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On the set of eigenvalues for some classes of coercive and noncoercive problems involving $(2, p(x))$-Laplacian-like operators

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Abstract. We consider a class of double-phase nonlinear eigenvalue problems driven by a $(2, \phi)$-Laplace-like operator:

$$-\Delta u - \varepsilon \operatorname{div}[\phi(x, |\nabla u|)\nabla u] = \lambda(u + \varepsilon)$$

in a domain Ω, subject to Dirichlet boundary conditions, where Ω is a bounded subset of $\mathbb{R}^N$ with a smooth boundary. Here, $\varepsilon > 0$, and the potential function ϕ exhibits $p(x)$-variable growth.

We establish several results on the existence and concentration of eigenvalues for this problem, focusing on the influence of the growth behavior of the potential function ϕ, specifically through the interaction between the variable growth exponent $p(x)$ and the constant growth exponent 2. The proofs rely on variational arguments based on the Direct Method in the Calculus of Variations, Ekeland's variational principle, and energy estimates.

Keywords: double-phase differential operator, continuous bounded or unbounded spectrum, variable exponent, eigenvalue problem, coercive and noncoercive case.

2020 Mathematics Subject Classification: 35P30, 49R05, 58C40.

1 Introduction

The study of variational problems with nonstandard growth conditions has been extensively developed in recent years. Moreover, as technological advancements in key areas such as robotics, aerospace engineering, and image restoration have accelerated, new mathematical models have emerged to obtain significant results. Notably, $(2, p)$-equations (i.e., equations driven by the sum of a Laplacian and a p-Laplacian) appear in mathematical physics, including quantum physics (see Benci, D'Avenia, Fortunato, Pisani [7]) and plasma physics (see Cherfils, Ilyasov [13]). Since equations driven by the sum of two differential operators with

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different structures (such as $(2, p)$-equations or, in our case, $(2, p(x))$-equations) arise in mathematical models of various physical processes, we refer to the work of Marano and Mosconi [24] for more details.

Problems involving Laplace operators with different homogeneity have been extensively studied over the years. We mention here some relevant works by Barile and Figueiredo [6], Motreanu and Tanaka [29], as well as Papageorgiou and Rădulescu [31]. A significant advancement in the study of problems involving sums of two differential operators has been made through the investigation of problems with the $p(x)$-Laplacian and $p(x)$-Laplacian like operators, which exhibit a more complex structure and lack homogeneity. There is strong motivation for studying equations with variable exponent growth conditions, as they model various phenomena in elastic mechanics (see Zhikov [44]), electrorheological fluids (see Acerbi, Mingione [3], Diening [15]), and image restoration (see Chen, Levine, Rao [11]). Additionally, recent trends in variational fractional analysis allow us to model problems arising in fluid mechanics (see Ragusa [34]), electromagnetism and electrochemistry (see Guariglia [19], Li, Dao, and Guo [23]), viscoelasticity (see Abbas and Ragusa [1], Li, Dao, and Guo [23]), and signal processing (see Guariglia [18]).

In this paper, we extend the results obtained in the aforementioned studies by replacing the p-Laplace operator with a $p(x)$-Laplacian-type operator driven by a potential function ϕ, which may take the following forms:

(1) $\phi(x,z) = z^{p(x)-2}$, which includes the weighted $p(x)$-Laplace operator;

(2) $\phi(x,z) = \sqrt{(1+|z|^2)^{p(x)-2}}$, corresponding to the generalized mean curvature operator;

(3) $\phi(x,z) = \left(1 + \frac{z^{p(x)}}{\sqrt{\left(1+z^{2p(x)}\right)}}\right)$, which describes the capillary phenomenon.

The last case has recently gained significant attention (see Avci [4], Uță [41], Vetro [42], Wang, Zhou [43]). This growing interest is driven not only by the intrinsic fascination with naturally occurring phenomena such as the motion of drops, bubbles, and waves but also by the importance of these studies in applied fields, including industrial, biomedical, pharmaceutical, and microfluidic systems.

Our study is based on newly developed differential operators introduced by Kim and Kim [21], which allow us to analyze problems that may lack uniform convexity. In this paper, we study the operators introduced by Kim and Kim from a different perspective, as our problems involve the $(2, p(x))$-Laplace-like operator, which has a more complex structure. Additionally, the fact that the reaction function, i.e., $f(x,z) = z + \varepsilon$, exhibits constant growth behavior introduces additional technical difficulties, especially due to the presence of the Laplace differential operator embedded in the structure of the differential operator.

To analyze our problem, we consider different growth behaviors for the variable exponent $p(x)$, as follows:

(1) The first case:
$$1 < p^- \leq p(x) \leq p^+ < 2 \quad \text{on } \overline{\Omega}. \tag{1.1}$$

(2) The second case:
$$2 < p^- \leq p(x) \leq p^+ < p^*(x) \quad \text{on } \overline{\Omega}. \tag{1.2}$$

(3) The third case:
$$1 < p^- < 2 < p^+ < p^*(x) \quad \text{on } \overline{\Omega}. \tag{1.3}$$

In both the second and third cases, $p^*(x) = \frac{Np(x)}{N-p(x)}$ represents the critical Sobolev exponent for variable exponent spaces.

