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Asymptotic stability analysis on a neutral Nicholson's blowflies equation with time-varying delays

Chuangxia Huang✉ [1, 2], **Xiaodan Ding**[2] and **Qian Wang**[3]

[1]College of Science, Hunan University of Science and Engineering,
Yongzhou 425199, Hunan, P. R. China
[2]School of Mathematics and Statistics, Changsha University of Science and Technology,
Changsha 410114, Hunan, P. R. China
[3]School of Finance and Mathematics, Huainan Normal University, Huainan 232038, P. R. China

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Abstract. This study is devoted to the stability issue of a time-varying delayed Nicholson's blowflies equation of neutral type. By the aid of the Lyapunov stability theory and novel analysis techniques, two sharp conditions ensuring separately the global asymptotic stability of the zero equilibrium and positive equilibrium are derived, which completely cover the corresponding ones in the associated non-neutral equation. In addition, a numerical example is taken to support the availability of the theoretical results.

Keywords: Nicholson's blowflies equation, neutral type, time-varying delay, global asymptotic stability.

MSC 2020 Subject Classification: 34K40, 34D23.

1 Introduction

The following Nicholson's blowflies equation

$$w'(t) = -\delta w(t) + \beta w(t-\tau)e^{-aw(t-\tau)}, \tag{1.1}$$

was proposed by Gurney et al. in [6] to describe the dynamics of the Australian sheep-blowfly. Here $w(t)$ labels the size of blowflies population at time t, δ, β, a, τ are positive constants with distinct biological significance, that is, δ signifies the average daily mortality rate among adult blowflies, β quantifies the maximal mean rate of oviposition per day, $\frac{1}{a}$ means the size at which the blowflies population reproduces at its maximum rate, and τ stands for the maturation delay. In the past four decades, extensive research has elucidated the qualitative dynamics and stability characteristics of Eq. (1.1) and its generalized forms (see [1, 8, 9, 11, 14] and the references therein). In particular, if the biological parameters satisfy $\frac{\beta}{\delta} \le 1$ and $1 < \frac{\beta}{\delta} \le e^2$, the global asymptotic stability has been respectively proven for the trivial equilibrium and positive equilibrium to Eq. (1.1) in [1, 8, 9, 11, 14]. Furthermore, Ref. [17] revealed that the

✉Corresponding author. Tel./fax: +860746-6381425. E-mail:huangchuangxia@sina.com

positive equilibrium of Eq. (1.1) admits global attractivity if $\frac{\beta}{\delta} > 1$ and the delay τ is small. Meanwhile, the authors in [19] and [5] established the global asymptotical stability for the positive equilibrium of Eq. (1.1) if $1 < \frac{\beta}{\delta} \le e^2$. It is noteworthy that Yang and So in [18] showed that the positive equilibrium is unstable and Hopf bifurcation appears when $\frac{\beta}{\delta} > e^2$ and the delay τ is large. It is demonstrated from aforementioned findings that $\frac{\beta}{\delta} \le 1$ and $1 < \frac{\beta}{\delta} \le e^2$ are two sharp conditions which respectively assure the global asymptotic stability of trivial equilibrium and positive equilibrium of Eq. (1.1).

In the realistic world, biological populations have complex dynamic characteristics, and the existing population models cannot accurately describe the properties of a population evolution process. Consequently, it is both natural and interesting that biological models should incorporate some information about the derivative of past states to better depict and model the dynamics of such complex population evolution processes. This has drawn extensive attention to neutral type delayed population dynamic models [3, 7, 15, 21]. In particular, due to the effect of the population age distribution, Eq. (1.1) can be commonly generalized to the following neutral functional differential equation (NFDE),

$$(w(t) - cw(t-\alpha))' = -\delta w(t) + c\delta w(t-\alpha) + \beta w(t-\tau)e^{-aw(t-\tau)}, \quad t \in [0, +\infty), \tag{1.2}$$

where $a, \delta, \beta, \alpha, \tau \in (0, +\infty), c \in [0, 1)$. For more details on the biological ecology background and the derivation of Eq. (1.2), one can refer to [2, 12]. It should be pointed out that, if the population function $w(t)$ is continuously differentiable, Eq. (1.2) can be rewritten as the following special case,

$$w'(t) - cw'(t-\alpha) = -\delta w(t) + c\delta w(t-\alpha) + \beta w(t-\tau)e^{-aw(t-\tau)}, \quad t \in [0, +\infty). \tag{1.3}$$

Recently, the dynamics including Hopf bifurcation, the stability of (almost) periodic solutions of Eq. (1.3) and its generalizations have been extensively studied in [10, 12, 13, 15, 20, 22]. Meanwhile, delays in population and ecology models may depend on the time-varying environmental conditions and climate, and hence the neutral delayed Nicholson's blowflies equation (1.2) can be naturally extrapolated to the following non-autonomous form:

$$(w(t) - cw(t-\tau_1(t)))' = -\delta w(t) + c\delta w(t-\tau_1(t)) + \beta w(t-\tau_2(t))e^{-aw(t-\tau_2(t))}, \tag{1.4}$$

where $c \in [0, 1)$, $t \ge t_0 \in \mathbb{R}$, the delay functions $\tau_i(t)$ $(i = 1, 2)$ possess continuity and boundedness. In addition, according to the definition of initial set [23], we always assume that there exists $r_i \in (0, +\infty)$ satisfying

$$\{t - \tau_i(t) : t - \tau_i(t) \le t_0, \ t \ge t_0\} \cup \{t_0\} = [t_0 - r_i, \ t_0], \quad i = 1, 2, \tag{1.5}$$

and

$$r = r_2 \ge r_1, \quad \text{and} \quad \sigma := \min\left\{\inf_{t\in\mathbb{R}} \tau_1(t), \inf_{t\in\mathbb{R}} \tau_2(t)\right\} > 0. \tag{1.6}$$

