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Positive periodic solutions for a second-order damped differential equation with an indefinite singularity

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Abstract. We prove the existence of at least one positive periodic solution for a second-order damped nonlinear differential equation with an indefinite singularity by assuming the corresponding linear equation have a positive Green's function. Our results are applicable to a weak singularity as well as a strong singularity. The proof is based on the Leray–Schauder alternative principle.

Keywords: Leray–Schauder alternative principle, Green's function, indefinite singularity, positive periodic solution.

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1 Introduction

The main purpose of this paper is to investigate the existence of a positive periodic solution for the following damped indefinite singular equation

$$x'' + p(t)x' + q(t)x = \frac{b(t)}{x^\rho} + e(t), \tag{1.1}$$

where ρ is a real constant and $\rho > 0$, q and $e \in C(\mathbb{R}/\omega\mathbb{Z})$ are positive, $p \in C(\mathbb{R}/\omega\mathbb{R})$, $b \in C^1(\mathbb{R}/\omega\mathbb{Z})$, the weight term b may have zero, even it may change sign. Moreover, note that when $p(t) \equiv 0$, equation (1.1) reduces to

$$x'' + q(t)x = \frac{b(t)}{x^\rho} + e(t). \tag{1.2}$$

According to the relevant literature [21], the singular term $\frac{b(t)}{x^\rho}$ represents a singularity of repulsive type in the case that $b(t) > 0$ for all $t \in [0, \omega]$, and a singularity of attractive type in the case that $b(t) < 0$ for all $t \in [0, \omega]$. Additionally, equation (1.1) is said to satisfy the strong force condition if $\rho \geq 1$ and the weak force condition if $0 < \rho < 1$.

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In recent years, Schauder's fixed point theorem [10,23,26], the method of lower and upper solutions [15,17,27,28], the Leray–Schauder alternative principle [9,20,22], Krasnoselskii's fixed point theorem in cones [29,30], and coincidence degree theory [5,7,32] have been employed to investigate the existence of positive periodic solutions for equations (1.1) and (1.2) with the weight term $b(t) > 0$ or $b(t) < 0$ for all $t \in [0, \omega]$(i.e. singularity of repulsive or attractive type).

Recently, several intriguing results have been obtained on differential equations with an indefinite singularity [1,8,13,18,24], which are significant both in theory and in practice. In this paper, we aim to establish the existence of a positive periodic solution for equation (1.1) by applying the Leray–Schauder alternative principle.

Depending on the sign of the weight term b, equation (1.1) may exhibit both attractive and repulsive singularities depending on the variable t, even the singularity can vanish in certain subintervals where the weight term b equals zero. Since the weight term b of the above papers [7–10,20,22,23,26,27,29,30,32] does not change sign, the methods employed in these works are no longer applicable to proving the existence of a positive periodic solution for equation (1.1) with an indefinite singularity. Thus, an alternative approach is required to overcome these difficulties. To address the issue of an indefinite singularity (i.e., the singular term that may change sign), we introduce the change of variable $x = u^{\alpha}$ with $\alpha = \frac{1}{\rho+1}$ and $\alpha < 1$. This transformation simplifies equation (1.1) into a more easily treatable singular equation

$$u'' + p(t)u' + \frac{1}{\alpha}q(t)u = \frac{e(t)}{\alpha}u^{1-\alpha} + (1-\alpha)\frac{u'^2}{u} + \frac{b(t)}{\alpha}. \tag{1.3}$$

It is readily seen that the existence of a positive periodic solution to equation (1.1) reduces to proving the existence of a positive periodic solution for equation (1.3). A similar transformation method has also appeared in the literature [12].

Our purpose is to show that the Leray–Schauder alternative principle can be applied to indefinite singular equation (1.1). The rest of this paper is organized as follows. In section 2, Green's function is provided, and its positivity is obtained. In section 3, in order to facilitate the study of equation (1.3), we first study the following damped singular differential equation

$$u'' + p(t)u' + \ell(t)u = f(t, u, u') + m(t), \tag{1.4}$$

where $\ell(t) := \frac{1}{\alpha}q(t)$, $m(t) := \frac{b(t)}{\alpha}$, $f \in C(\mathbb{R} \times (0, +\infty) \times \mathbb{R}, \mathbb{R})$ is ω-periodic with respect to t and exhibits a singularity of repulsive type at $u = 0$. By applying the Leray–Schauder alternative principle, we prove that equation (1.4) has at least one positive periodic solution. Afterwards, we obtain the existence of a positive periodic solution for equation (1.1), and the results are applicable to both strong and weak singularities.

2 Positivity of Green's function

We consider the following nonhomogeneous linear differential equation

$$\begin{cases} u'' + p(t)u' + \ell(t)u = h(t), \\ u(0) = u(\omega), \ u'(0) = u'(\omega), \end{cases} \tag{2.1}$$

where $h \in C(\mathbb{R}/\omega\mathbb{Z})$. Equation (2.1) has a unique ω-periodic solution which can be written as

$$u(t) = \int_0^{\omega} G(t, s)h(s)ds,$$

where $G(t,s)$ is the Green's function of equation (2.1). Throughout this paper, we assume that

(A) The Green's function $G(t,s)$ of equation (2.1) is positive for all $(t,s) \in [0,\omega] \times [0,\omega]$.

The rest part of this section is to make a brief on some known sufficient conditions to guarantee condition (A) is satisfied. We will discuss the following three cases.

Case I. The general case $p \in C(\mathbb{R}/\omega\mathbb{R})$.

Define functions

$$\varsigma(p)(t) = \exp\left(\int_0^t p(s)ds\right)$$

and

$$\varsigma_1(p)(t) = \varsigma(p)(\omega)\int_0^t \varsigma(p)(s)ds + \int_t^\omega \varsigma(p)(s)ds.$$

Lemma 2.1 (see [16]). *Assume that $\ell(t) \geq 0$ and $\ell \not\equiv 0$ and the following two inequalities are satisfied*

$$\int_0^\omega \ell(s)\varsigma(p)(s)\varsigma_1(-p)(s)ds \geq 0 \tag{2.2}$$

and

$$\sup_{0\leq t\leq\omega}\left\{\int_t^{t+\omega}\varsigma(-p)(s)ds, \int_t^{t+\omega}\ell(s)\varsigma(p)(s)ds\right\} \leq 4. \tag{2.3}$$

Then condition (A) holds.

Case II. Special case $\bar{p} := \frac{1}{\omega}\int_0^\omega p(t)dt > 0$.

In 2005, Wang et al. [31, Lemma 2.4] discussed the positivity of Green's function $G(t,s)$ for all $(t,s) \in [0,\omega] \times [0,\omega]$ if the following conditions are satisfied:

(A_1) There are continuous ω-periodic functions $a_1(t)$ and $a_2(t)$ such that $\int_0^\omega a_1(t)dt > 0$, $\int_0^\omega a_2(t)dt > 0$ and

$$a_1(t) + a_2(t) = p(t), \qquad a_1'(t) + a_1(t)a_2(t) = \ell(t), \quad \text{for } t \in [0,\omega].$$

(A_2) $\bar{p}^2 \geq 4\omega^2 \exp\left(\frac{1}{\omega}\int_0^\omega \ln \ell(s)ds\right)$.

