, x_4 < x_2 < x_3, y_2 < y_v & \Rightarrow \simeq \text{Config. } \mathcal{P}.105.
\end{array}$$

So, we deduce that in the case $B_1 \neq 0$ and $\mathbf{D} = 0$ we get the unique configuration *Config. P.105*.

2: The subcase $B_1 = 0$. Considering (3.97) and (3.96), we conclude that due to $\zeta_9 \neq 0$ the condition $B_1 = 0$ is equivalent to

$$(u-8)(6+u)(13+u)(20+u)(5+2u)(4+3u) = 0.$$

However, we could decrease the number of factors.

Remark 3.14. We remark that in the case $u-1 \neq 0$ (i.e., when the second parabola exists), applying the transformation

$$x_1 = \frac{343}{(u-1)^3}y - \frac{21}{4(u-1)}, \quad y_1 = \frac{343}{(u-1)^3}x + \frac{147}{4(u-1)^2}, \quad t_1 = \frac{(u-1)^3}{343}t,$$

we arrive at a family of systems of the same form (3.95):

$$\begin{aligned}
\dot{x}_1 &= -\frac{3(u_1-1)^2(5+2u_1)}{2744} - \frac{u_1^2-1}{28}x_1 - \frac{3y_1}{2} + 2x_1^2 + x_1y_1, \\
\dot{y}_1 &= -\frac{3(u_1-1)^2(5+2u_1)}{1372}x_1 - \frac{u_1^2-1}{4}y_1 + x_1y_1 + 2y_1^2,
\end{aligned}$$

with the new parameter $u_1 = (48+u)/(u-1)$ (then $u = (48+u_1)/(u_1-1)$).

Considering Remark 3.14 and the relation $u = (48 + u_1)/(u_1 - 1)$, we calculate

$$u + 13 = \frac{7(5 + 2u_1)}{u_1 - 1}, \quad u + 20 = \frac{7(3u_1 + 4)}{u_1 - 1}.$$

So, to determine the configurations given by the condition $B_1 = 0$, it is sufficient to consider the conditions provided by the equality

$$(u - 8)(6 + u)(5 + 2u)(4 + 3u) = 0.$$

2.1: The possibility $u = -4/3$. This leads to the system

$$\dot{x} = (-1 - 2x + 144x^2 - 108y + 72xy)/72, \quad \dot{y} = (x + 2y)(-1 + 36y)/36,$$

possessing the invariant line $y = 1/36$ and two invariant parabolas:

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = 1 - 16x + 72y - 432y^2.$$

We determine that the configuration of the above system corresponds to *Config. P.106*.

2.2: The possibility $u = 8$. This leads to the system

$$\dot{x} = (2x - 3)(3 + 8x + 4y)/8, \quad \dot{y} = (x + 2y)(4y - 9)/4,$$

possessing three invariant lines $y = 9/4$, $y = x + 3/4$ and $x = 3/2$ and two invariant parabolas:

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = 3 + 16x + 24y - 16y^2.$$

We observe that all five invariant curves intersect at the singular point $M_4(3/2, 9/4)$. So we get the configuration *Config. P.107*.

2.3: The possibility $u = -5/2$. In this case we arrive at the system

$$\dot{x} = (-3x + 32x^2 - 24y + 16xy)/16, \quad \dot{y} = y(-3 + 8x + 16y)/4,$$

possessing the invariant line $y = 0$ and two invariant parabolas:

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = 3 - 32x + 96y - 256y^2.$$

We observe that the invariant line $y = 0$ is tangent to the parabola $\Phi_1(x, y) = 0$ at the point $M_1(0, 0)$ and intersects the second parabola at $M_2(3/32, 0)$. In this case, we have the configuration *Config. P.108*.

2.4: The possibility $u = -6$. In this case we get the system

$$\dot{x} = (3 - 10x + 16x^2 - 12y + 8xy)/8, \quad \dot{y} = (3x - 10y + 4xy + 8y^2)/4,$$

possessing the invariant line $y = x - 1/4$ and the invariant parabolas

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = 3 - 16x + 24y - 16y^2 = 0.$$

Considering the coordinates (3.98) of the singularities of systems (3.95), we observe that for $u = -6$ the singular points M_2 , M_3 and M_1 coalesce producing a triple singular point. Moreover, this triple singularity is a point of tangency of the invariant line $y = x - 1/4$ with both parabolas. As a result, we get the configuration *Config. P.109*.

On the other hand, for systems (3.95) we have

$$\begin{aligned}\zeta_9 &= \frac{2227500}{823543}(u-50)(u-22)(u^2-4)(u-1)^3(3u-10)(3u+46), \\ \zeta_6 &= -\frac{3}{1372}(6+u)(292-52u+5u^2), \quad \text{Discrim}[292-52u+5u^2] = -3136 < 0, \\ \zeta_{26} &= \frac{3}{941192}(u-50)(u^2-4)(u-1)(3u+46)(6+u)(13+u)(5+2u), \\ \zeta_{27} &= \frac{1}{2744}(u-8)(380+52u+9u^2), \quad \text{Discrim}[380+52u+9u^2] = -10976 < 0.\end{aligned}$$

We observe that, due to $\zeta_9 \neq 0$, the condition $\zeta_{26} \neq 0$ is equivalent to $(6+u)(13+u)(5+2u) \neq 0$. Moreover, considering Remark 3.14, we conclude that for $\zeta_{26} = 0$ we may assume $(6+u)(5+2u) = 0$ because the condition $13+u = 0$ could be brought to $5+2u = 0$ via an affine transformation and time rescaling.

Thus, in the case $B_1 = 0$, systems (3.95) possess the following configurations if and only if the corresponding conditions are satisfied:

$$\begin{aligned}\zeta_{26} \neq 0, \zeta_{27} \neq 0 \text{ (then } u = -4/3) &\Leftrightarrow \text{Config. } \mathcal{P}.106; \\ \zeta_{26} \neq 0, \zeta_{27} = 0 \text{ (then } u = 8) &\Leftrightarrow \text{Config. } \mathcal{P}.107; \\ \zeta_{26} = 0, \zeta_6 \neq 0 \text{ (then } u = -5/2) &\Leftrightarrow \text{Config. } \mathcal{P}.108; \\ \zeta_{26} = 0, \zeta_6 = 0 \text{ (then } u = -6) &\Leftrightarrow \text{Config. } \mathcal{P}.109.\end{aligned}$$

3.2.4 The statement ($\mathcal{B}_6$)

In this case the condition $\chi_4 = \zeta_9 = 0$ must be fulfilled. Considering (3.92), due to the condition $\zeta_5 \neq 0$, we obtain that $\zeta_9 = 0$ is equivalent to $\mathcal{W} = 0$. Straightforward calculations give us that the systems of equations $\chi_4 = 0$ and $\mathcal{W} = 0$ could have only the following solutions $\mathcal{S}_i = (m_i, n_i)$ ($i = 1, 2, 3, 4$):

$$\mathcal{S}_1 = (0, 0), \quad \mathcal{S}_2 = \left(-\frac{1}{8}, -\frac{1}{4}\right), \quad \mathcal{S}_3 = \left(-\frac{5}{72}, -\frac{13}{36}\right), \quad \mathcal{S}_4 = \left(-\frac{27}{2944}, -\frac{3}{8}\right).$$

However, we have

$$\begin{aligned}\chi_4(\mathcal{S}_i) &= \zeta_9(\mathcal{S}_i) = 0, \quad i = 1, 2, 3, 4, \\ \zeta_5(\mathcal{S}_1) &\neq 0, \quad \zeta_5(\mathcal{S}_2) \neq 0, \quad \zeta_5(\mathcal{S}_3) = \zeta_5(\mathcal{S}_4) = 0,\end{aligned}$$

and hence only the solutions $\mathcal{S}_1$ and $\mathcal{S}_2$ satisfy the conditions of statement ($\mathcal{B}_6$). Therefore, we examine only these two solutions.

We observe that each one of them gives us a concrete system (without parameters), and it remains to construct the corresponding system having a single fixed configuration of the invariant parabolas and lines.

For systems (3.77), we calculate:

$$\begin{aligned}\zeta_9 &= 3^6 17252510 [65536m^4 - 32m^3(6131 + 3252n) - 16m^2(-32110 - 7953n + 484n^2) \\ &\quad + 6m(10221 - 53292n + 5540n^2 + 4336n^3) \\ &\quad - 9(-2304 - 7857n - 12140n^2 + 836n^3 + 240n^4)].\end{aligned}$$

We obtain that $\zeta_9(\mathcal{S}_2) = 0$ and $\zeta_9(\mathcal{S}_1) \neq 0$, and then we examine two cases: $\zeta_9 \neq 0$ and $\zeta_9 = 0$.

The case $\zeta_9 \neq 0$. Then we consider the solution $\mathcal{S}_1$, i.e., $m = n = 0$. In this case, we arrive at the system

$$\dot{x} = -\frac{3y}{2} + 2x^2 + xy, \quad \dot{y} = y(x + 2y), \quad (3.99)$$

possessing the invariant line $y = 0$ and the invariant parabola $\Phi(x, y) = x^2 - y = 0$.

This system possesses the following two singular points: $M_1(0, 0)$ and $M_2(-1/2, 1/4)$. We observe that the point M_1 is the point of tangency of the invariant line with the parabola. Moreover, this point is a triple singularity of system (3.99), because we have

$$\mu_4 = \mu_3 = \mu_2 = 0, \quad \mu_1 = -3(x + 2y) \neq 0,$$

and, by [1, Lemma 5.2, statement (ii)], the point M_1 is of multiplicity exactly 3. As a result, we get the configuration *Config. P.40*.

The case $\zeta_9 = 0$. In this case we get the solution $\mathcal{S}_2$, i.e., $m = -1/8$ and $n = -1/4$. Then, we arrive at the system

$$\dot{x} = -\frac{1}{8} - \frac{x}{4} - \frac{3y}{2} + 2x^2 + xy, \quad \dot{y} = \frac{1}{4}(4y - 1)(x + 2y),$$

possessing the invariant lines $y = 1/4$ and $y = x - 1/4$ and the invariant parabola $\Phi(x, y) = x^2 - y = 0$.

As a result, we get the configuration *Config. P.110*.

3.2.5 The statement ($\mathcal{B}_7$)

In this case, the condition $\chi_4 = \zeta_5 = 0$ and $\zeta_6 \neq 0$ must be fulfilled. Straightforward calculations give us that the systems of equations $\chi_4 = 0$ and $\zeta_5 = 0$ could have only the following solutions $\tilde{\mathcal{S}}_i = (m_i, n_i)$ ($i = 1, \dots, 6$):

$$\begin{aligned} \tilde{\mathcal{S}}_1 &= \left(-\frac{27}{2744}, -\frac{3}{28}\right), & \tilde{\mathcal{S}}_2 &= \left(-\frac{2205}{8}, -\frac{357}{4}\right), & \tilde{\mathcal{S}}_3 &= \left(\frac{539}{72}, -\frac{301}{36}\right), \\ \tilde{\mathcal{S}}_4 &= \left(\frac{35}{8}, -\frac{37}{4}\right), & \tilde{\mathcal{S}}_5 &= \left(-\frac{5}{72}, -\frac{13}{36}\right), & \tilde{\mathcal{S}}_6 &= \left(-\frac{189}{8}, -\frac{69}{4}\right). \end{aligned}$$

We split these solutions into two sets:

$$\mathcal{G}_1 = \{\tilde{\mathcal{S}}_1, \tilde{\mathcal{S}}_2, \tilde{\mathcal{S}}_3\}, \quad \mathcal{G}_2 = \{\tilde{\mathcal{S}}_4, \tilde{\mathcal{S}}_5, \tilde{\mathcal{S}}_6\}.$$

Lemma 3.15. Assume that the conditions of statement ($\mathcal{B}_7$) are satisfied and then the system of equations $\chi_4 = \zeta_5 = 0$ generates six solutions $\tilde{\mathcal{S}}_i = (m_i, n_i)$ ($i = 1, \dots, 6$) given above. In this case, the invariant polynomial ζ_6 distinguishes the set $\mathcal{G}_1$ from the set $\mathcal{G}_2$.