Obviously, when $\tau_i(t)$ $(i = 1, 2)$ are constants, Eqs. (1.1)–(1.3) are special cases of Eq. (1.4). From the basic theory of functional differential equations (FDEs), it is evident that the initial value conditions for Eqs. (1.2) and (1.4) are fundamentally different, which yields theoretical difficulties in analyzing the dynamic behavior of Eq. (1.4). In addition, the existence of both neutral structure and time-varying delays also brings technical challenges to explore the kinetic properties of the non-autonomous NFDE. Therefore, a natural question is whether one could establish the sharp criteria separately ensuring the global asymptotic stability of trivial and positive equilibria of neutral time-varying delayed Nicholson's blowflies equation (1.4), and such a problem has not been reported up to now.

Inspired by the aforementioned discussions, the primary object of this paper is to establish sharp global asymptotic stability criteria for Eq. (1.4). Specifically, the main contributions of this study can be summarized as follows:

1) A kind of neutral Nicholson's blowflies equation with time-varying delays is proposed, the well-posedness including existence, uniqueness, boundedness and positiveness of solutions to the addressed model is first proved.

2) Global asymptotic stability criteria on the trivial and positive equilibria to Eq. (1.4) are established, respectively, which are sharp and substantially extend the existing stability results of the corresponding non-neutral ones.

3) A numerical example and some comparative analyses are presented to explicate the correctness and innovation of the theoretical findings.

The structure of this paper is outlined below. Section 2 introduces some preliminaries. Sections 3 and 4 establish respectively the global asymptotic stability of zero equilibrium and positive equilibrium. Section 5 affords a numerical example to exhibit effectiveness of the theoretical results. At last, the conclusion is conducted in Section 6.

2 Preliminaries

Firstly, we present some notations which are needed later. For convenience, denote the Banach space supplemented with supremum norm $\|\cdot\|$ by $\Gamma = C([-r,\ 0],\ \mathbb{R})$. Given that $\zeta \geq 0$, $t_0 \in \mathbb{R}$, and $W \in C([t_0 - r, t_0 + \zeta], \mathbb{R})$, then, for any $t \in [t_0, t_0 + \zeta]$, $W_t \in \Gamma$ is interpreted by $W_t(\nu) = W(t + \nu)$, $-r \leq \nu \leq 0$. For $t_0 \in \mathbb{R}$, denote

$$\Gamma_+ = \Big\{\varphi \in \Gamma \big| \varphi(0) - c\varphi(-\tau_1(t_0)) \geq 0,\ \varphi(\theta) \geq 0 \ \text{ for all } \ \theta \in [-r,\ 0]\Big\},$$

and let $w_t(t_0, \varphi)(w(t; t_0, \varphi))$ be the solution of Eq. (1.4) under the admissible initial conditions

$$w_{t_0} = \varphi, \qquad \varphi \in \Gamma_+. \tag{2.1}$$

Clearly, Eq. (1.4) only possesses a trivial equilibrium $N_0 = 0$ when $\frac{\beta}{\delta(1-c)} \leq 1$ and a unique positive equilibrium $N_* = \frac{1}{a} \ln \frac{\beta}{\delta(1-c)} \leq \frac{2}{a}$ when $1 < \frac{\beta}{\delta(1-c)} \leq e^2$. For simplicity of notation, we designate

$$w(t) = w(t; t_0, \varphi), \quad x(t) = w(t) - cw(t - \tau_1(t)),$$

and

$$y(t) = x(t) - (1-c)N_* = (w(t) - N_*) - c(w(t - \tau_1(t)) - N_*).$$

Moreover, for $\varphi \in \Gamma_+$, we extend the initial value function as follows:

$$\varphi(\theta - \tau_1(\theta)) = \varphi(-r) \quad \text{for all } \theta \in [-r,\ 0] \text{ and } \theta - \tau_1(\theta) < -r. \tag{2.2}$$

Now, we investigate some basic properties of the solutions for Eq. (1.4).

Lemma 2.1. *Let $\varphi \in \Gamma_+$. Then Eq. (1.4) has a unique, non-negative, ultimately bounded solution $w_t(t_0, \varphi)$ on $[t_0,\ +\infty)$. Meanwhile, for $\varphi \in \Gamma_+ \setminus \{0\}$, $w_t(t_0, \varphi)$ is ultimately positive.*

Proof. For any $t \in [t_0, t_0 + \sigma]$, multiplying both sides of Eq. (1.4) by $e^{\delta t}$ and integrating from t_0 to t, together with (2.1), gives us

$$x(t) = (\varphi(0) - c\varphi(-\tau_1(t_0)))e^{-\delta(t-t_0)} + \beta e^{-\delta t}\int_{t_0}^{t} \varphi(s - \tau_2(s) - t_0)e^{-a\varphi(s-\tau_2(s)-t_0)}e^{\delta s}ds \geq 0$$

and

$$w(t) = x(t) + c\varphi(t - \tau_1(t) - t_0) \geq 0.$$

Similarly, applying the step-by-step method, the case $x(t) \geq 0$ and $w(t) \geq 0$ on $[t_0 + \sigma, t_0 + 2\sigma]$ holding as follows. Therefore, $x(t) \geq 0$ and $w(t) \geq 0$ for all $t \geq t_0$. Consequently, $w_t(t_0, \varphi)$ exists and is unique on $[t_0, +\infty)$. In addition, $w_t(t_0, \varphi) \geq 0$ for all $t \in [t_0, +\infty)$.