We note that a problem similar to the case (1.1) has been studied by Costea and Mihăilescu [14] for a simpler $p(x)$-Laplace-like operator, using Banach's fixed point theorem. For further results on $(p_1(x), p_2(x))$-problems involving similar potentials, we refer to Rădulescu *et al.* [5, 9], and Uță [38] for the coercive case; Rădulescu [12] and Uță [39, 41] for the noncoercive case; Aberqi, Bennouna, Benslimane, and Ragusa [2], and Uță [37] for studies on manifolds; and Repovš [35] and Uță [40] for cases involving a lack of compactness.

The structure of this paper is as follows: In Section 2, we provide a brief description of the functional framework, including constant Lebesgue and Sobolev spaces for the first case and variable exponent Lebesgue and Sobolev spaces for the second and third cases. In Section 3, we introduce the basic hypotheses, some auxiliary results essential to our study, and we state the main results for each case. Section 4 is dedicated to proving the main results. Finally, in Section 5, we present some remarks, future research perspectives related to our methods, and open problems that may arise under different assumptions.

2 The functional framework

In this section, we introduce the basic properties of variable exponent spaces, which constitute the necessary functional framework for the study of problem $(P_{\lambda,\varepsilon})$.

These results are described in the following books: Diening, Hästö, Harjulehto, Ružička [20], Rădulescu, Repovš [33]. We also refer to the survey paper by Rădulescu [32].

Let Ω be a bounded domain in $\mathbb{R}^N$.

For a measurable function $p : \overline{\Omega} \to \mathbb{R}$, we define:

$$p^+ = \sup_{x\in\Omega} p(x), \quad p^- = \inf_{x\in\Omega} p(x).$$

Define:

$$C_+(\Omega) = \left\{ p \in C(\overline{\Omega}) : \ p(x) > 1, \forall x \in \overline{\Omega} \right\}.$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined as:

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega |u|^{p(x)} dx < \infty \right\},$$

with the norm:

$$|u|_{p(x)} = \inf \left\{ \mu > 0 : \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}.$$

Equipped with this norm, $L^{p(x)}(\Omega)$ becomes a Banach space whose dual is $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Remark 2.1. If $1 < p(x) < \infty$, then $L^{p(x)}(\Omega)$ is a reflexive Banach space. Moreover, if p is measurable and bounded, then $L^{p(x)}(\Omega)$ is also separable.

Remark 2.2. If $0 < |\Omega| < \infty$ and $h(x)$, $r(x)$ satisfy $h(x) < r(x)$ almost everywhere in Ω, then the following continuous embedding holds:

$$L^{r(x)}(\Omega) \hookrightarrow L^{h(x)}(\Omega).$$

Denoting $L^{p'(x)}(\Omega)$ as the dual space of $L^{p(x)}(\Omega)$, we have the following Hölder-type inequality for all $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$:

$$\left| \int_\Omega uv\, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(x)} |v|_{p'(x)} \leq 2 |u|_{p(x)} |v|_{p'(x)}. \tag{2.1}$$

A key role in studies involving variable exponent Lebesgue spaces is played by the modular of $L^{p(x)}(\Omega)$, denoted as $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ and defined by:

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

Remark 2.3. If $p(x) \not\equiv$ constant in Ω, then for u, $(u_n) \in L^{p(x)}(\Omega)$, the following relations hold:

$$|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}, \tag{2.2}$$

$$|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}, \tag{2.3}$$

$$|u|_{p(x)} = 1 \Rightarrow \rho_{p(x)}(u) = 1, \tag{2.4}$$

$$|u_n - u|_{p(x)} \to 0 \Leftrightarrow \rho_{p(x)}(u_n - u) \to 0. \tag{2.5}$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined as:

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

with the equivalent norms:

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)},$$

$$\|u\| = \inf \left\{ \mu : \int_\Omega \left(\left| \frac{\nabla u(x)}{\mu} \right|^{p(x)} + \left| \frac{u(x)}{\mu} \right|^{p(x)} \right) dx \leq 1 \right\}.$$

The space $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\| \cdot \|_{p(x)}$:

$$W_0^{1,p(x)}(\Omega) = \left\{ u : u|_{\partial\Omega} = 0, u \in L^{p(x)}(\Omega), |\nabla u| \in L^{p(x)}(\Omega) \right\}.$$

Taking into account [21], for $p \in C_+(\overline{\Omega})$, we have the $p(\cdot)$-Poincaré-type inequality:

$$|u|_{p(x)} \leq C |\nabla u|_{p(x)}, \tag{2.6}$$

where $C > 0$ is a constant depending on p and Ω.

Remark 2.4. If $p, q : \Omega \to (1, \infty)$ are Lipschitz continuous, with $p^+ < N$ and $p(x) \leq q(x) \leq p^*(x)$ for all $x \in \Omega$, where $p^*(x) = \frac{Np(x)}{N - p(x)}$, then the embedding:

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

is compact and continuous.