Proof. To prove this lemma it is sufficient to evaluate ζ_6 for the elements of each one of the sets. For systems (3.77), we calculate:

$$\zeta_6 = 2^9 3^4 2877985 m(1 + 4m + 2n)(-147 + 50m + 61n + 8n^2),$$

and we obtain

$$\zeta_6(\tilde{\mathcal{S}}_i) \neq 0, \quad i = 1, 2, 3, \quad \zeta_6(\tilde{\mathcal{S}}_j) = 0, \quad j = 4, 5, 6,$$

and we complete the proof of the lemma. $\square$

According to the above lemma, we discuss two cases: $\zeta_6 \neq 0$ and $\zeta_6 = 0$.

The case $\zeta_6 \neq 0$. Then we have to examine the elements of the first set $\mathcal{G}_1$.

1: The subcase $\tilde{\mathcal{S}}_1$. Then we have $m = -27/2744$ and $n = -3/28$, and we get the system

$$\dot{x} = 2x^2 + xy - \frac{3x}{28} - \frac{3y}{2} - \frac{27}{2744}, \quad \dot{y} = xy - \frac{27x}{1372} + 2y^2 - \frac{3y}{14}, \quad (3.100)$$

possessing the following three invariant parabolas: $\Phi_1(x, y) = x^2 - y = 0$ and

$$\Phi_2(x, y) = 3 + 112x + 1176y - 38416y^2 = 0, \quad \Phi_3(x, y) = -243 + 3024x - 10584y + 38416y^2 = 0.$$

We observe that the singular point $M_1(3/14, 9/196)$ is the point of intersection of all three invariant parabolas. So we get the configuration *Config. P.111*.

Next, we prove that the systems generated by $\tilde{\mathcal{S}}_2$ and $\tilde{\mathcal{S}}_3$ could be brought to system (3.100) via an affine transformation and a time rescaling.

Consider first the solution $\tilde{\mathcal{S}}_2$, i.e., $m = -2205/8$ and $n = -357/4$. This leads to the system

$$\dot{x} = 2x^2 + xy - \frac{357x}{4} - \frac{3y}{2} - \frac{2205}{8}, \quad \dot{y} = xy - \frac{2205x}{4} + 2y^2 - \frac{357y}{2},$$

which via the transformation

$$x_1 = -\frac{3}{28} + \frac{y}{343}, \quad y_1 = \frac{3}{196} + \frac{x}{343}, \quad t_1 = 343t,$$

is brought to the system (3.100).

Analogously, taking the solution $\tilde{\mathcal{S}}_3$, i.e., $m = 539/72$ and $n = -301/36$, we arrive at the system

$$\dot{x} = 2x^2 + xy - \frac{301x}{36} - \frac{3y}{2} + \frac{539}{72}, \quad \dot{y} = xy + \frac{539x}{36} + 2y^2 - \frac{301y}{18},$$

which via the transformation

$$x_1 = \frac{9}{28} - \frac{27y}{343}, \quad y_1 = \frac{27}{196} - \frac{27x}{343}, \quad t_1 = -\frac{343t}{27},$$

is brought to the system (3.100).

The case $\zeta_6 = 0$. Then we have to examine the elements of the second set $\mathcal{G}_2$.

1: The subcase $\tilde{\mathcal{S}}_4$. Then we have $m = 35/8$ and $n = -37/4$, and we get the system

$$\dot{x} = 2x^2 + xy - \frac{37x}{4} - \frac{3y}{2} + \frac{35}{8}, \quad \dot{y} = xy + \frac{35x}{4} + 2y^2 - \frac{37y}{2}, \quad (3.101)$$

possessing the invariant line $y = x - 1/4$ and three invariant parabolas: $\Phi_1(x, y) = x^2 - y = 0$ and

$$\Phi_2(x, y) = 5 - 12x + 3x^2 + y = 0, \quad \Phi_3(x, y) = -35 + 144x - 152y + 16y^2 = 0.$$

We observe that the singular point $M_1(1/2, 1/4)$ is the point of intersection of all four invariant curves. So we get the configuration *Config. P.112*.

Next, we prove that the systems generated by $\tilde{\mathcal{S}}_5$ and $\tilde{\mathcal{S}}_6$ could be brought to system (3.101) via an affine transformation and a time rescaling.

Consider first the solution $\tilde{\mathcal{S}}_5$, i.e., $m = -5/72$ and $n = -13/36$. This leads to the system

$$\dot{x} = 2x^2 + xy - \frac{13x}{36} - \frac{3y}{2} - \frac{5}{72}, \quad \dot{y} = -\frac{5x}{36} - \frac{13y}{18} + xy + 2y^2,$$

which via the transformation

$$x_1 = \frac{11}{4} - 9y, \quad y_1 = \frac{19}{4} - 9x, \quad t_1 = -\frac{t}{9},$$

is brought to the system (3.101).

Analogously, taking the solution $\tilde{S}_6$, i.e., $m = -189/8$ and $n = -69/4$, we arrive at the system

$$\dot{x} = 2x^2 + xy - \frac{69x}{4} - \frac{3y}{2} - \frac{189}{8}, \quad \dot{y} = xy - \frac{189x}{4} + 2y^2 - \frac{69y}{2},$$

which via the transformation

$$x_1 = 2 - \frac{x}{3}, \quad y_1 = 7 - \frac{y}{3}, \quad t_1 = -3t,$$

is brought to the system (3.101).

3.3 Configurations of systems in $\mathbf{QSP}_{(\eta < 0)}$

In what follows, we examine the configurations of the systems in $\mathbf{QSP}_{(\eta < 0)}$ for each of the cases provided by Proposition 2.9. According to this proposition, we consider the canonical form (2.5), i.e., the systems

$$\dot{x} = m + (2n - 1)x/2 + gx^2 - gy/2 - xy, \quad \dot{y} = 2mx - x^2 + 2ny + gxy - 2y^2, \quad (3.102)$$

with $C_2 = x(x^2 + y^2)$, possessing the invariant parabola $\Phi(x, y) = x^2 - y = 0$.

3.3.1 The statement ($\mathcal{E}_1$)

For systems (3.102), we calculate

$$\begin{aligned} \zeta_4 &= (25 + g^2)(3g + 9g^3 - 4m - 6gn)/16, \\ \mathcal{R}_1 &= 15(1 + g^2)(25 + g^2)(3g + 9g^3 - 4m - 6gn)/2. \end{aligned} \quad (3.103)$$

The case $B_1 \neq 0$. Then, according to Lemma 2.3, systems (3.102) cannot possess any invariant line.

Let us examine the finite singularities of these systems. Following [1, Proposition 5.1], we calculate the invariant polynomial $\mathbf{D} = 12F_1'^2 F_2'$, where

$$\begin{aligned} F_1' &= -2gm - 2g^3m + 4m^2 - n - g^2n - 4gmn + g^2n^2, \\ F_2' &= 8 - g^2 - 72gm + 8g^3m + 432m^2 - 48n + 4g^2n + 144gmn + 96n^2 - 4g^2n^2 - 64n^3, \end{aligned} \quad (3.104)$$

and we discuss two subcases: $\mathbf{D} \neq 0$ and $\mathbf{D} = 0$.

The subcase $\mathbf{D} \neq 0$. The systems (3.102) possess four finite singularities $M_i(x_i, y_i)$, ($i = 1, 2, 3, 4$), with coordinates

$$\begin{aligned} x_1 &= -\frac{gn - 2m}{g^2 + 1}, \quad y_1 = \frac{2gm + n}{g^2 + 1}; \quad x_2 = \frac{1}{6\mathcal{Y}^{1/3}} [\mathcal{Y}^{2/3} + \mathcal{Y}^{1/3}g - 3g^2 + 2\mathcal{Z}], \\ y_2 &= \frac{1}{6(\sqrt[3]{\mathcal{W}}(2\mathcal{Z} - 3g^2) + 4g\mathcal{W}^{2/3} + \mathcal{W})} [3\sqrt{3F'_2}\mathcal{Z} + 3\mathcal{W}^{2/3}(g^3 + 10gn - 5g + 12m) \\ &\quad + \mathcal{W}^{1/3}\mathcal{Z}(2\mathcal{Z} - 3g^2) + 3(g(g^4 + 22g^2n - 11g^2 + 84n^2 - 84n + 21) + 36m\mathcal{Z}) + g\mathcal{W}^{4/3}]; \\ x_3 &= \frac{1}{12\mathcal{Y}^{1/3}} [(-1 + i\sqrt{3})\mathcal{Y}^{2/3} + 2g\mathcal{Y}^{1/3} - (1 + i\sqrt{3})(2\mathcal{Z} - 3g^2)], \\ y_3 &= \frac{1}{-48g\mathcal{Y}^{2/3} + 6(1 + i\sqrt{3})\mathcal{Y}^{1/3}(2\mathcal{Z} - 3g^2) + 6(1 - i\sqrt{3})\mathcal{Y}} [-6\mathcal{Y}^{2/3}(g^3 + 10gn - 5g + 12m) \\ &\quad + (1 + i\sqrt{3})\mathcal{Y}^{4/3}g + (1 + i\sqrt{3})\mathcal{Y}^{1/3}\mathcal{Z}(2\mathcal{Z} - 3g^2) \\ &\quad + (1 - i\sqrt{3})\mathcal{Y}\mathcal{Z} + (1 - i\sqrt{3})g(2\mathcal{Z} - 3g^2)^2]; \\ x_4 &= \frac{1}{12\mathcal{Y}^{1/3}} [(-1 - i\sqrt{3})\mathcal{Y}^{2/3} + 2g\mathcal{Y}^{1/3} - (1 - i\sqrt{3})(2\mathcal{Z} - 3g^2)], \\ y_4 &= \frac{1}{-48g\mathcal{Y}^{2/3} + 6(1 - i\sqrt{3})\mathcal{Y}^{1/3}(2\mathcal{Z} - 3g^2) + 6(1 + i\sqrt{3})\mathcal{Y}} [-6\mathcal{Y}^{2/3}(g^3 + 10gn - 5g + 12m) \\ &\quad + (1 - i\sqrt{3})\mathcal{Y}^{4/3}g + (1 - i\sqrt{3})\mathcal{Y}^{1/3}\mathcal{Z}(2\mathcal{Z} - 3g^2) \\ &\quad + (1 + i\sqrt{3})\mathcal{Y}\mathcal{Z} + (1 + i\sqrt{3})g(2\mathcal{Z} - 3g^2)^2]. \end{aligned}$$

In the above expressions for the singularities, we use the following notations:

$$\begin{aligned} \mathcal{Y} &= g^3 + 18gn - 9g + 108m + 3\sqrt{\mathcal{X}}, \quad \mathcal{Z} = -3 + 2g^2 + 6n, \\ \mathcal{W} &= -9g + g^3 + 108m + 18gn + 3\sqrt{3F'_2}, \end{aligned}$$

where

$$\mathcal{X} = 24 - 3g^2 - 216gm + 24g^3m + 1296m^2 - 144n + 12g^2n + 432gmn + 288n^2 - 12g^2n^2 - 192n^3.$$

Calculations yield:

$$\Phi(x_2, y_2) = \Phi(x_3, y_3) = \Phi(x_4, y_4) = 0, \quad \Phi(x_1, y_1) = \frac{F'_1}{(1 + g^2)^2},$$

and therefore the three singularities M_2 , M_3 , and M_4 of systems (3.102) lie on the invariant parabola. Moreover, M_1 is located outside the parabola and would belong to the parabola if and only if the condition $F'_1 = 0$ holds, where F'_1 is given in (3.104). However, we have $\mathbf{D} = 12F_1'^2F_2' \neq 0$, and hence there are always exactly three simple singularities on the parabola.

On the other hand, according to [1, Proposition 5.1], if $\mathbf{D} > 0$, systems (3.102) possess two real and two complex finite singularities. For $\mathbf{D} < 0$, we could have either four real or four complex finite singularities. However, since M_1 is a real singular point for these systems, we conclude that in the case $\mathbf{D} < 0$, we must have four real finite distinct singularities.

Thus, since the real singularity M_1 is outside the invariant parabola and all three finite singularities on the parabola (whether real or complex) are distinct, and furthermore we cannot have any invariant line, we arrive at the configuration *Config. P.113* if $\mathbf{D} < 0$, and at *Config. P.114* if $\mathbf{D} > 0$.

The subcase $\mathbf{D} = 0$. This implies that $F'_1 F'_2 = 0$, and we calculate:

$$\zeta_1 = -6\zeta_4 F'_1.$$

Therefore, we deduce that, due to $\zeta_4 \neq 0$, the condition $F'_1 = 0$ is equivalent to $\zeta_1 = 0$. Thus, we examine two possibilities: $\zeta_1 \neq 0$ and $\zeta_1 = 0$.