On the other hand, the fact $\sup_{u\geq 0} ue^{-u} = \frac{1}{e}$ leads to

$$\begin{aligned} x(t) =& (\varphi(0) - c\varphi(-\tau_1(t_0)))e^{-\delta(t-t_0)} \\ &+ e^{-\delta t}\int_{t_0}^{t} \beta w(s - \tau_2(s))e^{-aw(s-\tau_2(s))}e^{\delta s}ds \\ \leq& (\varphi(0) - c\varphi(-\tau_1(t_0)))e^{-\delta(t-t_0)} + \frac{\beta}{ae}e^{-\delta t}\int_{t_0}^{t} e^{\delta s}ds \\ =& (\varphi(0) - c\varphi(-\tau_1(t_0)))e^{-\delta(t-t_0)} + \frac{\beta}{\delta ae}\left[1 - e^{-\delta(t-t_0)}\right] \quad \text{for all } t \in [t_0, +\infty), \end{aligned}$$

and hence

$$\limsup_{t\to+\infty} x(t) \leq \frac{\beta}{\delta ae},$$

which, together with the following fact

$$\limsup_{t\to+\infty} w(t) = \limsup_{t\to+\infty}[x(t) + cw(t - \tau_1(t))] \leq \limsup_{t\to+\infty} x(t) + c\limsup_{t\to+\infty} w(t),$$

gives us

$$\limsup_{t\to+\infty} w(t) \leq \frac{1}{1-c}\limsup_{t\to+\infty} x(t) \leq \frac{1}{1-c}\frac{\beta}{\delta ae}.$$

This implies that $w(t)$ is ultimately uniformly bounded.

Now, let $\varphi \in \Gamma_+ \setminus \{0\}$. We state that there is a $t_1 \in [t_0, +\infty)$ satisfying $x(t_1) > 0$. Otherwise, $x(t) \equiv 0$ on $[t_0, +\infty)$. Consequently, $\varphi \not\equiv 0$, this allows one can choose $t^* \in [t_0 - r, t_0)$ satisfying $w(t^*) = \varphi(t^* - t_0) > 0$. This, together with (1.5) and (1.6), shows that there is $\xi^{**} \in (t_0, +\infty)$ such that

$$t^* = \xi^{**} - \tau_2(\xi^{**}) \quad \text{and} \quad w(\xi^{**} - \tau_2(\xi^{**})) = w(t^*) = \varphi(t^* - t_0) > 0,$$

which yields

$$\begin{aligned} 0 =& x'(\xi^{**}) \\ =& -\delta x(\xi^{**}) + \beta w(\xi^{**} - \tau_2(\xi^{**}))e^{-aw(\xi^{**}-\tau_2(\xi^{**}))} \\ =& \beta w(\xi^{**} - \tau_2(\xi^{**}))e^{-aw(\xi^{**}-\tau_2(\xi^{**}))} \\ >& 0. \end{aligned}$$

This is a contradiction, and hence

$$\begin{aligned} w(t) &\geq x(t) \\ &= x(t_1)e^{-\delta(t-t_1)} + e^{-\delta t}\int_{t_1}^{t} \beta w(s-\tau_2(s))e^{-aw(s-\tau_2(s))}e^{\delta s}ds \\ &> 0 \quad \text{for all } \ t \in [t_1, \ +\infty). \end{aligned}$$

In other words, $x(t)$ and $w(t)$ are ultimately positive. □

To establish the next main theorems, we make use of the following results.

Lemma 2.2. *If $W \in C([t_0, \ +\infty), [0, \ +\infty))$ is bounded, and there exist constants $p, q \in \mathbb{R}$ satisfying that $0 \leq q < 1$ and*

$$\lim_{t\to+\infty} \left[W(t) - qW(t-\tau_1(t))\right] = p,$$

then

$$\lim_{t\to+\infty} W(t) = \frac{p}{1-q}.$$

Proof. Let $A = \liminf_{t\to+\infty} W(t)$ and $B = \limsup_{t\to+\infty} W(t)$, then $0 \leq A \leq B$, and there exist two monotonically increasing sequences $\{Z_n^1\}_{n\geq1}$ and $\{Z_n^2\}_{n\geq1}$ such that

$$\lim_{n\to+\infty} Z_n^1 = +\infty, \quad \lim_{n\to+\infty} Z_n^2 = +\infty, \quad \lim_{n\to+\infty} W(Z_n^1) = A, \quad \lim_{n\to+\infty} W(Z_n^2) = B,$$

and

$$\lim_{n\to+\infty} W(Z_n^i - \tau_1(Z_n^i)) = C_i \in [A, \ B], \quad i = 1, 2.$$