Remark 2.5. If $0 < |\Omega| < \infty$ and $p_2(x) < p_1(x)$ in Ω, then the following continuous embedding holds:

$$W_0^{1,p_1(x)}(\Omega) \hookrightarrow W_0^{1,p_2(x)}(\Omega).$$

Remark 2.6. If $p(x) = p \equiv$ constant for every $x \in \Omega$, then the modular $\rho_{p(x)}(u)$ becomes $\|u\|_p^p$, and thus we work within the framework of constant exponent spaces:

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega |u|^p \, dx < \infty \right\},$$

endowed with the norm

$$|u|_p = \left(\int_\Omega |u(x)|^p \, dx \right)^{\frac{1}{p}}.$$

Similarly, the Sobolev space is defined as

$$W_0^{1,p}(\Omega) = \{ u : \Omega \to \mathbb{R} \text{ measurable} : u|_{\partial\Omega} = 0, \ u \in L^p(\Omega), \ |\nabla u| \in L^p(\Omega) \},$$

endowed with the norm

$$\|u\|_p = \left(\int_\Omega |\nabla u(x)|^p \, dx \right)^{\frac{1}{p}}.$$

3 Basic hypotheses

We will study the problem

$$\begin{cases} -\Delta u - \varepsilon \operatorname{div} [\phi(x, |\nabla u|) \nabla u] = \lambda \, (u + \varepsilon) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$P_{\lambda,\varepsilon}$}$$

In order to state more precisely our results we have that:

(ϕ_1) $\phi : \Omega \times [0, \infty) \to [0, \infty)$ fulfill the following assumptions:

- $\phi(\cdot, z)$ is measurable on Ω for all $z \geq 0$;
- $\phi(x, \cdot)$ is locally absolutely continuous on $[0, \infty)$ for almost all $x \in \Omega$.

(ϕ_2) There exist a function $\alpha \in L^{p'(x)}(\Omega)$ and a positive constant β such that

$$|\phi(x, |z|) z| \leq \alpha(x) + \beta |z|^{p(x)-1}$$

for almost all $x \in \Omega$ and for all $z \in \mathbb{R}^N$.

(ϕ_3) There is a positive constant c such that the following hypotheses hold for almost all $x \in \Omega$:

- $\phi(x, z) \geq c z^{p(x)-2}$;
- $z \dfrac{\partial \phi}{\partial z}(x, z) + \phi(x, z) \geq c z^{p(x)-2}$, for almost all $z > 0$.

Definition 3.1. We say that $\lambda \in \mathbb{R}$ is an eigenvalue of the problem ($P_{\lambda,\varepsilon}$) if there exists $u \in W \setminus \{0\}$ such that:

$$\int_\Omega \nabla u \nabla \varphi dx + \varepsilon \int_\Omega [\phi(x, |\nabla u|) \nabla u \nabla \varphi] \, dx = \lambda \int_\Omega (u + \varepsilon) \, \varphi dx$$

for all $\varphi \in W$.

Remark 3.2. As we are interested in finding weak solutions to problem ($P_{\lambda,\varepsilon}$), and our differential operator exhibits a double-phase behavior (being driven by both the Laplace and the ϕ-Laplace operator) the natural function space in which we seek solutions depends on the relationship between the growth behaviors of these two differential operators.

In the first case, we aim to find solutions in the functional space $W_0^{1,2}(\Omega) := H_0^1(\Omega)$, considering that $1 < p^- \leq p(x) \leq p^+ < 2$ on $\overline{\Omega}$. In the second and third cases, the appropriate space for finding solutions is $W_0^{1,p(x)}(\Omega)$, taking into account that either $2 < p^- \leq p(x) \leq p^+$ and $2 < p^*(x)$, or $p^- < 2 < p^+$ and $2 < p^*(x)$ on $\overline{\Omega}$.

Throughout this paper, we denote the appropriate functional framework by W, where $W = W_0^{1,2}(\Omega) := H_0^1(\Omega)$ in the first case, and $W = W_0^{1,p(x)}(\Omega)$ in the second and third cases. For further details, we refer to [22, Theorem 2.8].

In what follows we set:

$$S_0(x,t) = \int_0^t \phi(x,z)z dz.$$

An important role in our variational approach is played by the fact that the following assumption holds true for the potential ϕ:

(ϕ_4) For all $x \in \overline{\Omega}$, all $z \in \mathbb{R}^N$, the following estimate is true:

$$0 \leq \phi(x,|z|)|z|^2 \leq p^+ S_0(x,|z|).$$

In order to identify the eigenvalues of problem ($P_{\lambda,\varepsilon}$), we consider the following energy functional and demonstrate that its critical points correspond to weak solutions of our problem. Let $T_{\lambda,\varepsilon} : W \to \mathbb{R}$ be defined as follows:

$$T_{\lambda,\varepsilon}(u) = S(u) - \lambda R(u),$$

where

$$S(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \varepsilon \int_\Omega S_0(x,|\nabla u|) dx,$$
$$R(u) = \frac{\lambda}{2}\int_\Omega u^2 dx + \lambda\varepsilon \int_\Omega u dx.$$

Now, using [21, Lemmas 3.2, 3.4], by standard arguments we can see that $T_{\lambda,\varepsilon} \in C^1(W,\mathbb{R})$ and

$$\langle T'_{\lambda,\varepsilon}(u), \varphi\rangle = \int_\Omega \nabla u \nabla \varphi dx + \varepsilon \int_\Omega \phi(x,|\nabla u|)\nabla u \nabla \varphi dx - \lambda \int_\Omega (u+\varepsilon)\, dx,$$

hence we can say that the critical points of $T_{\lambda,\varepsilon}$ are weak solutions for the problem ($P_{\lambda,\varepsilon}$).

