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On a coupled nonlocal Schrödinger–Kirchhoff system with singular exponential nonlinearity in $\mathbb{R}^N$

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Abstract. This paper is concerned with the existence of solutions for parameters dependent Schrödinger–Kirchhoff system driven by nonlocal integro-differential operators with singular Trudinger–Moser nonlinearity in the whole Euclidean space $\mathbb{R}^N$. These parameters have a major impact on the produced analysis. It is noted that, we also study the asymptotic behaviour of solutions depending upon these parameters. The proofs of the existence results to the aforementioned system rely on the mountain pass theorem, the Ekeland variational principle, the classical deformation lemma, and the Krasnoselskii genus theory. The salient feature and novelty of this paper is that it also covers the so-called degenerate case of the Kirchhoff function, that is, it could vanish at zero.

Keywords: nonlocal integro-differential operator, mountain pass theorem, variational principle, genus theory.

2020 Mathematics Subject Classification: 35A15, 35A23, 35D30, 35R11, 35J60.

1 Introduction and main results

In this paper, we study the following nonlocal Schrödinger–Kirchhoff type system:

$$\begin{cases} M(\|(u,v)\|^p)(\mathcal{L}_p^s(u)+V(x)|u|^{p-2}u)=\dfrac{F_u(x,u,v)}{|x|^\gamma}+\lambda h(x)|u|^{q-2}u & \text{in } \mathbb{R}^N, \\ M(\|(u,v)\|^p)(\mathcal{L}_p^s(v)+V(x)|v|^{p-2}v)=\dfrac{F_v(x,u,v)}{|x|^\gamma}+\mu h(x)|v|^{q-2}v & \text{in } \mathbb{R}^N, \end{cases} \tag{$\mathcal{S}_{\lambda,\mu}$}$$

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where $N \geq 1$, $0 < s < 1$, $sp = N$, $1 < q < \infty$, $\gamma \in [0, N)$, λ and μ are two positive parameters, and the norm

$$\|(u,v)\| = \left(\|u\|^p_{W^{s,p}_{K,V}} + \|v\|^p_{W^{s,p}_{K,V}} \right)^{\frac{1}{p}},$$

where for the singular kernel $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$ with $\mathbb{R}^+ = (0, \infty)$ and $w \in \{u, v\}$, we define

$$\|w\|_{W^{s,p}_{K,V}} = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |w(x) - w(y)|^p K(x-y)\, \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|w|^p\, \mathrm{d}x \right)^{\frac{1}{p}}.$$

Consequently, $M : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ with $\mathbb{R}^+_0 = [0, \infty)$ is a Kirchhoff function, $V : \mathbb{R}^N \to \mathbb{R}$ is a scalar potential, $h : \mathbb{R}^N \to \mathbb{R}^+$ is a measurable function, the functions F_u and F_v are partial derivatives of a Carathéodory function F, of exponential type and $\mathcal{L}^s_p$ is the nonlocal fractional operator which, up to a normalization constant, is defined by

$$\mathcal{L}^s_p \phi(x) = 2 \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} |\phi(x) - \phi(y)|^{p-2} (\phi(x) - \phi(y)) K(x-y)\, \mathrm{d}y, \qquad \forall\, x \in \mathbb{R}^N,$$

along any $\phi \in C^\infty_0(\mathbb{R}^N)$, where $B_\epsilon(x) = \{y \in \mathbb{R}^N : |x-y| < \epsilon\}$. The singular kernel $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$ is a measurable function satisfying the following properties for $sp = N$ and $0 < s < 1$:

(a) $\eta K \in L^1(\mathbb{R}^N)$, where $\eta(x) = \min\{|x|^p, 1\}$.

(b) there exists $K_0 > 0$ such that $K(x) \geq K_0 |x|^{-(N+sp)}$ for any $x \in \mathbb{R}^N \setminus \{0\}$.

In addition, we make a note that from here onwards, $\cdot$ stands for the natural inner product in any Euclidean space $\mathbb{R}^d$ for any dimension $d \geq 1$ and $|\cdot|$ denotes the corresponding Euclidean norm.

Throughout the paper, without further mentioning, we have the following assumptions on the scalar potential V, and the Kirchhoff function M.

(V) The function $V : \mathbb{R}^N \to \mathbb{R}$ is assumed to be continuous and to satisfy:

- (V_1) There exists a constant $V_0 > 0$ such that $\inf_{\mathbb{R}^N} V \geq V_0$.
- (V_2) There exists $h > 0$ such that $\lim_{|y| \to \infty} \operatorname{meas}\big(\{x \in B_h(y) : V(x) \leq c\}\big) = 0$, $\forall\, c > 0$, where we denote the Lebesgue measure of any set $E \subset \mathbb{R}^N$ by $\operatorname{meas}(E)$.

(M) The function $M : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ is assumed to be continuous and to satisfy:

- (M_1) For any $\tau > 0$, there exists $\kappa = \kappa(\tau) > 0$ such that

$$M(t) \geq \kappa, \qquad \forall\, t \geq \tau.$$

- (M_2) There exists $\theta \geq 1$ such that

$$tM(t) \leq \theta \widehat{M}(t), \qquad \forall\, t \in \mathbb{R}^+_0,$$

where

$$\widehat{M}(t) = \int_0^t M(\xi)\, \mathrm{d}\xi.$$

Remark 1.1. The condition (V_2) is weaker than the coercivity assumption, that is, $V(x) \to \infty$ as $|x| \to \infty$.

On concerns about the nonlinearity term F, we have the following assumptions.

(F) The function $F : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function and to satisfy:

(F_1) $F(x,\cdot,\cdot) \in C^1(\mathbb{R}^2)$ for a.e. $x \in \mathbb{R}^N$, $F(x,\cdot,\cdot) \geq 0$ in $\mathbb{R}^2$, $F(x,0,0) = 0$ for a.e. $x \in \mathbb{R}^N$, $F_u(x,u,v) = 0$ for all $u \leq 0$ and $v \in \mathbb{R}$, $F_v(x,u,v) = 0$ for all $u \in \mathbb{R}$ and $v \leq 0$, $F_u(x,u,0) = 0$ for all $u \in \mathbb{R}$ and $F_v(x,0,v) = 0$ for all $v \in \mathbb{R}$. Moreover, for all $(u,v) \in \mathbb{R}^+ \times \mathbb{R}^+$, the function $F(\cdot,u,v)$ is strictly positive for a.e. $x \in \mathbb{R}^N$.

(F_2) There exists $\alpha_0 > 0$ with the property that for all $\varepsilon > 0$, there exists $\kappa_\varepsilon > 0$ such that for a.e. $x \in \mathbb{R}^N$ and all $z = (u,v) \in \mathbb{R}^2$ with $|z| = \sqrt{u^2+v^2}$, $\nabla F = (F_u, F_v)$ and $j_p = \min\{j \in N :\ j \geq p\}$, we have

$$|\nabla F(x,u,v)| \leq \varepsilon |z|^{\theta p-1} + \kappa_\varepsilon \Phi(\alpha_0 |z|^{p'}),$$

where

$$\Phi(t) = \exp(t) - \sum_{j=0}^{j_p-2} \frac{t^j}{j!} \quad \text{and} \quad p' = \frac{N}{N-s}.$$

(F_3) There holds

$$\nabla F(x,u,v) = o(|z|^{\theta p-1}) \quad \text{as } |z| \to 0^+ \text{ uniformly for } x \in \mathbb{R}^N.$$

(F_4) There exists $\sigma > \theta p$ such that

$$0 \leq \sigma F(x,u,v) \leq \nabla F(x,u,v).(u,v), \qquad \forall\ (x,u,v) \in \mathbb{R}^N \times \mathbb{R}^2.$$

In recent years, studying elliptic partial differential equations involving fractional Laplacian or more general nonlocal integro-differential operators has become a very interesting area of nonlinear analysis. Such types of operators occur naturally in several real-world applications, such as finance, optimization, game theory, image processing, multiple scattering, phase transition phenomena, population dynamics, continuum mechanics, ultra-relativistic limits of quantum mechanics, soft thin films, minimal surfaces, and the stochastic stability of Lévy processes. In this regard, we refer the readers to study [6,12–14,29,31] and related references. Indeed, the literature on nonlocal fractional operators and their applications is somewhat vast, as shown by the new monographs [2,34], the comprehensive work [19] and the references included therein.

In the context of fractional order Sobolev space, the Sobolev embedding states that for $sp < N$ with $s \in (0,1)$, the continuous embedding $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ holds for any $r \in [p, p_s^*]$, where $p_s^* = \frac{Np}{N-sp}$ is called the critical Sobolev exponent. In conclusion, to study the variational problems with subcritical and critical growth, the nonlinearity cannot exceed the polynomial of degree p_s^*. Despite this, in the Sobolev limiting case (commonly known as the Trudinger–Moser case), that is, $sp = N$, the continuous embedding $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ holds for any $q \in [p, \infty)$ but we cannot assume $q = \infty$ for such an embedding. Further, in this case, every polynomial growth is allowed. In this scenario, to deal with variational problems, many authors proved separately that the maximal growth of the nonlinearity is of exponential type, for a detailed study, we refer to [28,36,45]. The Trudinger–Moser inequalities have many applications, including extremal problems for determinants and zeta functions under conformal deformation of metric. For instance, on the four-dimensional sphere, the determinant of the conformal Laplacian is extremized under conformal deformation with fixed area by the standard metric. The most important aspect of the Moser–Trudinger inequality has been its connection to the Polyakov–Onofri log determinant variation formula, as well as its later development in conformal geometry and geometric analysis of conformally invariant operators on higher-dimensional manifolds. In this regard, we refer to see [8, 16]. It is worth noting that there has been significant progress in the Trudinger–Moser inequalities to analyse the existence, nonexistence, and multiplicity of solutions to nonlinear PDEs in the context of the factional Sobolev space in $\mathbb{R}^N$. For a detailed study, one may go through [10,46,47,50–52] and references therein.

During the last few decades, there has been a tremendous amount of attention focused by many authors towards the study of Kirchhoff-type problems driven by nonlocal fractional Laplacian operators due to their applicability in various models of physical and biological systems. A typical model of Kirchhoff function M can be considered by $M(t) = a + b\theta t^{\theta-1}$ for all $t \geq 0$, where $a, b \geq 0$, $a + b > 0$ and $\theta > 1$. We say that M is of degenerate type if $a = 0$, while it is called non-degenerate type if $a > 0$. Obviously, one can easily notice that assumptions (M_1)–(M_2) in this paper also cover the degenerate case that corresponds to $M(0) = 0$. In addition, we make a note that in the study of Kirchhoff-type problems, many authors often used M as a nondecreasing function on $\mathbb{R}_0^+$, for instance, see [24, 37]. But however, in view of the assumption (M_1), one can consider that M is not monotone in nature as M can be chosen $M(t) = (1 + t)^k + (1 + t)^{-1}$ for $t \geq 0$ with $0 < k < 1$. Moreover, it is worth mentioning that the degenerate case in Kirchhoff's theory is more interesting and significant than the non-degenerate case. From a physical point of view, M measures the change of the tension on the string generated by the change of its length during the vibration, while $M(0) = 0$ indicates that the base tension of the string is zero. The presence of the nonlinear coefficient M is crucial to be considered when the changes in tension during the motion cannot be neglected. The existence of solutions for non-degenerate fractional Kirchhoff stationary problems are discussed in [3–5, 24, 26, 30], whereas degenerate problems are addressed in [22, 40, 48] and the relevant references. To the best of our knowledge, most of the works on Kirchhoff-type problems are driven by nonlocal fractional Laplacian or more general integro-differential operators involve the nonlinear terms satisfying some polynomial growth, but there are only a few papers dealing with nonlinear terms satisfying exponential type of growth, this is one of the key motivation towards the study of this paper. In this context, we recommend the readers to study [9,32,33,43,49].

Nowadays, the study of elliptic systems involving fractional Laplacian or more general nonlocal integro-differential operators has gained much attention due to a wide range of applications in applied sciences. Indeed, if $w = (u, v)^T$ denotes a vector of concentration variable, $H = (f(x, u, v), g(x, u, v))^T$ describes a local reaction term related to source and loss process, then for $M \equiv 1$, $K(x) = |x|^{-(N+sp)}$ and $s = 1$, the system $(\mathcal{S}_{\lambda,\mu})$ derive from the following p-Laplacian reaction-diffusion elliptic system

$$z_t = \operatorname{div}(|\nabla z|^{p-2}\nabla z) + H(x, z)$$

with

$$\operatorname{div}(|\nabla z|^{p-2}\nabla z) = \begin{pmatrix} \operatorname{div}(|\nabla u|^{p-2}\nabla u) \\ \\ \operatorname{div}(|\nabla v|^{p-2}\nabla v) \end{pmatrix}.$$

Such equations can be frequently seen in physics, plasma physics, biophysics, chemical reaction design, etc. In many situations, $H = (f(x, u, v), g(x, u, v))^T$ has both components of polynomial type with variable coefficients, but for the Liouville–Bratu–Gelfand and the

Frank–Kamenetsky models H has exponential growth at infinity. We refer to [15,17] for other physical examples of such problems. On concerning nonlocal elliptic systems with critical and subcritical growth one can see to [23,30,38] and references therein. However, concerning about nonlocal elliptic systems with exponential growth the literature is very limited (see [18,20,35]). This is another key motivation towards the study of this paper.

Motivated by the above-cited works, especially by [35,45,49], we study for the first time in the literature to solve a coupled Schrödinger–Kirchhoff elliptic system with singular exponential growth driven by the nonlocal integro-differential operator in $\mathbb{R}^N$.

To establish our main results, we first define the weak solution of the system $(\mathcal{S}_{\lambda,\mu})$.

Definition 1.2. We say that $(u,v) \in \mathbf{X}$ (see (2.1) for its definition) is a (weak) solution for the system $(\mathcal{S}_{\lambda,\mu})$, if for all $(\varphi,\psi) \in \mathbf{X}$, we have

$$\begin{aligned} M(\|(u,v)\|^p)\Big(\langle u,\varphi\rangle_{K_p,V} + \langle v,\psi\rangle_{K_p,V}\Big) = \int_{\mathbb{R}^N} \frac{\nabla F(x,u,v).(\varphi,\psi)}{|x|^\gamma}\,\mathrm{d}x + \lambda \int_{\mathbb{R}^N} h(x)|u|^{q-2}u\varphi\,\mathrm{d}x \\ + \mu \int_{\mathbb{R}^N} h(x)|v|^{q-2}v\psi\,\mathrm{d}x, \end{aligned}$$

where for any w_1 and w_2, we define

$$\begin{aligned} \langle w_1,w_2\rangle_{K_p,V} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |w_1(x)-w_1(y)|^{p-2}(w_1(x)-w_1(y))(w_2(x)-w_2(y))K(x-y)\,\mathrm{d}x\mathrm{d}y \\ + \int_{\mathbb{R}^N} V(x)|w_1|^{p-2}w_1w_2\,\mathrm{d}x. \end{aligned}$$

Our main results for this paper are listed below.

Theorem 1.3. *Suppose V satisfies $(V_1)-(V_2)$, M satisfies $(M_1)-(M_2)$ and (F) fulfills $(F_1)-(F_4)$. If $1 < q < p$ and $h \in L^\eta(\mathbb{R}^N)$ with $\eta = \frac{N}{N-sq}$, then there exists $\widetilde{\lambda} > 0$ such that for all $(\lambda,\mu) \in (0,\widetilde{\lambda}) \times (0,\widetilde{\lambda})$, the system $(\mathcal{S}_{\lambda,\mu})$ admits at least one nontrivial nonnegative solution $(u_{\lambda,\mu},v_{\lambda,\mu})$ in $\mathbf{X}$. In addition, there holds*

$$\lim_{(\lambda,\mu)\to(0^+,0^+)} \|(u_{\lambda,\mu},v_{\lambda,\mu})\| = 0.$$

Theorem 1.4. *Suppose V satisfies (V_1)–(V_2), M satisfies (M_1)–(M_2) and (F) fulfills (F_1)–(F_4). If $q > \theta p$ and $h \in L^\infty(\mathbb{R}^N)$, then there exists $\widehat{\lambda} > 0$ such that for all $(\lambda,\mu) \in (\widehat{\lambda},\infty) \times (\widehat{\lambda},\infty)$, the system $(\mathcal{S}_{\lambda,\mu})$ admits at least one nontrivial nonnegative solution $(u_{\lambda,\mu},v_{\lambda,\mu})$ in $\mathbf{X}$. Further, there holds*

$$\lim_{(\lambda,\mu)\to(\infty,\infty)} \|(u_{\lambda,\mu},v_{\lambda,\mu})\| = 0.$$

Theorem 1.5. *Suppose V satisfies (V_1)–(V_2) and (F) fulfills (F_2)–(F_4). In addition, we assume that F and M satisfy the following conditions:*

- (F_1') *$F(x,\cdot,\cdot) \in C^1(\mathbb{R}^2)$ for a.e. $x \in \mathbb{R}^N$, $F(x,\cdot,\cdot) \geq 0$ in $\mathbb{R}^2$ and $F(x,0,0) = 0$ for a.e. $x \in \mathbb{R}^N$. Consequently, we assume that $F(x,u,v) = F(x,-u,-v)$ for all $(x,u,v) \in \mathbb{R}^N \times \mathbb{R}^2$.*
- (M') *the Kirchhoff function is of type $M(t) = a + b\theta t^{\theta-1}$ for all $t \geq 0$, where $a,b \geq 0$, $a+b > 0$ and $\theta > 1$.*

Then for $1 < q < p$ and $h \in L^\eta(\mathbb{R}^N)$ with $\eta = \frac{N}{N-sq}$, there exists $\bar{\lambda} > 0$ such that for all $(\lambda,\mu) \in (0,\bar{\lambda}) \times (0,\bar{\lambda})$, the system $(\mathcal{S}_{\lambda,\mu})$ has infinitely many solutions in $\mathbf{X}$.

The rest of the paper is organized as follows: In Section 2, we discuss some preliminary results useful for the next main sections and the variational structure of the system $(\mathcal{S}_{\lambda,\mu})$. Section 3 is devoted to proving Theorem 1.3 via the Ekeland variational principle and using the standard topological tools. In Section 4, we prove Theorem 1.4 with the help of the mountain pass theorem. Finally, in Section 5, by introducing a truncated functional and using the deformation lemma along with the Krasnoselskii genus theory, we prove Theorems 1.5.

Notations. From now on in this paper, we have the following notations:

- For any Banach space $(X, \|\cdot\|_X)$, we denote its continuous dual by $(X^*, \|\cdot\|_{X^*})$.
- $o_n(1)$ denotes the real sequence such that $o_n(1) \to 0$ as $n \to \infty$.
- $\rightharpoonup$ means weak convergence and $\to$ means strong convergence.
- $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$.

2 Preliminary results

In this section, we shall discuss about some basic properties of fractional Sobolev spaces and related lemmas, which are used in the sequel of this paper. Note that throughout this paper, we use $N \geq 1$, $s \in (0,1)$ and $sp = N$.