1: The possibility $\zeta_1 \neq 0$. In this case, the condition $\mathbf{D} = 0$ implies $F'_2 = 0$. Since this polynomial is quadratic in the parameter m , we calculate

$$\text{Discrim}[F'_2, m] = 64(g^2 + 12n - 6)^3.$$

Therefore, as the parameters m , n , and g of systems (3.102) must be real, the condition $g^2 + 12n - 6 \geq 0$ must hold. Introducing a new parameter v such that $g^2 + 12n - 6 = v^2 \geq 0$, we get $n = (6 - g^2 + v^2)/12$. Then, we compute

$$F'_2 = \frac{1}{108} [216m - (g+v)^2(g-2v)] [216m - (g-v)^2(g+2v)] = 0,$$

and, due to the symmetry under the change $v \mapsto -v$, we may assume the first factor vanishes. This yields:

$$m = (g-2v)(g+v)^2/216.$$

Considering the expressions for the parameters m and n , we arrive at the following two-parameter family of systems:

$$\begin{aligned} \dot{x} &= \frac{(g-2v)(g+v)^2}{216} - \frac{g^2-v^2}{12}x - \frac{g}{2}y + gx^2 - xy, \\ \dot{y} &= \frac{(g-2v)(g+v)^2}{108}x + \frac{6-g^2+v^2}{6}y + gxy - 2y^2, \end{aligned} \quad (3.105)$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y = 0$.

We observe that for the above systems, the following conditions on the parameters g and v hold:

$$\begin{aligned} \zeta_1 \neq 0 &\Leftrightarrow (8g-v)^2(4g+v)(2g^2-8gv-v^2+18)(g^2+2gv+v^2+9) \neq 0; \\ B_1 \neq 0 &\Leftrightarrow (2g-v)(4g+v)(36+4g^2-4gv+v^2)(g^2+2gv+v^2+9) \\ &\quad \times (g^2-4gv+4v^2+9) \neq 0. \end{aligned} \quad (3.106)$$

We determine that systems (3.105) possess three finite singularities $M_i(x_i, y_i)$ ($i = 1, 2, 3$) with coordinates:

$$\begin{aligned} x_1 &= \frac{-27g+5g^3-6gv^2-v^3}{54(1+g^2)}, & y_1 &= \frac{54-9g^2+g^4+9v^2-3g^2v^2-2gv^3}{108(1+g^2)}, \\ x_2 &= \frac{g-2v}{6}, & y_2 &= \frac{(g-2v)^2}{36}, & x_3 &= \frac{g+v}{6}, & y_3 &= \frac{(g+v)^2}{36}. \end{aligned}$$

We calculate:

$$\Phi(x_2, y_2) = \Phi(x_3, y_3) = 0, \quad \Phi(x_1, y_1) = -\frac{(2g^2-8gv-v^2+18)(g^2+2gv+v^2+9)^2}{2916(1+g^2)^2},$$

and conclude that the singular points M_2 and M_3 lie on the invariant parabola, while M_1 lies outside the parabola due to the conditions (3.106).

We claim that M_3 is a multiple singularity of systems (3.105). Indeed, applying the corresponding translation, we can place M_3 at the origin, arriving at the systems:

$$\begin{aligned}\dot{x} &= \frac{1}{18}(g+v)(4g+v)x - \frac{1}{6}(4g+v)y + gx^2 - xy, \\ \dot{y} &= \frac{1}{54}(g+v)(2g^2 + gv - v^2 - 18)x + \frac{1}{18}(v^2 - 2g^2 - gv + 18)y + gxy - x^2 - 2y^2,\end{aligned}$$

where $M_0(0,0)$ corresponds to the singularity M_3 .

Following [1], we calculate the following invariant polynomials: $\mu_4 = \mu_3 = 0$, and

$$\mu_2 = \frac{1}{324}v[(g+v)^2 + 9][(v-2g)x + 6y][(3-g^2-gv)x + (4g+v)y].$$

By [1, Lemma 5.2, statement (ii)], the point M_0 has multiplicity at least 2. We observe that, due to the condition $B_1 \neq 0$, we have $\mu_2 = 0$ if and only if $v = 0$. In this case, we calculate:

$$\mu_2 = 0, \quad \mu_1 = -\frac{1}{27}[(5g^4 + 27)x - 32g^3y] \neq 0.$$

Thus, according to [1, Lemma 5.2, statement (ii)], we have a double point if $v \neq 0$ and a triple point if $v = 0$.

On the other hand, for systems (3.105), we calculate:

$$\xi_2 = \frac{1}{209952}v^2(18 + 2g^2 - 8gv - v^2)^2(9 + g^2 + 2gv + v^2)^2,$$

and due to (3.106), we conclude that the condition $v = 0$ is equivalent to $\xi_2 = 0$.

Thus, for systems (3.105), we obtain configuration *Config. P.115* if $\xi_2 \neq 0$ and *Config. P.116* if $\xi_2 = 0$.

2: The possibility $\xi_1 = 0$. We obtain $F'_1 = 0$, and since this polynomial is quadratic in the parameter m , we calculate:

$$\text{Discrim}[F'_1, m] = 4(1 + g^2)^2(g^2 + 4n).$$

It is clear that for the existence of real solutions of the equation $F'_1 = 0$, the condition $g^2 + 4n \geq 0$ must hold.

Thus, we introduce a new parameter u such that $g^2 + 4n = u^2 \geq 0$, leading to $n = (u^2 - g^2)/4$. Then, calculations yield:

$$F'_1 = -\frac{1}{16}[8m - (g-u)(2+g^2-gu)][8m - (g+u)(2+g^2+gu)] = 0,$$

and, due to symmetry under $u \mapsto -u$, we may assume the first factor vanishes. This gives:

$$m = (g-u)(2+g^2-gu)/8.$$

Considering the expressions for m and n , we arrive at the following two-parameter family of systems:

$$\begin{aligned}\dot{x} &= \frac{(g-u)(2+g^2-gu)}{8} + \frac{u^2-g^2-2}{4}x - \frac{g}{2}y + gx^2 - xy, \\ \dot{y} &= \frac{(g-u)(2+g^2-gu)}{4}x - \frac{g^2-u^2}{2}y + gxy - 2y^2,\end{aligned}\tag{3.107}$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y = 0$.

The systems (3.107) possess three finite singularities $M_i(x_i, y_i)$ ($i = 1, 2, 3$) with coordinates:

$$\begin{aligned} x_1 &= \frac{g-u}{2}, \quad y_1 = \frac{(g-u)^2}{4}; \\ x_{2,3} &= \frac{1}{4} \left(u \pm \sqrt{Y_1} \right), \quad y_{2,3} = \frac{1}{8} \left[u^2 + 2gu - 2g^2 - 4 \pm u\sqrt{Y_1} \right], \\ Y_1 &= u^2 + 4gu - 4g^2 - 8. \end{aligned}$$

We calculate:

$$\Phi(x_1, y_1) = \Phi(x_2, y_2) = \Phi(x_3, y_3) = 0,$$

and therefore all three singularities are located on the invariant parabola.

Moreover, we point out that M_1 is a singularity of systems (3.107) with multiplicity at least 2. Indeed, applying a translation that places M_1 at the origin, we arrive at the system:

$$\begin{aligned} \dot{x} &= \frac{1}{2}(g^2 - gu - 1)x + \frac{1}{2}(u - 2g)y + gx^2 - xy, \\ \dot{y} &= \frac{1}{2}(g - u)(g^2 - gu - 1)x + \frac{1}{2}(g - u)(2g - u)y - x^2 + gxy - 2y^2, \end{aligned}$$

where $M_0(0, 0)$ is the singularity corresponding to M_1 .

Following [1], we calculate the following invariant polynomials: $\mu_4 = \mu_3 = 0$, and:

$$\mu_2 = \frac{1}{2}(g^2 + 1) [(g - u)^2 + 1] [(g^2 - gu + 1)x^2 + (u - 2g)xy + 2y^2].$$

We observe that $\mu_2 \neq 0$, and by [1, Lemma 5.2, statement (ii)], the point M_0 has multiplicity exactly 2.

On the other hand, the singularities M_2 and M_3 may be either real or complex, depending on the value of Y_1 . To determine the position of the double singularity M_1 relative to M_2 and M_3 when they are real (i.e., $Y_1 > 0$), we calculate:

$$(x_2 - x_1)(x_3 - x_1) = \frac{(g - u)^2 + 1}{2} > 0.$$

Therefore, in the case $Y_1 > 0$, both singularities M_2 and M_3 lie on the same side of the double point M_1 .

It is clear that for $Y_1 = 0$, the points M_2 and M_3 coalesce, and we obtain two double points located on the invariant parabola.

For systems (3.107), calculations yield:

$$\xi_2 = \frac{1}{8}(1 + g^2)^2 [1 + (g - u)^2]^2 Y_1,$$

and hence $\text{sign}(\xi_2) = \text{sign}(Y_1)$ whenever $\xi_2 \neq 0$.

Thus, we conclude that systems (3.107) possess configuration *Config. P.117* if $\xi_2 < 0$, *Config. P.118* if $\xi_2 > 0$, and *Config. P.119* if $\xi_2 = 0$.

The case $B_1 = 0$. For systems (3.102), we calculate:

$$B_1 = -\frac{1}{64} [g + g^3 + 4m - 2gn] [(g - 8m)^2 + (1 - 4n)^2] \Psi(g, m, n),$$

where

$$\Psi(g, m, n) = 16m^2 + 8gm(3 + 2n) + (4 + g^2)(1 + g^2 + 4n + 4n^2).$$

On the other hand, we calculate:

$$\begin{aligned}\zeta_4 &= \frac{13125}{2}(25 + g^2)(3g + 9g^3 - 4m - 6gn)[(g - 8m)^2 + (1 - 4n)^2], \\ \zeta_4 &= \frac{(25 + g^2)(3g + 9g^3 - 4m - 6gn)}{16},\end{aligned}$$

and, due to $\zeta_4 \neq 0$, we deduce that $\zeta_4 = 0$ is equivalent to $(g - 8m)^2 + (1 - 4n)^2 = 0$.

Thus, we examine two subcases: $\zeta_4 \neq 0$ and $\zeta_4 = 0$.

The subcase $\zeta_4 \neq 0$. Then, the condition $B_1 = 0$ implies either:

$$g + g^3 + 4m - 2gn = 0, \quad \text{or} \quad \Psi(g, m, n) = 0.$$

We state the following lemma.

Lemma 3.16. *For systems (3.102), if $\zeta_4 \neq 0$, then the condition $B_1 = B_2 = 0$ is equivalent to $\Psi(g, m, n) = 0$.*

Proof. Assume first that the condition $\Psi(g, m, n) = 0$ holds. We calculate the discriminant of Ψ with respect to m :

$$\text{Discrim}[\Psi, m] = -64(g^2 - 4n - 2)^2 \leq 0.$$

Thus, for real solutions in m , the necessary condition is $g^2 - 4n - 2 = 0$, which gives $n = (g^2 - 2)/4$. Substituting back, we obtain:

$$\Psi(g, m) = \frac{1}{4}(4g + g^3 + 8m)^2, \quad B_2 = (4g + g^3 + 8m)^2 \phi(g, m, x, y),$$

where $\phi(g, m, x, y)$ is a quartic polynomial in x and y . Thus, clearly, $\Psi(g, m, n) = 0$ implies $B_2 = 0$.

Conversely, assume now that for systems (3.102), the conditions $B_1 = B_2 = 0$ and $\zeta_4 \neq 0$ hold, but suppose (for contradiction) that $\Psi(g, m, n) \neq 0$. Then, the condition $B_1 = 0$ yields $g + g^3 + 4m - 2gn = 0$, and solving for m , we get $m = -g(g^2 - 2n + 1)/4$.

Next, we calculate:

$$\begin{aligned}\Psi(g, n) &= (1 + g^2)(g^2 - 4n - 2)^2, \\ B_2 &= -\frac{81}{2}(1 + g^2)^2(g^2 - 4n - 2)^2[4g^4 + g^2(8 - 16n) + (1 - 4n)^2]x^4, \\ \text{Discrim}[4g^4 + g^2(8 - 16n) + (1 - 4n)^2, n] &= -256g^2 < 0,\end{aligned}$$

where the last inequality holds because $\zeta_4 = g(g^2 + 25)(5g^2 - 4n + 2)/8 \neq 0$. Therefore, for $B_2 = 0$, it must follow that $\Psi(g, n) = 0$, leading to a contradiction with our assumption that $\Psi(g, m, n) \neq 0$. This completes the proof of Lemma 3.16. $\square$

Therefore, in what follows we discuss two possibilities: $B_2 \neq 0$ and $B_2 = 0$.