Therefore,

$$p = \lim_{n\to+\infty} \left[W(Z_n^1) - qW(Z_n^1 - \tau_1(Z_n^1))\right] = A - qC_1 \leq A(1-q)$$

and

$$p = \lim_{n\to+\infty} [W(Z_n^2) - qW(Z_n^2 - \tau_1(Z_n^2))] = B - qC_2 \geq B(1-q),$$

which assures that $\lim_{t\to+\infty} W(t) = A = B = \frac{p}{1-q}$. □

Lemma 2.3 (see [4, Lemma 2.3])**.** *If $r^* \in (0, 2]$, then*

$$\left|\mu e^{-\mu} - r^* e^{-r^*}\right| < e^{-r^*}|\mu - r^*| \quad \text{for all } \mu > 0 \text{ and } \mu \neq r^*. \tag{2.3}$$

3 Global asymptotic stability of $N_0 = 0$ when $\beta \leq \delta(1-c)$

In this section, we deduce a new criterion for global asymptotic stability on $N_0 = 0$ of Eq. (1.4), which extends the previous stability findings in its special cases. This result can be stated in the following theorem.

Theorem 3.1. *If $\beta \leq \delta(1-c)$, then the trivial equilibrium N_0 of Eq. (1.4) is globally asymptotically stable. Particularly, when $\beta < \delta(1-c)$, N_0 is globally exponentially stable.*

Proof. First, we demonstrate the stability of $N_0 = 0$. For any $\varepsilon > 0$, denote $H = (1-c)\varepsilon$ and $\|\varphi\| < H$ with $\varphi \in \Gamma_+$. We shall demonstrate that $w(t) < \varepsilon$ for all $t \in [t_0 - r,\ +\infty)$. Then, by (2.1), (2.2), and Lemma 2.1, we have

$$x(t) \geq 0 \quad \text{for any } t \geq t_0,$$

and

$$x(t) = w(t) - cw(t-\tau_1(t)) = \varphi(t-t_0) - c\varphi(t-\tau_1(t)-t_0) < H \text{ for any } t \in [t_0 - r,\ t_0].$$

Now, we claim that

$$0 \leq x(t) < H \quad \text{for any } t > t_0. \tag{3.1}$$

Otherwise, there exists $G^* > t_0$ such that

$$x(G^*) = H \quad \text{and} \quad x(t) < H \quad \text{for any } t \in [t_0 - r,\ G^*), \tag{3.2}$$

it follows that

$$\begin{aligned} w(\theta) &= x(\theta) + cw(\theta - \tau_1(\theta)) \\ &\leq x(\theta) + c \sup_{\min\limits_{s\in[t_0-r,\theta]}(s-\tau_1(s)) \leq u \leq \theta} w(u - \tau_1(u)) \\ &\leq x(\theta) + c \sup_{t_0 - r \leq u \leq t} w(u) \\ &< H + c \sup_{t_0-r\leq u\leq t} w(u), \end{aligned}$$

for any $\theta \in [t_0 - r,\ t]$ and $t \in [t_0 - r,\ G^*)$. Hence

$$w(t) \leq \sup_{t_0-r\leq u\leq t} w(u) < \frac{H}{1-c} \quad \text{for any } t \in [t_0 - r,\ G^*). \tag{3.3}$$

Moreover, it indicates from Eqs. (1.4) and (3.3) that

$$\begin{aligned} 0 &\leq x'(G^*) \\ &= -\delta x(G^*) + \beta w(G^* - \tau_2(G^*))e^{-aw(G^*-\tau_2(G^*))} \\ &\leq -\delta x(G^*) + \beta w(G^* - \tau_2(G^*)) \\ &< H\Big(-\delta + \frac{\beta}{1-c}\Big) \\ &\leq 0, \end{aligned}$$

we yield the contradiction and find that assertion (3.1) is true. Utilizing a similar approach as in the identification of (3.3), we obtain

$$w(t) \leq \sup_{t_0-r\leq u\leq t} w(u) < \frac{H}{1-c} = \varepsilon \quad \text{for any } t \geq t_0,$$

which shows that N_0 is stable.

Second, we prove the global attractivity of N_0. Since

$$\limsup_{t\to+\infty} w(t) \leq \limsup_{t\to+\infty} x(t) + c\limsup_{t\to+\infty} w(t-\tau_1(t)) \leq \limsup_{t\to+\infty} x(t) + c\limsup_{t\to+\infty} w(t),$$

one has

$$\limsup_{t\to+\infty} w(t) \le \frac{1}{1-c} \limsup_{t\to+\infty} x(t). \tag{3.4}$$

It suffices to verify $\limsup_{t\to+\infty} x(t) = 0$. We find from the fluctuation lemma [16, Lemma A.1.] that there exists a monotonically increasing sequence $\{Z_n\}_{n\ge1}$ agreeing with

$$\lim_{n\to+\infty} Z_n = +\infty, \quad \lim_{n\to+\infty} x'(Z_n) = 0, \quad \text{and} \quad \lim_{n\to+\infty} x(Z_n) = \limsup_{t\to+\infty} x(t),$$

and $\lim_{n\to+\infty} w(Z_n - \tau_2(Z_n)) = \eta$ holds. Then (3.4) yields

$$\eta = \lim_{n\to+\infty} w(Z_n - \tau_2(Z_n)) \le \limsup_{t\to+\infty} w(t) \le \frac{1}{1-c} \limsup_{t\to+\infty} x(t). \tag{3.5}$$