Let $r \in (1, \infty)$ and $L^r(\mathbb{R}^N)$ denotes the standard Lebesgue space with the norm $\|\cdot\|_r$. Moreover, for nonnegative measurable function $V : \mathbb{R}^N \to \mathbb{R}$, the space $L^r_V(\mathbb{R}^N)$, consisting of all real-valued measurable functions, with $V(x)|u|^r \in L^1(\mathbb{R}^N)$, equipped with the seminorm

$$\|u\|_{r,V} = \left(\int_{\mathbb{R}^N} V(x)|u|^r \,\mathrm{d}x \right)^{\frac{1}{r}},$$

which is a norm, thanks to (V_1). The space $\left(L^r_V(\mathbb{R}^N), \|\cdot\|_{r,V}\right)$ is a uniform convex Banach space (see [39]), thanks to $(V1)$. Consequently, under the assumption of (V_1), the embedding $L^r_V(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ is continuous.

For $\gamma \in [0, N)$, we define the space $L^r(\mathbb{R}^N, |x|^{-\gamma}\mathrm{d}x)$, consisting of all real-valued measurable functions, with $|u|^r|x|^{-\gamma} \in L^1(\mathbb{R}^N)$, equipped with the norm

$$\|u\|_{r,\gamma} = \left(\int_{\mathbb{R}^N} |u|^r|x|^{-\gamma} \,\mathrm{d}x \right)^{\frac{1}{r}}.$$

Define the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ by

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\},$$

where $[u]_{s,p}$ denotes the Gagliardo seminorm, defined by

$$[u]_{s,p} = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \,\mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}}.$$

Under the following norm

$$\|u\|_{W^{s,p}} = (\|u\|_p^p + [u]_{s,p}^p)^{\frac{1}{p}},$$

the space $\left(W^{s,p}(\mathbb{R}^N), \|\cdot\|_{W^{s,p}}\right)$ is a uniformly convex Banach space and hence a reflexive Banach space (see [39]).

Due to (V_1), the weighted fractional Sobolev space $W_V^{s,p}(\mathbb{R}^N)$ makes sense and defined by

$$W_V^{s,p}(\mathbb{R}^N) = \{u \in W^{s,p}(\mathbb{R}^N) : u \in L_V^p(\mathbb{R}^N)\},$$

endowed with the norm

$$\|u\|_{W_V^{s,p}} = \left(\|u\|_{p,V}^p + [u]_{s,p}^p\right)^{\frac{1}{p}}.$$

It is well-known that the space $\left(W_V^{s,p}(\mathbb{R}^N), \|\cdot\|_{W_V^{s,p}}\right)$ is a uniformly convex Banach space and $C_0^\infty(\mathbb{R}^N)$ is dense in $W_V^{s,p}(\mathbb{R}^N)$. Consequently, in virtue of (V_1), the embedding $W_V^{s,p}(\mathbb{R}^N) \hookrightarrow W^{s,p}(\mathbb{R}^N)$ is continuous and there holds $\min\{1, V_0\}\|u\|_{W^{s,p}}^p \leq \|u\|_{W_V^{s,p}}^p$ for all $u \in W_V^{s,p}(\mathbb{R}^N)$ (see [39]).

Let $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$ be the singular kernel, which is stated in the introductory part of this paper. Now we define the generalized fractional Sobolev space $W_K^{s,p}(\mathbb{R}^N)$ by

$$W_K^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,K_p} < \infty\},$$

equipped with the following norm

$$\|u\|_{W_K^{s,p}} = \left(\|u\|_p^p + [u]_{s,K_p}^p\right)^{\frac{1}{p}},$$

where

$$[u]_{s,K_p} = \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x) - u(y)|^p K(x-y)\, \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}.$$

The space $\left(W_K^{s,p}(\mathbb{R}^N), \|\cdot\|_{W^{s,p}}\right)$ is a uniformly convex Banach space and hence a reflexive Banach space (see [45]).

Under the assumption (V_1), the weighted fractional generalized Sobolev space $W_{V,K}^{s,p}(\mathbb{R}^N)$ makes sense and defined by

$$W_{K,V}^{s,p}(\mathbb{R}^N) = \{u \in W_K^{s,p}(\mathbb{R}^N) : u \in L_V^p(\mathbb{R}^N)\},$$

endowed with the norm

$$\|u\|_{W_{K,V}^{s,p}} = \left(\|u\|_{p,V}^p + [u]_{s,K_p}^p\right)^{\frac{1}{p}}.$$

It is easy to see that the space $\left(W_{K,V}^{s,p}(\mathbb{R}^N), \|\cdot\|_{W_{K,V}^{s,p}}\right)$ is a uniformly convex Banach space and hence reflexive (see [45]).

The natural function space to study the system $(\mathcal{S}_{\lambda,\mu})$ is the generalized vectorial fractional Sobolev space $\mathbf{X}$, defined by

$$\mathbf{X} = W_{V,K}^{s,p}(\mathbb{R}^N) \times W_{V,K}^{s,p}(\mathbb{R}^N), \tag{2.1}$$

endowed with the norm

$$\|(u,v)\| = \left([(u,v)]_{s,K_p}^p + \|(u,v)\|_{p,V}^p\right)^{\frac{1}{p}},$$

where

$$[(u,v)]_{s,K_p} = \left([u]_{s,K_p}^p + [v]_{s,K_p}^p\right)^{\frac{1}{p}} \quad \text{and} \quad \|(u,v)\|_{p,V} = \left(\|u\|_{p,V}^p + \|v\|_{p,V}^p\right)^{\frac{1}{p}}.$$

Consequently, we note that the space $\left(\mathbf{X}, \|(\cdot,\cdot)\|\right)$ is a uniform convex Banach space, and thus it is reflexive. Similarly, we can define the norm of the Banach space $W^{s,p}(\mathbb{R}^N) \times W^{s,p}(\mathbb{R}^N)$. Now we list some technical lemmas that will be used later in this paper. The following first three lemmas are direct consequences of [45, Lemma 2, 4 and 6].

Lemma 2.1. *Under the assumption of* (V_1)*, the following chain of embeddings*

$$\mathbf{X} \hookrightarrow W^{s,p}(\mathbb{R}^N) \times W^{s,p}(\mathbb{R}^N) \hookrightarrow L^\nu(\mathbb{R}^N) \times L^\nu(\mathbb{R}^N)$$

are continuous for all $\nu \in [p,\infty)$ *and there holds* $\min\{V_0, K_0\}\|(u,v)\|^p_{W^{s,p}\times W^{s,p}} \leq \|(u,v)\|^p$ *for all* $(u,v) \in \mathbf{X}$*. Moreover, if* $\nu \in [1,\infty)$*, then the embeddings* $\mathbf{X} \hookrightarrow L^\nu(B_R) \times L^\nu(B_R)$ *is compact for any* $R > 0$*.*

Lemma 2.2. *Suppose* (V_1) *and* (V_2) *holds. Then the embedding* $\mathbf{X} \hookrightarrow L^\nu(\mathbb{R}^N) \times L^\nu(\mathbb{R}^N)$ *is compact for all* $\nu \in [p,\infty)$*. Consequently, there holds* $\|(u,v)\|_{L^\nu(\mathbb{R}^N)\times L^\nu(\mathbb{R}^N)} \leq \mathcal{A}_\nu^{-1}\|(u,v)\|$ *for all* $(u,v) \in \mathbf{X}$*, where* $\mathcal{A}_\nu$ *is the best constant in the embedding* $\mathbf{X} \hookrightarrow L^\nu(\mathbb{R}^N) \times L^\nu(\mathbb{R}^N)$ *and defined by*

$$\mathcal{A}_\nu = \inf_{(u,v)\in\mathbf{X}\setminus\{(0,0)\}} \frac{\|(u,v)\|}{\|(u,v)\|_{L^\nu(\mathbb{R}^N)\times L^\nu(\mathbb{R}^N)}} \text{ with } \|(u,v)\|_{L^\nu(\mathbb{R}^N)\times L^\nu(\mathbb{R}^N)} = \left(\|u\|_\nu^\nu + \|v\|_\nu^\nu\right)^{\frac{1}{\nu}}.$$

Lemma 2.3. *For any* $\nu \in [p,\infty)$ *and* $\gamma \in [0,N)$*, the embedding* $\mathbf{X} \hookrightarrow L^\nu(\mathbb{R}^N, |x|^{-\gamma}\,\mathrm{d}x) \times L^\nu(\mathbb{R}^N, |x|^{-\gamma}\,\mathrm{d}x)$ *is compact. In addition, there holds*

$$\|(u,v)\|_{L^\nu(\mathbb{R}^N,|x|^{-\gamma}\,\mathrm{d}x)\times L^\nu(\mathbb{R}^N,|x|^{-\gamma}\,\mathrm{d}x)} \leq \mathcal{B}_{\nu,\gamma}^{-1}\|(u,v)\|, \qquad \forall\, (u,v) \in \mathbf{X},$$

where $\mathcal{B}_{\nu,\gamma}$ *is the best constant in the embedding* $\mathbf{X} \hookrightarrow L^\nu(\mathbb{R}^N, |x|^{-\gamma}\,\mathrm{d}x) \times L^\nu(\mathbb{R}^N, |x|^{-\gamma}\,\mathrm{d}x)$*, which is defined by*

$$\mathcal{B}_{\nu,\gamma} = \inf_{(u,v)\in\mathbf{X}\setminus\{(0,0)\}} \frac{\|(u,v)\|}{\|(u,v)\|_{L^\nu(\mathbb{R}^N,|x|^{-\gamma}\,\mathrm{d}x)\times L^\nu(\mathbb{R}^N,|x|^{-\gamma}\,\mathrm{d}x)}}$$

with

$$\|(u,v)\|_{L^\nu(\mathbb{R}^N,|x|^{-\gamma}\,\mathrm{d}x)\times L^\nu(\mathbb{R}^N,|x|^{-\gamma}\,\mathrm{d}x)} = \left(\|u\|^\nu_{\nu,\gamma} + \|v\|^\nu_{\nu,\gamma}\right)^{\frac{1}{\nu}}.$$

Lemma 2.4. *Let* $1 < q < p$ *and* $h \in L^\eta(\mathbb{R}^N)$ *with* $\eta = \frac{N}{N-sq}$*, then the embedding* $L^p(\mathbb{R}^N) \hookrightarrow L^q_h(\mathbb{R}^N)$ *is continuous and there holds*

$$\|u\|_{q,h} \leq \|h\|_\eta^{\frac{1}{q}}\|u\|_p, \quad \forall\, u \in L^p(\mathbb{R}^N) \quad \text{with} \quad \|u\|^q_{q,h} = \int_{\mathbb{R}^N} h(x)|u|^q\,\mathrm{d}x.$$

Further, the embedding $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^q_h(\mathbb{R}^N)$ *is compact. In addition, due to [45, Lemma 4], the embedding* $W^{s,p}_{K,V}(\mathbb{R}^N) \hookrightarrow L^q_h(\mathbb{R}^N)$ *is compact.*

Proof. Let $q \in (1,p)$ and $h \in L^\eta(\mathbb{R}^N)$ with $\eta = \frac{N}{N-sq}$. Observe that $\frac{1}{\eta} + \frac{q}{p} = 1$. Consequently, by the Hölder's inequality, we have

$$\|u\|_{q,h} \leq \|h\|_\eta^{\frac{1}{q}}\|u\|_p, \qquad \forall\, u \in L^p(\mathbb{R}^N).$$

It follows that the embedding $L^p(\mathbb{R}^N) \hookrightarrow L^q_h(\mathbb{R}^N)$ is continuous. To complete the proof, we only have to show that if $u_n \rightharpoonup u$ in $W^{s,p}(\mathbb{R}^N)$ as $n \to \infty$, then $u_n \to u$ in $L^q_h(\mathbb{R}^N)$ as $n \to \infty$. For this, first we assume that $u_n \rightharpoonup u$ in $W^{s,p}(\mathbb{R}^N)$ as $n \to \infty$. Due to [45, Lemma 4], the sequence $\{u_n - u\}_n$ is bounded in $L^p(\mathbb{R}^N)$ and thus there exists $m > 0$ such that $\|u_n - u\|_p \leq m$ for all $n \in \mathbb{N}$. Further, since $h \in L^\eta(\mathbb{R}^N)$ and using the fact that every integrable function is tight, we can assume that for every $\epsilon > 0$, there exists $R_\epsilon > 0$ large enough such that

$$\int_{\mathbb{R}^N\setminus B_{R_\epsilon}} |h(x)|^\eta\,\mathrm{d}x < \left(\frac{\epsilon}{2m^q}\right)^\eta.$$

This together with the Hölder's inequality implies at once that

$$\int_{\mathbb{R}^N\setminus B_{R_\epsilon}} h(x)|u_n-u|^q\,dx \le \left(\int_{\mathbb{R}^N\setminus B_{R_\epsilon}} |h(x)|^\eta\,dx\right)^{\frac{1}{\eta}} \|u_n-u\|_p^q < \frac{\epsilon}{2}.$$

In virtue of [45, Lemma 4], we also have $u_n \to u$ in $L^p(B_{R_\epsilon})$. Therefore, using the the Hölder's inequality, we get

$$\int_{B_{R_\epsilon}} h(x)|u_n-u|^q\,dx \le \|h\|_\eta \left(\int_{B_{R_\epsilon}} |u_n-u|^p\,dx\right)^{\frac{1}{p}} = o_n(1) \quad \text{as } n\to\infty.$$

In conclusion, there exists $n_0\in\mathbb{N}$ and $\epsilon>0$ such that $\int_{B_{R_\epsilon}} h(x)|u_n-u|^q\,dx < \frac{\epsilon}{2}$ for all $n\ge n_0$. Now gathering all the above information, we obtain for all $n\ge n_0$ that

$$\|u_n-u\|_{q,h}^q = \int_{\mathbb{R}^N\setminus B_{R_\epsilon}} h(x)|u_n-u|^q\,dx + \int_{B_{R_\epsilon}} h(x)|u_n-u|^q\,dx < \epsilon,$$

and thus we conclude the proof. □

An immediate byproduct of the above lemma, we have the following result.

Lemma 2.5. *Let $1<q<p$ and $h\in L^\eta(\mathbb{R}^N)$ with $\eta=\frac{N}{N-sq}$, then the embedding*

$$L^p(\mathbb{R}^N)\times L^p(\mathbb{R}^N)\hookrightarrow L_h^q(\mathbb{R}^N)\times L_h^q(\mathbb{R}^N)$$

is continuous. In addition, the embedding $W^{s,p}(\mathbb{R}^N)\times W^{s,p}(\mathbb{R}^N)\hookrightarrow L_h^q(\mathbb{R}^N)\times L_h^q(\mathbb{R}^N)$ is compact. Further, due to Lemma 2.1, the embedding $\mathbf{X}\hookrightarrow L_h^q(\mathbb{R}^N)\times L_h^q(\mathbb{R}^N)$ is compact and there holds $\|(u,v)\|_{L_h^q(\mathbb{R}^N)\times L_h^q(\mathbb{R}^N)} \le \mathcal{S}_{q,h}^{-1}\|(u,v)\|$ for all $(u,v)\in\mathbf{X}$, where $\mathcal{S}_{q,h}$ is the best constant in the embedding $\mathbf{X}\hookrightarrow L_h^q(\mathbb{R}^N)\times L_h^q(\mathbb{R}^N)$, which is defined by

$$\mathcal{S}_{q,h} = \inf_{(u,v)\in\mathbf{X}\setminus\{(0,0)\}} \frac{\|(u,v)\|}{\|(u,v)\|_{L_h^q(\mathbb{R}^N)\times L_h^q(\mathbb{R}^N)}} \quad \textit{with} \quad \|(u,v)\|_{L_h^q(\mathbb{R}^N)\times L_h^q(\mathbb{R}^N)} = \left(\|u\|_{q,h}^q+\|v\|_{q,h}^q\right)^{\frac{1}{q}}.$$

Lemma 2.6 (cf. [11,35])**.** *The function $\Phi(t)=\exp(t)-\sum_{j=0}^{j_p-2}\frac{t^j}{j!}$ is increasing and convex in $[0,\infty)$. Moreover, for any $\alpha>0$ and $r>1$, there exists a constant $C=C(r)$ such that for $j_p=\min\{j\in N:\ j\ge p\}$, we have*

$$\left(\exp(\alpha|s|^{p'})-\sum_{j=0}^{j_p-2}\frac{\alpha^j|s|^{jp'}}{j!}\right)^r \le C\left(\exp(\alpha r|s|^{p'})-\sum_{j=0}^{j_p-2}\frac{\alpha^j r^j|s|^{jp'}}{j!}\right), \qquad \forall\, s\in\mathbb{R}.$$

The following theorem was proved by Nguyen in [45], which is called singular Trudinger–Moser inequality in the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$. It can be read as follows:

Theorem 2.7 (Singular Trudinger–Moser inequality in $\mathbb{R}^N$)**.** *For any $\alpha>0$, $\gamma\in[0,N)$ and $u\in W^{s,p}(\mathbb{R}^N)$ with $s\in(0,1)$ and $sp=N$, there holds*

$$\frac{\Phi(\alpha|u|^{p'})}{|x|^\gamma}\in L^1(\mathbb{R}^N),$$

where Φ is defined as in (F_2). Moreover, there exists $\beta_*>0$ such that for any $0\le\alpha\le\beta_*<\alpha_*$ with $\alpha_*\le\alpha_{s,N}^*$, the following inequality holds true:

$$\sup_{u\in W^{s,p}(\mathbb{R}^N),\|u\|_{W^{s,p}}\le 1} \int_{\mathbb{R}^N} \frac{\Phi(\alpha|u|^{p'})}{|x|^\gamma}\,dx < +\infty,$$

where

$$\alpha^*_{s,N} = N\left(\frac{2(N\omega_N)^2\Gamma(p+1)}{N!}\sum_{k=0}^{+\infty}\frac{(N+k-1)!}{k!}\frac{1}{(N+2k)^p}\right)^{\frac{s}{N-s}} \quad \textit{with } \omega_N = \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N+2}{2}\right)}.$$

In addition, the above inequality is sharp for $\alpha > \alpha^*_{s,N}$, that is, the supremum is infinity.

Since we are interested in studying the nonnegative solutions of the system $(\mathcal{S}_{\lambda,\mu})$, we define the associated Euler–Lagrange variational functional $J_{\lambda,\mu} : \mathbf{X} \to \mathbb{R}$ by

$$J_{\lambda,\mu}(u,v) = \frac{1}{p}\widehat{M}(\|(u,v)\|^p) - \int_{\mathbb{R}^N}\frac{F(x,u,v)}{|x|^\gamma}\,\mathrm{d}x - \frac{1}{q}(\lambda\|u^+\|^q_{q,h} + \mu\|v^+\|^q_{q,h}), \ \ \forall\,(u,v)\in\mathbf{X}. \quad (2.2)$$

In virtue of the assumption (F_2) and Theorem 2.7, one can easily verify that $J_{\lambda,\mu}$ is well-defined, of class $C^1(\mathbf{X},\mathbb{R})$ and its Gâteaux derivative is given by

$$\begin{aligned}\langle J'_{\lambda,\mu}(u,v),(\varphi,\psi)\rangle = M(\|(u,v)\|^p)\big(\langle u,\varphi\rangle_{K_p,V} + \langle v,\psi\rangle_{K_p,V}\big) - \int_{\mathbb{R}^N}\frac{\nabla F(x,u,v).(\varphi,\psi)}{|x|^\gamma}\,\mathrm{d}x \\ - \int_{\mathbb{R}^N} h(x)\{\lambda(u^+)^{q-1}\varphi + \mu(v^+)^{q-1}\psi\}\,\mathrm{d}x, \qquad \forall\,(\varphi,\psi)\in\mathbf{X}, \qquad (2.3)\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathbf{X}^*$ and $\mathbf{X}$. It follows that the critical points of $J_{\lambda,\mu}$ are exactly the weak solutions of the system $(\mathcal{S}_{\lambda,\mu})$.