Remark 3.3. Throughout this paper by

$$\lambda_1(\Delta) := \min_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx}$$

we will denote the principal eigenvalue of the Laplace operator.

We are now ready to state our existence theorems.

Theorem 3.4. *Assume that hypotheses (ϕ_1)–(ϕ_4), (1.1) and $\varepsilon > 0$ hold true, then every $\lambda \in (0, \lambda_1(\Delta))$ is an eigenvalue of the problem $(P_{\lambda,\varepsilon})$.*

Theorem 3.5. *Assume that hypotheses (ϕ_1)–(ϕ_4), (1.2) and $\varepsilon > 0$ hold true, then there exists a constant $\lambda_0 > 0$, such that any $\lambda \in (0, \lambda_0)$ is an eigenvalue of the problem $(P_{\lambda,\varepsilon})$.*

Theorem 3.6. *Assume that hypotheses (ϕ_1)–(ϕ_4), (1.2) and $\varepsilon > 0$ hold true, then every $\lambda \in \mathbb{R}$, $\lambda > 0$ is an eigenvalue of the problem $(P_{\lambda,\varepsilon})$.*

Theorem 3.7. *Assume that hypotheses (ϕ_1)–(ϕ_4), (1.3) and $\varepsilon > 0$ hold true, then every $\lambda \in (0, \lambda_1(\Delta))$ is an eigenvalue for the problem $(P_{\lambda,\varepsilon})$.*

Theorem 3.8. *Assume that hypotheses (ϕ_1)–(ϕ_4), (1.3) and $\varepsilon > 0$ hold true, then there exists a constant $\lambda^* > 0$, such that any $\lambda \in (0, \lambda^*)$ is an eigenvalue of the problem $(P_{\lambda,\varepsilon})$.*

4 Main results

4.1 Proof of Theorem 3.4

Since (1.1) holds, we have $1 < p^- \leq p(x) \leq p^+ < 2$, which implies the following continuous embeddings hold:

$$\begin{aligned} W_0^{1,p(x)}(\Omega) &\hookrightarrow W_0^{1,2}(\Omega) := H_0^1(\Omega), \\ L^{p(x)}(\Omega) &\hookrightarrow L^2(\Omega), \end{aligned} \tag{4.1}$$

and by W we will denote $H_0^1(\Omega)$. As $\lambda \in (0, \lambda_1(\Delta))$ we can remark the following.

Remark 4.1. There exists a constant $C_\lambda > 0$ such that, for every $u \in W$, the following holds:

$$C_\lambda \int_\Omega |\nabla u|^2 dx \leq \int_\Omega \left(|\nabla u|^2 - \lambda |u|^2\right) dx. \tag{4.2}$$

Therefore we have that:

$$\begin{aligned} T_{\lambda,\varepsilon}(u) &= S(u) - \lambda R(u) \\ &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \varepsilon \int_\Omega S_0(x, |\nabla u|) dx - \frac{\lambda}{2}\int_\Omega u^2 dx - \lambda\varepsilon \int_\Omega u dx \\ &\overset{(4.2)}{\geq} \frac{1}{2} C_\lambda \int_\Omega |\nabla u|^2 dx + \varepsilon \int_\Omega S_0(x, |\nabla u|) dx - \lambda\varepsilon \int_\Omega u dx \\ &\overset{(\phi_4)}{\geq} \frac{1}{2} C_\lambda \int_\Omega |\nabla u|^2 dx + \frac{\varepsilon}{p^+}\int_\Omega \phi(x, |\nabla u|)|\nabla u|^2 dx - \lambda\varepsilon \int_\Omega u dx \\ &\overset{(\phi_3)}{\geq} \frac{1}{2} C_\lambda \|u\|_W^2 + \frac{c\varepsilon}{p^+}\int_\Omega |\nabla u|^{p(x)} dx - \lambda\varepsilon |u|_1 \\ &\geq \frac{1}{2} C_\lambda \|u\|_W^2 + \frac{c\varepsilon C^p}{p^+} \min\left\{\|u\|_W^{p^-}, \|u\|_W^{p^+}\right\} - \lambda\varepsilon C_1 \|u\|_W, \end{aligned}$$

where $C^p = \max\{C^{p^-}, C^{p^+}\}$, and $C^{p^\pm} > 0$ are some constants obtained from the continuous embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow H_0^1(\Omega)$, and $C_1 > 0$ is a constant obtained from the continuous embedding $H_0^1(\Omega) \hookrightarrow L^1(\Omega)$.