Taking the limit on both sides of the following equation:

$$x'(Z_n) = -\delta x(Z_n) + \beta w(Z_n - \tau_2(Z_n))e^{-aw(Z_n-\tau_2(Z_n))}$$

produces

$$0 = -\delta \limsup_{t\to+\infty} x(t) + \beta\eta e^{-a\eta}. \tag{3.6}$$

For the sake of contradiction, assume that $\limsup_{t\to+\infty} x(t) > 0$, (3.5) and (3.6) give us

$$0 < \eta \le \frac{1}{1-c} \limsup_{t\to+\infty} x(t),$$

and the contradiction

$$0 = -\delta \limsup_{t\to+\infty} x(t) + \beta\eta e^{-a\eta} < \left(-\delta + \beta\frac{1}{1-c}\right) \limsup_{t\to+\infty} x(t) \le 0,$$

and thus

$$\lim_{t\to+\infty} w(t) = N_0 = 0.$$

Finally, we show the global exponential stability of N_0 when $\beta < \delta(1-c)$, we only need to prove its globally exponential attractivity. To do this, we pick a sufficiently small $\lambda > 0$ obeying

$$1 - ce^{\lambda r} > 0 \quad \text{and} \quad -(\delta - \lambda) + \frac{\beta e^{\lambda r}}{1 - ce^{\lambda r}} < 0. \tag{3.7}$$

Since $\varphi(\theta - \tau_1(\theta)) = \varphi(-r)$ for all $\theta \in [-r,\ 0]$ with $\theta - \tau_1(\theta) < -r$, we can denote

$$x(t)e^{\lambda t} = x(t_0 - r)e^{\lambda(t_0 - r)} \quad \text{for any } t \in (-\infty,\ t_0 - r],$$

$$z(t) = x(t)e^{\lambda t} \quad \text{for any } t \in [t_0 - r,\ +\infty),$$

and

$$M_\varphi = \max_{t_0 - r \le t \le t_0} x(t)e^{\lambda t} = \max_{t_0 - r \le t \le t_0} \left[\varphi(t - t_0) - c\varphi(t - \tau_1(t) - t_0)\right]e^{\lambda t}.$$

Clearly, $z(t) < M_\varphi + 1$ for any $t \in [t_0 - r,\ t_0]$. We hence claim

$$z(t) < M_\varphi + 1 \quad \text{for all } t \ge t_0. \tag{3.8}$$

Otherwise, there exists $t_2 > t_0$ satisfying

$$z(t_2) = M_\varphi + 1, \quad 0 \le z(t) < M_\varphi + 1 \quad \text{for all } t \in [t_0 - r, t_2). \tag{3.9}$$

For any $\theta \in [t_0 - r,\ t]$ and $t \in [t_0 - r,\ t_2)$, we have

$$\begin{aligned} w(\theta)e^{\lambda\theta} &= x(\theta)e^{\lambda\theta} + cw(\theta - \tau_1(\theta))e^{\lambda(\theta-\tau_1(\theta))}e^{\lambda\tau_1(\theta)} \\ &\le z(\theta) + c \sup_{\min\limits_{s\in[t_0-r,\theta]}(s-\tau_1(s))\le u\le\theta} w(u-\tau_1(u))e^{\lambda(u-\tau_1(u))}e^{\lambda r} \\ &\le z(\theta) + c \sup_{t_0-r\le u\le t} w(u)e^{\lambda u}e^{\lambda r} \\ &< M_\varphi + 1 + c \sup_{t_0-r\le u\le t} w(u)e^{\lambda u}e^{\lambda r}, \end{aligned}$$

and thus

$$w(t)e^{\lambda t} \le \sup_{t_0-r\le u\le t} w(u)e^{\lambda u} < \frac{M_\varphi+1}{1-ce^{\lambda r}} \quad \text{for any } t \in [t_0 - r,\ t_2). \tag{3.10}$$

It follows immediately from (1.4), (3.7), (3.9) and (3.10) that

$$\begin{aligned} 0 &\le z'(t_2) \\ &= x'(t_2)e^{\lambda t_2} + \lambda x(t_2)e^{\lambda t_2} \\ &= -(\delta-\lambda)x(t_2)e^{\lambda t_2} + \beta w(t_2 - \tau_2(t_2))e^{\lambda(t_2-\tau_2(t_2))}e^{\lambda\tau_2(t_2)}e^{-aw(t_2-\tau_2(t_2))} \\ &< \Big[-(\delta-\lambda) + \frac{\beta e^{\lambda r}}{1-ce^{\lambda r}}\Big](M_\varphi+1) \\ &< 0, \end{aligned}$$

which results in a contradiction, and claim (3.8) holds. Furthermore, by the same manner as those in establishing (3.10), it follows from (3.8) that

$$w(t) \le \frac{M_\varphi+1}{1-ce^{\lambda r}}e^{-\lambda t} \quad \text{for any } t \ge t_0,$$

which confirms the global exponential attractivity of N_0. The evidence of Theorem 3.1 is finished. □

4 Global asymptotic stability of $N_* = \frac{1}{a}\ln\frac{\beta}{\delta(1-c)}$ when $1 < \frac{\beta}{\delta(1-c)} \le e^2$

We are now ready to derive the conditions under which the positive equilibrium N_* of Eq. (1.4) has global asymptotic stability.