3 Proof of Theorem 1.3

In this section, for the sake of simplicity, we assume without further mentioning that the structural assumptions required in Theorem 1.3 hold.

The following lemma shows that every nontrivial (weak) solution of the system $(\mathcal{S}_{\lambda,\mu})$ is nonnegative.

Lemma 3.1. *For all $\lambda > 0$ and $\mu > 0$, any nontrivial solution of the system $(\mathcal{S}_{\lambda,\mu})$ is nonnegative in $\mathbb{R}^N$.*

Proof. Suppose $\lambda > 0$ and $\mu > 0$ are fixed and $(u,v)\in\mathbf{X}\setminus\{(0,0)\}$ is a solution of the system $(\mathcal{S}_{\lambda,\mu})$. Notice that $u = u^+ - u^-$ and $v = v^+ - v^-$ and thus testing (2.3) by $(-u^-,-v^-)$, we have

$$\langle J'_{\lambda,\mu}(u,v),(-u^-,-v^-)\rangle = 0.$$

Due to (F_1), we obtain the following estimates

$$\int_{\mathbb{R}^N}\frac{\nabla F(x,u,v).(u^-,v^-)}{|x|^\gamma}\,\mathrm{d}x = 0$$

and

$$\int_{\mathbb{R}^N} h(x)\{\lambda(u^+)^{q-1}u^- + \mu(v^+)^{q-1}v^-\}\,\mathrm{d}x = 0.$$

On the other hand, we also have

$$\int_{\mathbb{R}^N} V(x)\big(|u|^{p-2}u(-u^-) + |v|^{p-2}v(-v^-)\big)\,\mathrm{d}x = \int_{\mathbb{R}^N} V(x)\big(|u^-|^p + |v^-|^p\big)\,\mathrm{d}x.$$

Invoking [32, Lemma 2.1], for a.e. $x, y \in \mathbb{R}^N$ and $w \in \{u, v\}$, we infer that

$$|w^-(x) - w^-(y)|^p \leq |w(x) - w(y)|^{p-2}(w(x) - w(y))(-w^-(x) + w^-(y)).$$

Gathering all the above information, we obtain from $\langle J'_{\lambda,\mu}(u,v), (-u^-, -v^-)\rangle = 0$ that

$$M(\|(u,v)\|^p)(\|(u^-,v^-)\|^p) \leq 0.$$

Hence by using the fact that $\|(u,v)\| > 0$ and (M_1), we get $u^- = v^- = 0$ a.e. in $\mathbb{R}^N$. This shows that $u = u^+$ and $v = v^+$ a.e. in $\mathbb{R}^N$, and we finish the proof. □

To apply the minimization argument, we first prove the following geometrical structures of the functional $J_{\lambda,\mu}$.

Lemma 3.2. *There exists $\rho \in (0,1]$ and two positive numbers λ_* and j, depending upon ρ, such that $J_{\lambda,\mu}(u,v) \geq j$ for all $(u,v) \in \mathbf{X}$ with $\|(u,v)\| = \rho$ and for all $\lambda > 0$ and $\mu > 0$ such that* $\max\{\lambda,\mu\} \leq \lambda_*$.

Proof. It follows from (M_2) that

$$\widehat{M}(t) \geq \widehat{M}(1)t^\theta, \qquad \forall\, t \in [0,1].$$

By using (F_3), it follows that for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|\nabla F(x,u,v)| \leq \varepsilon |z|^{\theta p - 1}, \qquad \forall\, x \in \mathbb{R}^N \text{ and } |z| \leq \delta. \tag{3.1}$$

Further, by using (F_2) and Lemma 2.6, we get for any $\vartheta > \theta p$ that

$$|\nabla F(x,u,v)| \leq \tilde{\kappa}_\varepsilon |z|^{\vartheta-1}\Phi(\alpha_0|z|^{p'}), \qquad \forall\, x \in \mathbb{R}^N \text{ and } |z| \geq \delta, \tag{3.2}$$

where there exists $r > 0$ such that $\vartheta = \theta p + r$ and $\tilde{\kappa}_\varepsilon = \frac{\varepsilon}{\delta^r \Phi(\alpha_0 \delta^{p'})} + \frac{\kappa_\varepsilon}{\delta^{\vartheta-1}}$. Combining (3.1) and (3.2), we obtain

$$|\nabla F(x,u,v)| \leq \varepsilon|z|^{\theta p-1} + \tilde{\kappa}_\varepsilon |z|^{\vartheta-1}\Phi(\alpha_0|z|^{p'}), \qquad \forall\, (x,z) \in \mathbb{R}^N \times \mathbb{R}^2 \text{ and } \vartheta > \theta p. \tag{3.3}$$

Now we obtain from (3.3) and Lemma 2.6 that

$$\begin{aligned}
|F(x,u,v)| &= \left| \int_0^1 \frac{d}{dt} F(x,tu,tv)\, dt \right| = \left| \int_0^1 \nabla F(x,tu,tv).(u,v)\, dt \right| \\
&\leq \varepsilon |z|^{\theta p} \int_0^1 t^{\theta p - 1}\, dt + \tilde{\kappa}_\varepsilon |z|^\vartheta \int_0^1 t^{\vartheta-1}\Phi(\alpha_0 t^{p'}|z|^{p'})\, dt \\
&\leq \varepsilon|z|^{\theta p} + \tilde{\kappa}_\varepsilon |z|^\vartheta \Phi(\alpha_0|z|^{p'}).
\end{aligned}$$

This yields at once that

$$|F(x,u,v)| \leq \varepsilon|z|^{\theta p} + \tilde{\kappa}_\varepsilon|z|^\vartheta\Phi(\alpha_0|z|^{p'}), \qquad \forall\, (x,z) \in \mathbb{R}^N \times \mathbb{R}^2 \text{ and } \vartheta > \theta p. \tag{3.4}$$

Suppose $\bar{\delta} \in (0,1]$ is sufficiently small enough and there holds $0 < \|(u,v)\| \leq \bar{\delta}$. It is easy to see that $|z| = \sqrt{u^2+v^2} \leq \psi := |u| + |v|$ and thus by using Lemma 2.1, we get

$$\begin{aligned}
\|\psi\|_{W^{s,p}} &\leq \|u\|_{W^{s,p}} + \|v\|_{W^{s,p}} \leq 2\big(\|u\|^p_{W^{s,p}} + \|v\|^p_{W^{s,p}}\big)^{\frac{1}{p}} \\
&= 2\|(u,v)\|_{W^{s,p}\times W^{s,p}} \leq 2(\min\{V_0, K_0\})^{-\frac{1}{p}}\|(u,v)\|.
\end{aligned} \tag{3.5}$$

Choose $t, t' > 1$ satisfying $\frac{1}{t} + \frac{1}{t'} = 1$ and $\widetilde{\psi} = \psi / \|\psi\|_{W^{s,p}}$. Due to (3.5) and the fact that $\bar{\delta} \in (0,1]$ is small enough such that $0 < \|(u,v)\| \leq \bar{\delta}$, we can assume that $\alpha_0 t' \|\psi\|^{p'}_{W^{s,p}} \leq (2\bar{\delta})^{p'} (\min\{V_0, K_0\})^{-\frac{1}{p-1}} \alpha_0 t' \leq \beta_* < \alpha_*$ with $\alpha_* \leq \alpha^*_{s,N}$. Therefore, by applying the Hölder's inequality, Lemma 2.3 and Lemma 2.6, we obtain from (3.4) that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{|F(x,u,v)|}{|x|^\gamma}\, \mathrm{d}x \\
&\leq \varepsilon \int_{\mathbb{R}^N} \frac{|z|^{\theta p}}{|x|^\gamma}\, \mathrm{d}x + \tilde{\kappa}_\varepsilon \int_{\mathbb{R}^N} \frac{|z|^{\vartheta} \Phi(\alpha_0 |z|^{p'})}{|x|^\gamma}\, \mathrm{d}x \\
&\leq 2^{\theta p} \mathcal{B}^{-\theta p}_{\theta p,\gamma} \varepsilon \|(u,v)\|^{\theta p} + \tilde{\kappa}_\varepsilon \left(\int_{\mathbb{R}^N} \frac{|z|^{\vartheta t}}{|x|^\gamma}\, \mathrm{d}x \right)^{\frac{1}{t}} \left(\int_{\mathbb{R}^N} \frac{C\, \Phi(\alpha_0 t' |z|^{p'})}{|x|^\gamma}\, \mathrm{d}x \right)^{\frac{1}{t'}} \\
&\leq 2^{\theta p} \mathcal{B}^{-\theta p}_{\theta p,\gamma} \varepsilon \|(u,v)\|^{\theta p} + 2^{\vartheta} C^{\frac{1}{t'}} \mathcal{B}^{-\vartheta}_{\vartheta t,\gamma} \tilde{\kappa}_\varepsilon \|(u,v)\|^{\vartheta} \left(\int_{\mathbb{R}^N} \frac{\Phi(\alpha_0 t' \|\psi\|^{p'}_{W^{s,p}} |\widetilde{\psi}|^{p'})}{|x|^\gamma}\, \mathrm{d}x \right)^{\frac{1}{t'}} \\
&\leq 2^{\theta p} \mathcal{B}^{-\theta p}_{\theta p,\gamma} \varepsilon \|(u,v)\|^{\theta p} + 2^{\vartheta} D C^{\frac{1}{t'}} \mathcal{B}^{-\vartheta}_{\vartheta t,\gamma} \tilde{\kappa}_\varepsilon \|(u,v)\|^{\vartheta},
\end{aligned}$$

where $D > 0$ is a constant, thanks to Theorem 2.7. Gathering all the above information and using Lemma 2.5, we obtain for $0 < \|(u,v)\| \leq \bar{\delta}$ that

$$J_{\lambda,\mu}(u,v) \geq \left(\frac{\widehat{M}(1)}{p} - 2^{\theta p} \mathcal{B}^{-\theta p}_{\theta p,\gamma} \varepsilon \right) \|(u,v)\|^{\theta p} - \widetilde{C} \|(u,v)\|^{\vartheta} - \frac{1}{q} \max\{\lambda, \mu\} \mathcal{S}^{-q}_{q,h} \|(u,v)\|^q, \tag{3.6}$$

where $\widetilde{C} = 2^{\vartheta} D C^{\frac{1}{t'}} \mathcal{B}^{-\vartheta}_{\vartheta t,\gamma} \tilde{\kappa}_\varepsilon$. Take $\varepsilon = \widehat{M}(1) / 2^{\theta p+1} p \mathcal{B}^{-\theta p}_{\theta p,\gamma}$, and consider the function $g : [0, \bar{\delta}] \to \mathbb{R}$ as follows:

$$g(\ell) = \frac{\widehat{M}(1)}{2(p+1)^2} \ell^{\theta p} - \widetilde{C} \ell^{\vartheta}, \qquad \forall\, \ell \in [0, \bar{\delta}].$$

Notice that g admits a positive maximum j in $[0, \bar{\delta}]$ at a point $\rho \in (0, \bar{\delta}]$. This shows that for all $(u,v) \in \mathbf{X}$ with $\|(u,v)\| = \rho$, one has

$$J_{\lambda,\mu}(u,v) \geq \frac{\widehat{M}(1)}{2p} \rho^{\theta p} - \widetilde{C} \rho^{\vartheta} - \frac{1}{q} \max\{\lambda, \mu\} \mathcal{S}^{-q}_{q,h} \rho^q \geq g(\rho) = j > 0,$$

for all $\lambda > 0$ and $\mu > 0$ with $\max\{\lambda, \mu\} \leq \lambda_*$, where λ_* is given as follows

$$\lambda_* = \frac{q \widehat{M}(1)}{2(p+1) \mathcal{S}^{-q}_{q,h}} \rho^{\theta p - q}.$$

This finishes the proof. □

Lemma 3.3. *For all $\lambda > 0$ and $\mu > 0$ with $\max\{\lambda, \mu\} \leq \lambda_*$, there holds*

$$m_{\lambda,\mu} = \inf\{J_{\lambda,\mu}(u,v) : (u,v) \in \overline{B}_\rho\} < 0.$$

where

$$B_\rho = \{(u,v) \in \mathbf{X} : \|(u,v)\| < \rho\}.$$

Finally, there exists a nonnegative sequence $\{(u_n, v_n)\}_n$ and some nonnegative function $(u_{\lambda,\mu}, v_{\lambda,\mu})$ in $\overline{B}_\rho$ such that for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\|(u_n,v_n)\| < \rho,\ m_{\lambda,\mu} \leq J_{\lambda,\mu}(u_n,v_n) \leq m_{\lambda,\mu} + \frac{1}{n},\ (u_n,v_n) \rightharpoonup (u_{\lambda,\mu}, v_{\lambda,\mu}) \quad \text{in } \mathbf{X}, \\
&(u_n,v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu}) \quad \text{a.e. in} \quad \mathbb{R}^N \quad \text{and} \quad J'_{\lambda,\mu}(u_n,v_n) \to 0 \quad \text{in } \mathbf{X}^* \quad \text{as } n \to \infty.
\end{aligned} \tag{3.7}$$

Proof. Fix $\lambda > 0$ and $\mu > 0$ with $\max\{\lambda, \mu\} \le \lambda_*$ and a pair $(u, v) \in \mathbf{X}$ such that $\|(u, v)\| \le \rho$. Let $\tau \in [0, 1]$, then by using (F_1), we have

$$J_{\lambda,\mu}(\tau u, \tau v) \le \frac{1}{p}\widehat{M}(\|\tau(u, v)\|^p) - \frac{\tau^q}{q}(\lambda\|u^+\|^q_{q,h} + \mu\|v^+\|^q_{q,h})$$
$$\le \tau^p\left(\frac{\rho^p}{p}\max_{s\in[0,\rho^p]} M(s)\right) - \frac{\tau^q}{q}(\lambda\|u^+\|^q_{q,h} + \mu\|v^+\|^q_{q,h}).$$

Since $1 < q < p$, it follows that $J_{\lambda,\mu}(\tau u, \tau v) < 0$ for $\tau \in (0, 1]$ sufficiently small enough. This together with (3.6) implies that

$$-\infty < m_{\lambda,\mu} = \inf\{J_{\lambda,\mu}(u, v) : (u, v) \in \overline{B}_\rho\} < 0. \tag{3.8}$$

This shows that the functional $J_{\lambda,\mu}$ is bounded from below and of class C^1 on $\overline{B}_\rho$. In addition, we also know that $\overline{B}_\rho$ is a complete metric space with the metric defined by the norm of $\mathbf{X}$. Due to Lemma 3.2, we obtain that

$$\inf_{\partial B_\rho} J_{\lambda,\mu}(u, v) \ge j > 0. \tag{3.9}$$

In virtue of (3.8) and (3.9), for n large enough, we can assume that

$$\frac{1}{n} \in \left(0, \inf_{\partial B_\rho} J_{\lambda,\mu}(u, v) - \inf_{\overline{B}_\rho} J_{\lambda,\mu}(u, v)\right). \tag{3.10}$$

Now applying Ekeland variational principle [21] to the functional $J_{\lambda,\mu} : \overline{B}_\rho \to \mathbb{R}$, we can find a sequence $\{(u_n, v_n)\}_n$ in $\overline{B}_\rho$ such that

$$m_{\lambda,\mu} \le J_{\lambda,\mu}(u_n, v_n) \le m_{\lambda,\mu} + \frac{1}{n} \quad \text{and} \quad J_{\lambda,\mu}(u_n, v_n) \le J_{\lambda,\mu}(u, v) + \frac{1}{n}\|(u_n, v_n) - (u, v)\| \tag{3.11}$$

for all $(u, v) \in \overline{B}_\rho$ satisfying $(u, v) \ne (u_n, v_n)$ for each $n \in \mathbb{N}$. Consequently, we obtain from (3.10) and (3.11) that

$$J_{\lambda,\mu}(u_n, v_n) \le m_{\lambda,\mu} + \frac{1}{n} = \inf_{\overline{B}_\rho} J_{\lambda,\mu}(u, v) + \frac{1}{n} < \inf_{\partial B_\rho} J_{\lambda,\mu}(u, v).$$

It follows that $\{(u_n, v_n)\}_n \subset B_\rho$, that is, $\|(u_n, v_n)\| < \rho$ for all $n \in \mathbb{N}$. Suppose $(\varphi, \psi) \in S$, where $S = \{(\varphi, \psi) \in \mathbf{X} : \|(\varphi, \psi)\| = 1\}$, and $t > 0$ sufficiently small such that $(u_n + t\varphi, v_n + t\psi) \in \overline{B}_\rho$ for all $n \in \mathbb{N}$. By using (3.11), we can notice that

$$\langle J'_{\lambda,\mu}(u_n, v_n), (\varphi, \psi)\rangle = \lim_{t\to 0^+} \frac{J_{\lambda,\mu}(u_n + t\varphi, v_n + t\psi) - J_{\lambda,\mu}(u_n, v_n)}{t} \ge -\frac{1}{n}, \qquad \forall\, (\varphi, \psi) \in S.$$

The arbitrariness of $(\varphi, \psi) \in S$ infers that

$$|\langle J'_{\lambda,\mu}(u_n, v_n), (\varphi, \psi)\rangle| \le \frac{1}{n}, \qquad \forall\, (\varphi, \psi) \in S.$$

This together with (3.11) implies at once that

$$J_{\lambda,\mu}(u_n, v_n) \to m_{\lambda,\mu} \quad \text{and} \quad J'_{\lambda,\mu}(u_n, v_n) \to 0 \quad \text{in } \mathbf{X}^* \quad \text{as } n \to \infty. \tag{3.12}$$

On the other hand, it is easy to see that the sequence $\{(u_n, v_n)\}_n$ is bounded, and thus up to a subsequence still denoted itself and $(u_{\lambda,\mu}, v_{\lambda,\mu}) \in \overline{B}_\rho$ such that $(u_n, v_n) \rightharpoonup (u_{\lambda,\mu}, v_{\lambda,\mu})$ in $\mathbf{X}$ as

$n \to \infty$. In view of Lemma 2.2, we have $(u_n, v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu})$ in $L^\nu(\mathbb{R}^N) \times L^\nu(\mathbb{R}^N)$ as $n \to \infty$ for all $\nu \in [p, \infty)$. Further, since the maps $u \mapsto u^\pm$ are continuous from $L^\nu(\mathbb{R}^N)$ into itself, therefore we conclude that $u_n \to u_{\lambda,\mu}, u_n^\pm \to u_{\lambda,\mu}^\pm, v_n \to v_{\lambda,\mu}$, and $v_n^\pm \to v_{\lambda,\mu}^\pm$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. In addition, from (3.12), we infer that

$$\langle J'_{\lambda,\mu}(u_n, v_n), (-u_n^-, -v_n^-)\rangle \to 0 \quad \text{as } n \to \infty.$$

Now using a similar strategy developed in Lemma 3.1, we obtain

$$M(\|(u_n, v_n)\|^p)(\|(u_n^-, v_n^-)\|^p) = o_n(1) \quad \text{as } n \to \infty. \tag{3.13}$$

Hence we have two possibilities, that is, either $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = d > 0$ or $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = 0$.