Now letting $\|u\|_W \to \infty$ we obtain that

$$\lim_{\|u\|_W \to \infty} T_{\lambda,\varepsilon}(u) = \infty,$$

hence we get that the energy functional $T_{\lambda,\varepsilon}$ is coercive. By the properties of the potential function ϕ and using relation (1.1) we have that $T_{\lambda,\varepsilon}$ is weakly lower semicontinuous, i.e. $T_{\lambda,\varepsilon}(u) \leq \liminf_{n\to\infty} T_{\lambda,\varepsilon}(u_n)$, for any sequence $(u_n)_{n\geq 1} \subset W$ for which we have $u_n \rightharpoonup u$ in W (for more details we refer to [21, Lemma 4.3]).

In conclusion, using [36, Theorem 1.2], we get that for each $\lambda \in (0, \lambda_1(\Delta))$ we obtain at least one nontrivial critical point of $T_{\lambda,\varepsilon}$, which is in fact a global minimum point of our energy functional. $\square$

Remark 4.2. In order to enhance clarity and improve the readability of the paper, we present Figure 4.1 illustrating a simple function that exhibits the same geometrical properties as our energy functional. For simplicity, we omit the x-dependence of f and define $f : \mathbb{R} \to \mathbb{R}$ as

$$f(z) = \frac{1}{2}|z|^2 + \varepsilon|z|^{p(z)} - \lambda\left(\frac{1}{2}|z|^2 + \varepsilon \cdot z\right).$$

We consider the following parameters $\varepsilon = 3, \lambda = 0.2, p(x) = 1.5 + 0.49 \cdot \sin(6\pi x)$ for:

(a) the behavior near the origin;

(b) the behavior away from the origin.

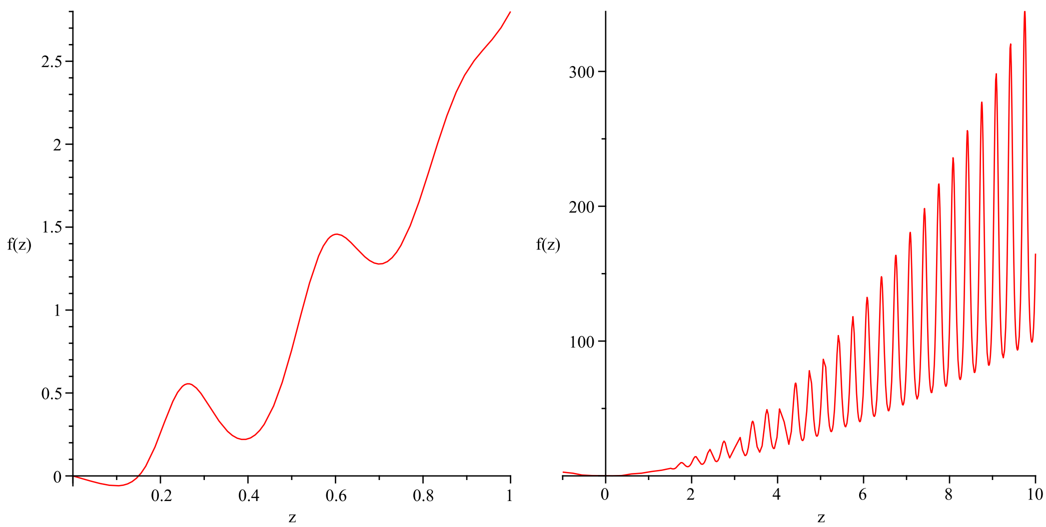

Figure 4.1: For the parameters specified above, the first figure illustrates the existence of a global minimum near the origin, while the second figure demonstrates the coercivity of the function f.

Remark 4.3. It is obvious that the critical point of our energy functional, described as

$$T_{\lambda,\varepsilon}(u_\lambda) = \min_{u\in W} T_{\lambda,\varepsilon}(u)$$

is nontrivial since $u = 0$ is not a solution of the problem $(P_{\lambda,\varepsilon})$.

Remark 4.4. For similar results in the anisotropic case we refer to [8, Theorem 3.2], [26, Theorem 3], [27, Theorem 1.1(c)], [30, Theorem 1] and to [33, Chapter 5].

4.2 Proof of Theorem 3.5

Since from now on (1.2) holds true, we have that $2 < p^- \leq p(x) < p^+$ and $2 < p^*(x) = \frac{Np(x)}{N-p(x)}$.

Given that we are in the second case of study, we will consider $W := W_0^{1,p(x)}(\Omega)$ as the appropriate functional framework.

Since the proof of the present theorem is longer and involves more technical difficulties than Theorem 3.4, we will split it into several steps.

Firstly we shall prove some geometric properties of the energy functional associated to our problem.

Step 1. We will prove the existence of a "mountain" near the origin, that is the existence of some constants $\rho > 0$ and $\theta > 0$ such that

$$T_{\lambda,\varepsilon}(u) \geq \theta > 0$$

for any $u \in W$ with $\|u\|_W = \rho$.