Theorem 4.1. *If $1 < \frac{\beta}{\delta(1-c)} \le e^2$, then Eq. (1.4) admits a unique positive equilibrium $N_* = \frac{1}{a}\ln\frac{\beta}{\delta(1-c)}$, which possesses global asymptotic stability.*

Proof. Clearly, Eq. (1.4) admits a unique positive equilibrium $N_* = \frac{1}{a}\ln\frac{\beta}{\delta(1-c)} \le \frac{2}{a}$. Now, we reveal the stability of N_*. For any $\varepsilon > 0$, let $\bar{H} = \frac{(1-c)}{1+c}\varepsilon$ and $\|\varphi - N_*\| < \bar{H}$ with $\varphi \in \Gamma_+$, it suffices to check that

$$|w(t) - N_*| < \varepsilon \quad \text{for all } t \in [t_0 - r,\ +\infty).$$

In fact, one finds from (2.1) and (2.2) that

$$\begin{aligned}|y(t)| &= |x(t)-(1-c)N_*| \\ &= |w(t)-N_*-c(w(t-\tau_1(t))-N_*)| \\ &= |\varphi(t-t_0)-N_*-c(\varphi(t-\tau_1(t)-t_0)-N_*)| \\ &< (1+c)\bar{H} \quad \text{for all } t\in[t_0-r,\ t_0],\end{aligned}$$

which allows us to assert that

$$|y(t)| < (1+c)\bar{H} \quad \text{for any } t > t_0. \tag{4.1}$$

Otherwise, there exists $S_2 > t_0$ satisfying

$$|y(S_2)| = (1+c)\bar{H} \quad \text{and} \quad |y(t)| < (1+c)\bar{H} \quad \text{for any } t\in[t_0-r,\ S_2), \tag{4.2}$$

which gives us

$$\begin{aligned}|w(\theta)-N_*| &= |y(\theta)+c(w(\theta-\tau_1(\theta))-N_*)| \\ &\le |y(\theta)|+c \sup_{\min\limits_{s\in[t_0-r,\theta]}(s-\tau_1(s))\le u\le\theta} |w(u-\tau_1(u))-N_*| \\ &\le |y(\theta)|+c\sup_{t_0-r\le u\le t}|w(u)-N_*| \\ &< (1+c)\bar{H}+c\sup_{t_0-r\le u\le t}|w(u)-N_*|,\end{aligned} \tag{4.3}$$

for any $\theta\in[t_0-r,\ t]$ and $t\in[t_0-r,\ S_2)$. A simple calculation yields

$$|w(t)-N_*| \le \sup_{t_0-r\le u\le t}|w(u)-N_*| < \frac{(1+c)\bar{H}}{1-c} \quad \text{for any } t\in[t_0-r,\ S_2). \tag{4.4}$$

In view of Eq. (1.4), $y(t)$ satisfies

$$y'(t) = -\delta y(t)+\beta\Big[w(t-\tau_2(t))e^{-aw(t-\tau_2(t))}-N_*e^{-aN_*}\Big], \quad t\ge t_0\in\mathbb{R}. \tag{4.5}$$

By calculating the Dini derivative of $|y(t)|$, (4.2), (4.4) and (4.5) and Lemma 2.3 imply that

$$\begin{aligned}0 &\le D^-|y(t)|\big|_{t=S_2} \\ &\le -\delta|y(S_2)|+\beta|w(S_2-\tau_2(S_2))e^{-aw(S_2-\tau_2(S_2))}-N_*e^{-aN_*}| \\ &\le -\delta|y(S_2)|+\beta e^{-aN_*}|w(S_2-\tau_2(S_2))-N_*| \\ &< \Big[-\delta+\beta e^{-aN_*}\frac{1}{1-c}\Big](1+c)\bar{H} \\ &= 0,\end{aligned}$$

it follows a contradiction and implies that (4.1) is true. One can follow a similar argument as above to deduce that (4.4) holds, we obtain

$$|w(t)-N_*| \le \sup_{t_0-r\le u\le t}|w(u)-N_*| < \frac{(1+c)\bar{H}}{1-c} = \varepsilon \quad \text{for any } t\ge t_0.$$

Hence, the positive equilibrium N_* admits stability.

Next, we prove the global attractivity of N_*. Label

$$L = \limsup_{t\to+\infty} y(t) \quad \text{and} \quad l = \liminf_{t\to+\infty} y(t).$$

In view of Lemma 2.2, the global attractivity of N_* is equivalent to discover

$$\max\{|L|, |l|\} = 0. \tag{4.6}$$

To prove (4.6), we argue by contradiction. Assume $\max\{|L|, |l|\} = L > 0$ (the case of $\max\{|L|, |l|\} = -l > 0$ is similar). Based on the fluctuation lemma [16, Lemma A.1.], we know that there exists a sequence $\{F_k\}_{k=1}^{+\infty}$ satisfying

$$\lim_{k\to+\infty} F_k = +\infty, \quad \lim_{k\to+\infty} y(F_k) = L, \quad \text{and} \quad \lim_{k\to+\infty} y'(F_k) = 0. \tag{4.7}$$

In general, we may suppose that $\lim\limits_{k\to+\infty} w(F_k - \tau_2(F_k))$ exists as well.