Case 1. Let $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = d > 0$. Denote $\kappa = \kappa(\tau)$ as the number corresponding to $\tau = d^p$ in (M_1) such that

$$M(\|(u_n, v_n)\|^p) \geq \kappa, \qquad \forall\, n \in \mathbb{N}. \tag{3.14}$$

In virtue of (3.13) and (3.14), we conclude that $(u_n^-, v_n^-) \to (0, 0)$ in $\mathbf{X}$. It follows that $u_{\lambda,\mu}^- = v_{\lambda,\mu}^- = 0$ a.e. in $\mathbb{R}^N$ and thus the pair $(u_{\lambda,\mu}, v_{\lambda,\mu})$ is nonnegative in $\mathbb{R}^N$. Consequently, without loss of generality, we can assume that $(u_n, v_n) = (u_n^+, v_n^+)$, thanks to $(u_n^-, v_n^-) \to (0, 0)$ in $\mathbf{X}$. This shows that the sequence $\{(u_n, v_n)\}_n$ is nonnegative.

Case 2. Let $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = 0$, then either $(0, 0)$ is an accumulation point of the sequence $\{(u_n, v_n)\}_n$ or $(0, 0)$ is an isolated point of the sequence $\{(u_n, v_n)\}_n$. If $(0, 0)$ is an accumulation point of $\{(u_n, v_n)\}_n$, then up to a subsequence still denoted by itself such that it strongly converges to $(u_{\lambda,\mu}, v_{\lambda,\mu}) = (0, 0)$. This situation is impossible. Indeed, if not, then $0 = J_{\lambda,\mu}(0, 0) = m_{\lambda,\mu} < 0$, which is a contradiction. On the other hand, if $(0, 0)$ is an isolated point of $\{(u_n, v_n)\}_n$, then there exists a subsequence still denoted by itself such that $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| > 0$. In this situation, we can proceed as in Case 1 to conclude the proof.

From the above discussions, we infer that the sequence $\{(u_n, v_n)\}_n$ and $(u_{\lambda,\mu}, u_{\lambda,\mu})$ are nonnegative. This finishes the proof. □

Proof of Theorem 1.3. Due to Lemma 3.3, there exists a nonnegative sequence $\{(u_n, v_n)\}_n$ in $\overline{B}_\rho$ such that (3.7) holds, that is, we have

$$J_{\lambda,\mu}(u_n, v_n) \to m_{\lambda,\mu}\ (< 0) \quad \text{and} \quad J'_{\lambda,\mu}(u_n, v_n) \to 0 \quad \text{in } \mathbf{X}^* \quad \text{as } n \to \infty.$$

Evidently, we have two situations according to the behaviour of the Kirchhoff function M: either $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = d > 0$ or $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = 0$. Hence we divide the proof into two parts.

Case 1. Let $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = d > 0$. Denote $\kappa = \kappa(\tau)$ as the number corresponding to $\tau = d^p$ in (M_1) such that

$$M(\|(u_n, v_n)\|^p) \geq \kappa, \qquad \forall\, n \in \mathbb{N}. \tag{3.15}$$

Let $\lambda > 0$ and $\mu > 0$ be such that $\max\{\lambda, \mu\} < \widetilde{\lambda}$ with $\widetilde{\lambda} = \min\{\lambda_*, \lambda_0\}$, where λ_* as in Lemma 3.2, while λ_0 is defined by

$$\lambda_0 = \frac{q\kappa(\sigma - \theta p)\big(\min\{V_0, K_0\}\big)^{\frac{p-q}{p}}}{2^{p-q}(\sigma - q)\theta p \mathcal{S}_{q,h}^{-q}} \left(\frac{\beta_*}{2\alpha_0}\right)^{\frac{p-q}{p'}} > 0. \tag{3.16}$$

By using (3.7) and Lemma 2.2, Lemma 2.3 and Lemma 2.5, one has

$$\begin{cases} (u_n, v_n) \rightharpoonup (u_{\lambda,\mu}, v_{\lambda,\mu}) & \text{in } \mathbf{X}; \\ (u_n, v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu}) & \text{in } L^{\nu}(\mathbb{R}^N) \times L^{\nu}(\mathbb{R}^N),\ \forall\, \nu \in [p, \infty); \\ (u_n, v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu}) & \text{in } L^{\xi}(\mathbb{R}^N, |x|^{-\gamma}\,\mathrm{d}x) \times L^{\xi}(\mathbb{R}^N, |x|^{-\gamma}\,\mathrm{d}x),\ \forall\, \xi \in [p, \infty); \\ (u_n, v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu}) & \text{in } L^q_h(\mathbb{R}^N) \times L^q_h(\mathbb{R}^N); \\ (u_n, v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu}) & \text{a.e. in } \mathbb{R}^N \quad \text{as} \quad n \to \infty. \end{cases} \tag{3.17}$$

For each $(u, v) \in \mathbf{X}$, we define the functional $L(u, v) : \mathbf{X} \to \mathbb{R}$ by

$$\langle L(u, v), (\varphi, \psi) \rangle = \langle u, \varphi \rangle_{K_p, V} + \langle v, \psi \rangle_{K_p, V}, \qquad \forall\, (\varphi, \psi) \in \mathbf{X}.$$

Recalling the elementary inequality as follows

$$a^{\alpha} c^{1-\alpha} + b^{\alpha} d^{1-\alpha} \le (a+b)^{\alpha} (c+d)^{1-\alpha}, \qquad \forall\, a, b, c, d \ge 0 \quad \text{and} \quad \alpha \in (0, 1). \tag{3.18}$$

By applying the Hölder's inequality and (3.18), it is not difficult to show that

$$\left| \langle L(u, v), (\varphi, \psi) \rangle \right| \le \|(u, v)\|^{p-1} \|(\varphi, \psi)\|, \qquad \forall\, (\varphi, \psi) \in \mathbf{X}.$$

Thus, the definition of L implies that for each $(u, v) \in \mathbf{X}$, $L(u, v)$ is a continuous linear functional on $\mathbf{X}$. Consequently, the weak convergence of $\{(u_n, v_n)\}_n$ in $\mathbf{X}$ gives that

$$\langle L(u_{\lambda,\mu}, v_{\lambda,\mu}), (u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu}) \rangle = o_n(1) \quad \text{as } n \to \infty. \tag{3.19}$$

Furthermore, due to (3.17), there exists $l_p \ge 0$ such that up to a subsequence still denoted by same symbol, we have $\|(u_n, v_n)\| \to l_p$ as $n \to \infty$. In virtue of (3.7), we can also deduce that

$$\langle J'_{\lambda,\mu}(u_n, v_n), (u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu}) \rangle = o_n(1) \quad \text{as } n \to \infty.$$

On simplifying the above convergence, we have

$$\begin{aligned} & M(\|(u_n, v_n)\|^p) \langle L(u_n, v_n), (u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu}) \rangle \\ & \quad - \int_{\mathbb{R}^N} \frac{\nabla F(x, u_n, v_n).(u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})}{|x|^{\gamma}} \,\mathrm{d}x - \lambda \int_{\mathbb{R}^N} h(x) u_n^{q-1} (u_n - u_{\lambda,\mu}) \,\mathrm{d}x \\ & \quad - \mu \int_{\mathbb{R}^N} h(x) v_n^{q-1} (v_n - v_{\lambda,\mu}) \,\mathrm{d}x = o_n(1) \quad \text{as } n \to \infty. \end{aligned} \tag{3.20}$$

Using the notation of Lemma 2.5, (F_4), (M_2) and (3.15), we obtain from (3.7) that as $n \to \infty$

$$\begin{aligned} 0 > m_{\lambda,\mu} &= J_{\lambda,\mu}(u_n, v_n) - \frac{1}{\sigma} \langle J'_{\lambda,\mu}(u_n, v_n), (u_n, v_n) \rangle + o_n(1) \\ &\ge \frac{1}{p} \widehat{M}(\|(u_n, v_n)\|^p) - \frac{1}{\sigma} M(\|(u_n, v_n)\|^p) \|(u_n, v_n)\|^p \\ &\quad - \max\{\lambda, \mu\} \left(\frac{1}{q} - \frac{1}{\sigma} \right) \mathcal{S}_{q,h}^{-q} \|(u_n, v_n)\|^q + o_n(1) \\ &\ge \left(\frac{1}{\theta p} - \frac{1}{\sigma} \right) \kappa \|(u_n, v_n)\|^p - \max\{\lambda, \mu\} \left(\frac{1}{q} - \frac{1}{\sigma} \right) \mathcal{S}_{q,h}^{-q} \|(u_n, v_n)\|^q + o_n(1). \end{aligned}$$

Define $z_n = (u_n, v_n)$ and $z = (u_{\lambda,\mu}, v_{\lambda,\mu})$. Then it is easy to see that $|z_n| = \sqrt{u_n^2 + v_n^2} \le \psi_n := |u_n| + |v_n|$. By direct calculation, one has

$$\|\psi_n\|_{W^{s,p}} \le 2(\min\{V_0, K_0\})^{-\frac{1}{p}} \|(u_n, v_n)\|.$$

Hence from the above two inequalities and (3.16), we can easily deduce that

$$\limsup_{n\to\infty} \|\psi_n\|_{W^{s,p}}^{p'} \leq \left(\frac{2^{p-q}(\sigma-q)\theta p \mathcal{S}_{q,h}^{-q} \max\{\lambda,\mu\}}{q\kappa(\sigma-\theta p)\big(\min\{V_0,K_0\}\big)^{\frac{p-q}{p}}} \right)^{\frac{p'}{p-q}} < \frac{\beta_*}{2\alpha_0}. \tag{3.21}$$

By using the Hölder's inequality, (3.17) and (3.18), we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} h(x)\{\lambda u_n^{q-1}(u_n-u_{\lambda,\mu}) + \mu v_n^{q-1}(v_n-v_{\lambda,\mu})\}\, \mathrm{d}x \right| \\
&\quad \leq \max\{\lambda,\mu\}\left[\|u_n\|_{q,h}^{q-1}\|u_n-u_{\lambda,\mu}\|_{q,h} + \|v_n\|_{q,h}^{q-1}\|v_n-v_{\lambda,\mu}\|_{q,h}\right] \\
&\quad \leq \max\{\lambda,\mu\}\|(u_n,v_n)\|_{L_h^q(\mathbb{R}^N)\times L_h^q(\mathbb{R}^N)}^{q-1}\|(u_n-u_{\lambda,\mu},v_n-v_{\lambda,\mu})\|_{L_h^q(\mathbb{R}^N)\times L_h^q(\mathbb{R}^N)} \\
&\quad \leq \max\{\lambda,\mu\}\mathcal{S}_{q,h}^{-(q-1)}\sup_{n\in\mathbb{N}}\|(u_n,v_n)\|^{q-1}\|(u_n-u_{\lambda,\mu},v_n-v_{\lambda,\mu})\|_{L_h^q(\mathbb{R}^N)\times L_h^q(\mathbb{R}^N)} \\
&\quad \to 0 \quad \text{as } n\to\infty,
\end{aligned}$$

thanks to the boundedness of the sequence $\{(u_n,v_n)\}_n$ in $\mathbf{X}$. It follows at once that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} h(x)\{\lambda u_n^{q-1}(u_n-u_{\lambda,\mu}) + \mu v_n^{q-1}(v_n-v_{\lambda,\mu})\}\, \mathrm{d}x = 0. \tag{3.22}$$

Set

$$I_1 = \varepsilon\int_{\mathbb{R}^N} \frac{|z_n|^{\theta p-1}|z_n-z|}{|x|^\gamma}\, \mathrm{d}x \quad \text{and} \quad I_2 = \kappa_\varepsilon \int_{\mathbb{R}^N} \frac{\Phi(\alpha_0|z_n|^{p'})|z_n-z|}{|x|^\gamma}\, \mathrm{d}x.$$

Invoking the Hölder's inequality, we obtain by using (3.17) and the boundedness of $\{(u_n,v_n)\}_n$ in $\mathbf{X}$ that

$$\begin{aligned}
I_1 &\leq \varepsilon\left(\int_{\mathbb{R}^N}\frac{|z_n|^{\theta p}}{|x|^\gamma}\,\mathrm{d}x\right)^{\frac{\theta p-1}{\theta p}}\left(\int_{\mathbb{R}^N}\frac{|z_n-z|^{\theta p}}{|x|^\gamma}\,\mathrm{d}x\right)^{\frac{1}{\theta p}} \\
&\leq 2^{\theta p}\varepsilon\|(u_n,v_n)\|_{L^{\theta p}(\mathbb{R}^N,|x|^{-\gamma}\,\mathrm{d}x)\times L^{\theta p}(\mathbb{R}^N,|x|^{-\gamma}\,\mathrm{d}x)}^{\theta p-1} \\
&\qquad \times \|(u_n-u_{\lambda,\mu},v_n-v_{\lambda,\mu})\|_{L^{\theta p}(\mathbb{R}^N,|x|^{-\gamma}\,\mathrm{d}x)\times L^{\theta p}(\mathbb{R}^N,|x|^{-\gamma}\,\mathrm{d}x)} \\
&\leq 2^{\theta p}\varepsilon\mathcal{B}_{\theta p,\gamma}^{-(\theta p-1)}\sup_{n\in\mathbb{N}}\|(u_n,v_n)\|^{\theta p-1} \\
&\qquad \times \|(u_n-u_{\lambda,\mu},v_n-v_{\lambda,\mu})\|_{L^{\theta p}(\mathbb{R}^N,|x|^{-\gamma}\,\mathrm{d}x)\times L^{\theta p}(\mathbb{R}^N,|x|^{-\gamma}\,\mathrm{d}x)} \\
&\to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

Define $\tilde{\psi}_n = \psi_n/\|\psi_n\|_{W^{s,p}}$. Suppose $t\geq p$ and $t' = \frac{t}{t-1} > 1$ such that $\frac{1}{t}+\frac{1}{t'} = 1$. Due to (3.21), there exist $m>0$ and $n_0\in\mathbb{N}$ such that $\|\psi_n\|_{W^{s,p}}^{p'} < m < \beta_*/2\alpha_0$ for all $n\geq n_0$. Now choose $t'>1$ close to 1 in such a way that there holds $\alpha_0 t'\|\psi_n\|_{W^{s,p}}^{p'} < t'\beta_*/2 < \alpha_*$ for all $n\geq n_0$ with

$\alpha_* \leq \alpha^*_{s,N}$. Using the Hölder's inequality, Lemma 2.6 and (3.17), we get

$$\begin{aligned}
I_2 &\leq \kappa_\varepsilon \left\| \frac{\Phi(\alpha_0 |z_n|^{p'})}{|x|^{\frac{\gamma}{t'}}} \right\|_{t'} \left(\int_{\mathbb{R}^N} \frac{|z_n - z|^t}{|x|^\gamma}\, \mathrm{d}x \right)^{\frac{1}{t}} \\
&\leq C^{\frac{1}{t'}} \kappa_\varepsilon \left(\int_{\mathbb{R}^N} \frac{\Phi(\alpha_0 t' |\psi_n|^{p'})}{|x|^\gamma}\, \mathrm{d}x \right)^{\frac{1}{t'}} \left(\int_{\mathbb{R}^N} \frac{|z_n - z|^t}{|x|^\gamma}\, \mathrm{d}x \right)^{\frac{1}{t}} \\
&\leq 2C^{\frac{1}{t'}} \kappa_\varepsilon \left(\int_{\mathbb{R}^N} \frac{\Phi(\alpha_0 t' \|\psi_n\|^{p'}_{W^{s,p}} |\tilde{\psi}_n|^{p'})}{|x|^\gamma}\, \mathrm{d}x \right)^{\frac{1}{t'}} \\
&\quad \times \|(u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})\|_{L^t(\mathbb{R}^N, |x|^{-\gamma}\, \mathrm{d}x) \times L^t(\mathbb{R}^N, |x|^{-\gamma}\, \mathrm{d}x)} \\
&\leq \hat{C} \|(u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})\|_{L^t(\mathbb{R}^N, |x|^{-\gamma}\, \mathrm{d}x) \times L^t(\mathbb{R}^N, |x|^{-\gamma}\, \mathrm{d}x)} \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

where

$$\hat{C} = 2C^{\frac{1}{t'}} \kappa_\varepsilon \left(\sup_{n \geq n_0} \int_{\mathbb{R}^N} \frac{\Phi(\alpha_0 t' \|\psi_n\|^{p'}_{W^{s,p}} |\tilde{\psi}_n|^{p'})}{|x|^\gamma}\, \mathrm{d}x \right)^{\frac{1}{t'}} < +\infty,$$

thanks to Theorem 2.7. Consequently, we obtain from (F_2) that

$$\left| \int_{\mathbb{R}^N} \frac{\nabla F(x, u_n, v_n).(u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})}{|x|^\gamma}\, \mathrm{d}x \right| \leq I_1 + I_2 \to 0 \quad \text{as } n \to \infty.$$

It follows that

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{\nabla F(x, u_n, v_n).(u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})}{|x|^\gamma}\, \mathrm{d}x = 0. \tag{3.23}$$

Passing limit $n \to \infty$ in (3.20) and using (3.22) as well as (3.23), we get

$$M(\|(u_n, v_n)\|^p) \langle L(u_n, v_n), (u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu}) \rangle = o_n(1) \quad \text{as } n \to \infty.$$

It follows from $\inf_{n \in \mathbb{N}} \|(u_n, v_n)\| = d > 0$ that $l_p > 0$. This shows that $M(\|(u_n, v_n)\|^p) \to M(l_p^p) > 0$ as $n \to \infty$, thanks to the continuity of M. In conclusion, from the above convergence, we deduce that

$$\big\langle L(u_n, v_n), (u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu}) \big\rangle = o_n(1) \quad \text{as} \quad n \to \infty. \tag{3.24}$$

On combining (3.19) and (3.23), we obtain

$$\big\langle L(u_n, v_n) - L(u_{\lambda,\mu}, v_{\lambda,\mu}), (u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu}) \big\rangle = o_n(1) \quad \text{as } n \to \infty. \tag{3.25}$$

Recall the well-known Simon inequality [42] as follows:

$$|\xi - \eta|^p \leq \begin{cases} C_p(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta).(\xi - \eta) & \text{for } p \geq 2, \\ \widetilde{C}_p[(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta).(\xi - \eta)]^{\frac{p}{2}} (|\xi|^p + |\eta|^p)^{\frac{2-p}{2}} & \text{for } 1 < p < 2, \end{cases} \tag{3.26}$$

where $\xi, \eta \in \mathbb{R}^N$ and C_p as well as $\widetilde{C}_p$ are positive constants depending only upon p. In virtue of the Simon inequality, we distinguish the following two situations :

Situation 1. Let $p \geq 2$, then we obtain from (3.25) and (3.26) that

$$\begin{aligned}
\|(u_n, v_n) - (u_{\lambda,\mu}, v_{\lambda,\mu})\|^p &\leq C_p \big\langle L(u_n, v_n) - L(u_{\lambda,\mu}, v_{\lambda,\mu}), (u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu}) \big\rangle \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

It follows that $(u_n, v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu})$ in $\mathbf{X}$ as $n \to \infty$ for any $p \geq 2$.