1: The possibility $B_2 \neq 0$. Then, by Lemma 3.16, we have $\Psi(g, m, n) \neq 0$, and the condition $B_1 = 0$ implies $g + g^3 + 4m - 2gn = 0$. Solving for m , we obtain $m = -g(g^2 - 2n + 1)/4$, leading to the two-parameter family of systems:

$$\begin{aligned}\dot{x} &= -\frac{1}{4}g(g^2 - 2n + 1) + \frac{1}{2}(2n - 1)x - \frac{g}{2}y + gx^2 - xy, \\ \dot{y} &= -\frac{1}{2}g(g^2 - 2n + 1)x + 2ny - x^2 + gxy - 2y^2,\end{aligned}\tag{3.108}$$

possessing the invariant line $L_1(x, y) = 2x + g = 0$, in addition to the invariant parabola $\Phi(x, y) = x^2 - y = 0$.

For these systems, we calculate:

$$\zeta_4 = \frac{g(g^2 + 25)(5g^2 - 4n + 2)}{8}, \quad \theta = -8(g^2 + 9).$$

Since $\theta \neq 0$, by Lemma 2.4, systems (3.108) cannot have an invariant line parallel to $2x + g = 0$. Moreover, by Lemma 2.3, there cannot be invariant lines in other directions because $B_2 \neq 0$.

The systems (3.108) possess four finite singularities $M_i(x_i, y_i)$, $i = 1, 2, 3, 4$, with coordinates:

$$\begin{aligned}x_1 &= -\frac{g}{2}, \quad y_1 = \frac{g^2}{4}; \quad x_2 = -\frac{g}{2}, \quad y_2 = \frac{2n - g^2}{2}; \\ x_{3,4} &= \frac{1}{2}(g \pm \sqrt{Y_2}), \quad y_{3,4} = \frac{1}{2}(2n - 1 \pm g\sqrt{Y_2}), \quad Y_2 = 4n - g^2 - 2.\end{aligned}$$

We compute:

$$\begin{aligned}\Phi(x_1, y_1) &= \Phi(x_3, y_3) = \Phi(x_4, y_4) = L_1(x_1, y_1) = L_1(x_2, y_2) = 0, \\ \Phi(x_2, y_2) &= (3g^2 - 4n)/4,\end{aligned}$$

therefore, M_1 is the intersection point of the invariant line with the invariant parabola. Moreover, M_2 lies on the invariant line and belongs to the invariant parabola if and only if $3g^2 - 4n = 0$. Finally, M_3 and M_4 may be real, complex, or coinciding, depending on the value of Y_2 , and they always lie on the invariant parabola.

To determine the relative positions of the finite singularities, we calculate:

$$(x_3 - x_1)(x_4 - x_1) = (5g^2 - 4n + 2)/4 = \gamma_1/4, \quad y_2 - y_1 = (4n - 3g^2)/4 = \delta_1/4.$$

Thus, when $Y_2 > 0$, M_2 and M_3 lie on the same side (respectively, on opposite sides) of M_1 if $\gamma_1 > 0$ (respectively, $\gamma_1 < 0$). Notice that $\gamma_1 \neq 0$ due to $\zeta_4 \neq 0$.

Also, $y_2 > y_1$ if $\delta_1 > 0$, $y_2 < y_1$ if $\delta_1 < 0$, and $y_2 = y_1$ if $\delta_1 = 0$, in which case the intersection point of the invariant line and the parabola is a double singular point of systems (3.108).

We compute the invariant polynomial $\mathbf{D}$ responsible for the existence of multiple finite singularities:

$$\begin{aligned}\mathbf{D} &= -3(g^2 + 1)^4 Y_2 \gamma_1^2 \delta_1^2 / 4, \quad \zeta_4 = g(25 + g^2) \gamma_1 / 8, \\ \zeta_7 &= 4698510000 g^2 (1 + g^2)^2 Y_2 \gamma_1 \delta_1^2, \quad \zeta_8 = -247290000 g^2 (1 + g^2)^2 Y_2 \delta_1 \gamma_1^2.\end{aligned}$$

Thus, since $\zeta_4 \neq 0$, in the case $\mathbf{D} \neq 0$ we have:

$$\text{sign}(\mathbf{D}) = -\text{sign}(Y_2), \quad \text{sign}(\zeta_7) = \text{sign}(Y_2 \gamma_1), \quad \text{sign}(\zeta_8) = -\text{sign}(Y_2 \delta_1),$$

and we examine two cases: $\mathbf{D} \neq 0$ and $\mathbf{D} = 0$.

1.1: The case $\mathbf{D} \neq 0$. Then $Y_2 \neq 0$, and systems (3.108) have four distinct finite singularities.

Remark 3.17. We observe that $\gamma_1 + \delta_1 = 2(g^2 + 1) > 0$. Therefore, the conditions $\gamma_1 < 0$ and $\delta_1 < 0$ are incompatible.

Considering Remark 3.17, in the case $\mathbf{D} \neq 0$, systems (3.108) exhibit the following configurations:

$$\begin{aligned} \mathbf{D} < 0, \zeta_7 < 0 &\Rightarrow (x_3 - x_1)(x_4 - x_1) < 0, y_2 > y_1 \Rightarrow \text{Config. } \mathcal{P}.120; \\ \mathbf{D} < 0, \zeta_7 > 0, \zeta_8 < 0 &\Rightarrow (x_3 - x_1)(x_4 - x_1) > 0, y_2 > y_1 \Rightarrow \text{Config. } \mathcal{P}.121; \\ \mathbf{D} < 0, \zeta_7 > 0, \zeta_8 > 0 &\Rightarrow (x_3 - x_1)(x_4 - x_1) > 0, y_2 < y_1 \Rightarrow \text{Config. } \mathcal{P}.122; \\ \mathbf{D} > 0, \zeta_8 < 0 &\Rightarrow y_2 < y_1 \Rightarrow \text{Config. } \mathcal{P}.123; \\ \mathbf{D} > 0, \zeta_8 > 0 &\Rightarrow y_2 > y_1 \Rightarrow \text{Config. } \mathcal{P}.124. \end{aligned}$$

1.2: The case $\mathbf{D} = 0$. Since $\gamma_1 \neq 0$ (due to $\zeta_4 \neq 0$), this condition implies $Y_2\delta_1 = 0$, and we calculate:

$$\zeta_1 = -3g(1 + g^2)^2(25 + g^2)\gamma_1\delta_1/16.$$

Thus, since $\gamma_1 \neq 0$, the condition $\delta_1 = 0$ is equivalent to $\zeta_1 = 0$. We consider two subcases: $\zeta_1 \neq 0$ and $\zeta_1 = 0$.

1.2.1: The subcase $\zeta_1 \neq 0$. Then $Y_2 = 0$ (i.e., $n = (g^2 + 2)/4$), so M_3 and M_4 coalesce into a double singular point on the parabola. Moreover, the position of M_2 is determined by δ_1 . For these systems, with $n = (g^2 + 2)/4$, we calculate:

$$\zeta_1\zeta_8 = -3g^4(1 + g^2)^3(25 + g^2)\delta_1/2, \quad \zeta_4 = g^3(25 + g^2)/2.$$

Since $\zeta_4 \neq 0$, we have $\text{sign}(\zeta_1\zeta_8) = -\text{sign}(\delta_1)$. Therefore, in the case $\mathbf{D} = 0$ and $\zeta_1 \neq 0$, we have configuration *Config. P.125* if $\zeta_1\zeta_8 < 0$, and *Config. P.126* if $\zeta_1\zeta_8 > 0$.

1.2.2: The subcase $\zeta_1 = 0$. This condition implies $n = (3g^2)/4$, where M_2 and M_1 coalesce. Note that in this case, $\gamma_1 = 2(g^2 + 1) > 0$.

For $n = 3g^2/4$, we have:

$$Y_2 = 2(g^2 - 1), \quad \zeta_2 = (g^2 - 1)(1 + g^2)^4,$$

and clearly, $\zeta_2 = 0$ if and only if $Y_2 = 0$, which corresponds to M_3 and M_4 also coalescing, yielding two double singularities on the invariant parabola.

Thus, for $\mathbf{D} = \zeta_1 = 0$, we get *Config. P.127* if $\zeta_2 \neq 0$, and *Config. P.128* if $\zeta_2 = 0$.

2: The possibility $B_2 = 0$. Thus, $B_1 = B_2 = 0$, and by Lemma 3.16, the condition $\Psi(g, m, n) = 0$ holds. Referring to the proof of Lemma 3.16, we get:

$$g^2 - 4n - 2 = 0 \Rightarrow n = (g^2 - 2)/4, \quad 4g + g^3 + 8m = 0 \Rightarrow m = -g(g^2 + 4)/8.$$

This leads to the one-parameter family of systems:

$$\begin{aligned} \dot{x} &= -\frac{1}{8}g(g^2 + 4) + \frac{1}{4}(g^2 - 4)x - \frac{g}{2}y + gx^2 - xy, \\ \dot{y} &= -\frac{1}{4}g(g^2 + 4)x + \frac{1}{2}(g^2 - 2)y - x^2 + gxy - 2y^2, \end{aligned} \tag{3.109}$$

possessing three invariant lines:

$$L_1(x, y) = 2x + g = 0, \quad L_{2,3}(x, y) = 4(y \pm ix) - g(g \mp 2i) = 0,$$

besides the invariant parabola.

We find that systems (3.109) possess four finite singularities $M_i(x_i, y_i)$, $i = 1, 2, 3, 4$, with coordinates:

$$\begin{aligned} x_1 &= -\frac{g}{2}, \quad y_1 = \frac{g^2}{4}; \quad x_2 = -\frac{g}{2}, \quad y_2 = -\frac{g^2+2}{4}; \\ x_{3,4} &= \frac{1}{2}(g \pm 2i), \quad y_{3,4} = \frac{1}{4}(g \pm 2i)^2. \end{aligned}$$

We calculate:

$$\begin{aligned} \Phi(x_1, y_1) &= \Phi(x_3, y_3) = \Phi(x_4, y_4) = 0, \quad \Phi(x_2, y_2) = (g^2 + 1)/2, \\ L_1(x_1, y_1) &= L_1(x_2, y_2) = L_2(x_1, y_1) = L_2(x_3, y_3) = L_3(x_4, y_4) = 0. \end{aligned}$$

Thus, M_1 is the intersection point of all three invariant lines with the invariant parabola. Additionally, M_2 lies on the invariant line but not on the parabola, since $g^2 + 1 \neq 0$. Moreover, since $y_2 - y_1 = -(g^2 + 1)/2 < 0$, M_2 lies below M_1 along the vertical line $L_1 = 0$.

Hence, systems (3.109) possess the unique configuration *Config. P.129*.

The subcase $\zeta_4 = 0$. This condition implies $(g - 8m)^2 + (1 - 4n)^2 = 0$, yielding $m = g/8$ and $n = 1/4$. Thus, we obtain the following one-parameter family of systems:

$$\begin{aligned} \dot{x} &= \frac{g}{8} - \frac{x}{4} - \frac{gy}{2} + gx^2 - xy, \\ \dot{y} &= \frac{gx}{4} + \frac{y}{2} - x^2 + gxy - 2y^2, \end{aligned} \tag{3.110}$$

which, in addition to the invariant parabola $\Phi(x, y) = x^2 - y = 0$, possess two complex invariant lines given by $L_{1,2}(x, y) = 4(y \pm ix) - 1 = 0$.

For these systems, we calculate:

$$\zeta_4 = g(25 + g^2)(1 + 9g^2)/16, \quad B_3 = -3g(1 + g^2)(x^2 + y^2)^2/4,$$

and since $\zeta_4 \neq 0$ (i.e., $g \neq 0$), we have $B_3 \neq 0$. Thus, according to Lemma 2.3, these systems cannot have an invariant line in the third (real) direction.