Consider $u \in W$ such that $\|u\|_W < 1$, we have that

$$\begin{aligned}
T_{\lambda,\varepsilon}(u) &= S(u) - \lambda R(u) \\
&= \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \varepsilon \int_\Omega S_0(x, |\nabla u|)dx - \frac{\lambda}{2}\int_\Omega u^2 dx - \lambda\varepsilon \int_\Omega u dx \\
&\geq \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{c\varepsilon}{p^+}\int_\Omega |\nabla u|^{p(x)} dx - \frac{\lambda}{2}\int_\Omega u^2 dx - \lambda\varepsilon \int_\Omega u dx \\
&\quad \text{(using hypotheses } (\phi_3), (\phi_4)) \\
&\geq \frac{1}{2}C_{2p}\|u\|_W^2 + \frac{c\varepsilon}{p^+}\|u\|_W^{p^+} - \frac{\lambda}{2}|u|_2^2 - \lambda\varepsilon|u|_1 \\
&\geq \frac{c\varepsilon}{p^+}\rho^{p^+} - \lambda\left(\frac{C'_{2p}}{2}\rho^2 + \varepsilon C_{1p}\rho\right) \\
&\geq \rho\left[\frac{c\varepsilon}{p^+}\rho^{p^+-1} - \lambda\left(\frac{C'_{2p}}{2} + \varepsilon C_{1p}\right)\right]
\end{aligned}$$

where $C_{2p} > 0$, $C'_{2p} > 0$, $C_{1p} > 0$ are some constants obtained using the following continuous embeddings $W \hookrightarrow H_0^1(\Omega)$, $W \hookrightarrow L^2(\Omega)$, respectively $W \hookrightarrow L^1(\Omega)$.

Therefore, we can set

$$\lambda_0 = \frac{c\varepsilon\rho^{p^+-1}}{p^+} \cdot \frac{2}{C'_{2p} + 2\varepsilon C_{1p}} \tag{4.3}$$

then for any $\lambda \in (0, \lambda_0)$, $\varepsilon > 0$,we may find a constant $\theta = \theta(\lambda_0, \varepsilon) > 0$ such that

$$T_{\lambda,\varepsilon}(u) \geq \theta > 0,$$

for any $u \in W$, with $\|u\|_W = \rho$.

Step 2. We shall prove in what follows the existence of a "valley" at a suitable distance from the origin. Hence, we will highlight the existence of an element $v \in W$ such that

$$T_{\lambda,\varepsilon}(zv) < 0, \tag{4.4}$$

provided that $z > 0$ is sufficiently small.

So, let $v \in C_0^\infty(\Omega)$, $v(x) \geq 0$, then there exists $\overline{\Omega}_0 \subset \Omega$ such that for any $x \in \overline{\Omega}_0$ we have $v(x) > 0$. For more details on similar arguments we refer to [28, Lemma 2.3] and [25, Lemma 2(iii)] (see also [26, proof of Theorem 4], [27, proof of Theorem 1.1(b)]). Then for $z \in (0,1)$ sufficiently small we have that:

$$\begin{aligned} T_{\lambda,\varepsilon}(zv) &= S(zv) - \lambda R(zv) \\ &\leq C_v \varepsilon z^{p^-} + \frac{z^2}{2}\int_\Omega |\nabla v|^2 dx - \lambda \frac{z^2}{2}\int_\Omega |v|^2 dx - \lambda \varepsilon z \int_{\Omega_0} v dx \\ &\leq C_v \varepsilon z^{p^-} + \frac{z^2}{2}\int_\Omega |\nabla v|^2 dx - \lambda \varepsilon z \int_{\Omega_0} v dx \\ &\leq z^2 \left(C_v \varepsilon + \frac{1}{2}\int_\Omega |\nabla v|^2 dx \right) - \lambda \varepsilon z \int_{\Omega_0} v dx \\ &\quad \text{(due to the fact that } 2 < p^- \text{ and } z \in (0,1)), \end{aligned}$$

where $C_v = 2C_\phi |\alpha|_{p'(x)} \|v\|_W^{p^-} + \frac{\beta}{p^-}\|v\|_W^{p^-}$, and $C_\phi > 0$ is a constant which depends on the potential ϕ.

Since z is chosen to be small enough we get that

$$T_{\lambda,\varepsilon}(zv) < 0,$$

provided by any $z < z_0$, where

$$0 < z_0 < \min\left\{1, \frac{\lambda\varepsilon \int_{\Omega_0} v dx}{C_v \varepsilon + \frac{1}{2}\int_\Omega |\nabla v|^2 dx}\right\}.$$

Remark 4.5. We may observe from the results obtained in Step 2 that our energy functional does not have the appropriate geometry for applying the mountain pass theorem, as the valley obtained is not sufficiently far from the origin as required.

Step 3. In what follows, we aim to prove that the geometrical properties of the energy functional established in the previous steps can still guarantee the existence of a nontrivial critical point. Therefore, we seek to find a Palais–Smale sequence that converges to this critical point.