Situation 2. Let $1 < p < 2$. First, recall the following elementary inequality:

$$(a+b)^{\frac{2-p}{2}} \leq a^{\frac{2-p}{2}} + b^{\frac{2-p}{2}}, \qquad \forall\, a, b \geq 0 \quad \text{and} \quad 1 < p < 2. \tag{3.27}$$

Using the Hölder's inequality, (3.25), (3.26) and (3.27), we obtain

$$\begin{aligned}
&\|(u_n, v_n) - (u_{\lambda,\mu}, v_{\lambda,\mu})\|^p \\
&\quad \leq \widetilde{C}_p[\langle L(u_n, v_n) - L(u_{\lambda,\mu}, v_{\lambda,\mu}), (u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})\rangle]^{\frac{p}{2}} \\
&\qquad \times \left(\|(u_n, v_n)\|^p + \|(u_{\lambda,\mu}, v_{\lambda,\mu})\|^p\right)^{\frac{2-p}{2}} \\
&\quad \leq \widetilde{C}_p[\langle L(u_n, v_n) - L(u_{\lambda,\mu}, v_{\lambda,\mu}), (u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})\rangle]^{\frac{p}{2}} \\
&\qquad \times \left(\|(u_n, v_n)\|^{\frac{(2-p)p}{2}} + \|(u_{\lambda,\mu}, v_{\lambda,\mu})\|^{\frac{(2-p)p}{2}}\right) \\
&\quad \leq C[\langle L(u_n, v_n) - L(u_{\lambda,\mu}, v_{\lambda,\mu}), (u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})\rangle]^{\frac{p}{2}} \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

where

$$C = \widetilde{C}_p\left(\sup_{n\in\mathbb{N}} \|(u_n, v_n)\|^{\frac{(2-p)p}{2}} + \|(u_{\lambda,\mu}, v_{\lambda,\mu})\|^{\frac{(2-p)p}{2}}\right) < +\infty,$$

thanks to the uniform boundedness of the sequence

$$\left\{\|(u_n, v_n)\|^{\frac{(2-p)p}{2}} + \|(u_{\lambda,\mu}, v_{\lambda,\mu})\|^{\frac{(2-p)p}{2}}\right\}_n \quad \text{in } \mathbb{R}.$$

This shows at once that $(u_n, v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu})$ in $\mathbf{X}$ as $n \to \infty$ for any $1 < p < 2$.

From the above discussions, we conclude that $(u_n, v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu})$ in $\mathbf{X}$ as $n \to \infty$. Now we shall discuss about the situation when $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = 0$.

Case 2. Let $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = 0$. If $(0, 0)$ is an isolated point of the sequence $\{(u_n, v_n)\}_n$, then there exists a subsequence still denoted by the same symbol such that

$$\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = d > 0.$$

In this scenario, we can proceed as before. On the other hand, if $(0, 0)$ is an accumulation point of $\{(u_n, v_n)\}_n$, then up to a subsequence still denoted by itself such that it strongly converges to $(u_{\lambda,\mu}, v_{\lambda,\mu}) = (0, 0)$. This situation is impossible. Indeed, if not, then $0 = J_{\lambda,\mu}(0, 0) = m_{\lambda,\mu} < 0$, which is a contradiction.

By using $(u_n, v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu})$ in $\mathbf{X}$ as $n \to \infty$, we deduce that $J_{\lambda,\mu}(u_{\lambda,\mu}, v_{\lambda,\mu}) = m_{\lambda,\mu}$ and $J'_{\lambda,\mu}(u_{\lambda,\mu}, v_{\lambda,\mu}) = 0$, thanks to $J_{\lambda,\mu} \in C^1(\mathbf{X}, \mathbb{R})$. This shows that $(u_{\lambda,\mu}, v_{\lambda,\mu}) \in \overline{B}_\rho$ and thus the solution $(u_{\lambda,\mu}, v_{\lambda,\mu})$ of the system $(\mathcal{S}_{\lambda,\mu})$ is also a minimizer of the functional $J_{\lambda,\mu}$ in $\overline{B}_\rho$. Consequently, we deduce that

$$J_{\lambda,\mu}(u_{\lambda,\mu}, v_{\lambda,\mu}) = m_{\lambda,\mu} < 0 < j \leq J_{\lambda,\mu}(u, v), \qquad \forall\, (u, v) \in \partial B_\rho,$$

thanks to Lemma 3.2. It follows that $(u_{\lambda,\mu}, v_{\lambda,\mu}) \in B_\rho$ and hence $(u_{\lambda,\mu}, v_{\lambda,\mu})$ is a nontrivial nonnegative solution of the system $(\mathcal{S}_{\lambda,\mu})$ for all $\lambda > 0$ and $\mu > 0$ such that $\max\{\lambda, \mu\} < \widetilde{\lambda}$.

On the other hand, one can see that $\{(u_{\lambda,\mu}, v_{\lambda,\mu})\}_{(\lambda,\mu)\in(0,\widetilde{\lambda})\times(0,\widetilde{\lambda})}$ is uniformly bounded in $\mathbf{X}$, thanks to the fact that $\rho > 0$, which is independent of $\lambda > 0$ and $\mu > 0$, as specified in

Lemma 3.2. In virtue of (F_4), (M_2), (3.15) and using $(u_n, v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu})$ in $\mathbf{X}$ as $n \to \infty$, one has

$$\begin{aligned}
0 > m_{\lambda,\mu} &= \lim_{n\to\infty} \left[J_{\lambda,\mu}(u_n, v_n) - \frac{1}{\sigma} \langle J'_{\lambda,\mu}(u_n, v_n), (u_n, v_n) \rangle \right] \\
&\geq \frac{1}{p} \widehat{M}(\|(u_{\lambda,\mu}, v_{\lambda,\mu})\|^p) - \frac{1}{\sigma} M(\|(u_{\lambda,\mu}, v_{\lambda,\mu})\|^p) \|(u_{\lambda,\mu}, v_{\lambda,\mu})\|^p \\
&\quad - \max\{\lambda, \mu\} \left(\frac{1}{q} - \frac{1}{\sigma} \right) \mathcal{S}_{q,h}^{-q} \|(u_{\lambda,\mu}, v_{\lambda,\mu})\|^q \\
&\geq \left(\frac{1}{\theta p} - \frac{1}{\sigma} \right) \kappa \|(u_{\lambda,\mu}, v_{\lambda,\mu})\|^p - \max\{\lambda, \mu\} \mathcal{D}_{q,h},
\end{aligned}$$

where

$$\mathcal{D}_{q,h} = \left(\frac{1}{q} - \frac{1}{\sigma} \right) \mathcal{S}_{q,h}^{-q} \sup_{(\lambda,\mu)\in(0,\widetilde{\lambda})\times(0,\widetilde{\lambda})} \|(u_{\lambda,\mu}, v_{\lambda,\mu})\|^q < +\infty.$$

This yields at once that

$$0 \geq \limsup_{(\lambda,\mu)\to(0^+,0^+)} m_{\lambda,\mu} \geq \limsup_{(\lambda,\mu)\to(0^+,0^+)} \left(\frac{1}{\theta p} - \frac{1}{\sigma} \right) \kappa \|(u_{\lambda,\mu}, v_{\lambda,\mu})\|^p \geq 0.$$

This implies that

$$\lim_{(\lambda,\mu)\to(0^+,0^+)} \|(u_{\lambda,\mu}, v_{\lambda,\mu})\| = 0,$$

and thus we conclude the proof. □

4 Proof of Theorem 1.4

Throughout this section, we assume without further mentioning that the structural assumptions required in Theorem 1.4 hold. To prove Theorem 1.4, we need the mountain pass theorem and some basic definitions, as stated below.

Definition 4.1 (Palais–Smale compactness condition)**.** Let X be a Banach space and $\mathcal{I} : X \to \mathbb{R}$ be a functional of class $C^1(X, \mathbb{R})$. We say that $\mathcal{I}$ satisfies the Palais–Smale compactness condition at a suitable level $c \in \mathbb{R}$, if for any sequence $\{u_n\}_n \subset X$ such that

$$\mathcal{I}(u_n) \to c \quad \text{and} \quad \sup_{\|\varphi\|_X=1} |\langle \mathcal{I}'(u_n), \varphi \rangle| \to 0 \quad \text{as } n \to \infty \tag{4.1}$$

has a strongly convergent subsequence in X. Note that if this strongly convergent subsequence exists only for some values of c, we say that $\mathcal{I}$ satisfies a local Palais–Smale condition. We also remark that the sequence $\{u_n\}_n \subset X$ satisfying (4.1) is known as a Palais–Smale sequence at level $c \in \mathbb{R}$ [$(PS)_c$ sequence in short].

Theorem 4.2 (The mountain pass theorem, cf. [41, 44])**.** *Let X be a real Banach space and $I \in C^1(X, \mathbb{R})$. Suppose $I(0) = 0$ and*

(a) there exist two constants $\alpha, \rho > 0$ such that $I(u) \geq \alpha$ for all $u \in X$ with $\|u\| = \rho$;

(b) there exists $e \in X$ satisfying $\|e\| > \rho$ such that $I(e) < 0$.

Define

$$\Gamma = \{\gamma_* \in C^1([0,1], X) : \ \gamma_*(0) = 1, \ \gamma_*(1) = e\}.$$

Then

$$c = \inf_{\gamma_* \in \Gamma} \max_{t \in [0,1]} I(\gamma_*(t)) \geq \alpha$$

and there exists a $(PS)_c$ sequence $\{u_n\}_n$ for I in X. Consequently, if I satisfies the Palais–Smale condition, then I possesses a critical value $c \geq \alpha$.

Now we shall verify the validity of the mountain pass geometrical structure of the functional $J_{\lambda,\mu}$.

Lemma 4.3 (Mountain Pass Geometry-I). *For all $\lambda > 0$ and $\mu > 0$, there exist numbers $j_0 > 0$ and $\rho_0 \in (0,1]$ very small enough such that $J_{\lambda,\mu}(u,v) \geq j_0$ for all $(u,v) \in \mathbf{X}$ with $\|(u,v)\| = \rho_0$.*

Proof. The assumption (M_2) gives that

$$\widehat{M}(t) \geq \widehat{M}(1)t^{\vartheta}, \qquad \forall\, t \in [0,1].$$

Suppose $\bar{\delta} \in (0,1]$ small enough and there holds $0 < \|(u,v)\| \leq \bar{\delta}$. Then by using the notation of Lemma 3.2, we can easily deduce that

$$\int_{\mathbb{R}^N} \frac{|F(x,u,v)|}{|x|^{\gamma}}\, \mathrm{d}x \leq 2^{\theta p} \mathcal{B}_{\theta p,\gamma}^{-\theta p} \varepsilon \|(u,v)\|^{\theta p} + 2^{\vartheta} D C^{\frac{1}{t'}} \mathcal{B}_{\vartheta t,\gamma}^{-\vartheta} \tilde{\kappa}_{\varepsilon} \|(u,v)\|^{\vartheta}$$

for any $0 < \|(u,v)\| \leq \bar{\delta}$. Consequently, from the above information, we obtain for any $0 < \|(u,v)\| \leq \bar{\delta}$ that

$$J_{\lambda,\mu}(u,v) \geq A\|(u,v)\|^{\theta p} - B\|(u,v)\|^{\vartheta} - C\|(u,v)\|^{q}, \tag{4.2}$$

where

$$A = \frac{\widehat{M}(1)}{p} - 2^{\theta p} \mathcal{B}_{\theta p,\gamma}^{-\theta p} \varepsilon, \quad B = 2^{\vartheta} D C^{\frac{1}{t'}} \mathcal{B}_{\vartheta t,\gamma}^{-\vartheta} \tilde{\kappa}_{\varepsilon} \quad \text{and} \quad C = \frac{1}{q} \max\{\lambda,\mu\} \mathcal{A}_q^{-q} \|h\|_{\infty}.$$

Choose $\varepsilon > 0$ sufficiently small such that $A > 0$. Suppose $\xi = \min\{\vartheta, q\}$, then using the fact that $0 < \|(u,v)\| \leq \bar{\delta} \leq 1$, we obtain from (4.2) that

$$J_{\lambda,\mu}(u,v) \geq A\|(u,v)\|^{\theta p} - (B+C)\|(u,v)\|^{\xi}, \quad \forall\, (u,v) \in \mathbf{X} \text{ with } 0 < \|(u,v)\| \leq \bar{\delta}. \tag{4.3}$$

Define the function

$$f(\ell) = A\ell^{\theta p} - (B+C)\ell^{\xi}, \qquad \forall\, \ell \in [0,\bar{\delta}].$$

Observe that f admits a positive maximum j_0 in $[0,\bar{\delta}]$ at a point $\rho_0 \in (0,\bar{\delta}]$. This shows that for all $(u,v) \in \mathbf{X}$ with $\|(u,v)\| = \rho_0$, we obtain from (4.3) that

$$J_{\lambda,\mu}(u,v) \geq A\rho_0^{\theta p} - (B+C)\rho_0^{\xi} = f(\rho_0) = j_0 > 0.$$

This completes the proof. □

Lemma 4.4 (Mountain Pass Geometry-II). *For all $\lambda > 0$ and $\mu > 0$, there exist a nonnegative nontrivial couple $(e_1,e_2) \in C_0^{\infty}(\mathbb{R}^N) \times C_0^{\infty}(\mathbb{R}^N)$ with both components nontrivial, and also independent of λ and μ such that $J_{\lambda,\mu}(e_1,e_2) < 0$ for all $\|(e_1,e_2)\| > \rho_0$, where ρ_0 is stated as in Lemma 4.3.*

Proof. Due to the assumption (M_2), we obtain

$$\widehat{M}(t) \leq \widehat{M}(1)t^{\theta}, \qquad \forall\, t \in [1,\infty).$$

Let $(u,v) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$ be a nonnegative nontrivial couple with both components nontrivial such that $\|(u,v)\| = 1$ and there holds $\int_{\mathbb{R}^N} \frac{F(x,u,v)}{|x|^\gamma}\,\mathrm{d}x > 0$, thanks to (F_1).

Define the map

$$\zeta : [1,\infty) \to \mathbb{R}$$

by

$$\zeta(t) = t^{-\sigma}F(x,tu,tv) - F(x,u,v),\ \forall\, t \geq 1 \quad \text{and} \quad u,v \in \mathbb{R}^+.$$

By direct computation, we have

$$\begin{aligned}
\zeta'(t) &= t^{-\sigma}F_u(x,tu,tv)u + t^{-\sigma}F_v(x,tu,tv)v - \sigma t^{-\sigma-1}F(x,tu,tv)\\
&= t^{-\sigma-1}[F_u(x,tu,tv)tu + F_v(x,tu,tv)tv - \sigma F(x,tu,tv)]\\
&= t^{-\sigma-1}[\nabla F(x,tu,tv).(tu,tv) - \sigma F(x,tu,tv)].
\end{aligned}$$

This together with (F_4) implies at once that $\zeta'(t) \geq 0$ for all $t \geq 1$ and $u,v \in \mathbb{R}^+$. It follows that the map $[1,\infty) \ni t \mapsto \zeta(t)$ is monotonically increasing. In conclusion, we have

$$F(x,tu,tv) \geq t^{\sigma}F(x,u,v), \quad \forall\, t \geq 1 \quad \text{and} \quad u,v \in \mathbb{R}^+.$$

Choosing $t > \rho_0$ sufficiently large enough, we get by using the above information that

$$\begin{aligned}
J_{\lambda,\mu}(tu,tv) &\leq \frac{\widehat{M}(1)}{p}t^{\theta p}\|(u,v)\|^{\theta p} - \int_{\mathbb{R}^N} \frac{F(x,tu,tv)}{|x|^\gamma}\,\mathrm{d}x\\
&\leq \frac{\widehat{M}(1)}{p}t^{\theta p} - t^{\sigma}\int_{\mathbb{R}^N} \frac{F(x,u,v)}{|x|^\gamma}\,\mathrm{d}x \to -\infty \quad \text{as } t \to \infty,
\end{aligned}$$

thanks to $\sigma > \theta p$. Hence by taking $(e_1,e_2) = (tu,tv)$ with $t > \rho_0$ sufficiently large enough, we get $J_{\lambda,\mu}(e_1,e_2) < 0$ for all $\|(e_1,e_2)\| > \rho_0$ and we conclude the proof. □

In view of Lemma 4.3 and Lemma 4.4, we notice that the functional $J_{\lambda,\mu}$ satisfies the geometrical assumptions required in Theorem 4.2. Hence we infer that there exists a $(PS)_{c_{\lambda,\mu}}$ sequence $\{(u_n,v_n)\}_n \subset \mathbf{X}$ such that

$$J_{\lambda,\mu}(u_n,v_n) \to c_{\lambda,\mu} \quad \text{and} \quad J'_{\lambda,\mu}(u_n,v_n) \to 0 \quad \text{in } \mathbf{X}^* \quad \text{as } n \to \infty,$$

where

$$c_{\lambda,\mu} = \inf_{\gamma_*\in\Gamma}\max_{t\in[0,1]} I(\gamma_*(t)) \geq j_0$$

with

$$\Gamma = \{\gamma_* \in C^1([0,1],\mathbf{X}) :\ \gamma_*(0) = (0,0),\ \gamma_*(1) = (e_1,e_2)\}.$$

It is obvious that $c_{\lambda,\mu} > 0$, thanks to Lemma 4.3. Now we will discuss about the asymptotic behaviour of the mountain pass level $c_{\lambda,\mu}$.