The systems (3.110) possess four finite singularities $M_i(x_i, y_i)$, $i = 1, 2, 3, 4$, with coordinates:

$$x_1 = 0, \quad y_1 = \frac{1}{4}; \quad x_2 = \frac{g}{2}, \quad y_2 = \frac{g^2}{4}; \quad x_{3,4} = \pm \frac{i}{2}, \quad y_{3,4} = -\frac{1}{4}.$$

We compute:

$$\Phi(x_2, y_2) = \Phi(x_3, y_3) = \Phi(x_4, y_4) = 0, \quad \Phi(x_1, y_1) = -\frac{1}{4},$$

and observe that the real singularity M_1 is the intersection point of the two complex invariant lines and lies outside the invariant parabola. The second real singular point M_2 lies on the parabola, and its position depends on the real parameter $g \neq 0$.

As a result, we arrive at a single configuration: *Config. P.130*.

3.3.2 The statement ($\mathcal{E}_2$)

According to Proposition 2.9, in this case the conditions $\zeta_4 = 0$ and $\mathcal{R}_7\zeta_5 \neq 0$ must hold. From (3.103), the condition $\zeta_4 = 0$ implies:

$$3g + 9g^3 - 4m - 6gn = 0 \quad \Rightarrow \quad m = 3g(3g^2 - 2n + 1)/4,$$

which leads to the following family of systems:

$$\begin{aligned}\dot{x} &= \frac{3}{4}g(3g^2 - 2n + 1) + \frac{1}{2}(2n - 1)x - \frac{gy}{2} + gx^2 - xy, \\ \dot{y} &= \frac{3}{2}g(3g^2 - 2n + 1)x + 2ny - x^2 + gxy - 2y^2,\end{aligned}\tag{3.111}$$

possessing two invariant parabolas: the canonical one, $\Phi_1(x, y) = x^2 - y = 0$, and a second parabola,

$$\Phi_2(x, y) = (3g^2 - 4n)(1 + 3g^2 - 2n) + 2g(3g^2 - 4n)x - 4(1 + g^2)x^2 + 2(2 + 5g^2 - 4n)y = 0.$$

For the systems (3.111), we calculate:

$$\begin{aligned}\zeta_5 &= \frac{19}{4}(g^2 + 25)(3g^2 - 4n)^2, \quad \mathcal{R}_7 = 16120(3g^2 + 1)(5g^2 - 4n + 2), \\ \theta &= -8(g^2 + 9), \quad B_1 = -g(g^2 + 1)(9g^2 + 1)(5g^2 - 4n + 2)\Psi_2\Psi_3/32,\end{aligned}\tag{3.112}$$

where

$$\Psi_2(g, n) = 81g^4 + g^2(28 - 72n) + 4(1 + 2n)^2, \quad \Psi_3(g, n) = 36g^4 + g^2(16 - 48n) + (1 - 4n)^2.$$

According to Lemma 2.3, systems (3.111) may possess at least one invariant line only if $B_1 = 0$. Thus, we proceed to examine two possibilities: $B_1 \neq 0$ and $B_1 = 0$.

The possibility $B_1 \neq 0$. The systems (3.111) possess four finite singularities $M_i(x_i, y_i)$ ($i = 1, 2, 3, 4$) with coordinates:

$$\begin{aligned}x_1 &= \frac{3g}{2}, \quad y_1 = \frac{9g^2}{4}; \\ x_2 &= \frac{g(3 + 9g^2 - 8n)}{2(1 + g^2)}, \quad y_2 = \frac{9g^4 + g^2(3 - 6n) + 2n}{2(1 + g^2)}; \\ x_{3,4} &= \frac{1}{2}(-g \pm \sqrt{Y_3}), \quad y_{3,4} = \frac{1}{2}(2n - 1 - 2g^2 \mp g\sqrt{Y_3}), \\ Y_3 &= 4n - 5g^2 - 2.\end{aligned}\tag{3.113}$$

We calculate:

$$\Phi_1(x_1, y_1) = \Phi_1(x_3, y_3) = \Phi_1(x_4, y_4) = 0, \quad \Phi_2(x_2, y_2) = \Phi_2(x_3, y_3) = \Phi_2(x_4, y_4) = 0,$$

therefore, the singularities M_3 and M_4 are the points of intersection of both invariant parabolas. Moreover, the point M_1 lies on the parabola $\Phi_1 = 0$, while M_2 lies on $\Phi_2 = 0$.

To determine the relative positions of M_1 and M_2 with respect to M_3 and M_4 (when $Y_3 > 0$), we calculate:

$$\begin{aligned}(x_3 - x_1)(x_4 - x_1) &= (2 + 21g^2 - 4n)/4 = \gamma_2/4, \\ (x_3 - x_2)(x_4 - x_2) &= -\frac{Y_3}{4(1 + g^2)^2} [21g^4 + 2g^2(5 - 8n) + 1] = -\frac{Y_3}{4(1 + g^2)^2} \delta_2, \\ (x_3 - x_1) + (x_4 - x_1) &= -4g, \quad (x_3 - x_2) + (x_4 - x_2) = \frac{2gY_3}{1 + g^2}.\end{aligned}$$

We observe the following sign relations:

$$\text{sign}((x_3 - x_1) + (x_4 - x_1)) = -\text{sign}(g), \quad \text{sign}((x_3 - x_2) + (x_4 - x_2)) = \text{sign}(gY_3). \quad (3.114)$$

Thus, we deduce that the point M_1 (respectively M_2) lies on the invariant parabola $\Phi_1 = 0$ (respectively $\Phi_2 = 0$) between M_3 and M_4 if and only if $\gamma_2 < 0$ (respectively $Y_3\delta_2 > 0$).

From these relations, we state the following remark.

Remark 3.18. Assume that, for the singularities given in (3.113), the following conditions hold:

$$(x_3 - x_1)(x_4 - x_1) > 0, \quad (x_3 - x_2)(x_4 - x_2) > 0.$$

Then, when $Y_3 > 0$, the condition

$$[(x_3 - x_1) + (x_4 - x_1)][(x_3 - x_2) + (x_4 - x_2)] > 0$$

is impossible.

This follows directly from the sign relations in (3.114).

For systems (3.111), we also calculate:

$$\mathbf{D} = -3(3g^2 - 4n)^2 Y_3 \gamma_2^2 \delta_2^2 / 4, \quad \xi_{14} = 1235 \gamma_2 \delta_2 / 2, \quad \xi_{30} = 1235 [Y_3 \delta_2 - (1 + g^2)^2 \gamma_2] / 2. \quad (3.115)$$

We observe that, when $\mathbf{D} \neq 0$,

$$\text{sign}(\mathbf{D}) = -\text{sign}(Y_3), \quad \text{sign}(\xi_{14}) = \text{sign}(\gamma_2 \delta_2).$$

Moreover, in the case where $\xi_{14} < 0$ (i.e., $\gamma_2 \delta_2 < 0$) and $\mathbf{D} < 0$ (i.e., $Y_3 > 0$), we find:

$$\text{sign}(\xi_{30}) = \text{sign}[Y_3 \delta_2 - (1 + g^2)^2 \gamma_2] = \text{sign}(\delta_2).$$

Thus, we proceed to analyze two cases: $\mathbf{D} \neq 0$ and $\mathbf{D} = 0$.

The case $\mathbf{D} \neq 0$. We observe that, in the case $\mathbf{D} > 0$, the singular points M_3 and M_4 are complex, and therefore in this situation it is not necessary to distinguish the signs of the polynomials γ_2 and δ_2 .

Thus, taking into account Remark 3.18 and the information discussed above, we detect that in the case $\mathbf{D} \neq 0$ the systems admit the following configurations:

$$\begin{aligned} \mathbf{D} < 0, \gamma_2 < 0, \delta_2 > 0 &\Rightarrow (x_3 - x_1)(x_4 - x_1) < 0, (x_3 - x_2)(x_4 - x_2) < 0 \Rightarrow \text{Config. } \mathcal{P}.132; \\ \mathbf{D} < 0, \gamma_2 > 0, \delta_2 < 0 &\Rightarrow (x_3 - x_1)(x_4 - x_1) > 0, (x_3 - x_2)(x_4 - x_2) > 0 \Rightarrow \text{Config. } \mathcal{P}.131; \\ \mathbf{D} < 0, \gamma_2 < 0, \delta_2 < 0 &\Rightarrow (x_3 - x_1)(x_4 - x_1) < 0, (x_3 - x_2)(x_4 - x_2) > 0 \Rightarrow \text{Config. } \mathcal{P}.133; \\ \mathbf{D} < 0, \gamma_2 > 0, \delta_2 > 0 &\Rightarrow (x_3 - x_1)(x_4 - x_1) > 0, (x_3 - x_2)(x_4 - x_2) < 0 \Rightarrow \simeq \text{Config. } \mathcal{P}.133; \\ \mathbf{D} > 0 &\Rightarrow \Rightarrow \text{Config. } \mathcal{P}.134. \end{aligned}$$

Note that in the case $\gamma_2 \delta_2 > 0$, both conditions lead to equivalent configurations to *Config. P.133*. Thus, we deduce that for $\mathbf{D} \neq 0$, the systems (3.111) realize the following configurations if and only if the corresponding invariant conditions hold:

$$\begin{aligned} \mathbf{D} < 0, \xi_{14} < 0, \xi_{30} < 0 &\Rightarrow \text{Config. } \mathcal{P}.131; \\ \mathbf{D} < 0, \xi_{14} < 0, \xi_{30} > 0 &\Rightarrow \text{Config. } \mathcal{P}.132; \\ \mathbf{D} < 0, \xi_{14} > 0 &\Rightarrow \text{Config. } \mathcal{P}.133; \\ \mathbf{D} > 0 &\Rightarrow \text{Config. } \mathcal{P}.134. \end{aligned}$$

The case $\mathbf{D} = 0$. Considering the values of the invariant polynomials obtained above and the conditions $\zeta_5 \neq 0$ (i.e., $3g^2 - 4n \neq 0$) and $\mathcal{R}_7 \neq 0$ (i.e., $Y_3 \neq 0$), we deduce that the condition $\mathbf{D} = 0$ implies $\gamma_2\delta_2 = 0$. From (3.115), this is equivalent to $\xi_{14} = 0$.

We claim that, in the case $\xi_{14} = 0$, the systems (3.111) exhibit configuration *Config. P.135* if $\xi_{30} < 0$, configuration *Config. P.136* if $\xi_{30} > 0$, and configuration *Config. P.137* if $\xi_{30} = 0$.

Indeed, assume $\xi_{14} = 0$, that is, $\gamma_2\delta_2 = 0$. To prove our claim, we examine both possible subcases.

1: The subcase $\delta_2 = 0$. This condition implies:

$$n = \frac{1 + 10g^2 + 21g^4}{16g^2}, \quad \gamma_2 = \frac{(7g^2 - 1)(9g^2 + 1)}{4g^2}.$$

We determine that, under this condition, the singular point M_2 coalesces with M_4 . If in addition $\gamma_2 = 0$, then M_1 coalesces with M_3 , producing two double singularities on the parabola.

Thus, for systems (3.111), if $\gamma_2 > 0$, they realize *Config. P.135*, if $\gamma_2 < 0$, they realize *Config. P.136*, and if $\gamma_2 = 0$, they realize *Config. P.137*.

Considering (3.115), for $\delta_2 = 0$ we observe that $\text{sign}(\xi_{30}) = -\text{sign}(\gamma_2)$, which confirms the claim for this subcase.

2: The subcase $\gamma_2 = 0$. This implies $n = (2 + 21g^2)/4$, and under this condition, the singular point M_1 coalesces with M_3 . We calculate:

$$\delta_2 = (1 - 7g^2)(1 + 9g^2).$$

It is straightforward to verify that, in this case, if $\delta_2 < 0$, the configuration is *Config. P.135*, if $\delta_2 > 0$, the configuration is *Config. P.136*, and if $\delta_2 = 0$, the configuration is *Config. P.137*.

Finally, from (3.115), for $\gamma_2 = 0$ we find $Y_3 = 16g^2 > 0$, so $\text{sign}(\xi_{30}) = \text{sign}(\delta_2)$, which completes the proof of our claim.

The possibility $B_1 = 0$. From (3.112), this condition implies $g(5g^2 - 4n + 2)\Psi_2\Psi_3 = 0$. We claim that, due to the condition $\mathcal{R}_7 \neq 0$, the equality $B_1 = 0$ is equivalent to $g = 0$.

Indeed, assuming $g \neq 0$, we compute:

$$\text{Discrim}[\Psi_2, n] = -4096g^2 < 0, \quad \text{Discrim}[\Psi_3, n] = -256g^2 < 0,$$

and hence the equations $\Psi_2 = 0$ and $\Psi_3 = 0$ cannot have real solutions with respect to the parameter n . This completes the proof of our claim.