By the assumption that $\lambda \in (0, \lambda_0)$ and taking into account Step 1, we obtain

$$\inf_{u \in \partial B(0,\rho)} T_{\lambda,\varepsilon}(u) > 0, \tag{4.5}$$

where $B(0,\rho)$ denotes the ball centered at 0 with radius ρ in W, and $\partial B(0,\rho)$ represents its boundary.

Moreover, by Step 2, there exists $v \in W$ such that $T_{\lambda,\varepsilon}(zv) < 0$, provided that $z > 0$ is sufficiently small.

Hence, by the fact that $2 < p^*(x)$ and by relations (2.2), (2.3) we obtain the following continuous embeddings

$$W \hookrightarrow L^2(\Omega) \hookrightarrow L^1(\Omega),$$

and

$$T_{\lambda,\varepsilon}(u) \geq \frac{1}{2}C_{2p}\|u\|_W^2 + \frac{c\varepsilon}{p^+}\|u\|_W^{p^+} - \frac{\lambda}{2}C'_{2p}|u|_W^2 - \lambda\varepsilon C_{1p}|u|_W. \tag{4.6}$$

So, we obtain that there exists a constant $\underline{c} < 0$ such that:

$$-\infty < \underline{c} := \inf_{\overline{B(0,\rho)}} T_{\lambda,\varepsilon} < 0. \tag{4.7}$$

Using (4.5), (4.6), and (4.7), let $\omega > 0$ be a constant such that:

$$\omega < \inf_{\partial B(0,\rho)} T_{\lambda,\varepsilon} - \inf_{B(0,\rho)} T_{\lambda,\varepsilon},$$

and by applying Ekeland's variational principle (see [17]) to the energy functional $T_{\lambda,\varepsilon} : \overline{B(0,\rho)} \to \mathbb{R}$, we obtain the existence of a function $u_\omega \in \overline{B(0,\rho)}$ such that

$$\begin{aligned} T_{\lambda,\varepsilon}(u_\omega) &\leq \inf_{\overline{B(0,\rho)}} T_{\lambda,\varepsilon} + \omega, \\ T_{\lambda,\varepsilon}(u_\omega) &\leq T_{\lambda,\varepsilon}(u) + \omega \|u - u_\omega\|_W, \quad u \neq u_\omega. \end{aligned}$$

Hence,

$$T_{\lambda,\varepsilon}(u_\omega) \leq \inf_{\overline{B(0,\rho)}} T_{\lambda,\varepsilon} + \omega \leq \inf_{B(0,\rho)} T_{\lambda,\varepsilon} + \omega < \inf_{\partial B(0,\rho)} T_{\lambda,\varepsilon},$$

therefore we point out that $\|u_\omega\| < \rho$. In what follows, let $\mathcal{E}$ be an energy functional, defined as:

$$\begin{aligned} &\mathcal{E} : \overline{B(0,\rho)} \to \mathbb{R}, \\ &\mathcal{E}(u) = T_{\lambda,\varepsilon}(u) + \omega \|u - u_\omega\|_W. \end{aligned} \tag{4.8}$$

So, by (4.8) we can say that

$$\begin{aligned} \mathcal{E}(u_\omega) &= T_{\lambda,\varepsilon}(u_\omega) < T_{\lambda,\varepsilon}(u) + \omega \|u - u_\omega\|_W \\ &= \mathcal{E}(u), u \neq u_\omega. \end{aligned} \tag{4.9}$$

Now, using (4.9), one can observe that u_ω is a minimum point for $\mathcal{E}$, so it follows that (using the same arguments as in [5, 21, 28])

$$\frac{\mathcal{E}(u_\omega + z\gamma) - \mathcal{E}(u_\omega)}{z} \geq 0 \tag{4.10}$$

for $z > 0$ small and every $\gamma \in W$ with $\|\gamma\|_W < 1$.

Therefore, using (4.10) we obtain that

$$\frac{T_{\lambda,\varepsilon}(u_\omega + z\gamma) - T_{\lambda,\varepsilon}(u_\omega)}{z} + \omega \|\gamma\|_W \geq 0.$$

Letting $z \to 0$, it follows that

$$\begin{aligned} &\langle T'_{\lambda,\varepsilon}(u_\omega), \gamma \rangle > -\omega \|\gamma\|_W, \\ &\langle T'_{\lambda,\varepsilon}(u_\omega), \gamma \rangle > -\omega. \end{aligned}$$

Thus, we obtain that $\|T'_{\lambda,\varepsilon}(u_\omega)\|_W \leq \omega$.

By the above relations we obtain the existence of a sequence $(u_n)_{n \geq 1} \subset B(0,\rho)$ such that

$$\begin{aligned} T_{\lambda,\varepsilon}(u_n) &\to \underline{c} \\ T'_{\lambda,\varepsilon}(u_n) &\to 0. \end{aligned} \tag{4.11}$$

By the fact that $(u_n)_{n \geq 1} \subset B(0,\rho)$ we have that

$$\|u_n\|_W \leq \rho, \forall\, n \geq 1,$$

so, $(u_n)_{n\geq 1}$ is bounded in W.