Lemma 4.5. *There holds*

$$\lim_{(\lambda,\mu)\to(\infty,\infty)} c_{\lambda,\mu} = 0.$$

Proof. Let (e_1, e_2) be a couple as in Lemma 4.4 and thus $J_{\lambda,\mu}(te_1, te_2) \to -\infty$ as $t \to \infty$. So, there exists $t_{\lambda,\mu} > 0$ such that

$$J_{\lambda,\mu}(t_{\lambda,\mu}e_1, t_{\lambda,\mu}e_2) = \max_{t \geq 0} J_{\lambda,\mu}(te_1, te_2).$$

Consequently, we have

$$\langle J'_{\lambda,\mu}(t_{\lambda,\mu}e_1, t_{\lambda,\mu}e_2), (e_1, e_2)\rangle = 0.$$

On simplifying, we deduce by using (F_4) that

$$\begin{aligned} M(\|t_{\lambda,\mu}(e_1, e_2)\|^p)\|t_{\lambda,\mu}(e_1, e_2)\|^p &= \int_{\mathbb{R}^N} \frac{\nabla F(x, t_{\lambda,\mu}e_1, t_{\lambda,\mu}e_2).(t_{\lambda,\mu}e_1, t_{\lambda,\mu}e_2)}{|x|^\gamma} \, \mathrm{d}x \\ &\quad + t^q_{\lambda,\mu} \int_{\mathbb{R}^N} h(x)\{\lambda e_1^q + \mu e_2^q\} \, \mathrm{d}x \\ &\geq \sigma \int_{\mathbb{R}^N} \frac{F(x, t_{\lambda,\mu}e_1, t_{\lambda,\mu}e_2)}{|x|^\gamma} \, \mathrm{d}x. \end{aligned} \tag{4.4}$$

From this we conclude that $\{t_{\lambda,\mu}\}_{(\lambda,\mu)}$ is a bounded sequence of real numbers. Indeed, if not, let there exists a subsequence of $\{t_{\lambda,\mu}\}_{(\lambda,\mu)}$ still denoted by itself such that $t_{\lambda,\mu} \to \infty$ as $(\lambda, \mu) \to (\infty, \infty)$. Hence without loss of generality, we can assume $t_{\lambda,\mu} \geq 1$ for sufficiently large values of λ and μ. In addition, due to (M_2) and using similarly approach as in Lemma 4.4, we can deduce that

$$\widehat{M}(t) \leq \widehat{M}(1)t^\theta, \ \forall \ t \geq 1 \quad \text{and} \quad F(x, t_{\lambda,\mu}e_1, t_{\lambda,\mu}e_2) \geq t^\sigma_{\lambda,\mu}F(x, e_1, e_2). \tag{4.5}$$

In virtue of (M_2) and using (4.4) as well as (4.5), we get

$$\begin{aligned} \theta\widehat{M}(1)t^\theta_{\lambda,\mu}\|(e_1, e_2)\|^p \geq \theta\widehat{M}(\|t_{\lambda,\mu}(e_1, e_2)\|^p) &\geq M(\|t_{\lambda,\mu}(e_1, e_2)\|^p)\|t_{\lambda,\mu}(e_1, e_2)\|^p \\ &\geq \sigma t^\sigma_{\lambda,\mu} \int_{\mathbb{R}^N} \frac{F(x, e_1, e_2)}{|x|^\gamma} \, \mathrm{d}x. \end{aligned}$$

This yields at once that

$$\frac{1}{\sigma}\theta\widehat{M}(1)t^{\theta p-\sigma}_{\lambda,\mu}\|(e_1, e_2)\|^p \geq \int_{\mathbb{R}^N} \frac{F(x, e_1, e_2)}{|x|^\gamma} \, \mathrm{d}x.$$

Letting $(\lambda, \mu) \to (\infty, \infty)$ in the above inequality and using the fact that

$$\sigma > \theta p \quad \text{and} \quad \int_{\mathbb{R}^N} \frac{F(x, e_1, e_2)}{|x|^\gamma} \, \mathrm{d}x > 0,$$

we arrive at a contradiction, thanks to (F_1). This shows that $\{t_{\lambda,\mu}\}_{(\lambda,\mu)}$ is a bounded sequence of real numbers. Now, we fix a sequence $\{(\lambda_n, \mu_n)\}_n \subset \mathbb{R}^+ \times \mathbb{R}^+$ such that $(\lambda_n, \mu_n) \to (\infty, \infty)$ as $n \to \infty$. From the above arguments, we know that $\{t_{\lambda_n,\mu_n}\}_n$ is a bounded sequence of real numbers. This shows that there exists $t_0 \geq 0$ and a subsequence of $\{(\lambda_n, \mu_n)\}_n$ still denoted by itself such that $t_{\lambda_n,\mu_n} \to t_0$ as $n \to \infty$. In conclusion, due to the continuity of M, we infer that $\{M(\|t_{\lambda_n,\mu_n}(e_1, e_2)\|^p)\|t_{\lambda_n,\mu_n}(e_1, e_2)\|^p\}_n$ is bounded. So, there exists a constant $C > 0$ such that

$$M(\|t_{\lambda_n,\mu_n}(e_1, e_2)\|^p)\|t_{\lambda_n,\mu_n}(e_1, e_2)\|^p \leq C, \qquad \forall \ n \in \mathbb{N}.$$

We claim that $t_0 = 0$. Indeed, if not, let $t_0 > 0$. Consequently, we obtain from the above inequality and (4.4) that for all $n \in \mathbb{N}$, the following estimate holds

$$\int_{\mathbb{R}^N} \frac{\nabla F(x, t_{\lambda_n,\mu_n} e_1, t_{\lambda_n,\mu_n} e_2).(t_{\lambda_n,\mu_n} e_1, t_{\lambda_n,\mu_n} e_2)}{|x|^\gamma} \,\mathrm{d}x + t^q_{\lambda_n,\mu_n} (\lambda_n \|e_1\|^q_{q,h} + \mu_n \|e_2\|^q_{q,h}) \leq C. \tag{4.6}$$

Due to (3.3) and Theorem 2.7, we obtain from the Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{\nabla F(x, t_{\lambda_n,\mu_n} e_1, t_{\lambda_n,\mu_n} e_2).(t_{\lambda_n,\mu_n} e_1, t_{\lambda_n,\mu_n} e_2)}{|x|^\gamma} \,\mathrm{d}x = \int_{\mathbb{R}^N} \frac{\nabla F(x, t_0 e_1, t_0 e_2).(t_0 e_1, t_0 e_2)}{|x|^\gamma} \,\mathrm{d}x.$$

The above convergence together with $\|e_i\|^q_{q,h} > 0$ for $i = 1, 2$ implies that

$$\lim_{n\to\infty} \left[\int_{\mathbb{R}^N} \frac{\nabla F(x, t_{\lambda_n,\mu_n} e_1, t_{\lambda_n,\mu_n} e_2).(t_{\lambda_n,\mu_n} e_1, t_{\lambda_n,\mu_n} e_2)}{|x|^\gamma} \,\mathrm{d}x + t^q_{\lambda_n,\mu_n} (\lambda_n \|e_1\|^q_{q,h} + \mu_n \|e_2\|^q_{q,h} \right] = \infty.$$

In virtue of (4.6), we arrive at a contradiction. It follows that $t_0 = 0$ and $t_{\lambda,\mu} \to 0$ as $(\lambda, \mu) \to (\infty, \infty)$, thanks to the arbitrariness of $\{(\lambda_n, \mu_n)\}_n$. Now consider the path $\xi(t) = t(e_1, e_2)$ with $t \in [0, 1]$, belongs to Γ. Using the continuity of $\widehat{M}$, we get

$$\begin{aligned}
0 < c_{\lambda,\mu} &\leq \max_{t\geq 0} J_{\lambda,\mu}(\xi(t)) = J_{\lambda,\mu}(t_{\lambda,\mu} e_1, t_{\lambda,\mu} e_2) \\
&\leq \frac{1}{p} \widehat{M}(\|t_{\lambda,\mu}(e_1, e_2)\|^p) \to 0 \quad \text{as } (\lambda, \mu) \to (\infty, \infty),
\end{aligned}$$

thanks to the couple (e_1, e_2), which does not depend on λ and μ. This completes the proof. □

Lemma 4.6. *There exists $\widehat{\lambda} > 0$ such that for all $(\lambda, \mu) \in (\widehat{\lambda}, \infty) \times (\widehat{\lambda}, \infty)$, the functional $J_{\lambda,\mu}$ satisfies the $(PS)_{c_{\lambda,\mu}}$ condition on $\mathbf{X}$.*

Proof. Let $\{(u_n, v_n)\}_n \subset \mathbf{X}$ be a $(PS)_{c_{\lambda,\mu}}$ for $J_{\lambda,\mu}$. It follows at once that

$$\left\langle J'_{\lambda,\mu}(u_n, v_n), \frac{(u_n, v_n)}{\|(u_n, v_n)\|} \right\rangle = o_n(1) \quad \text{and} \quad J_{\lambda,\mu}(u_n, v_n) = c_{\lambda,\mu} + o_n(1) \quad \text{as } n \to \infty. \tag{4.7}$$

Possibly, we have two situations based on the nature of the Kirchhoff function M: either $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = d > 0$ or $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = 0$. Hence we have to discuss these situations separately.

Case 1. Let $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = d > 0$. Denote $\kappa = \kappa(\tau)$ as the number corresponding to $\tau = d^p$ in (M_1) such that

$$M(\|(u_n, v_n)\|^p) \geq \kappa, \qquad \forall\, n \in \mathbb{N}. \tag{4.8}$$

Now we claim to prove $\{(u_n, v_n)\}_n$ is bounded in $\mathbf{X}$. For this, we first consider the case when $\theta p < \sigma \leq q$. In this case, we obtain from (4.7), (4.8), (F_4) and (M_2) that as $n \to \infty$

$$\begin{aligned}
c_{\lambda,\mu} + o_n(1) + o_n(1)\|(u_n, v_n)\| &= J_{\lambda,\mu}(u_n, v_n) - \frac{1}{\sigma} \langle J'_{\lambda,\mu}(u_n, v_n), (u_n, v_n) \rangle \\
&\geq \frac{1}{p} \widehat{M}(\|(u_n, v_n)\|^p) - \frac{1}{\sigma} M(\|(u_n, v_n)\|^p) \|(u_n, v_n)\|^p \\
&\geq \left(\frac{1}{\theta p} - \frac{1}{\sigma} \right) \kappa \|(u_n, v_n)\|^p.
\end{aligned}$$

Since $p > 1$, we conclude from the above inequality that $\{(u_n, v_n)\}_n$ is bounded in $\mathbf{X}$ and thus we have

$$\|(u_n, v_n)\|^{p'} \leq \left(\frac{\sigma\theta p\, c_{\lambda,\mu}}{(\sigma - \theta p)\kappa} + o_n(1) \right)^{p'}. \tag{4.9}$$

Define $z_n = (u_n, v_n)$ and $z = (u_{\lambda,\mu}, v_{\lambda,\mu})$. Then it is easy to see that $|z_n| = \sqrt{u_n^2 + v_n^2} \leq \psi_n := |u_n| + |v_n|$. By direct calculation, we can deduce that

$$\|\psi_n\|_{W^{s,p}} \leq 2(\min\{V_0, K_0\})^{-\frac{1}{p}} \|(u_n, v_n)\|. \tag{4.10}$$

In virtue of (4.9) and (4.10), we infer that

$$\limsup_{n\to\infty} \|\psi_n\|_{W^{s,p}}^{p'} \leq \frac{2^{p'}}{(\min\{V_0, K_0\})^{\frac{1}{p-1}}} \left[\frac{\sigma\theta p\, c_{\lambda,\mu}}{(\sigma - \theta p)\kappa} \right]^{p'}. \tag{4.11}$$

Similarly, when $\sigma > q > \theta p$, we obtain from (4.7), (4.8), (F_4) and (M_2) that as $n \to \infty$

$$\begin{aligned} c_{\lambda,\mu} + o_n(1) + o_n(1)\|(u_n, v_n)\| &= J_{\lambda,\mu}(u_n, v_n) - \frac{1}{q}\langle J'_{\lambda,\mu}(u_n, v_n), (u_n, v_n)\rangle \\ &\geq \frac{1}{p}\widehat{M}(\|(u_n, v_n)\|^p) - \frac{1}{q}M(\|(u_n, v_n)\|^p)\|(u_n, v_n)\|^p \\ &\geq \left(\frac{1}{\theta p} - \frac{1}{q}\right)\kappa\|(u_n, v_n)\|^p. \end{aligned}$$

This shows that $\{(u_n, v_n)\}_n$ is bounded in $\mathbf{X}$. Using a similar procedure as discussed above, we have

$$\limsup_{n\to\infty} \|\psi_n\|_{W^{s,p}}^{p'} \leq \frac{2^{p'}}{(\min\{V_0, K_0\})^{\frac{1}{p-1}}} \left[\frac{q\theta p\, c_{\lambda,\mu}}{(q - \theta p)\kappa} \right]^{p'}. \tag{4.12}$$

On combining (4.11) and (4.12), we get

$$\limsup_{n\to\infty} \|\psi_n\|_{W^{s,p}}^{p'} \leq \frac{(2c_{\lambda,\mu})^{p'}}{(\min\{V_0, K_0\})^{\frac{1}{p-1}}} \left(\left[\frac{\sigma\theta p}{(\sigma - \theta p)\kappa} \right]^{p'} + \left[\frac{q\theta p}{(q - \theta p)\kappa} \right]^{p'} \right). \tag{4.13}$$

By using Lemma 4.5, there exists $\widehat{\lambda} > 0$ such that for all $(\lambda, \mu) \in (\widehat{\lambda}, \infty) \times (\widehat{\lambda}, \infty)$, we can assume

$$0 < c_{\lambda,\mu} < \left[\frac{\beta_*(\min\{V_0, K_0\})^{\frac{1}{p-1}}}{2^{p'+1}\alpha_0} \right]^{\frac{1}{p'}} \left(\left[\frac{\sigma\theta p}{(\sigma - \theta p)\kappa} \right]^{p'} + \left[\frac{q\theta p}{(q - \theta p)\kappa} \right]^{p'} \right)^{-\frac{1}{p'}}.$$

From the above inequality and (4.13), we obtain

$$\limsup_{n\to\infty} \|\psi_n\|_{W^{s,p}}^{p'} < \frac{\beta_*}{2\alpha_0}, \qquad \forall\, (\lambda, \mu) \in (\widehat{\lambda}, \infty) \times (\widehat{\lambda}, \infty).$$

Due to the boundedness of $\{(u_n, v_n)\}_n$ in $\mathbf{X}$, there exists $(u_{\lambda,\mu}, v_{\lambda,\mu}) \in \mathbf{X}$ such that up to a subsequence still denoted by itself, we can assume that $(u_n, v_n) \rightharpoonup (u_{\lambda,\mu}, v_{\lambda,\mu})$ in $\mathbf{X}$ as $n \to \infty$. In addition, by using the notations used in the proof of Theorem 1.3, it is easy to see that (3.17) holds, there exists $l_{\lambda,\mu} \geq 0$ such that up to a subsequence, we have $\|(u_n, v_n)\| \to l_{\lambda,\mu}$ as $n \to \infty$ and also we can easily obtain from (4.7) that

$$\langle J'_{\lambda,\mu}(u_n, v_n), (u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})\rangle = o_n(1) \quad \text{as } n \to \infty,$$

thanks to the boundedness of $\{(u_n, v_n)\}_n$ in $\mathbf{X}$. Moreover, by using the Hölder's inequality, (3.17) and (3.18), one has

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} h(x)\{\lambda(u_n^+)^{q-1}(u_n - u_{\lambda,\mu}) + \mu(v_n^+)^{q-1}(v_n - v_{\lambda,\mu})\}\,\mathrm{d}x\right| \\
&\quad \le \max\{\lambda,\mu\}\|h\|_\infty \left[\|u_n\|_q^{q-1}\|u_n - u_{\lambda,\mu}\|_q + \|v_n\|_q^{q-1}\|v_n - v_{\lambda,\mu}\|_q\right] \\
&\quad \le \max\{\lambda,\mu\}\|h\|_\infty \|(u_n,v_n)\|^{q-1}_{L^q(\mathbb{R}^N)\times L^q(\mathbb{R}^N)}\|(u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})\|_{L^q(\mathbb{R}^N)\times L^q(\mathbb{R}^N)} \\
&\quad \le \max\{\lambda,\mu\}\|h\|_\infty \mathcal{A}_{q,h}^{-(q-1)} \sup_{n\in\mathbb{N}} \|(u_n,v_n)\|^{q-1}\|(u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})\|_{L^q(\mathbb{R}^N)\times L^q(\mathbb{R}^N)} \\
&\quad \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

It follows at once that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} h(x)\{\lambda(u_n^+)^{q-1}(u_n - u_{\lambda,\mu}) + \mu(v_n^+)^{q-1}(v_n - v_{\lambda,\mu})\}\,\mathrm{d}x = 0. \tag{4.14}$$

Using all the above information and proceeding likewise as in the proof of Theorem 1.3, we can prove that $(u_n, v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu})$ in $\mathbf{X}$ as $n \to \infty$. This together with $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = d > 0$ implies that $l_{\lambda,\mu} = \|(u_{\lambda,\mu}, v_{\lambda,\mu})\| > 0$. This shows that $M(\|(u_n, v_n)\|^p) \to M(l^p_{\lambda,\mu}) > 0$ as $n \to \infty$, thanks to the continuity of M. Next, we claim that

$$\lim_{(\lambda,\mu)\to(\infty,\infty)} \|(u_{\lambda,\mu}, v_{\lambda,\mu})\| = 0.$$

Indeed, if not, let there exists a sequence $\{(\lambda_k, \mu_k)\}_k$ with $(\lambda_k, \mu_k) \to (\infty, \infty)$ as $k \to \infty$ such that $l_{\lambda_k,\mu_k} \to l_0 > 0$ as $k \to \infty$. It is easy to see that for $\varrho \in \{\sigma, q\}$ (according to the situations: either $\theta p < \sigma \le q$ or $\theta p < q < \sigma$), one has

$$c_{\lambda_k,\mu_k} = \lim_{n\to\infty}\left[J_{\lambda,\mu}(u_n, v_n) - \frac{1}{\varrho}\langle J'_{\lambda,\mu}(u_n, v_n), (u_n, v_n)\rangle\right] \ge \left(\frac{1}{\theta p} - \frac{1}{\varrho}\right) M(l^p_{\lambda_k,\mu_k}) l^p_{\lambda_k,\mu_k}.$$

Letting $k \to \infty$ in the above inequality and using Lemma 4.5, we get

$$0 \ge \left(\frac{1}{\theta p} - \frac{1}{\varrho}\right) M(l_0^p) l_0^p > 0,$$

which is not possible. It follows that

$$\lim_{(\lambda,\mu)\to(\infty,\infty)} \|(u_{\lambda,\mu}, v_{\lambda,\mu})\| = \lim_{(\lambda,\mu)\to(\infty,\infty)} l_{\lambda,\mu} = 0.$$

Now we shall discuss about the situation when $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = 0$.

Case 2. Let $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = 0$. If $(0, 0)$ is an isolated point of the sequence $\{(u_n, v_n)\}_n$, then there exists a subsequence still denoted by the same symbol such that

$$\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = d > 0.$$

In this situation, we can proceed as in Case 1. Moreover, if $(0, 0)$ is an accumulation point of $\{(u_n, v_n)\}_n$, then up to a subsequence still denoted by itself such that it strongly converges to $(u_{\lambda,\mu}, v_{\lambda,\mu}) = (0, 0)$. This situation is impossible. Indeed, if not, then $0 = J_{\lambda,\mu}(0, 0) = c_{\lambda,\mu} > 0$, which is a contradiction.