Thus, we conclude $g = 0$ and arrive at the following one-parameter family of systems:

$$\dot{x} = \frac{1}{2}x(2n - 2y - 1), \quad \dot{y} = -x^2 + 2ny - 2y^2, \quad (3.116)$$

which possess the invariant line $x = 0$ and the invariant parabolas:

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = x^2 + (2n - 1)y - n(2n - 1) = 0.$$

For these systems, we calculate:

$$\zeta_5 = 1900n^2, \quad \mathcal{R}_7 = -32240(2n - 1), \quad B_2 = -162(2n + 1)^2(4n - 1)^2x^4.$$

We now discuss two cases: $B_2 \neq 0$ and $B_2 = 0$.

The case $B_2 \neq 0$. By Lemma 2.3, in this case the system cannot possess invariant lines in any other direction. Considering the condition $g = 0$ in (3.113), we find that the systems (3.116) possess four finite singularities $M_i(x_i, y_i)$ ($i = 1, 2, 3, 4$) with coordinates:

$$\begin{aligned} x_1 = 0, \quad y_1 = 0; \quad x_2 = 0, \quad y_2 = n; \\ x_{3,4} = \frac{1}{2}(\pm \sqrt{Y_3}), \quad y_{3,4} = \frac{1}{2}(2n - 1), \quad Y_3 = 2(2n - 1). \end{aligned}$$

We observe that the invariant line $x = 0$ intersects the invariant parabola $\Phi_1 = 0$ at point M_1 , and intersects the parabola $\Phi_2 = 0$ at point M_2 .

For these systems we have:

$$\mathbf{D} = -12n^2Y_3^3 \Rightarrow \text{sign}(\mathbf{D}) = -\text{sign}(Y_3).$$

Since all singular points are fixed as intersection points of invariant curves, and $\mathbf{D} \neq 0$ (due to $\zeta_5\mathcal{R}_7 \neq 0$), we conclude that these systems realize configuration *Config. P.138* if $\mathbf{D} < 0$, and configuration *Config. P.139* if $\mathbf{D} > 0$.

The case $B_2 = 0$. This condition implies $(2n + 1)(4n - 1) = 0$.

Assume first $4n - 1 = 0$, that is, $n = 1/4$. Then, we arrive at the system:

$$\dot{x} = -\frac{1}{4}x(4y + 1), \quad \dot{y} = \frac{1}{2}(-2x^2 - 4y^2 + y), \quad (3.117)$$

which possesses three invariant lines: $L_1(x, y) = x = 0$ and $L_{2,3}(x, y) = y \pm ix - 1/4 = 0$, along with the invariant parabolas $\Phi_1(x, y) = x^2 - y = 0$ and $\Phi_2(x, y) = -4x^2 + 2y - 1/2 = 0$.

We observe that the point $M_2(0, 1/4)$ corresponds to the intersection of the above complex lines. As a result, this system exhibits the configuration *Config. P.140*.

Now, if $2n + 1 = 0$ (i.e., $n = -1/2$), we arrive at the system:

$$\dot{x} = -x(1 + y), \quad \dot{y} = -x^2 - y - 2y^2,$$

which can be transformed into system (3.117) via the affine transformation and time rescaling:

$$x_1 = x/2, \quad y_1 = y/2 + 1/4, \quad t_1 = 2t,$$

and hence also realizes *Config. P.140*.

3.3.3 The statement ($\mathcal{E}_3$)

According to Proposition 2.9, in this case the conditions $\zeta_4 = \zeta_5 = 0$ and $\mathcal{R}_7 \neq 0$ hold. Considering (3.112), the condition $\zeta_5 = 0$ implies:

$$3g^2 - 4n = 0 \Rightarrow n = \frac{3g^2}{4},$$

and we obtain the following one-parameter family of systems:

$$\begin{aligned} \dot{x} &= \frac{3}{8}g(3g^2 + 2) + \frac{1}{4}(3g^2 - 2)x - \frac{g}{2}y + gx^2 - xy, \\ \dot{y} &= \frac{3}{4}g(3g^2 + 2)x + \frac{3g^2}{2}y - x^2 + gxy - 2y^2, \end{aligned} \quad (3.118)$$

which, according to Proposition 2.9, possess $\Phi(x, y) = x^2 - y = 0$ as a double invariant parabola.

For these systems we compute:

$$\begin{aligned}\zeta_4 = \zeta_5 = 0, \quad \mathcal{R}_7 = 32240(1 + g^2)(1 + 3g^2) \neq 0, \\ \theta = -8(g^2 + 9) \neq 0, \quad B_1 = -\frac{g(1 + g^2)^4(1 + 9g^2)^3}{4}.\end{aligned}$$

According to Lemma 2.3, systems (3.118) can possess at least one invariant line only if $B_1 = 0$. Thus, we discuss two cases: $B_1 \neq 0$ and $B_1 = 0$.

The case $B_1 \neq 0$. Systems (3.118) possess the following three finite singularities $M_i(x_i, y_i)$ ($i = 1, 2, 3$) with coordinates:

$$\begin{aligned}x_1 = \frac{3g}{2}, \quad y_1 = \frac{9g^2}{4}; \\ x_{2,3} = -\frac{1}{2} \left[g \pm i\sqrt{2(1 + g^2)} \right], \quad y_{2,3} = \frac{1}{4} \left[-2 - g^2 \pm 2gi\sqrt{2(1 + g^2)} \right].\end{aligned}\tag{3.119}$$

It is clear that the real singular point M_1 is a double point, as it lies on the double invariant parabola. This fact can be checked directly.

Moreover, the complex singular points M_2 and M_3 also lie on the invariant parabola, though this is not relevant for the classification of the configuration. We deduce that in the case $B_1 \neq 0$, systems (3.118) realize a single configuration: *Config. P.141*.

The case $B_1 = 0$. This condition implies $g = 0$. Thus, system (3.118) with $g = 0$ possesses an additional invariant line: $x = 0$. Considering the singularities (3.119) evaluated at $g = 0$, we arrive at the unique configuration *Config. P.142*.

3.3.4 The statement ($\mathcal{E}_4$)

According to Proposition 2.9, in this case the conditions $\zeta_4 = \mathcal{R}_7 = 0$ and $\zeta_5 \neq 0$ hold. Considering (3.112), the condition $\mathcal{R}_7 = 0$ implies:

$$5g^2 - 4n + 2 = 0 \quad \Rightarrow \quad n = \frac{2 + 5g^2}{4}.$$

Thus, we arrive at the following one-parameter family of systems:

$$\begin{aligned}\dot{x} &= \frac{1}{8}(g + 2x)(3g^2 + 4gx - 4y), \\ \dot{y} &= \frac{3g^3x}{4} + \frac{1}{2}(5g^2 + 2)y - x^2 + gx y - 2y^2,\end{aligned}\tag{3.120}$$

which possess the invariant parabola $\Phi(x, y) = x^2 - y = 0$ and the invariant line $2x + g = 0$. For these systems we calculate:

$$B_2 = -648(1 + g^2)^5(1 + 9g^2)x^4 \neq 0,$$

and by Lemma 2.3, no additional invariant lines can exist.

Systems (3.120) have three finite singularities $M_i(x_i, y_i)$ ($i = 1, 2, 3$) with coordinates:

$$x_1 = -\frac{g}{2}, \quad y_1 = \frac{g^2}{4}; \quad x_2 = -\frac{g}{2}, \quad y_2 = \frac{1}{4}(3g^2 + 2); \quad x_3 = \frac{3g}{2}, \quad y_3 = \frac{9g^2}{2}.$$

We claim that the singular point M_1 is a multiple singularity of systems (3.120). Indeed, applying a suitable translation to place M_1 at the origin, we arrive at the system:

$$\begin{aligned} \dot{x} &= x(gx - y), \\ \dot{y} &= g(1 + g^2)x - x^2 + (1 + g^2)y + gxy - 2y^2, \end{aligned}$$

where $M_0(0, 0)$ corresponds to the singularity M_1 .

Following [1], we compute the following invariant polynomials:

$$\begin{aligned} \mu_4 &= \mu_3 = 0, \quad \mu_2 = 2g(1 + g^2)^2x(gx - y), \\ \mu_1 &= -(1 + g^2)(x + 5g^2x - 4gy), \end{aligned}$$

and by [1, Lemma 5.2, statement (ii)], the point M_0 has multiplicity at least 2. Notice that $\mu_2 = 0$ if and only if $g = 0$, but in this case $\mu_1 \neq 0$. Therefore, according to [1, Lemma 5.2, statement (ii)], M_1 is a double point if $g \neq 0$ and a triple point if $g = 0$. This classification is governed by the invariant polynomial $\zeta_3 = 32g^2$.

We also note that the multiple singularity M_1 is the intersection point between the invariant line $2x + g = 0$ and the invariant parabola. Moreover, the singularity M_3 lies on the same invariant parabola, and M_3 coalesces with M_1 when $g = 0$, producing a triple finite singularity in systems (3.120).

On the other hand, the singularity M_2 lies on the invariant line above the point M_1 , since $y_2 - y_1 = (g^2 + 1)/2 > 0$. Therefore, we conclude that systems (3.120) realize configuration *Config. P.143* if $\zeta_3 \neq 0$, and *Config. P.144* if $\zeta_3 = 0$.

As all the cases have now been examined, we conclude that statement (B) of the Main Theorem is fully proved.

3.4 Geometric invariants and the proof of the statement (C)

In this subsection, we complete the proof of the Main Theorem by showing that all 144 configurations of invariant parabolas and invariant lines that we have constructed are non-equivalent according to Definition 1.3. To achieve this, we define the invariants that distinguish the configurations within the family $\mathbf{QSP}_{(\eta \neq 0)}$ into 144 *distinct* cases. We believe that these invariants are among the most suitable for describing the geometric phenomena specific to this class.

The basic algebraic-geometric notions that we will use here include the concept of an integer-valued r -cycle and its type:

Definition 3.19. Let V be an irreducible algebraic variety of dimension n over a field K . A cycle of dimension r , or an r -cycle, on V is a formal sum

$$\sum_W m(W)W,$$

where each W is a subvariety of V of dimension r that is not contained in the singular locus of V , $m(W) \in \mathbb{Z}$, and only finitely many of the coefficients $m(W)$ are nonzero. The *degree* of an r -cycle is the sum $\sum_W m(W)$. An $(n - 1)$ -cycle is called a *divisor*.

Definition 3.20. The *type of an r -cycle* is the set of all ordered pairs (n_1, n_2) , where n_1 is a coefficient appearing in the r -cycle (i.e., $n_1 = m(W)$ for some W), and n_2 is the number of subvarieties W in the cycle for which $m(W) = n_1$.

Definition 3.21. We define a set of numerical and geometric invariants that allow us to distinguish the obtained configurations:

- (a) $(M_{\mathcal{P}}, m_{\mathcal{L}})$ = (total multiplicity of invariant parabolas, total multiplicity of invariant affine lines);
- (b) $\tau_{\mathcal{P}}^f$ = the type of the divisor of the multiplicities of finite singularities of the system located on the parabolas;
- (c) $t_{\mathcal{P} \cap \mathcal{L}}^f$ = the type of the divisor of the intersection multiplicities (in the finite plane) between $\mathcal{P}$ and $\mathcal{L}$;
- (d) τ_s^∞ = the type of the divisor of the multiplicities of singularities on the line at infinity;
- (e) $M_{\mathcal{L} \cap \mathcal{P}}^n$ = the maximum number of real finite intersection points between an invariant line and a parabola;
- (f) $M_{\mathcal{P}}^m$ = the maximum multiplicity among the invariant parabolas;
- (g) Consider the finite segment of a parabola delimited by two singularities s_1 and s_2 of the system, having inside another singularity s_0 . Let $m(s_0)$ denote the multiplicity of s_0 . We define the invariant $I(s_0)$ for this configuration, assigning it the value 1 if s_0 is an intersection point between the parabola and a line, and 0 otherwise;
- (h) $m^f(\mathcal{P} \cap \mathcal{L})$ = the multiplicity of the intersection point (considered as a singularity of the system) between the parabola and an affine line;
- (i) Consider an invariant line passing through the infinite point of the parabola. This line has another finite intersection point with the parabola, splitting it into two branches. We introduce the invariant $\{n_{\mathcal{P}}^1, n_{\mathcal{P}}^2\}$, where $n_{\mathcal{P}}^i$ denotes the number of singularities of the system located on each open branch of the parabola;
- (j) $m_s^{\mathbb{R}}(\mathcal{P})$ = the total multiplicity of real singularities of the system located on the parabola;
- (k) $n_{in}^s(\mathcal{P})$ = the total number of real singularities located inside the union of all domains delimited by the parabolas;
- (l) If an invariant line intersects a parabola at two distinct real finite points, we obtain a domain $\mathcal{D}_{\mathcal{P} \cap \mathcal{L}}^f$ bounded by the line segment and the arc of the parabola not passing through its infinite point. We denote by $\tau(\mathcal{D}_{\mathcal{P} \cap \mathcal{L}}^f)$ the type of the divisor of the of the multiplicities of singularities of the system located on the boundary of $\mathcal{D}_{\mathcal{P} \cap \mathcal{L}}^f$;
- (m) $n_s^f(\mathcal{F})$: the number of finite singular points of the curve

$$\mathcal{F} : \mathcal{P}(X, Y, Z) \cdot L_1(X, Y, Z) \cdot L_2(X, Y, Z) = 0$$

that have multiplicity at least 2.