Obviously, condition (A_2) is hard restrictive for the positivity of the Green's function. After that, Cheng and Ren [7] in 2018 discussed the positivity of the Green's function for all $(t,s) \in [0,\omega] \times [0,\omega]$ if only condition (A_1) is satisfied.

Case III. Special case $\bar{p} = 0$.

Define

$$D(\iota) = \begin{cases} \left(\frac{2\pi}{\iota}\right)^{1/2}\left(\frac{2}{2+\iota}\right)^{1/2-1/\iota}\frac{\Gamma(1/\iota)}{\Gamma(1/2+1/\iota)}, & \text{if } 1 \leq \iota < \infty,\\ 2, & \text{if } \iota = \infty,\end{cases}$$

where Γ is the Gamma function with $\Gamma(t) = \int_0^{+\infty} x^{t-1}e^{-x}dx$.

Lemma 2.2 (see [3, Theorem 5.1]). *Assume that $\bar{p} = 0$ and $\int_0^\omega \varsigma(p)(t)\ell(t)dt > 0$. Suppose further that there exists $1 \leq \xi \leq \infty$ such that*

$$(\Upsilon_1(\omega))^{1+1/\iota}\|\Upsilon_2\|_\xi < D^2(2\iota), \tag{2.4}$$

where $\Upsilon_1(\omega) = \int_0^\omega \varsigma(-p)(t)dt$, $\Upsilon_2(t) = \ell_+(t)(\varsigma(p)(t))^{2-1/\xi}$, $\|\Upsilon_2\|_\xi = \left(\int_0^\omega |\Upsilon_2(t)|^\xi dt\right)^{\frac{1}{\xi}}$. Then condition (A) holds.

In follows what, we consider the case that $p(t) \equiv 0$.

Remark 2.3 (see [25, Corollary 2.3])**.** In the case $p(t) \equiv 0$, define

$$\widetilde{D}(\tilde{\iota}) = \begin{cases} \frac{2\pi}{\tilde{\iota}\omega^{1+2/\tilde{\iota}}} \left(\frac{2}{2+\tilde{\iota}}\right)^{1-2/\tilde{\iota}} \left(\frac{\Gamma(1/\tilde{\iota})}{\Gamma(1/2+1/\tilde{\iota})}\right)^2, & \text{if } 1 \le \tilde{\iota} < \infty, \\ \frac{4}{\omega}, & \text{if } \tilde{\iota} = \infty, \end{cases}$$

Assume that $\ell \in L^{\zeta}(0,\omega)$ for $1 \le \zeta \le \infty$ and $\ell(t) \ge 0$ for almost every $t \in [0,\omega]$. If

$$\|\ell\|_{\zeta} := \left(\int_0^{\omega} |\ell(t)|^{\zeta} dt\right)^{\frac{1}{\zeta}} < \widetilde{D}(2\zeta^*),$$

where $\zeta^* = \frac{\zeta}{\zeta-1}$ if $1 \le \zeta < \infty$ and $\zeta^* = 1$ if $\zeta = +\infty$, then the Green's function $G(t,s)$ is positive for all $(t,s) \in [0,\omega] \times [0,\omega]$.

Remark 2.4 (see [19, Lemma 2.5])**.** In the case $p(t) \equiv 0$ and $\ell(t) = \delta^2$ with $\delta > 0$ and $\delta \neq \frac{2k\pi}{\omega}$ for any natural k, the Green's function has the form

$$G_1(t,s) = \begin{cases} \dfrac{\cos\delta(t-s-\frac{\omega}{2})}{2\delta\sin\frac{\delta\omega}{2}}, & 0 \le s \le t \le \omega, \\ \dfrac{\cos\delta(t-s+\frac{\omega}{2})}{2\delta\sin\frac{\delta\omega}{2}}, & 0 \le t < s \le \omega. \end{cases}$$

If $\delta < \frac{\pi}{\omega}$, then Green's function $G_3(t,s)$ is positive for all $(t,s) \in [0,\omega] \times [0,\omega]$.

In this context, the general mechanism for the construction of Green functions in this context is described in [2,4].

3 Main results

In this section, we prove the existence of a positive periodic solution for equation (1.1). First, defined $X := \{u \in C^1 : u(t+\omega) \equiv u(t), \text{ for all } t \in (R)\}$ with the norm $\|u\| := \max_{t\in\mathbb{R}} |u(t)|$, X is a normed linear space (not complete). Our proof is based on the following Leray–Schauder alternative principle, which can be found in [14, p. 120-130].

Lemma 3.1. *Assume that Ω is an open subset of a convex set K in a normed linear space X and $\beta \in \Omega$. Let $T : \bar{\Omega} \to K$ be a compact map. Then one of the following two conclusions holds:*

(I) T has at least one fixed point in $\bar{\Omega}$.

(II) There exist $u \in \partial\Omega$ and $0 < \lambda < 1$ such that $u = \lambda Tu + (1-\lambda)\beta$.

From condition (A), we denote

$$\mathcal{A} := \min_{0\le s,t\le\omega} G(t,s), \qquad \mathcal{B} := \max_{0\le s,t\le\omega} G(t,s), \qquad \sigma := \frac{\mathcal{A}}{\mathcal{B}}, \qquad \iota := \frac{\max\limits_{0\le s,t\le\omega} \left|\frac{\partial G(t,s)}{\partial t}\right|}{\mathcal{A}}. \tag{3.1}$$

It is clear that $0 < \mathcal{A} \le \mathcal{B}$ and $0 < \sigma \le 1$.

Define the function $\gamma : \mathbb{R} \to \mathbb{R}$ by

$$\gamma(t) := \int_0^{\omega} G(t,s)m(s)ds,$$

which is the unique ω-periodic solution of the following equation

$$u''(t) + p(t)u'(t) + \ell(t)u(t) = m(t).$$

Denote

$$\gamma_* := \min_{t\in\mathbb{R}} \gamma(t) \quad \text{and} \quad \gamma^* := \max_{t\in\mathbb{R}} \gamma(t).$$

3.1 The case $\gamma_* \geq 0$

Theorem 3.2. *Assume that condition (A) and $0 \leq \bar{p}\omega < 1$ hold. Furthermore, assume that the following conditions are satisfied:*

(H_1) *For each constant $L > 0$, there exists a continuous function $\phi_L \succ 0$ such that $f(t, u, y) \geq \phi_L(t)$ for all $(t, u, y) \in [0, \omega] \times (0, L] \times \mathbb{R}$, where $\phi_L(t) \succ 0$ represents $\phi_L(t) \geq 0$ for almost every $t \in [0, \omega]$.*

(H_2) *There exist continuous non-negative functions k, φ, ϱ and continuous positive function g such that*

$$0 \leq f(t, u, y) \leq g(u)\varrho(|y|) + k(t)\varphi(u) \quad \text{for all } (t, u, y) \in [0, \omega] \times (0, \infty) \times \mathbb{R},$$

and $g(u)$ is non-increasing, $\varphi(u)/g(u)$ is non-decreasing and $\varrho(\cdot)$ is non-decreasing in $(0, +\infty)$.