By using $(u_n, v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu})$ in $\mathbf{X}$ as $n \to \infty$ and the fact that $J_{\lambda,\mu} \in C^1(\mathbf{X}, \mathbb{R})$, we get $J_{\lambda,\mu}(u_{\lambda,\mu}, v_{\lambda,\mu}) = c_{\lambda,\mu} > 0$ and $J'_{\lambda,\mu}(u_{\lambda,\mu}, v_{\lambda,\mu}) = 0$. It follows from Lemma 3.1 that $(u_{\lambda,\mu}, v_{\lambda,\mu})$ is a nontrivial nonnegative solution of the system $(\mathcal{S}_{\lambda,\mu})$. This completes the proof. □

Proof of Theorem 1.4. Due to Lemma 4.3–4.6, we infer that $J_{\lambda,\mu}$ enjoys all the assumptions of Theorem 4.2. In addition, by Lemma 4.6, there exists $\widehat{\lambda} > 0$ such that for all $(\lambda, \mu) \in (\widehat{\lambda}, \infty) \times (\widehat{\lambda}, \infty)$, the functional $J_{\lambda,\mu}$ admits a nontrivial nonnegative critical point $(u_{\lambda,\mu}, v_{\lambda,\mu}) \in \mathbf{X}$, which is a mountain pass solution of the system $(\mathcal{S}_{\lambda,\mu})$. Consequently, there also holds $\|(u_{\lambda,\mu}, v_{\lambda,\mu})\| \to 0$ as $(\lambda, \mu) \to (\infty, \infty)$. This finishes the proof. $\square$

5 Proof of Theorem 1.5

In this section, we shall prove Theorem 1.5 using the Krasnoselskii genus theory. For the sake of simplicity, we assume that the structural assumptions required in Theorem 1.5 hold. For the convenience of the readers, we first summarize some basic properties and definition of the genus.

Let X be a Banach space and A be a subset of X. A is said to be symmetric w.r.t. the origin if $u \in A$ implies $-u \in A$. Define

$$\mathbf{\Sigma} = \{A \subset X \setminus \{0\} : \ A \text{ closed and symmetric w.r.t. the origin}\}.$$

For $A \in \mathbf{\Sigma}$, we denote the genus of A by $\gamma(A)$, which is defined by

$$\gamma(A) = \min\{k \in \mathbb{N} : \ \exists \ \varphi \in C(A, \mathbb{R}^k \setminus \{0\}), \ \varphi(x) = -\varphi(-x)\}.$$

Moreover, we define $\gamma(\emptyset) = 0$. In addition, if such k does not exist, we set $\gamma(A) = \infty$. The following properties of the genus can be found in [7,25,27,41,49].

Proposition 5.1. *Let $A, B \in \mathbf{\Sigma}$. Then the following results hold:*

(a) If there exists an odd map $f \in C(A, B)$, then $\gamma(A) = \gamma(B)$;

(b) If $A \subset B$, then $\gamma(A) \leq \gamma(B)$;

(c) If there exists an odd homeomorphism from A onto B, then $\gamma(A) = \gamma(B)$;

(d) If $\gamma(A) \geq 2$, then A has infinitely many points;

(e) If $\mathbb{S}$ is a sphere centered at the origin in $\mathbb{R}^k$, then $\gamma(\mathbb{S}) = k$;

(f) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$;

(g) If $\gamma(B) < \infty$, then $\gamma(\overline{A \backslash B} \geq \gamma(A) - \gamma(B)$;

(h) If A is compact, then $\gamma(A) < \infty$, and $\exists \ \delta > 0$ such that $N_\delta(A) \in \mathbf{\Sigma}$ and $\gamma(A) = \gamma(N_\delta(A))$, where

$$N_\delta(A) = \{x \in X : \ d(x, A) \leq \delta\} \quad \textit{and} \quad d(x, A) = \inf\{\|x - u\| : \ u \in A\};$$

(i) If X_0 is a subspace of X with codimension k, and $\gamma(A) > k$, then $A \cap X_0 \neq \emptyset$.

Next, we recall the classical deformation lemma, established by A. Ambrosetti and P. H. Rabinowitz in [1]. It can be read as follows:

Proposition 5.2. *Let X be a Banach space and $I \in C^1(X,\mathbb{R})$ satisfying the (PS) condition. If $c \in \mathbb{R}$ and N is any neighbourhood of*

$$K_c = \{u \in X : \ I(u) = c, \ I'(u) = 0\},$$

then there exist $\eta(t,u) = \eta_t(u) \in C([0,1] \times X, X)$ and constants $\bar{\epsilon} > \epsilon > 0$ such that

(a) $\eta_0(u) = u$ *for all* $u \in X$;

(b) $\eta_t(u) = u$ *for all* $u \notin I^{-1}[c-\bar{\epsilon}, c+\bar{\epsilon}]$ *and all* $t \in [0,1]$;

(c) η_t *is a homeomorphism of* X *onto* X *for all* $t \in [0,1]$;

(d) $I(\eta_t(u)) \le I(u)$ *for all* $u \in X$ *and all* $t \in [0,1]$;

(e) $\eta_1(I^{c+\epsilon} \backslash N) \subset I^{c-\epsilon}$, *where* $I^c = \{u \in X : \ I(u) \le c\}$ *for all* $c \in \mathbb{R}$;

(f) If $K_c = \emptyset$, *then there holds* $\eta_1(I^{c+\epsilon}) \subset I^{c-\epsilon}$;

(g) If I *is even, then* η_t *is odd in* u.

Remark 5.3. We note that Proposition 5.2 is also valid if I satisfies the $(PS)_c$ condition for $c < c_0$ for some $c_0 \in \mathbb{R}$.

To study the infinitely many solutions of the system $(\mathcal{S}_{\lambda,\mu})$, we define the functional $J : \mathbf{X} \to \mathbb{R}$ by

$$J(u,v) = \frac{1}{p}\widehat{M}(\|(u,v)\|^p) - \int_{\mathbb{R}^N} \frac{F(x,u,v)}{|x|^\gamma}\,\mathrm{d}x - \frac{1}{q}(\lambda\|u\|^q_{q,h} + \mu\|v\|^q_{q,h}), \qquad \forall\ (u,v) \in \mathbf{X}. \tag{5.1}$$

By using (F_2) and Theorem 2.7, it follows that J is well-defined, of class $C^1(\mathbf{X},\mathbb{R})$ and the critical points are weak solutions of the system $(\mathcal{S}_{\lambda,\mu})$.

Note that by arguing similar arguments as in Lemma 4.4, we can prove that J is not bounded from below. Hence we have some mathematical difficulty in studying the multiplicity of critical points of the functional J. To avoid this difficulty, we use the technique used in [25]. For this, we shall construct the truncated functional I corresponding to the functional J and study the behaviour of solutions to the system $(\mathcal{S}_{\lambda,\mu})$.

Suppose that $\|(u,v)\| \le 1$. Observe that for $\ell \in [0,1]$, we obtain from the definition of $\widehat{M}$ that

$$\widehat{M}(\ell) \ge \widehat{M}(1)\ell^\theta = (a+b)\ell^\theta.$$

Using the notations as stated in Lemma 3.2, for $\|(u,v)\| \le \bar{\delta}$ with $\bar{\delta} \in (0,1]$, we have

$$J(u,v) \ge \left(\frac{a+b}{p} - 2^{\theta p}\mathcal{B}^{-\theta p}_{\theta p,\gamma}\varepsilon\right)\|(u,v)\|^{\theta p} - \widetilde{C}\|(u,v)\|^{\vartheta} - \frac{1}{q}\max\{\lambda,\mu\}\mathcal{S}^{-q}_{q,h}\|(u,v)\|^q,$$

where $\widetilde{C} = 2^\vartheta D C^{\frac{1}{t'}} \mathcal{B}^{-\vartheta}_{\vartheta t,\gamma}\tilde{\kappa}_\varepsilon$. Choosing $\varepsilon = a + b/2^{\theta p+1}p\mathcal{B}^{-\theta p}_{\theta p,\gamma}$, we obtain from the above inequality that

$$J(u,v) \ge \left(\frac{a+b}{2p}\right)\|(u,v)\|^{\theta p} - \widetilde{C}\|(u,v)\|^{\vartheta} - \frac{1}{q}\max\{\lambda,\mu\}\mathcal{S}^{-q}_{q,h}\|(u,v)\|^q \tag{5.2}$$

for all $(u,v) \in \mathbf{X}$ with $\|(u,v)\| \le 1$. Define the map $H_{\lambda,\mu} : [0,\infty) \to \mathbb{R}$ as follows

$$H_{\lambda,\mu}(\ell) = \left(\frac{a+b}{2p}\right)\ell^{\theta p} - \widetilde{C}\ell^{\vartheta} - \frac{1}{q}\max\{\lambda,\mu\}\mathcal{S}^{-q}_{q,h}\ell^q, \qquad \forall\ \ell \in [0,\infty)$$

and write

$$H_{\lambda,\mu}(\ell) = \ell^q \widetilde{H}_{\lambda,\mu}(\ell),$$

where

$$\widetilde{H}_{\lambda,\mu}(\ell) = \left(\frac{a+b}{2p}\right)\ell^{\theta p - q} - \widetilde{C}\ell^{\vartheta - q} - \frac{1}{q}\max\{\lambda,\mu\}\mathcal{S}_{q,h}^{-q}.$$

Hence one can easily notice that $J(u,v) \geq H_{\lambda,\mu}(\|(u,v)\|)$ for all $(u,v) \in \mathbf{X}$ with $\|(u,v)\| \leq 1$. In addition, we have

$$\widetilde{H}_{\lambda,\mu}(0) < 0 \quad \text{and} \quad \widetilde{H}_{\lambda,\mu}(\ell) \to -\infty \quad \text{as } \ell \to \infty. \tag{5.3}$$

By direct calculation, one has

$$\widetilde{H}'_{\lambda,\mu}(\ell) = 0 \implies \ell = \left[\frac{(a+b)(\theta p - q)}{2p\widetilde{C}(\vartheta - q)}\right]^{\frac{1}{\vartheta - \theta p}} =: T.$$

Further, it is easy to see that $\widetilde{H}'_{\lambda,\mu}(\ell) > 0$ for all $\ell \in (0,T)$ and $\widetilde{H}'_{\lambda,\mu}(\ell) < 0$ for all $\ell \in (T,\infty)$. This shows that the map $\ell \mapsto \widetilde{H}_{\lambda,\mu}(\ell)$ is strictly increasing on $(0,T)$ and strictly decreasing on (T,∞). In virtue of (5.3), we deduce that, if $\widetilde{H}_{\lambda,\mu}(T) > 0$, then

$$\widetilde{H}_{\lambda,\mu}(0)\widetilde{H}_{\lambda,\mu}(T) < 0 \quad \text{and} \quad \widetilde{H}_{\lambda,\mu}(T)\widetilde{H}_{\lambda,\mu}(\ell) < 0 \quad \text{for } \ell \text{ large enough.}$$

Consequently, by the intermediate value theorem, there exists at least one root of $\widetilde{H}_{\lambda,\mu}$ in between 0 and T. Since the map $\ell \mapsto \widetilde{H}_{\lambda,\mu}(\ell)$ is strictly increasing on $(0,T)$, we infer that there exists a unique root $T_0(\lambda,\mu)$ in between 0 and T for $\widetilde{H}_{\lambda,\mu}$. Similarly, we can prove that there exists a unique root $T_1(\lambda,\mu)$ in between T and ∞ for $\widetilde{H}_{\lambda,\mu}$. In conclusion, we can say, if $\widetilde{H}_{\lambda,\mu}(T) > 0$, then there exist two unique real numbers $T_0(\lambda,\mu)$ and $T_1(\lambda,\mu)$ with

$$0 < T_0(\lambda,\mu) < T < T_1(\lambda,\mu) < \infty \quad \text{such that} \quad \widetilde{H}_{\lambda,\mu}(T_0(\lambda,\mu)) = \widetilde{H}_{\lambda,\mu}(T_1(\lambda,\mu)) = 0.$$

Define

$$\boldsymbol{\lambda}^1 = \frac{q(\vartheta - \theta p)}{\mathcal{S}_{q,h}^{-q}}\left(\frac{a+b}{2p(\vartheta - q)}\right)^{\frac{\vartheta - q}{\vartheta - \theta p}}\left(\frac{\theta p - q}{\widetilde{C}}\right)^{\frac{\theta p - q}{\vartheta - \theta p}},$$

then if $\max\{\lambda,\mu\} < \boldsymbol{\lambda}^1$, there holds

$$\widetilde{H}_{\lambda,\mu}(T) = (\vartheta - \theta p)\left(\frac{a+b}{2p(\vartheta - q)}\right)^{\frac{\vartheta - q}{\vartheta - \theta p}}\left(\frac{\theta p - q}{\widetilde{C}}\right)^{\frac{\theta p - q}{\vartheta - \theta p}} - \frac{1}{q}\max\{\lambda,\mu\}\mathcal{S}_{q,h}^{-q} > 0,$$

so that, since we have $H_{\lambda,\mu}(T_0(\lambda,\mu)) = H_{\lambda,\mu}(T_1(\lambda,\mu)) = 0$, thanks to $H_{\lambda,\mu}(\ell) = \ell^q\widetilde{H}_{\lambda,\mu}(\ell)$. In addition, there holds

$$\begin{aligned} &H_{\lambda,\mu}(\ell) > 0, \ \forall\, \ell \in \big(T_0(\lambda,\mu), T_1(\lambda,\mu)\big) \quad \text{and} \\ &H_{\lambda,\mu}(\ell) \leq 0, \ \forall\, \ell \in \big[0, T_0(\lambda,\mu)\big] \cup \big[T_1(\lambda,\mu), \infty\big). \end{aligned} \tag{5.4}$$

Corollary 5.4. *The following holds*

$$\lim_{(\lambda,\mu)\to(0^+,0^+)} T_0(\lambda,\mu) = 0.$$

Proof. It follows from $H_{\lambda,\mu}(T_0(\lambda,\mu)) = 0$ and $H'_{\lambda,\mu}(T_0(\lambda,\mu)) > 0$ that

$$\left(\frac{a+b}{2p}\right)(T_0(\lambda,\mu))^{\theta p} = \widetilde{C}(T_0(\lambda,\mu))^{\vartheta} + \frac{1}{q}\max\{\lambda,\mu\}\mathcal{S}_{q,h}^{-q}(T_0(\lambda,\mu))^{q} \tag{5.5}$$

and

$$\left(\frac{a+b}{2p}\right)\theta p(T_0(\lambda,\mu))^{\theta p-1} > \widetilde{C}\vartheta(T_0(\lambda,\mu))^{\vartheta-1} + \max\{\lambda,\mu\}\mathcal{S}_{q,h}^{-q}(T_0(\lambda,\mu))^{q-1}. \tag{5.6}$$

On solving (5.5) and (5.6), we obtain

$$T_0(\lambda,\mu) < \left[\frac{(a+b)(\theta p-q)}{2p\widetilde{C}(\vartheta-q)}\right]^{\frac{1}{\vartheta-\theta p}}.$$

This shows that $T_0(\lambda,\mu)$ is uniformly bounded w.r.t. λ and μ. Let us choose a sequence $\{(\lambda_n,\mu_n)\}_n$ with $(\lambda_n,\mu_n) \to (0^+,0^+)$ as $n\to\infty$. Using the fact that $\{T_0(\lambda_n,\mu_n)\}_n$ is uniformly bounded, therefore up to a subsequence still denoted by itself such that

$$T_0(\lambda_n,\mu_n) \to T_0\ (\geq 0)\ \text{(say)} \quad \text{as } n\to\infty.$$

Consequently, by replacing λ_n in place of λ and μ_n in place of μ in (5.5) and (5.6), respectively and letting $n\to\infty$, we have

$$\left(\frac{a+b}{2p}\right)T_0^{\theta p} = \widetilde{C}T_0^{\vartheta} \quad \text{and} \quad \left(\frac{a+b}{2p}\right)\theta p T_0^{\theta p-1} \geq \widetilde{C}\vartheta T_0^{\vartheta-1}.$$

This directly implies at once that

$$\widetilde{C}(\vartheta-\theta p)T_0^{\vartheta-1} \leq 0.$$

Hence we deduce that $T_0 = 0$. Due to the arbitrariness of $\{(\lambda_n,\mu_n)\}_n$, we conclude that $T_0(\lambda,\mu)\to 0$ as $(\lambda,\mu)\to(0^+,0^+)$ and thus we conclude the proof. □

By Corollary 5.4, we can find $\boldsymbol{\lambda}^2 > 0$ small enough such that there holds $T_0(\lambda,\mu) < 1$ for all $(\lambda,\mu)\in(0,\boldsymbol{\lambda}^2)\times(0,\boldsymbol{\lambda}^2)$. It follows that $T_0(\lambda,\mu) < \min\{T_1(\lambda,\mu),1\}$ for all $(\lambda,\mu)\in(0,\boldsymbol{\lambda}^2)\times(0,\boldsymbol{\lambda}^2)$. Suppose that $(\lambda,\mu)\in(0,\min\{\boldsymbol{\lambda}^1,\boldsymbol{\lambda}^2\})\times(0,\min\{\boldsymbol{\lambda}^1,\boldsymbol{\lambda}^2\})$ and take a nonincreasing cut-off function $\Psi\in C_0^\infty([0,\infty),[0,1])$, which is defined by

$$\Psi(\ell) = \begin{cases} 1 & \text{if } \ell\in[0,T_0(\lambda,\mu)], \\ 0 & \text{if } \ell\in[\min\{T_1(\lambda,\mu),1\},\infty). \end{cases}$$

Define the truncated energy functional $I:\mathbf{X}\to\mathbb{R}$ by

$$I(u,v) = \frac{1}{p}\widehat{M}(\|(u,v)\|^p) - \Psi(\|(u,v)\|)\int_{\mathbb{R}^N}\frac{F(x,u,v)}{|x|^\gamma}\,\mathrm{d}x - \frac{1}{q}(\lambda\|u\|_{q,h}^q + \mu\|v\|_{q,h}^q),\ \forall\,(u,v)\in\mathbf{X}.$$

By the regularity of Ψ and J, we conclude that $I\in C^1(\mathbf{X},\mathbb{R})$. In addition, one can notice that I is coercive and bounded from below on $\mathbf{X}$. We also mention that the following results hold:

$$J(u,v) = I(u,v),\ \forall\,(u,v)\in\mathbf{X} \quad \text{with} \quad \|(u,v)\|\leq T_0(\lambda,\mu) < \min\{T_1(\lambda,\mu),1\}$$

and

$$J(u,v) \geq G_{\lambda,\mu}(\|(u,v)\|),\ \forall\,(u,v)\in\mathbf{X} \quad \text{with} \quad \|(u,v)\|\leq 1,$$

where

$$G_{\lambda,\mu}(\ell) = \left(\frac{a+b}{2p}\right)\ell^{\theta p} - \widetilde{C}\Psi(\ell)\ell^{\vartheta} - \frac{1}{q}\max\{\lambda,\mu\}\mathcal{S}_{q,h}^{-q}\ell^q.$$

Lemma 5.5. *There exists $\bar{\lambda} > 0$ such that for all $(\lambda, \mu) \in (0, \bar{\lambda}) \times (0, \bar{\lambda})$, we have*

(a) If $I(u, v) < 0$, then $\|(u, v)\| < T_0(\lambda, \mu)$ and $I(\varphi, \psi) = J(\varphi, \psi)$ for all (φ, ψ) in a small neighbourhood of (u, v);

(b) For all $c < 0$, the functional I satisfies a local $(PS)_c$ condition.