- (n) Suppose an affine line intersects the parabola $x^2 - y = 0$ at two finite points or has a point of tangency with it. Let Δ^b denote the curvilinear triangle on the Poincaré disk, with vertex at the infinite point of the parabola, bounded by the closed branch of the parabola, the adjacent segment of the invariant line, and the open arc a_∞ containing a singular point at infinity in its interior. We denote by $n(\Delta^b)$ (respectively, $m(\Delta^b)$) the number (respectively, the total multiplicity) of singularities located on Δ^b ;
- (o) Assume we have a real system possessing two real invariant parabolas. In this case, two distinct possibilities occur in the real projective plane:
- (1) The points at infinity of the two parabolas are distinct,
 - (2) The points at infinity coincide.

Clearly, the positions of the points at infinity (P_1^∞, P_2^∞) separate these two cases. If $P_1^\infty \neq P_2^\infty$, we are in case (1). If $P_1^\infty = P_2^\infty$, we are in case (2), which further splits into two subcases on the Poincaré disk (PD):

- (i) The two points at infinity coincide on PD,
- (ii) The two points are opposite on PD.

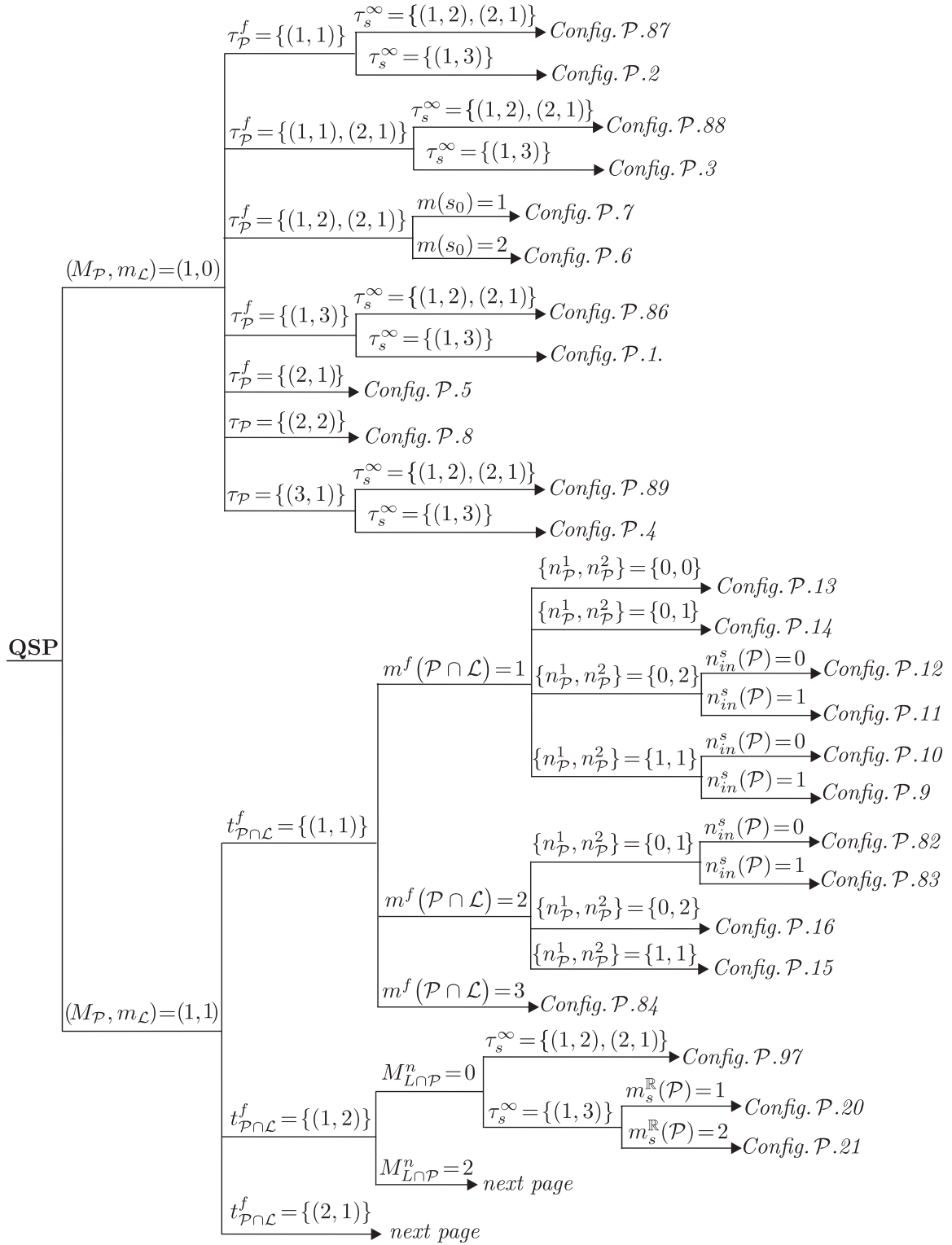
We define the invariant $J(P_1^\infty, P_2^\infty)$ of the group acting on the systems viewed on the Poincaré disk, assigning the value 0 to case (1), the value 1 to case (2.i), and the value 2 to case (2.ii);

- (p) If two invariant parabolas intersect at two distinct finite points, we obtain a finite domain $\mathcal{D}_{\mathcal{P}_1 \cap \mathcal{P}_2}^f$ delimited by the closed branches of the parabolas. We denote by $\tau(\mathcal{D}_{\mathcal{P}_1 \cap \mathcal{P}_2}^f)$ the type of the divisor of the multiplicities of singularities of the system located on the boundary of $\mathcal{D}_{\mathcal{P}_1 \cap \mathcal{P}_2}^f$;
- (q) Assume that two real invariant parabolas have two distinct points at infinity, $\mathcal{P}_1^\infty$ and $\mathcal{P}_2^\infty$, which are not opposite on PD. Let a_1^∞ and a_2^∞ denote the open arcs on the circumference of PD determined by $\mathcal{P}_1^\infty$ and $\mathcal{P}_2^\infty$. Assume that within the interior of arc a_1^∞ (respectively, a_2^∞), there are n_1 (respectively, n_2) singularities at infinity. We define a new invariant: $m(\mathcal{P}_1^\infty, \mathcal{P}_2^\infty) = \min\{n_1, n_2\}$.

The set of numerical and geometric invariants introduced in Definition 3.21 provides the necessary criteria to distinguish all 144 non-equivalent configurations of systems in the family **QSP**. Each invariant encodes specific geometric information regarding the arrangement of invariant parabolas, affine lines, and singularities, both finite and at infinity.

Figures 3.1 and 3.2 illustrate the complete bifurcation diagram associated with the Main Theorem. The diagrams are organized according to the sign of the parameter η , with Figure 3.1 representing the case $\eta > 0$ and Figure 3.2 corresponding to $\eta < 0$.

Each branch in these diagrams reflects a bifurcation step governed by one or more of the invariants from Definition 3.21, capturing the sequence of geometric changes that distinguish one configuration from another. Together, the invariants and the diagrams provide a complete and systematic classification of all phase portraits in the family **QSP**.

Diagram 3.1: Non-equivalent configurations of systems in **QSP** (the case $\eta > 0$).

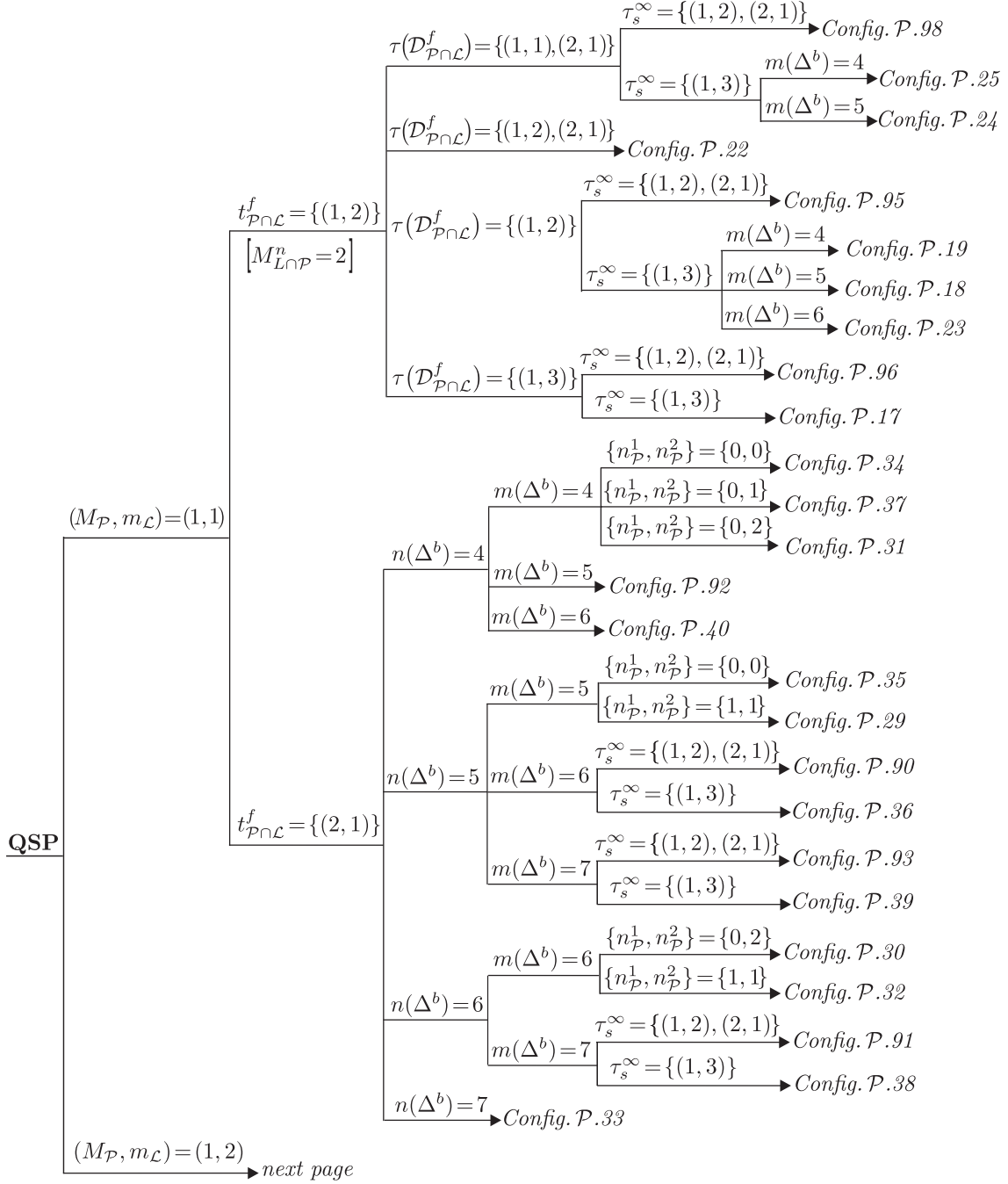


Diagram 3.1 (continued): Non-equivalent configurations of systems in **QSP** (the case $\eta > 0$).