(H_3) *There exists a positive real constant r such that*

$$\frac{r}{g(\sigma r + \gamma_*)\left(\varrho\left(\left|\frac{r\bar{\ell}\omega}{1-\bar{p}\omega} + \gamma'^*\right|\right) + \frac{\|k\|\varphi(r+\gamma^*)}{g(r+\gamma^*)}\right)} > \Lambda^*,$$

where $\Lambda(t) := \int_0^\omega G(t, s)ds$ and $\|k\| = \max_{t\in[0,\omega]} k(t)$.

If $\gamma_* \geq 0$, then equation (1.4) has at least one positive ω-periodic solution u with $u(t) > \gamma(t)$ for all $t \in [0, \omega]$ and $0 < \|u - \gamma\| < r$.

Proof. Consider equation

$$u''(t) + p(t)u'(t) + \ell(t)u(t) = f(t, u(t) + \gamma(t), u'(t) + \gamma'(t)). \tag{3.2}$$

It is easy to see that if equation (3.2) has a positive ω-periodic solution u satisfying $u(t) + \gamma(t) > 0$ for $t \in [0, \omega]$ and $0 < \|u\| < r$, then $v(t) := u(t) + \gamma(t)$ is a positive ω-periodic solution of equation (1.4) with $0 < \|v - \gamma\| < r$. So we just need to consider equation (3.2).

Since condition (H_3) holds, we choose $n_0 \in \{1, 2, \ldots\}$ such that $\frac{1}{n_0} < \sigma r + \gamma_*$ and

$$\Lambda^* g(\sigma r + \gamma_*)\left(\varrho\left(\left|\frac{r\bar{\ell}\omega}{1 - \bar{p}\omega} + \gamma'^*\right|\right) + \frac{\|k\|\varphi(r + \gamma^*)}{g(r + \gamma^*)}\right) + \frac{1}{n_0} < r.$$

Let $N_0 = \{n_0, n_0 + 1, \ldots\}$. Consider the family of equations

$$u'' + p(t)u' + \ell(t)u = \lambda f_n(t, u(t) + \gamma(t), u'(t) + \gamma'(t)) + \frac{\ell(t)}{n}, \tag{3.3}$$

where $\lambda \in [0, 1]$, and

$$f_n(t, u, y) = \begin{cases} f(t, u, y) & \text{if } u \geq \frac{1}{n}, \\ f(t, \frac{1}{n}, y) & \text{if } u < \frac{1}{n}. \end{cases}$$

An ω-periodic solution of equation (3.3) is just a fixed point of operator equation

$$u = \lambda T_n u + (1-\lambda)\beta, \tag{3.4}$$

where $\beta := \frac{1}{n}$ and T_n is a continuous operator defined by

$$(T_n u)(t) := \int_0^\omega G(t,s) f_n(s, u(s)+\gamma(s), u'(s)+\gamma'(s))ds + \frac{1}{n}, \tag{3.5}$$

where we used the fact $\int_0^\omega G(t,s)\ell(s)ds \equiv 1$ from [25, Corollary 2.3] and $G(t,s) > 0$ for all $(t,s) \in [0,\omega] \times [0,\omega]$ from condition (A).

Define

$$K := \{u \in X : u(t) > 0 \text{ for all } t \in \mathbb{R}\},$$

and

$$B_r := \{u \in K : \|u\| < r\},$$

where X and r are defined in Section 3 and Theorem 3.2, respectively. Obviously, B_r is an open subset in K. Further, for any $u \in \overline{B_r}$, it follows from (A) and (H_1) that

$$(T_n u)(t) = \int_0^\omega G(t,s) f_n(s, u(s)+\gamma(s), u'(s)+\gamma'(s))ds + \frac{1}{n} \geq \frac{1}{n} > 0,$$

which implies $T_n(\overline{B_r}) \subset K$. Besides, by using the Arzelà–Ascoli Theorem, it is easy to verify that $T_n : \overline{B_r} \to K$ is completely continuous.

Next, we claim that any fixed point u of equation (3.4) for any $\lambda \in (0,1)$ must satisfy $\|u\| \neq r$. Otherwise, assume that u is a fixed point of equation (3.4) for $\lambda \in (0,1)$ such that $\|u\| = r$. Then, we give

$$\begin{aligned} u(t) - \frac{1}{n} &= \lambda \int_0^\omega G(t,s) f_n(s, u(s)+\gamma(s), u'(s)+\gamma'(s))ds \\ &\geq \lambda \mathcal{A} \int_0^\omega f_n(s, u(s)+\gamma(s), u'(s)+\gamma'(s))ds \\ &= \lambda \sigma \mathcal{B} \int_0^\omega f_n(s, u(s)+\gamma(s), u'(s)+\gamma'(s))ds \\ &\geq \sigma \max_{t\in\mathbb{R}} \left\{ \lambda \int_0^\omega G(t,s) f_n(s, u(s)+\gamma(s), u'(s)+\gamma'(s))ds \right\} \\ &= \sigma \left\| u - \frac{1}{n} \right\|. \end{aligned}$$

Therefore, we obtain

$$u(t) \geq \sigma \|u - \frac{1}{n}\| + \frac{1}{n} \geq \sigma \left(\|u\| - \frac{1}{n} \right) + \frac{1}{n} \geq \sigma r,$$

and then

$$u(t) + \gamma(t) \geq \sigma r + \gamma_* > \frac{1}{n},$$

since $\frac{1}{n} \leq \frac{1}{n_0} < \sigma r + \gamma_*$.

Further, we claim that

$$\|u'\| \leq \frac{r\bar{\ell}\omega}{1-\bar{p}\omega} \tag{3.6}$$

for any ω-periodic solution $u(t)$ of equation (3.3). Integrating equation (3.3) over the interval $[0,\omega]$, we see that

$$\int_0^\omega p(t)u'(t)dt + \int_0^\omega \ell(t)u(t)dt = \lambda \int_0^\omega f_n(t,u(t)+\gamma(t),u'(t)+\gamma'(t))dt + \int_0^\omega \frac{\ell(t)}{n}dt. \quad (3.7)$$