Proof. Suppose that $\|(u, v)\| \geq 1$, then we have $\Psi(\|(u, v)\|) = 0$. In addition, we obtain

$$
\begin{aligned}
I(u, v) &= \frac{1}{p}\widehat{M}(\|(u, v)\|^p) - \frac{1}{q}(\lambda\|u\|^q_{q,h} + \mu\|v\|^q_{q,h}) \geq \left(\frac{a+b}{p}\right)\|(u, v)\|^p \\
&\quad - \frac{1}{q}\max\{\lambda, \mu\}\mathcal{S}^{-q}_{q,h}\|(u, v)\|^q \\
&=: \widetilde{g}_{\lambda,\mu}(\|(u, v)\|)
\end{aligned}
$$

for all $\|(u, v)\| \geq 1$, where $\widetilde{g}_{\lambda,\mu} : [0, \infty) \to \mathbb{R}$ is given by

$$
\widetilde{g}_{\lambda,\mu}(\ell) = \left(\frac{a+b}{p}\right)\ell^p - \frac{1}{q}\max\{\lambda, \mu\}\mathcal{S}^{-q}_{q,h}\ell^q.
$$

By direct computation, one can notice that $\widetilde{g}_{\lambda,\mu}$ has a global minimum point at

$$
\ell_0 = \left(\frac{\max\{\lambda, \mu\}\mathcal{S}^{-q}_{q,h}}{a+b}\right)^{\frac{1}{p-q}}
$$

and

$$
\widetilde{g}_{\lambda,\mu}(\ell_0) = \left(\frac{\max\{\lambda, \mu\}\mathcal{S}^{-q}_{q,h}}{a+b}\right)^{\frac{q}{p-q}} \max\{\lambda, \mu\}\mathcal{S}^{-q}_{q,h}\left(\frac{1}{p} - \frac{1}{q}\right) < 0,
$$

thanks to $1 < q < p$. Further, it is easy to see that

$$
\widetilde{g}_{\lambda,\mu}(\ell) \geq 0 \iff \ell \geq \left(\frac{\max\{\lambda, \mu\}p\mathcal{S}^{-q}_{q,h}}{q(a+b)}\right)^{\frac{1}{p-q}} =: \ell_1.
$$

This shows that

$$
I(u, v) \geq \widetilde{g}_{\lambda,\mu}(\|(u, v)\|) \geq 0, \qquad \forall\ \|(u, v)\| \geq 1
$$

with $\ell_1 < 1$, that is, we have

$$
\max\{\lambda, \mu\} < \frac{q(a+b)}{p\mathcal{S}^{-q}_{q,h}} =: \boldsymbol{\lambda}^3.
$$

Next, we define a positive constant $\boldsymbol{\lambda}^4$ as follows

$$
\boldsymbol{\lambda}^4 = \frac{qb(\sigma - \theta p)\left(\min\{V_0, K_0\}\right)^{\frac{\theta p - q}{p}}}{2^{\theta p - q}(\sigma - q)p\mathcal{S}^{-q}_{q,h}}\left(\frac{\beta_*}{2\alpha_0}\right)^{\frac{\theta p - q}{p'}}. \tag{5.7}
$$

Pick $\bar{\lambda} = \min\{\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^2, \boldsymbol{\lambda}^3, \boldsymbol{\lambda}^4\}$, then we conclude that for all $(\lambda, \mu) \in (0, \bar{\lambda}) \times (0, \bar{\lambda})$, we have $I(u, v) \geq 0$ for any $\|(u, v)\| \geq 1$. This implies at once that if

$$
I(u, v) < 0, \text{ then } \|(u, v)\| < 1,\ \forall\ (\lambda, \mu) \in (0, \bar{\lambda}) \times (0, \bar{\lambda}). \tag{5.8}
$$

Consequently, from the definitions of $H_{\lambda,\mu}$, $G_{\lambda,\mu}$ and Ψ, we get for all $(\lambda,\mu) \in (0,\bar{\lambda}) \times (0,\bar{\lambda})$ with $I(u,v) < 0$ that

$$H_{\lambda,\mu}(\|(u,v)\| \leq G_{\lambda,\mu}(\|(u,v)\|) \leq I(u,v) < 0.$$

This together with (5.4) implies that for all $(\lambda,\mu) \in (0,\bar{\lambda}) \times (0,\bar{\lambda})$ with $I(u,v) < 0$, we have

$$\|(u,v)\| \in (0,1) \cap [(0, T_0(\lambda,\mu)) \cup (T_1(\lambda,\mu),\infty)]. \tag{5.9}$$

Now we have following possibilities according to the nature of $T_0(\lambda,\mu)$ and $T_1(\lambda,\mu)$.

Situation 1. Let either $0 < T_0(\lambda,\mu) < 1 < T_1(\lambda,\mu)$ or $0 < T_0(\lambda,\mu) < T_1(\lambda,\mu) = 1$. In both cases, for all $(\lambda,\mu) \in (0,\bar{\lambda}) \times (0,\bar{\lambda})$ with $I(u,v) < 0$, it follows directly from (5.9) that $\|(u,v)\| \in (0, T_0(\lambda,\mu))$ and we are done.

Situation 2. Let $0 < T_0(\lambda,\mu) < T_1(\lambda,\mu) < 1$. Then for all $(\lambda,\mu) \in (0,\bar{\lambda}) \times (0,\bar{\lambda})$ with $I(u,v) < 0$, we deduce from (5.9) that $\|(u,v)\| \in (0, T_0(\lambda,\mu)) \cup (T_1(\lambda,\mu),1)$. Next, we claim that $\|(u,v)\| \notin (T_1(\lambda,\mu),1)$. Indeed, if not, let $T_1(\lambda,\mu) < \|(u,v)\| < 1$. By the definition of Ψ, we have $\Psi(\|(u,v)\|) = 0$. Consequently, one has

$$\begin{aligned}
I(u,v) &= \frac{1}{p}\widehat{M}(\|(u,v)\|^p) - \frac{1}{q}(\lambda\|u\|_{q,h}^q + \mu\|v\|_{q,h}^q) \\
&\geq \left(\frac{a+b}{p}\right)\|(u,v)\|^{\theta p} - \frac{1}{q}\max\{\lambda,\mu\}\mathcal{S}_{q,h}^{-q}\|(u,v)\|^q \\
&=: h_{\lambda,\mu}(\|(u,v)\|)
\end{aligned}$$

for all $T_1(\lambda,\mu) < \|(u,v)\| < 1$, where $h_{\lambda,\mu} : [0,\infty) \to \mathbb{R}$ is given by

$$h_{\lambda,\mu}(\ell) = \left(\frac{a+b}{p}\right)\ell^{\theta p} - \frac{1}{q}\max\{\lambda,\mu\}\mathcal{S}_{q,h}^{-q}\ell^q.$$

By simple computation, we can deduce that $h_{\lambda,\mu}$ has a global minimum point at

$$\widehat{\ell} = \left(\frac{\max\{\lambda,\mu\}\mathcal{S}_{q,h}^{-q}}{(a+b)\theta}\right)^{\frac{1}{\theta p - q}}$$

and

$$h_{\lambda,\mu}(\widehat{\ell}) = \left(\frac{\max\{\lambda,\mu\}\mathcal{S}_{q,h}^{-q}}{(a+b)\theta}\right)^{\frac{q}{\theta p - q}} \max\{\lambda,\mu\}\mathcal{S}_{q,h}^{-q}\left(\frac{1}{\theta p} - \frac{1}{q}\right) < 0,$$

we thank to $1 < q < \theta p$. In addition, one sees that

$$h_{\lambda,\mu}(\ell) \geq 0 \iff \ell \geq \left(\frac{\max\{\lambda,\mu\}p\mathcal{S}_{q,h}^{-q}}{q(a+b)}\right)^{\frac{1}{\theta p - q}} =: \ell_2.$$

Consequently, we infer that

$$I(u,v) \geq h_{\lambda,\mu}(\|(u,v)\|) \geq 0, \qquad \forall\ \|(u,v)\| \in (T_1(\lambda,\mu),1)$$

with $\ell_2 < T_1(\lambda,\mu)$, that is, we conclude that $\max\{\lambda,\mu\} < \boldsymbol{\lambda}^3$, which is a contradiction. This completes the proof of the claim. Hence in this case, we also have $\|(u,v)\| \in (0, T_0(\lambda,\mu))$.

From the above discussions, we deduce that $\|(u,v)\| < T_0(\lambda,\mu)$ and thus $I(u,v) = J(u,v)$. In addition, it follows that $I(\varphi,\psi) = J(\varphi,\psi)$ for all (φ,ψ) with $\|(\varphi,\psi)-(u,v)\| < T_0(\lambda,\mu) - \|(u,v)\|$. This shows that $I(\varphi,\psi) = J(\varphi,\psi)$ for all (φ,ψ) in a small neighbourhood of (u,v) and thus we finish the proof of part (a). Now our aim is to prove part (b). For this, first we take $c < 0$ and a $(PS)_c$ sequence $\{(u_n,v_n)\}_n \subset \mathbf{X}$ for the functional I. Therefore, we can assume that $I(u_n,v_n) < 0$ and $I'(u_n,v_n) = o_n(1)$ as $n \to \infty$ and thus by part (a), for any $(\lambda,\mu) \in (0,\bar{\lambda}) \times (0,\bar{\lambda})$, there holds

$$I(u_n,v_n) = J(u_n,v_n),\ I'(u_n,v_n) = J'(u_n,v_n) \quad \text{and} \quad \|(u_n,v_n)\| < T_0(\lambda,\mu).$$

In addition, since I is coercive on $\mathbf{X}$, we deduce that the $(PS)_c$ sequence $\{(u_n,v_n)\}_n \subset \mathbf{X}$ is bounded. Hence up to a subsequence still denoted by itself such that $(u_n,v_n) \rightharpoonup (u,v)$ in $\mathbf{X}$ as $n \to \infty$ for some couple $(u,v) \in \mathbf{X}$. Further, by using (F_4), we can notice that the following estimates hold as $n \to \infty$

$$\begin{aligned}
0 > c &= J(u_n,v_n) - \frac{1}{\sigma}\langle J'(u_n,v_n),(u_n,v_n)\rangle + o_n(1)\\
&\geq \frac{1}{p}\widehat{M}(\|(u_n,v_n)\|^p) - \frac{1}{\sigma}M(\|(u_n,v_n)\|^p)\|(u_n,v_n)\|^p\\
&\quad - \max\{\lambda,\mu\}\left(\frac{1}{q}-\frac{1}{\sigma}\right)\mathcal{S}_{q,h}^{-q}\|(u_n,v_n)\|^q + o_n(1)\\
&\geq \left(\frac{1}{p}-\frac{\theta}{\sigma}\right)b\|(u_n,v_n)\|^{\theta p} - \max\{\lambda,\mu\}\left(\frac{1}{q}-\frac{1}{\sigma}\right)\mathcal{S}_{q,h}^{-q}\|(u_n,v_n)\|^q + o_n(1).
\end{aligned}$$

Define $z_n = (u_n,v_n)$, then we can see that $|z_n| = \sqrt{u_n^2+v_n^2} \leq \psi_n := |u_n| + |v_n|$. By direct calculation, we have

$$\|\psi_n\|_{W^{s,p}} \leq 2(\min\{V_0,K_0\})^{-\frac{1}{p}}\|(u_n,v_n)\|.$$

It follows from the above two inequalities and (5.7) that

$$\limsup_{n\to\infty}\|\psi_n\|_{W^{s,p}}^{p'} \leq \left(\frac{2^{\theta p-q}(\sigma-q)p\mathcal{S}_{q,h}^{-q}\max\{\lambda,\mu\}}{qb(\sigma-\theta p)\big(\min\{V_0,K_0\}\big)^{\frac{\theta p-q}{p}}}\right)^{\frac{p'}{\theta p-q}} < \frac{\beta_*}{2\alpha_0},$$

thanks to the fact that $\max\{\lambda,\mu\} < \bar{\lambda} \leq \boldsymbol{\lambda}^4$, since we have

$$(\lambda,\mu) \in (0,\bar{\lambda}) \times (0,\bar{\lambda}) \quad \text{and} \quad \bar{\lambda} = \min\{\boldsymbol{\lambda}^1,\boldsymbol{\lambda}^2,\boldsymbol{\lambda}^3,\boldsymbol{\lambda}^4\}.$$

Using all of the above information, and arguing similarly as in the proof of Theorem 1.3, we can easily deduce that $(u_n,v_n) \to (u,v)$ in $\mathbf{X}$ as $n \to \infty$. This completes the proof of part (b). Hence the lemma is well established. □

It follows immediately from the above lemma that the following result holds.

Corollary 5.6. *For all $(\lambda,\mu) \in (0,\bar{\lambda}) \times (0,\bar{\lambda})$, the set K_c, which is given by*

$$K_c = \{(u,v) \in \mathbf{X} :\ I(u,v) = c < 0,\ I'(u,v) = 0\},\ \textit{is compact.}$$

For $\epsilon > 0$, we define

$$I^{-\epsilon} = \{(u,v) \in \mathbf{X} :\ I(u,v) \leq -\epsilon\}.$$

Lemma 5.7. *For any $k \in \mathbb{N}$, there exists $\epsilon_k > 0$ such that there holds $\gamma(I^{-\epsilon_k}) \geq k$.*

Proof. Suppose that $\mathbf{X}_k$ be a k-dimensional subspace of $\mathbf{X}$. For any $(u,v) \in \mathbf{X}_k \setminus \{(0,0)\}$, we define $(u,v) = r_k(\varphi,\psi)$ with $(\varphi,\psi) \in \mathbf{X}_k$, $\|(\varphi,\psi)\| = 1$ and $r_k = \|(u,v)\|$. Under the assumption of h, we known that $\|(\varphi,\psi)\|_{L^q_h(\mathbb{R}^N)\times L^q_h(\mathbb{R}^N)}$ is a norm of $\mathbf{X}_k$ for all $(\varphi,\psi) \in \mathbf{X}_k$. Since all the norms are equivalent in a finite-dimensional Banach space, therefore we infer that for each $(\varphi,\psi) \in \mathbf{X}_k$, there exists $d_k > 0$ such that

$$\|(\varphi,\psi)\|^q_{L^q_h(\mathbb{R}^N)\times L^q_h(\mathbb{R}^N)} \geq d_k.$$

Consequently, for $r_k \in (0, T_0(\lambda,\mu))$, we can easily deduce that

$$I(u,v) = J(u,v) \leq \left(\frac{a+b}{p}\right) r_k^p - \frac{1}{q}\min\{\lambda,\mu\} d_k r_k^q.$$

In virtue of $q < p$, without loss of generality, we can choose $r_k \in (0, T_0(\lambda,\mu))$ sufficiently small enough such that $I(u,v) \leq -\epsilon_k < 0$. Define $S_{r_k} = \{(u,v) \in \mathbf{X} : \|(u,v)\| = r_k\}$, then one has $S_{r_k} \cap \mathbf{X}_k \subset I^{-\epsilon_k}$. Now by using Proposition 5.1, we have $\gamma(I^{-\epsilon_k}) \geq \gamma(S_{r_k} \cap \mathbf{X}_k) = k$. This finishes the proof. □

Define

$$\mathbf{\Sigma}_k = \{A \in \mathbf{\Sigma} : \ \gamma(A) \geq k\}$$

and

$$c_k = \inf_{A \in \mathbf{\Sigma}_k} \sup_{(u,v)\in A} I(u,v).$$

It is obvious that the sequence $\{c_k\}_k$ is monotonically increasing in nature and there holds

$$-\infty < c_k \leq -\epsilon_k < 0 \quad \text{for each } k \in \mathbb{N},$$

thanks to the fact that I is bounded from below and $I^{-\epsilon_k} \in \mathbf{\Sigma}_k$. Due to Lemma 5.5, the functional I satisfies the $(PS)_c$ condition for $c < 0$, and we have $c_k < 0$ for each $k \in \mathbb{N}$, therefore by a standard argument, we infer that all c_k are critical values of I.

The next lemma shows that the set K_c, defined in Corollary 5.6, contains infinitely many critical points of I.

Lemma 5.8. *Let $(\lambda,\mu) \in (0,\bar{\lambda}) \times (0,\bar{\lambda})$. Then for $c = c_k = c_{k+1} = \cdots = c_{k+m}$ with some $m \in \mathbb{N}$, we have $\gamma(K_c) \geq m+1$.*

Proof. It will be proven by using the method of contradiction. For this, we first claim that $\gamma(K_c) \geq m+1$ holds. Indeed, if not, let $\gamma(K_c) \leq m$. In virtue of Corollary 5.6, the set K_c is compact and $K_c \in \mathbf{\Sigma}$. Consequently, by using Proposition 5.1, one has $\gamma(K_c) < \infty$ and there exists $\delta > 0$ such that $N_\delta(K_c) \in \mathbf{\Sigma}$ and $\gamma(K_c) = \gamma(N_\delta(K_c)) \leq m$. On the other hand, by using the assumption (F_1'), we infer that the functional I is even. Thus, due to Proposition 5.2, there exists an odd homeomorphism $\eta(t,(u,v)) = \eta_t(u,v) \in C([0,1]\times \mathbf{X}, \mathbf{X})$ such that

$$\eta_1(I^{c+\epsilon} \setminus N_\delta(K_c)) \subset I^{c-\epsilon} \quad \text{for some } \epsilon \in (0,-c).$$

From the hypothesis, we have $c = c_{k+m}$ and hence there exists a set $A \in \mathbf{\Sigma}_{k+m}$ such that $\sup_{(u,v)\in A} I(u,v) < c+\epsilon$, that is, $A \subset I^{c+\epsilon}$. This shows that $\eta_1(A \setminus N_\delta(K_c)) \subset \eta_1(I^{c+\epsilon} \setminus N_\delta(K_c)) \subset I^{c-\epsilon}$. Consequently, we deduce that

$$\sup_{(u,v)\in \eta_1(A\setminus N_\delta(K_c))} I(u,v) \leq c - \epsilon. \tag{5.10}$$

Due to Proposition 5.2, we have

$$\gamma(\overline{A \setminus N_\delta(K_c)}) \geq \gamma(A) - \gamma(N_\delta(K_c)) \geq k \quad \text{and} \quad \gamma(\overline{\eta_1(A \setminus N_\delta(K_c))}) \geq k.$$

It follows that $\overline{\eta_1(A \setminus N_\delta(K_c))} \in \mathbf{\Sigma}_k$ and thus we have

$$\sup_{(u,v) \in \overline{\eta_1(A \setminus N_\delta(K_c))}} I(u,v) \geq c_k = c,$$

which contradicts (5.10). This completes the proof of the claim and we finish the proof. □

Proof of Theorem 1.5. Let $(\lambda, \mu) \in (0, \bar{\lambda}) \times (0, \bar{\lambda})$. Notice that if we have $-\infty < c_1 < c_2 < \cdots < c_k < \cdots < 0$, then since c_k are critical values of I, we infer that I has infinitely many critical points. In virtue of Lemma 5.5, we have $J \equiv I$ if $I < 0$. This shows that the system $(\mathcal{S}_{\lambda,\mu})$ has infinitely many weak solutions.

On the other hand, if there exists $c_k = c_{k+m}$ for some $m \in \mathbb{N}$, then $c = c_k = c_{k+1} = \cdots = c_{k+m}$. In virtue of Lemma 5.8, we get $\gamma(K_c) \geq m + 1 \geq 2$ and thus by Proposition 5.1, one can notice that the set K_c has infinitely many points. In conclusion, we deduce that the system $(\mathcal{S}_{\lambda,\mu})$ has infinitely many weak solutions. This finishes the proof. □

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