If $\lim_{k\to+\infty} w(F_k - \tau_2(F_k)) \neq 0$, it follows from (2.3), (4.5) and (4.7) that

$$\begin{aligned}
0 =& \lim_{k\to+\infty} y'(F_k) \\
=& -\delta \lim_{k\to+\infty} y(F_k) + \beta\Big[\lim_{k\to+\infty} w(F_k - \tau_2(F_k)) e^{-a\lim_{k\to+\infty} w(F_k-\tau_2(F_k))} - N_* e^{-aN_*}\Big] \\
\leq& -\delta \lim_{k\to+\infty} y(F_k) + \frac{\beta}{a}\Big|a\lim_{k\to+\infty} w(F_k - \tau_2(F_k)) e^{-a\lim_{k\to+\infty} w(F_k-\tau_2(F_k))} - aN_* e^{-aN_*}\Big| \\
<& -\delta \lim_{k\to+\infty} y(F_k) + \beta e^{-aN_*}\lim_{k\to+\infty}\big|w(F_k - \tau_2(F_k)) - N_*\big| \\
\leq& L\Big[-\delta + \beta\frac{1}{1-c}e^{-aN_*}\Big] \\
=& 0.
\end{aligned}$$

This is a contradiction.

Likewise, if $\lim_{k\to+\infty} w(F_k - \tau_2(F_k)) = 0$, one sees readily the above contradiction. Accordingly

$$\max\{|L|, |l|\} = 0,$$

this completes the proof of Theorem 4.1. □

Remark 4.2. In references [12, 20], the stability analysis of (1.3) requires the solution $w(t)$ to be a differentiable function, with its derivative $w'(t-\alpha)$ existing. This imposes restrictions on the range of admissible initial functions and consequently narrows the scope of potential solutions. Moreover, the methodologies and strategies proposed in the aforementioned literature cannot be directly applied to analyze the stability and attractiveness of the non-autonomous neutral delayed Nicholson's blowflies equation (1.4) under the initial condition (2.1). Additionally, in contrast to the initial function set Γ in [3], which depends on the coefficient of the death term, the set Γ_+ introduced in this manuscript is independent of this coefficient. This distinction offers a more comprehensive and versatile framework for understanding the stability and attractiveness of solutions across different initial value ranges, providing deeper insights into the problem.

Remark 4.3. Evidently, when $c = 0$, Eq. (1.4) reduces to the standard Nicholson's blowflies equation Eq. (1.1), which has received widespread attention and in-depth research. Especially, the sharp criteria $\frac{\beta}{\delta} \le 1$ and $1 < \frac{\beta}{\delta} \le e^2$ for respectively ensuring the global asymptotic stability of trivial and positive equilibria were rigidly established in [5, 11, 17–19], which are special cases of the corresponding ones in Theorems 3.1 and 4.1 if $c = 0$. In this sense, global asymptotic stability criteria $\frac{\beta}{\delta(1-c)} \le 1$ and $1 < \frac{\beta}{\delta(1-c)} \le e^2$ established are also sharp to time-varying delayed Nicholson's blowflies equation of neutral type (1.4). On the other hand, since the delays are time varying, Eq. (1.4) is non-autonomous, which has brought theoretical and technical difficulties to the dynamic study of Eq. (1.4), and the stability problem for such a neutral model has not received attention until now. Hence, the theoretical results established in this paper are substantially new and enrich the theory of functional differential equations to some extent.

5 Numerical examples

In this section, numerical examples with simulation are afforded to illustrate the main theoretical findings.

Example 5.1. Consider a neutral time-varying delayed Nicholson's blowflies model with different parameters,

$$\begin{aligned}\Big[w(t) - \frac{1}{2}w(t-e^{-|\sin 2t|})\Big]' = &-4(w(t) - \frac{1}{2}w(t-e^{-|\sin 2t|}))\\ &+ w(t-2e^{-|\sin 2t|})e^{-2w(t-2e^{-|\sin 2t|})}, \quad t \ge t_0 = 0, \end{aligned} \tag{5.1}$$

$$\begin{aligned}\Big[w(t) - \frac{1}{2}w(t-e^{-|\sin 2t|})\Big]' = &-2(w(t) - \frac{1}{2}w(t-e^{-|\sin 2t|}))\\ &+ ew(t-2e^{-|\sin 2t|})e^{-2w(t-2e^{-|\sin 2t|})}, \quad t \ge t_0 = 0, \end{aligned} \tag{5.2}$$

and

$$\begin{aligned}\Big[w(t) - \frac{1}{2}w(t-e^{-|\sin 2t|})\Big]' = &-2(w(t) - \frac{1}{2}w(t-e^{-|\sin 2t|}))\\ &+ 100w(t-2e^{-|\sin 2t|})e^{-2w(t-2e^{-|\sin 2t|})}, \quad t \ge t_0 = 0. \end{aligned} \tag{5.3}$$

Obviously, $\tau_1(t) = e^{-|\sin 2t|}$, $\tau_2(t) = 2e^{-|\sin 2t|}$, $2 = r = r_2 > r_1 = 1$, let

$$\Gamma_+ = \big\{\varphi \in C([-2,\ 0],\ \mathbb{R}) \mid \varphi(0) - c\varphi(-\tau_1(0)) \ge 0,\ \varphi(\theta) \ge 0 \text{ for all } \theta \in [-2,\ 0]\big\}.$$