Since $u(0)=u(\omega)$, we know that there exists a point $t_1 \in (0,\omega)$ such that $u'(t_1)=0$. Therefore, we get

$$\begin{aligned} \|u'\| &= \max_{t\in\mathbb{R}} \left|\frac{1}{2}(u'(t)+u'(t-\omega))\right| \\ &= \max_{t\in\mathbb{R}} \frac{1}{2}\left|\int_{t_1}^t u''(s)ds - \int_{t-\omega}^{t_1} u''(s)ds\right| \\ &\le \frac{1}{2}\int_0^\omega |u''(s)|ds \qquad (3.8) \\ &= \frac{1}{2}\left(\int_0^\omega \left|\lambda f_n(t,u(t)+\gamma(t),u'(t)+\gamma'(t)) + \frac{\ell(t)}{n} - \ell(t)u(t) - p(t)u'(t)\right| dt\right) \\ &\le \frac{1}{2}\int_0^\omega \left(\lambda f_n(t,u(t)+\gamma(t),u'(t)+\gamma'(t)) + \frac{\ell(t)}{n} + \ell(t)u(t)\right) dt + \frac{1}{2}\|u'\|\bar{p}\omega \end{aligned}$$

since $f(t,u,u')>0$, $\ell(t)>0$ and $\bar{p}\ge 0$. Applying (3.7) to (3.8), we obtain

$$\begin{aligned} \|u'\| \le& \int_0^\omega \ell(t)u(t)dt + \frac{1}{2}\int_0^\omega p(t)u'(t)dt + \frac{1}{2}\|u'\|\int_0^\omega p(t)dt \\ =& r\bar{\ell}\omega + \|u'\|\bar{p}\omega. \end{aligned}$$

Since $1-\bar{p}\omega>0$, then we deduce

$$\|u'\| \le \frac{r\bar{\ell}\omega}{1-\bar{p}\omega},$$

which proves that (3.6) holds.

On the other hand, it follows from conditions (H_2) and (H_3) that

$$\begin{aligned} u(t) &= \lambda\int_0^\omega G(t,s)f_n(s,u(s)+\gamma(s),u'(s)+\gamma'(s))ds + \frac{1}{n} \\ &= \lambda\int_0^\omega G(t,s)f(s,u(s)+\gamma(s),u'(s)+\gamma'(s))ds + \frac{1}{n} \\ &\le \int_0^\omega G(t,s)(g(u(s)+\gamma(s))\varrho(|u'(s)+\gamma'(s)|) + k(s)\varphi(u(s)+\gamma(s)))ds + \frac{1}{n} \\ &= \int_0^\omega G(t,s)g(u(s)+\gamma(s))\left(\varrho(|u'(s)+\gamma'(s)|) + \frac{k(s)\varphi(u(s)+\gamma(s))}{g(u(s)+\gamma(s))}\right)ds + \frac{1}{n} \\ &\le g(\sigma r+\gamma_*)\left(\varrho\left(\left|\frac{r\bar{\ell}\omega}{1-\bar{p}\omega}+\gamma'^*\right|\right) + \frac{\|k\|\varphi(r+\gamma^*)}{g(r+\gamma^*)}\right)\Lambda^* + \frac{1}{n_0}, \end{aligned}$$

which implies

$$r = \|u\| \le g(\sigma r+\gamma_*)\left(\varrho\left(\left|\frac{r\bar{\ell}\omega}{1-\bar{p}\omega}+\gamma'^*\right|\right) + \frac{\|k\|\varphi(r+\gamma^*)}{g(r+\gamma^*)}\right)\Lambda^* + \frac{1}{n_0}.$$

This is a contradiction with the choice of n_0 and the claim is proved.

From this claim, Lemma 3.1 guarantees that

$$u = T_n u$$

has a fixed point, denoted by u_n in B_r. Then, we get

$$u'' + p(t)u' + \ell(t)u = f_n(t, u(t) + \gamma(t), u'(t) + \gamma'(t)) + \frac{\ell(t)}{n} \tag{3.9}$$

has an ω-periodic solution u_n with $\|u_n\| \le r$. Since $u_n(t) + \gamma(t) \ge \frac{1}{n} > 0$ for all $n > n_0$, $t \in \mathbb{R}$, u_n is actually a positive ω-periodic solution of equation (3.9).

From condition (H_1), there exists a continuous function $\phi_r(t) \ge 0$ such that $f(t, u, y) \ge \phi_r(t)$ for all $(t, u, y) \in [0, \omega] \times (0, r] \times \mathbb{R}$. Since $u_n(t) + \gamma(t) \ge \frac{1}{n} > 0$ for all $n > n_0$, $t \in \mathbb{R}$ and $\gamma_* \ge 0$, we have

$$\begin{aligned} u_n(t) + \gamma(t) &= \int_0^\omega G(t,s) f_n(s, u_n(s) + \gamma(s), u_n'(s) + \gamma'(s)) ds + \gamma(t) + \frac{1}{n} \\ &= \int_0^\omega G(t,s) f(s, u_n(s) + \gamma(s), u_n'(s) + \gamma'(s)) ds + \gamma(t) + \frac{1}{n} \\ &\ge \int_0^\omega G(t,s) \phi_{r+\gamma^*} ds + \gamma(t) \\ &\ge \Phi_* + \gamma_* =: \vartheta, \end{aligned} \tag{3.10}$$

where $\Phi(t) = \int_0^\omega G(t,s)\phi_{r+\gamma^*}(s) ds$. Since $G(t,s)$ is regular and $\phi_{r+\gamma^*}(s) > 0$, it follows that $\Phi_* > 0$. So we have $u_n(t) + \gamma(t) \ge \vartheta$.

Similar to the proof of equation (3.6), we get

$$\|u_n'\| \le \frac{r\bar{\ell}\omega}{1 - \bar{p}\omega}, \quad \text{for all } n \ge n_0. \tag{3.11}$$

Further, it follows from (3.9) and (3.11) that

$$\begin{aligned} \|u_n''\| &\le \left| p(t)u_n'(t) + \ell(t)u_n(t) + \lambda f_n(t, u_n(t) + \gamma(t), u_n'(t) + \gamma'(t)) + \frac{\ell(t)}{n} \right| \\ &\le \frac{r\|p\|\bar{\ell}\omega}{1 - \bar{p}\omega} + \|\ell\| r + \|f\| + \frac{\|\ell\|}{n_0} := M_1, \end{aligned}$$

where $\|f\| := \max_{\vartheta \le u_n + \gamma \le r + \gamma^*, \|u_n'\| \le \frac{r\bar{\ell}\omega}{1-\bar{p}\omega}} |f(t, u_n(t) + \gamma(t), u_n'(t) + \gamma'(t))|$.

In consequence, $\{u_n\}_{n \in N_0}$ and $\{u_n'\}_{n \in N_0}$ are bounded and equi-continuous family in C^1_ω. Now the Arzelà–Ascoli theorem guarantees that $\{u_n\}_{n \in N_0}$ has a subsequence $\{u_{n_j}\}_{j \in N_0}$, converging uniformly on $\mathbb{R}$ to a function $u \in C^1_\omega$. From the fact $\|u_n\| \le r$ and $\vartheta \le u_n + \gamma$, u satisfies $\vartheta \le u(t) + \gamma(t) \le r + \gamma^*$ for all t. Moreover, u_{n_j} satisfies the integral equation

$$u_{n_j}(t) = \int_0^\omega G(t,s) f(s, u_{n_j}(s) + \gamma(s), u_{n_j}'(s) + \gamma'(s)) ds + \frac{1}{n_j}.$$

Letting $j \to \infty$, we get

$$u(t) = \int_0^\omega G(t,s) f(s, u(s) + \gamma(s), u'(s) + \gamma'(s)) ds.$$

Therefore, u is a positive periodic solution of equation (3.2) and satisfies $0 < \|u\| \le r$. Besides, it is not difficult to show that $\|u\| < r$, by noting that if $\|u\| = r$, the argument similar to the proof of the first claim will lead to a contradiction. □

By Theorem 3.2, we get the following conclusion.