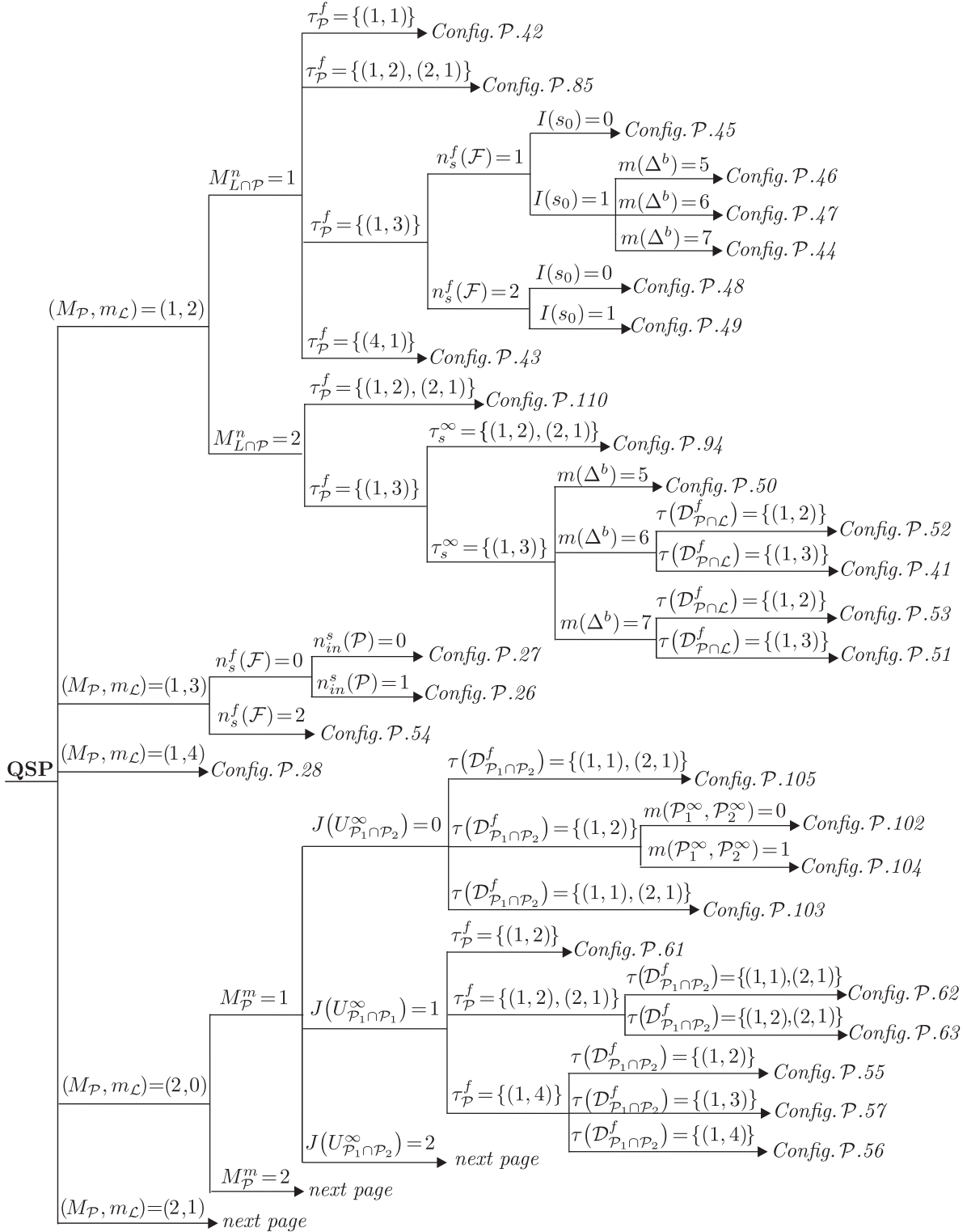


Diagram 3.1 (continued): Non-equivalent configurations of systems in QSP (the case $\eta > 0$).

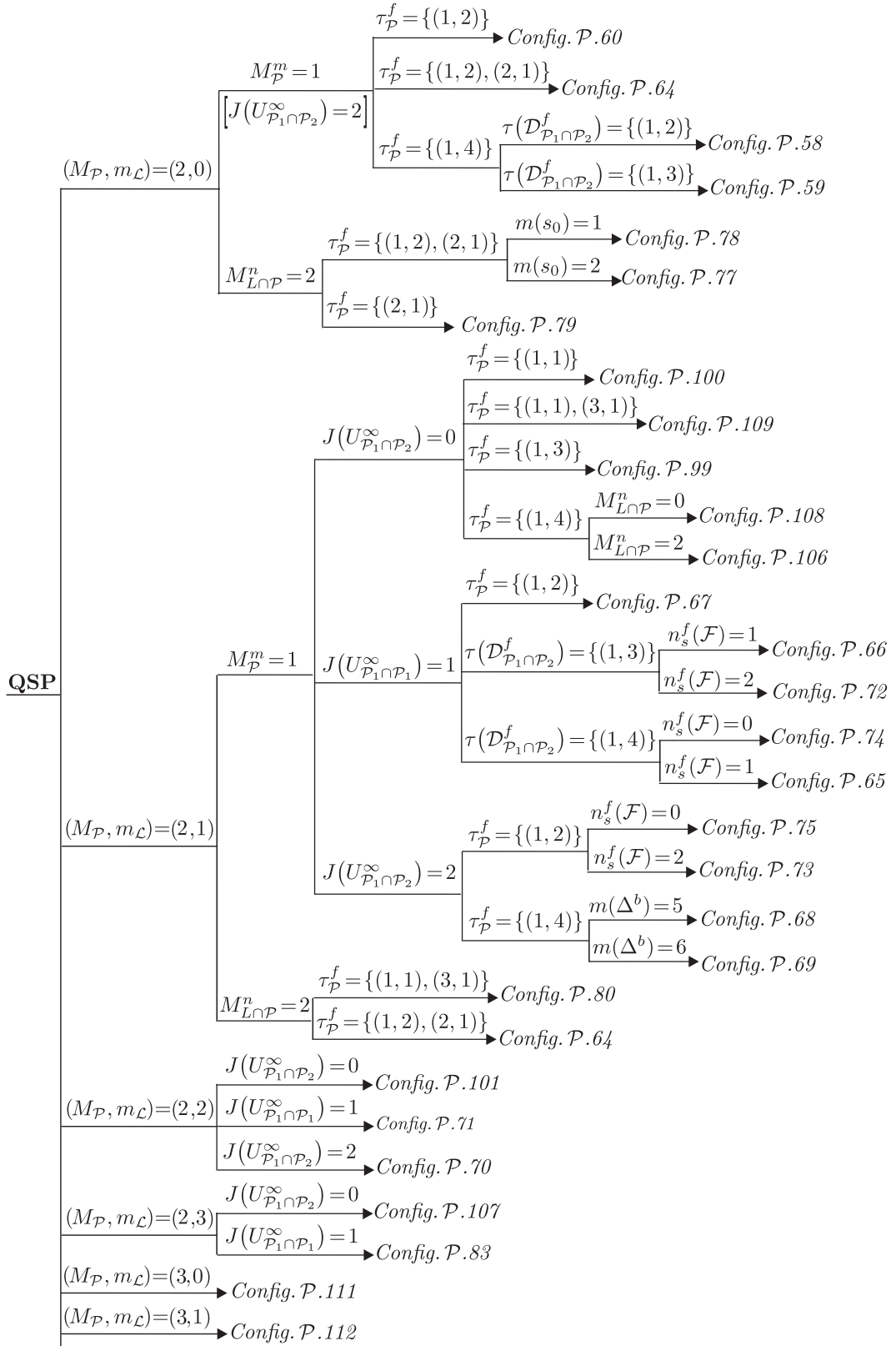
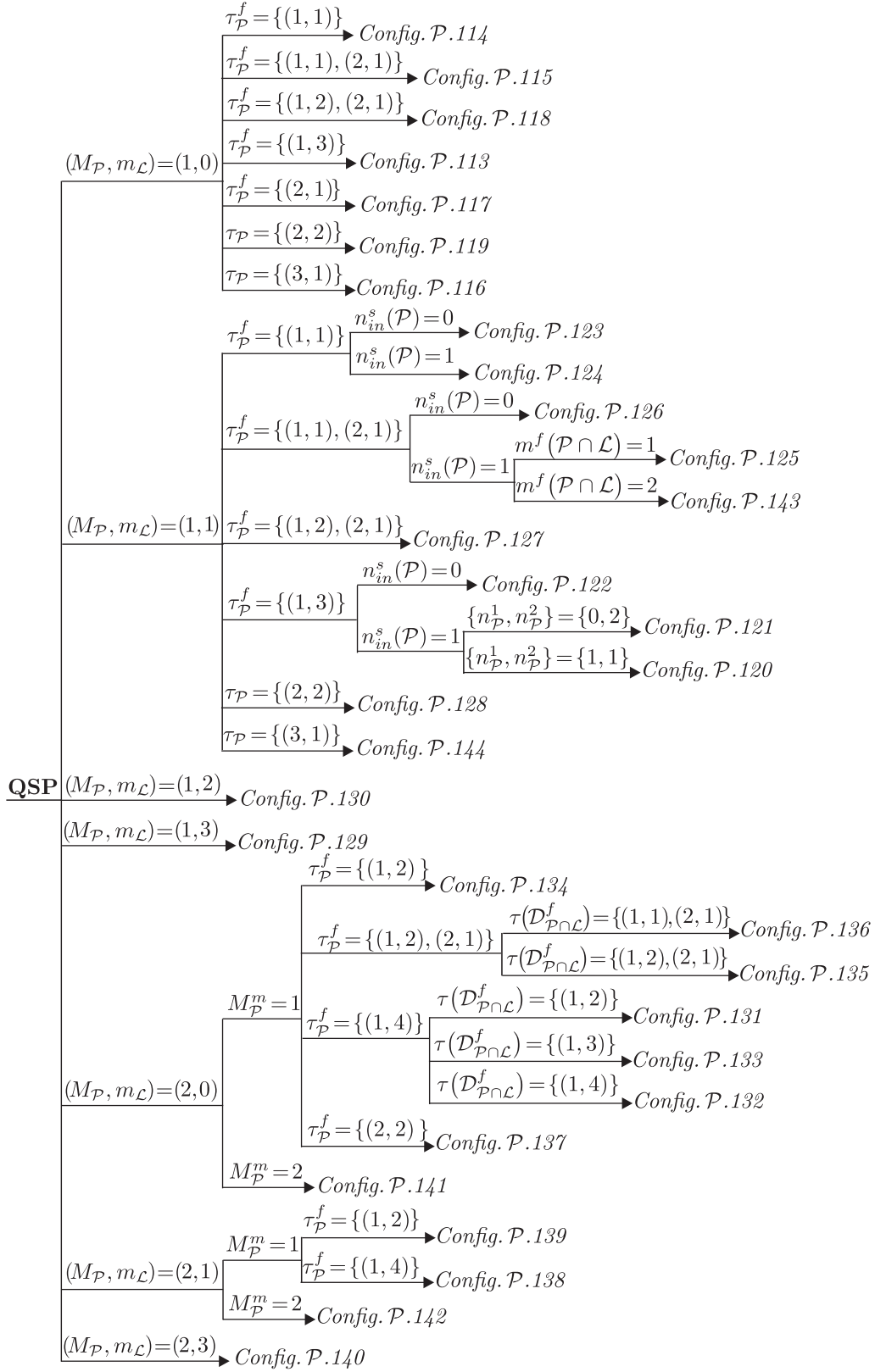


Diagram 3.1 (continued): Non-equivalent configurations of systems in QSP (the case $\eta > 0$).

Diagram 3.2: Non-equivalent configurations of systems in QSP (the case $\eta < 0$).

Acknowledgments

N. Vulpe thanks the Instituto de Ciências Matemáticas e de Computação (USP) and Universidade Federal de São Carlos (UFSCar), São Carlos, Brazil, for the hospitality and for the financial support during his visit to both institutes. R. D. S. Oliveira was supported by CNPq grant Projeto Universal 407454/2023-3, CNPq grant Produtividade em Pesquisa 310857/2023-6, by Projeto Temático FAPESP number 2019/21181-0, and by the European Union's Horizon Europe research and innovation programme under Grant Agreement No. 101138111 (DSYREKI). A. C. Rezende was partially supported by Projeto Temático FAPESP number 2019/21181-0 and Auxílio Regular FAPESP number 2024/13454-4. D. Schlomiuk and N. Vulpe were supported by the NSERC Grant RN000355. N. Vulpe was supported by the Program SADGET 011303 and partially supported by the grant 25.80012.5007.83SE, Moldova State University.

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