One can easily verify that $\frac{\beta}{\delta(1-c)} = \frac{1}{4(1-\frac{1}{2})} = \frac{1}{2} < 1$ and $1 < \frac{\beta}{\delta(1-c)} = \frac{e}{2\times(1-\frac{1}{2})} = e < e^2$ hold for Eqs. (5.1) and (5.2), respectively. It is concluded from Theorems 3.1 and 4.1 that the zero equilibrium of Eq. (5.1) and the positive equilibrium of Eq. (5.2) are all globally asymptotically stable (see Figs. 5.1 and 5.2). However, the stability criteria $\frac{\beta}{\delta(1-c)} \le 1$ and $1 < \frac{\beta}{\delta(1-c)} \le e^2$ are invalid for Eq. (5.3), which means that the positive equilibrium of Eq. (5.3) is not globally asymptotically stable, Fig. 5.3 indicates this fact.

Remark 5.2. It is not difficult to check that $\tau_1(t) = e^{-|\sin 2t|}$ and $\tau_2(t) = 2e^{-|\sin 2t|}$ in Eqs. (5.1)–(5.3) are time dependent, which do not satisfy the basic constraints of the delay terms in [2, 3, 7, 12, 21]. In addition, the sharp asymptotic stability conditions have not been touched in [10, 13, 20, 22]. Consequently, the results in above-mentioned literatures and their references could not directly applied to this example. This shows that the established theoretical results in this paper extend and improve existing ones.

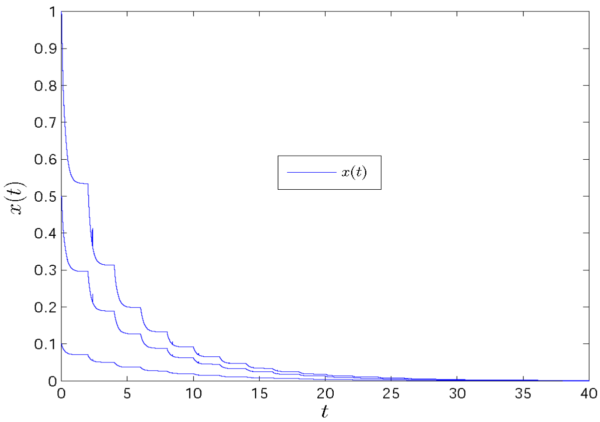

Figure 5.1: The state trajectories to Eq. (5.1) with initial values 0.1, 0.5, 1.

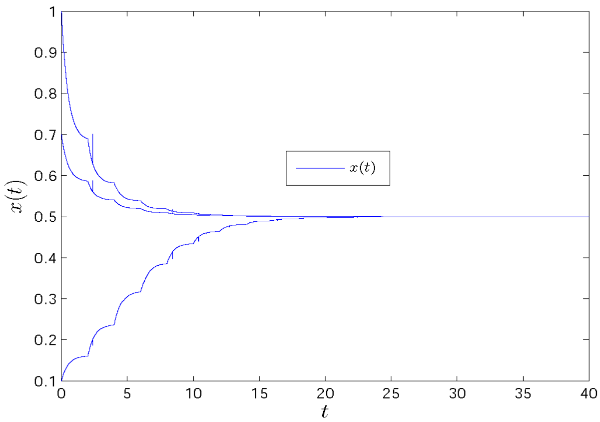

Figure 5.2: The state trajectories to Eq. (5.2) with initial values 0.1, 0.7, 1.

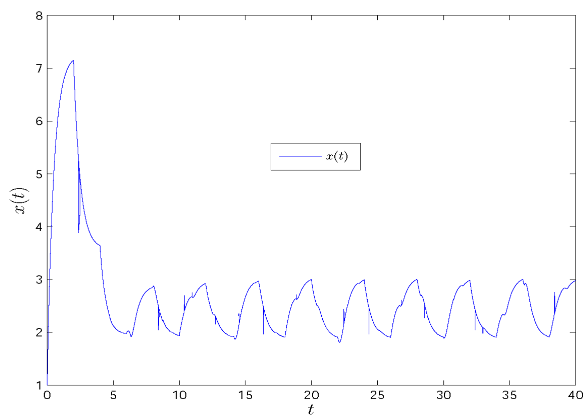

Figure 5.3: The state trajectory to Eq. (5.3) with initial value 1.

6 Conclusion

In this paper, the stability and attraction of the neutral Nicholson's blowflies equation with time-varying delays are addressed. By applying the analytical method such as Dini derivative and differential inequality techniques, sharp sufficient criteria substantially extend the existing results on global asymptotic stability, comparative analysis and simulations are also given to support the availability of the theoretical results.

It is also worth noting that the authors in [17] evidenced that the positive equilibrium point of the classical non-neutral delayed Nicholson's blowflies model attracts all solutions with nonnegative initial values under the delay-dependent condition $(e^{\delta\tau} - 1)\ln\frac{p}{\delta} \leq 1$. Whether or not our methods used in this paper are available to find delay-dependent criterion to ensure the global attraction of positive equilibrium to Eq. (1.4), it is an interesting problem and we leave it as an important topic of future researchers.

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