Corollary 3.3. *Assume that condition (A), $0 \le \bar{p}\omega < 1$ and $\gamma_* \ge 0$ hold. Furthermore, assume that the nonlinear term f satisfies the following condition:*

(F_1) *There exist positive constants κ, ν, μ, η, ϵ with $\epsilon \le \nu + 1$ such that*

$$f(t,u,y) = \kappa\frac{(|y|^\epsilon + 1)}{u^\nu} + \mu e(t)u^\eta, \quad \text{for all } (t,u,y) \in [0,\omega] \times (0,+\infty) \times \mathbb{R},$$

where $e(t) \in C(\mathbb{R}/\omega\mathbb{Z})$ is positive.

(i) If $\eta < 1$, then equation (1.4) has at least one positive ω-periodic solution for each $\mu > 0$.

(ii) If $\eta \ge 1$, then equation (1.4) has at least one positive ω-periodic solution for each $0 < \mu < \mu_1$, where μ_1 is a positive constant.

Proof. We apply Theorem 3.2. Take

$$\phi_L(t) = \frac{\kappa}{L^\nu}, \quad g(u) = \frac{1}{u^\nu}, \quad \varphi(u) = u^\eta, \quad k(t) = \mu e(t), \quad \varrho(|y|) = \kappa(|y|^\epsilon + 1).$$

Then conditions (H_1) and (H_2) are satisfied and the existence condition (H_3) becomes

$$\mu < \frac{r(\sigma r + \gamma_*)^\nu - \Lambda^*\kappa\left(\left|\frac{r\bar{\ell}\omega}{1-\bar{p}\omega} + \gamma'^*\right|^\epsilon + 1\right)}{\Lambda^*\|e\|(r+\gamma^*)^{\eta+\nu}}$$

for $r > 0$. Therefore, equation (1.4) has at least one positive periodic solution for

$$0 < \mu < \mu_1 := \sup_{r>0} \frac{r(\sigma r + \gamma_*)^\nu - \Lambda^*\kappa\left(\left|\frac{r\bar{\ell}\omega}{1-\bar{p}\omega} + \gamma'^*\right|^\epsilon + 1\right)}{\Lambda^*\|e\|(r+\gamma^*)^{\eta+\nu}}.$$

Note that $\mu_1 = \infty$ if $\eta < 1$ and $\mu_1 < \infty$ if $\eta \ge 1$, we have (i) and (ii). □

Theorem 3.4. *Assume that $\bar{p} = 0$, $\int_0^\omega \varsigma(p)(t)\ell(t)dt > 0$, inequality (2.4) and conditions (H_1), (H_2) hold. Furthermore, assume that the following condition is satisfied:*

(H_4) *There exists a positive real constant r such that*

$$\frac{r}{g(\sigma r + \gamma_*)\left(\varrho(|Qr + \gamma'^*|) + \frac{\|k\|\varphi(r+\gamma^*)}{g(r+\gamma^*)}\right)} > \Lambda^*,$$

where $Q := \frac{\int_0^\omega \varsigma(p)(t)\ell(t)dt}{\min_{t\in\mathbb{R}} \varsigma(p)(t)}$.

If $\gamma_* \ge 0$, then equation (1.4) has at least one positive ω-periodic solution u with $u(t) > \gamma(t)$ for all $t \in [0,\omega]$ and $0 < \|u - \gamma\| < r$.

Proof. We follow the same strategy and notations as in the proof of Theorem 3.2. Next, we claim

$$\|u'\| \le Qr \tag{3.12}$$

for any ω-periodic solution u of equation (3.3). Multiplying both sides of equation (3.3) by $\varsigma(p)(t)$, we get

$$(\varsigma(p)(t)u'(t))' + \varsigma(p)(t)\ell(t)u(t) = \varsigma(p)(t)\left(\lambda f_n(t, u(t)+\gamma(t), u'(t)+\gamma'(t)) + \frac{\ell(t)}{n}\right). \tag{3.13}$$

Integrating equation (3.13) over the interval $[0,\omega]$, we obtain

$$\int_0^\omega \varsigma(p)(t)\ell(t)u(t)dt = \int_0^\omega \varsigma(p)(t)\left(\lambda f_n(t,u(t)+\gamma(t),u'(t)+\gamma'(t)) + \frac{\ell(t)}{n}\right)dt$$

since $\bar{p}=0$. Because $u(0)=u(\omega)$, we know that there exists a point $t_2\in(0,\omega)$ such that $u'(t_2)=0$. Therefore, it is clear that

$$\begin{aligned}
&|\varsigma(p)(t)u'(t)| \\
&= \max_{t\in\mathbb{R}}\left|\frac{1}{2}(\varsigma(p)(t)u'(t)+\varsigma(p)(t)u'(t-\omega))\right| \\
&= \max_{t\in\mathbb{R}}\frac{1}{2}\left|\int_{t_2}^t(\varsigma(p)(s)u'(s))'ds - \int_{t-\omega}^{t_2}(\varsigma(p)(s)u'(s))'ds\right| \\
&\le \frac{1}{2}\int_0^\omega |(\varsigma(p)(t)u'(t))'|dt \\
&\le \frac{1}{2}\left(\int_0^\omega\left|\varsigma(p)(t)\left(\lambda f_n(t,u(t)+\gamma(t),u'(t)+\gamma'(t))+\frac{\ell(t)}{n}\right)-\varsigma(p)(t)\ell(t)u(t)\right|dt\right) \\
&= \frac{1}{2}\int_0^\omega \varsigma(p)(t)\left(\lambda f(t,u(t)+\gamma(t),u'(t)+\gamma'(t))+\frac{\ell(t)}{n}+\ell(t)u(t)\right)dt \\
&= \int_0^\omega \varsigma(p)(t)\ell(t)u(t)dt \\
&= r\int_0^\omega \varsigma(p)(t)\ell(t)dt,
\end{aligned}$$

where we used the assumption

$$\int_0^\omega \varsigma(p)(t)\ell(t)dt > 0.$$

Therefore, we obtain

$$\min_{t\in\mathbb{R}}\varsigma(p)(t)u'(t) \le r\int_0^\omega \varsigma(p)(t)\ell(t)dt,$$

which implies (3.12) holds.

The remaining part of the proof the same as in Theorem 3.2. □

By Theorem 3.2 and Corollary 3.3, we obtain the following conclusion.

Corollary 3.5. *Assume that* $\bar{p}=0$, $\int_0^\omega \varsigma(p)(t)\ell(t)dt>0$, *inequality* (2.4), *condition* (F_1) *and* $\gamma_*\ge 0$ *hold.*

(i) If $\eta<1$, *then equation* (1.4) *has at least one positive* ω*-periodic solution for each* $\mu>0$.

(ii) If $\eta\ge 1$, *then equation* (1.4) *has at least one positive* ω*-periodic solution for each* $0<\mu<\mu_1$, *where* μ_1 *is a positive constant.*

Remark 3.6. If the nonlinear term in equation (1.3) is

$$f(t,u,y) = (1-\alpha)\frac{y^2}{u} + \frac{e(t)}{\alpha}u^{1-\alpha}.$$

From Corollary 3.3, it is easy to verify that there is no continuous function $\phi_L(t)$ such that $f(t,u,y)\ge\phi_L(t)$, for all $(t,u,y)\in[0,\omega]\times\mathbb{R}^+\times\mathbb{R}$. Hence, condition (H_1) is not satisfied. In order to get around condition (H_1), we have to study the case $\gamma_*>0$ by applications of Theorem 3.2.

Remark 3.7. It is worth mentioning that, Cheng and Ren [6] investigated the following damped equation:

$$x'' + p(t)x' + q(t)x = f(t,x) + e(t), \tag{3.14}$$

where f has a repulsive singularity at the origin. In this paper, the singular term of equation (1.1) is sign-changing, which exhibits an indefinite singularity. This makes its study more complex compared to equation (3.14). Therefore, the results of this paper can be regarded as a generalization and refinement of those in [6].

3.2 The case $\gamma_* > 0$

Theorem 3.8. *Assume that conditions (A), (H_2), (H_3) and $0 \le \bar{p}\omega < 1$ hold. If $\gamma_* > 0$, then equation (1.4) has at least one positive ω-periodic solution u with $u(t) > \gamma(t)$ for all $t \in [0,\omega]$ and $0 < \|u - \gamma\| < r$.*

Proof. We follow the same strategy and notations as in the proof of Theorem 3.2. Next, we consider that $u_n(t) + \gamma(t)$ have a uniform positive lower bound, i.e., there exists a constant $\vartheta_1 > 0$, independent of $n \in N_0$, such that

$$\min_{t\in[0,\omega]} \{u_n(t) + \gamma(t)\} \ge \vartheta_1,$$

for all $n \in N_0$.

Because $u_n(t) + \gamma(t) \ge \frac{1}{n} > 0$ for all $n > n_0$, $t \in \mathbb{R}$, it follows from (H_2) and (3.10) that

$$\begin{aligned} u_n(t) + \gamma(t) &= \int_0^\omega G(t,s) f_n(s, u_n(s) + \gamma(s), u_n'(s) + \gamma'(s))ds + \gamma(t) + \frac{1}{n} \\ &\ge \int_0^\omega G(t,s) f(s, u_n(s) + \gamma(s), u_n'(s) + \gamma'(s))ds + \gamma(t) \\ &\ge \gamma_* := \vartheta_1, \end{aligned}$$

since $\gamma_* > 0$. So we have $u_n(t) + \gamma(t) \ge \vartheta_1$. The proof left is the same as Theorem 3.2. □

By Theorem 3.8, we get the following conclusion.

Corollary 3.9. *Assume that condition (A), $0 \le \bar{p}\omega < 1$ and $\gamma_* > 0$ hold. Furthermore, assume that the nonlinear term f satisfies the following condition:*

(F_2) *There exist positive constants κ, ν, μ', η, ϵ with $\epsilon \le \nu + 1$ such that*

$$f(t,u,y) = \kappa \frac{|y|^\epsilon}{u^\nu} + \mu' e(t) u^\eta, \quad \text{for all } (t,u,y) \in [0,\omega] \times (0,+\infty) \times \mathbb{R}.$$

- *(i) If $\eta < 1$, then equation (1.4) has at least one positive ω-periodic solution for each $\mu' > 0$.*
- *(ii) If $\eta \ge 1$, then equation (1.4) has at least one positive ω-periodic solution for each $0 < \mu' < \mu_1'$, where μ_1' is a positive constant.*

Proof. We apply Theorem 3.8, and follow the same strategy and notations as in the proof of Corollary 3.3. Take

$$g(u) = \frac{1}{u^\nu}, \quad \varphi(u) = u^\eta, \quad k(t) = \mu' e(t), \quad \varrho(|y|) = \kappa |y|^\epsilon.$$

Then condition (H_2) is satisfied and the existence condition (H_3) becomes

$$\mu' < \frac{r(\sigma r + \gamma_*)^\nu - \Lambda^* \kappa \left| \frac{r\bar{\ell}\omega}{1-\bar{p}\omega} + \gamma'^* \right|^\epsilon}{\Lambda^* \|e\| (r+\gamma^*)^{\eta+\nu}}$$

for $r > 0$. Therefore, equation (1.4) has at least one positive ω-periodic solution for

$$0 < \mu' < \mu'_1 := \sup_{r>0} \frac{r(\sigma r + \gamma_*)^\nu - \Lambda^* \kappa \left| \frac{r\bar{\ell}\omega}{1-\bar{p}\omega} + \gamma'^* \right|^\epsilon}{\Lambda^* \|e\| (r+\gamma^*)^{\eta+\nu}}. \qquad \square$$

Similar to the proof of Theorem 3.8, we get the following conclusions.

Theorem 3.10. *Assume that $\bar{p} = 0$, $\int_0^\omega \varsigma(p)(t)\ell(t)dt > 0$, inequality (2.4), conditions (H_2), (H_4) hold. If $\gamma_* > 0$, then equation (1.4) has at least one positive ω-periodic solution u with $u(t) > \gamma(t)$ for all $t \in [0, \omega]$ and $0 < \|u - \gamma\| < r$.*

By Corollary 3.9 and Theorem 3.10, we obtain the following conclusion.

Corollary 3.11. *Assume that $\bar{p} = 0$, $\int_0^\omega \varsigma(p)(t)\ell(t)dt > 0$, inequality (2.4), condition (F_2) and $\gamma_* > 0$ hold.*

(i) If $\eta < 1$, then equation (1.4) has at least one positive ω-periodic solution for each $\mu' > 0$.

(ii) If $\eta \geq 1$, then equation (1.4) has at least one positive ω-periodic solution for each $0 < \mu' < \mu'_1$, where μ'_1 is a positive constant.

Corollary 3.12. *Let the nonlinear term in equation (1.3) be*

$$f(t, u, y) = (1-\alpha)\frac{y^2}{u} + \frac{e(t)}{\alpha} u^{1-\alpha}.$$

Assume that condition (A), $0 \leq \bar{p}\omega < 1$ and $\gamma_* > 0$ hold. Then equation (1.1) has at least one positive ω-periodic solution.

Proof. We apply Corollary 3.9. Take

$$\kappa = 1-\alpha, \quad \epsilon = 2, \quad \nu = 1, \quad \mu' = \frac{1}{\alpha}, \quad \eta = 1-\alpha.$$

Then condition (F_2) is satisfied. Since $\eta = 1 - \alpha < 1$, equation (1.1) has at least one positive ω-periodic solution. $\square$

Corollary 3.13. *Let the nonlinear term in equation (1.3) be*

$$f(t, u, y) = (1-\alpha)\frac{y^2}{u} + \frac{e(t)}{\alpha} u^{1-\alpha}.$$

Assume that $\bar{p} = 0$, $\int_0^\omega \varsigma(p)(t)\ell(t)dt > 0$, inequality (2.4) and $\gamma_* > 0$ hold. Then equation (1.1) has at least one positive ω-periodic solution.

3.3 The case $p(t) \equiv 0$

Next, we prove the existence of a positive periodic solution for equation (1.2). We write equation (1.2) as

$$u'' + \ell(t)u = f(t,u,u') + m(t). \tag{3.15}$$

By Theorem 3.2, we obtain the following conclusion.

Theorem 3.14. *Assume that inequality (2.4), conditions (H_1), (H_2) and $\gamma_* \geq 0$ hold. Furthermore, assume that the following condition is satisfied:*

(H_5) *there exists a positive number $r > 0$ such that*

$$\frac{r}{g(\sigma r + \gamma_*)\left(\varrho\left(|r\bar{\ell}\omega + \gamma'^*|\right) + \frac{\|k\|\varphi(r+\gamma^*)}{g(r+\gamma^*)}\right)} > \Lambda^*.$$

Then equation (3.15) has at least one positive ω-periodic solution u with $u(t) > \gamma(t)$ for all $t \in [0,\omega]$ and $0 < \|u - \gamma\| < r$.

Proof. Consider the following equation

$$u''(t) + \ell(t)u(t) = f(t, u(t) + \gamma(t), u'(t) + \gamma'(t)). \tag{3.16}$$

We follow the same strategy and notations as the proof of Theorems 3.2 and 3.4. Now, we claim

$$\|u'\| \leq r\bar{\ell}\omega. \tag{3.17}$$

It follows from (3.8) and (3.16) that

$$\begin{aligned}
\|u'\| &\leq \frac{1}{2}\int_0^\omega |u''(t)|dt \\
&\leq \frac{1}{2}\int_0^\omega \left(\lambda f(t, u(t) + \gamma(t), u'(t) + \gamma'(t)) + \frac{\ell(t)}{n} + \ell(t)u(t)\right) dt \\
&\leq \int_0^\omega \ell(t)u(t)dt \\
&\leq r\bar{\ell}\omega,
\end{aligned}$$

since $f(t,u,u') > 0$ and $\ell(t) > 0$.

The proof left is the same as Theorem 3.2. □

By Theorems 3.8–3.14, Corollaries 3.3–3.13, we get the following conclusions.

Corollary 3.15. *Assume that inequality (2.4), condition (F_1) and $\gamma_* \geq 0$ hold.*

(i) If $\eta < 1$, then equation (3.15) has at least one positive ω-periodic solution for each $\mu > 0$.

(ii) If $\eta \geq 1$, then equation (3.15) has at least one positive ω-periodic solution for each

$$0 < \mu < \sup_{r>0} \frac{r(\sigma r + \gamma_*)^\nu - \Lambda^*\kappa(|r\bar{\ell}\omega + \gamma'^*|^\epsilon + 1)}{\Lambda^*\|e\|(r+\gamma^*)^{\eta+\nu}}.$$

Theorem 3.16. *Assume that inequality (2.4), conditions (H_2) and (H_5) hold. If $\gamma_* > 0$, then equation (3.15) has at least one positive ω-periodic solution u with $u(t) > \gamma(t)$ for all $t \in [0,\omega]$ and $0 < \|u - \gamma\| < r$.*

Corollary 3.17. *Assume that inequality* (2.4), *condition* (F_2) *and* $\gamma_* > 0$ *hold.*

(i) If $\eta < 1$, *then equation* (3.15) *has at least one positive* ω*-periodic solution for each* $\mu' > 0$.

(ii) If $\eta \geq 1$, *then equation* (3.15) *has at least one positive* ω*-periodic solution for each*

$$0 < \mu' < \sup_{r>0} \frac{r(\sigma r + \gamma_*)^\nu - \Lambda^* \kappa |r\bar{\ell}\omega + \gamma'^*|^\epsilon}{\Lambda^* \|e\| (r + \gamma^*)^{\eta+\nu}}.$$

Corollary 3.18. *Let the nonlinear term in equation* (3.15) *be*

$$f(t, u, y) = (1 - \alpha)\frac{y^2}{u} + \frac{e(t)}{\alpha} u^{1-\alpha}.$$

Assume that inequality (2.4) *and* $\gamma_* > 0$ *hold. Then equation* (1.2) *has at least one positive* ω*-periodic solution.*

Remark 3.19. It is worth mentioning that when $p(t) \equiv 0$, Chu et al. [11] studied the following equation:

$$x'' + p(t)x' = f(t, x) + e(t),$$

where f has a repulsive singularity at the origin. When $p(t) \equiv 0$, equation (1.1) exhibits an indefinite singularity, and it is evident that our results encompass those of Reference [11].

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References

[1] J. Bravo, P. Torres, Periodic solutions of a singular equation with indefinite weight, *Adv. Nonlinear Stud.* **10**(2010), No. 4, 927–938. https://doi.org/10.1515/ans-2010-0410; MR2683689; Zbl 1232.34064

[2] A. Cabada, *Green's functions in the theory of ordinary differential equations*, SpringerBriefs in Mathematics, Springer, New York, 2014. https://doi.org/10.1007/978-1-4614-9506-2; MR3155625; Zbl 1303.34001

[3] A. Cabada, J. Cid, On the sign of the Green's function associated to Hill's equation with an indefinite potential, *Appl. Math. Comput.* **205**(2008), No. 1, 303–308. https://doi.org/10.1016/j.amc.2008.08.008; MR2466634; Zbl 1161.34014

[4] A. Cabada, J. Á. Cid, B. Máquez-Villamarín, Computation of Green's functions for boundary value problems with *Mathematica*, *Appl. Math. Comput.* **219**(2012), No. 4, 1919–1936. https://doi.org/10.1016/j.amc.2012.08.035; MR2983897; Zbl 1298.34054

[5] Z. Cheng, Y. Qian, Q. Yuan, C. Kong, Ambrosetti–Prodi periodic problem of singular ϕ-Laplacian Rayleigh equations: Theory and numerical bifurcation analysis, *Discrete Contin. Dyn. Syst.* **45**(2025), 3458–3482. https://doi.org/10.3934/dcds.2025027

[6] Z. Cheng, J. Ren, Studies on a damped differential equation with repulsive singularity, *Math. Meth. Appl. Sci.* **36**(2013), No. 8, 983–992. https://doi.org/10.1002/mma.2659; MR3056951; Zbl 1282.34048

[7] Z. Cheng, P. Torres, New existence results of the planar L_p dual Minkowski problem, *J. Geom. Anal.* **34**(2024), No. 2, 1–16. https://doi.org/10.1007/s12220-023-01493-6; MR4678869; Zbl 1535.34036

[8] Z. Cheng, C. Xia, Q. Yuan, Existence and bifurcation of periodic solutions to the L_p-Minkowski problem with indefinite weight, *J. Math. Anal. Appl.* **534**(2024), No. 1, 1–20. https://doi.org/10.1016/j.jmaa.2023.128074; MR4688096; Zbl 1540.34087

[9] J. Chu, N. Fan, P. Torres, Periodic solutions for second order singular damped differential equations, *J. Math. Anal. Appl.* **388**(2012), No. 2, 665–675. https://doi.org/10.1016/j.jmaa.2011.09.061; MR2869777; Zbl 1232.34065

[10] J. Chu, P. Torres, Applications of Schauder's fixed point theorem to singular differential equations, *Bull. Lond. Math. Soc.* **39**(2007), No. 4, 653–660. https://doi.org/10.1112/blms/bdm040; MR2346946; Zbl 1128.34027

[11] J. Chu, P. J. Torres, M. Zhang, Periodic solutions of second order non-autonomous singular dynamical systems, *J. Differential Equations* **239**(2007), No. 1, 196–212. https://doi.org/10.1016/j.jde.2007.05.007; MR2341553; Zbl 1127.34023

[12] J. Á. Cid, G. Propst, M. Tvrdý, On the pumping effect in a pipe/tank flow configuration with friction, *Phys. D* **273/274**(2014), No. 4, 28–33. https://doi.org/10.1016/j.physd.2014.01.010; MR3173092; Zbl 1383.76174

[13] J. Godoy, M. Zamora, A general result to the existence of a periodic solution to an indefinite equation with a weak singularty, *J. Dynam. Differential Equations* **31**(2019), No. 1, 451–468. https://doi.org/10.1007/s10884-018-9704-9; MR3935151; Zbl 1447.34044

[14] A. Granas, J. Dugundji, *Fixed point theory*, Springer Monogr. Math., Springer-Verlag, New York, 2003. https://doi.org/10.1007/978-0-387-21593-8; MR1987179

[15] R. Hakl, P. Torres, On periodic solutions of second-order differential equations with attractive-repulsive singularities, *J. Differential Equations* **248**(2010), No. 1, 111–126. https://doi.org/10.1016/j.jde.2009.07.008; MR2557897; Zbl 1187.34049

[16] R. Hakl, P. Torres, Maximum and antimaximum principles for a second order differential operator with variable coefficients of indefinite sign, *Appl. Math. Comput.* **217**(2011), No. 19, 7599–7611. https://doi.org/10.1016/j.amc.2011.02.053; MR2799774; Zbl 1235.34064

[17] R. Hakl, M. Zamora, On the open problems connected to the results of Lazer and Solimini, *Proc. Roy. Soc. Edinburgh Sect. A* **144**(2014), No. 1, 109–118. https://doi.org/10.1017/S0308210512001862; MR3164538; Zbl 1417.34091

[18] R. Hakl, M. Zamora, Periodic solutions to second-order indefinite singular equations, *J. Differential Equations* **263**(2017), No. 2, 451–469. https://doi.org/10.1007/s12346-022-00583-0; MR4403246; Zbl 1367.34045

[19] W. Han, J. Ren, Some results on second-order neutral functional differential equations with infinite distributed delay, *Nonlinear Anal.* **70**(2009), No. 3, 1393–1406. https://doi.org/10.1016/j.na.2008.02.018; MR2474926; Zbl 1163.34051

[20] D. Jiang, J. Chu, M. Zhang, Multiplicity of positive periodic solutions to superlinear repulsive singular equations, *J. Differential Equations* **211**(2005), No. 2, 282–302. https://doi.org/10.1016/j.jde.2004.10.031; MR2125544; Zbl 1074.34048

[21] A. Lazer, S. Solimini, On periodic solutions of nonlinear differential equations with singularities, *Proc. Amer. Math. Soc.* **99**(1987), No. 1, 109–114. https://doi.org/10.2307/2046279; MR866438; Zbl 0616.34033

[22] S. Li, Y. Wang, Multiplicity of positive periodic solutions to second order singular dynamical systems, *Mediterr. J. Math.* **14**(2017), No. 5, 1–13. https://doi.org/10.1007/s00009-017-1000-0; MR3695372; Zbl 1391.34056

[23] X. Li, Z. Zhang, Periodic solutions for damped differential equations with a weak repulsive singularity, *Nonlinear Anal.* **70**(2009), No. 6, 2395–2399. https://doi.org/10.1016/j.na.2008.03.023; MR2498337; Zbl 1165.34349

[24] S. Lu, X. Yu, Periodic solutions for second order differential equations with indefinite singularities, *Adv. Nonlinear Anal.* **9**(2020), No. 2, 994–1007. https://doi.org/10.1007/s12220-023-01493-6; MR4678869; Zbl 1535.34036

[25] P. Torres, Existence of one signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed theorem, *J. Differential Equations* **190**(2003), No. 1, 643–662. https://doi.org/10.1515/anona-2020-0037; MR4012182; Zbl 1431.34055

[26] P. Torres, Weak singularities may help periodic solutions to exist, *J. Differential Equations* **232**(2007), No. 1, 277–284. https://doi.org/10.1016/j.jde.2006.08.006; MR2281196; Zbl 1116.34036

[27] X. Yu, Y. Song, S. Lu, J. Godoy, On the positive periodic solutions of a class of Liénard equations with repulsive singularities in degenerate case, *J. Differential Equations* **368**(2023), 1–25. https://doi.org/10.1016/j.jde.2023.05.039; MR4597483; Zbl 1526.34021

[28] X. Yu, Q. Yuan, Z. Cheng, Bifurcation and dynamics of periodic solutions to the Rayleigh–Plesset equation: Theory and numerical simulation, *Phys. D* **459**(2024), No. 134045, 1–17. https://doi.org/10.1016/j.physd.2023.134045; MR4683943; Zbl 1543.34043

[29] H. Wang, Positive periodic solutions of singular systems with a parameter, *J. Differential Equations* **249**(2010), No. 12, 2986–3002. https://doi.org/10.1016/j.jde.2010.08.027; MR2737417; Zbl 1364.34032

[30] H. Wang, Positive periodic solutions of singular systems of first order ordinary differential equations, *Appl. Math. Comput.* **218**(2011), No. 5, 2986–3002. https://doi.org/10.1016/j.amc.2011.06.038; MR 2831383; Zbl 1239.34043

[31] Y. Wang, H. Lian, W. Ge, Periodic solutions for a second order nonlinear functional differential equation, *Appl. Math. Lett.* **20**(2007), No. 2, 110–115. https://doi.org/10.1016/j.aml.2006.02.028; MR3847352; Zbl 1151.34056

[32] M. Zhang, Periodic solutions of Liénard equations with singular forces of repulsive type, *J. Math. Anal. Appl.* **203**(1996), No. 1, 254–269. https://doi.org/10.1006/jmaa.1996.0378; MR1412492; Zbl 0863.34039