Hence, there exists an element $u_0 \in W$ such that (passing eventually to a subsequence)

$$u_n \rightharpoonup u_0 \quad \text{in } W.$$

Now, by the compact embeddings

$$W \hookrightarrow L^{p(x)}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L^1(\Omega),$$

we get that

$$\begin{aligned} \lim_{n\to\infty} R(u_n) &= R(u_0) \\ \lim_{n\to\infty} \langle R'(u_n), u_n - u_0\rangle &= 0. \end{aligned} \tag{4.12}$$

Using relation (4.11) we obtain that

$$\lim_{n\to\infty} \langle T'_{\lambda,\varepsilon}(u_n), u_n - u_0\rangle = 0. \tag{4.13}$$

Using (4.12) and (4.13) we can obtain that

$$\lim_{n\to\infty} \langle S'(u_n) - S'(u_0), u_n - u_0\rangle \leq \lim_{n\to\infty} \langle T_{\lambda,\varepsilon}(u_n), u_n - u_0\rangle = 0,$$

thus (using same arguments as in [21, Lemma 3.4]) we get that

$$u_n \to u_0 \quad \text{in } W. \tag{4.14}$$

Now, combining relations (4.12), (4.14), and (4.11), we obtain that

$$T_{\lambda,\varepsilon}(u_0) = \underline{c} < 0 \quad \text{and} \quad T'_{\lambda,\varepsilon}(u_0) = 0,$$

so u_0 is a nontrivial critical point of the energy functional $T_{\lambda,\varepsilon}$. Hence, it follows that every $\lambda \in (0, \lambda_0)$ is an eigenvalue for the problem ($P_{\lambda,\varepsilon}$), with u_0 as its associated eigenfunction, thus completing our proof. □

4.3 Proof of Theorem 3.6

We now proceed to prove our second existence result for the case (1.2). Given that the relation (1.2) holds, we have that $2 < p^- \leq p(x) < p^+$, $2 < p^*(x) = \frac{Np(x)}{N-p(x)}$, and since we are in the second case of study, we will consider $W := W_0^{1,p(x)}(\Omega)$ as the appropriate functional framework.

In what follows we will show that the energy functional associated to the problem meets the hypotheses of [36, Theorem 1.2]. So, we have that:

$$\begin{aligned} T_{\lambda,\varepsilon}(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \varepsilon \int_\Omega S_0(x, |\nabla u|)dx - \frac{\lambda}{2}\int_\Omega u^2 dx - \lambda\varepsilon \int_\Omega u dx \\ &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{c\varepsilon}{p^+}\int_\Omega |\nabla u|^{p(x)} dx - \frac{\lambda}{2}\int_\Omega u^2 dx - \lambda\varepsilon \int_\Omega u dx \\ &\quad \text{(using hypotheses } (\phi_3) \text{ and } (\phi_4)) \\ &\geq \frac{C_{2p}}{2}\|u\|_W^2 + \frac{c\varepsilon}{p^+}\min\left\{\|u\|_W^{p^-}, \|u\|_W^{p^+}\right\} - \frac{\lambda C'_{2p}}{2}\|u\|_W^2 - \lambda\varepsilon C_{1p}\|u\|_W, \end{aligned}$$

where $C_{1p} > 0$, $C_{2p} > 0$, $C'_{2p} > 0$ are the same constants we used in the previous proofs.

Since $2 < p^- < p^+$, letting $\|u\|_W \to \infty$ we have that

$$\lim_{\|u\|_W \to \infty} T_{\lambda,\varepsilon}(u) = \infty,$$

so the energy functional $T_{\lambda,\varepsilon}$ is coercive.

Now, by the properties of the potential function ϕ and relation (1.2) we have that $T_{\lambda,\varepsilon}$ is weakly lower semicontinuous (see also [21, Lemmas 4.3, 4.4]), i.e.,

$$T_{\lambda,\varepsilon}(u) \leq \liminf_{n\to\infty} T_{\lambda,\varepsilon}(u_n)$$

for any sequence $(u_n)_{n\geq 1} \subset W$, such that $u_n \rightharpoonup u$ in W.

Therefore, as in the proof of Theorem 3.4, we only need to apply [36, Theorem 1.2], and we obtain that $T_{\lambda,\varepsilon}$ has a global minimum point, which is a solution to the problem ($P_{\lambda,\varepsilon}$). Thus, we have proved that for every $\lambda \in \mathbb{R}^*_+$, there exists a solution to the problem that is associated with the eigenvalue λ. □

Remark 4.6. In order to enhance clarity and improve the readability of the paper, we present Figure 4.2 illustrating a simple function that exhibits the same geometrical properties as our energy functional. For simplicity, we omit the x-dependence of f and define $f : \mathbb{R} \to \mathbb{R}$ as

$$f(z) = \frac{1}{2}|z|^2 + \varepsilon|z|^{p(z)} - \lambda\left(\frac{1}{2}|z|^2 + \varepsilon \cdot z\right).$$

We consider the following parameters:

(a) $\varepsilon = 10, \lambda = 0.4, p(x) = 2.51 + 0.5 \cdot \sin(6\pi x)$;

(b) $\varepsilon = 10, \lambda = 10, p(x) = 2.51 + 0.5 \cdot \sin(6\pi x)$.

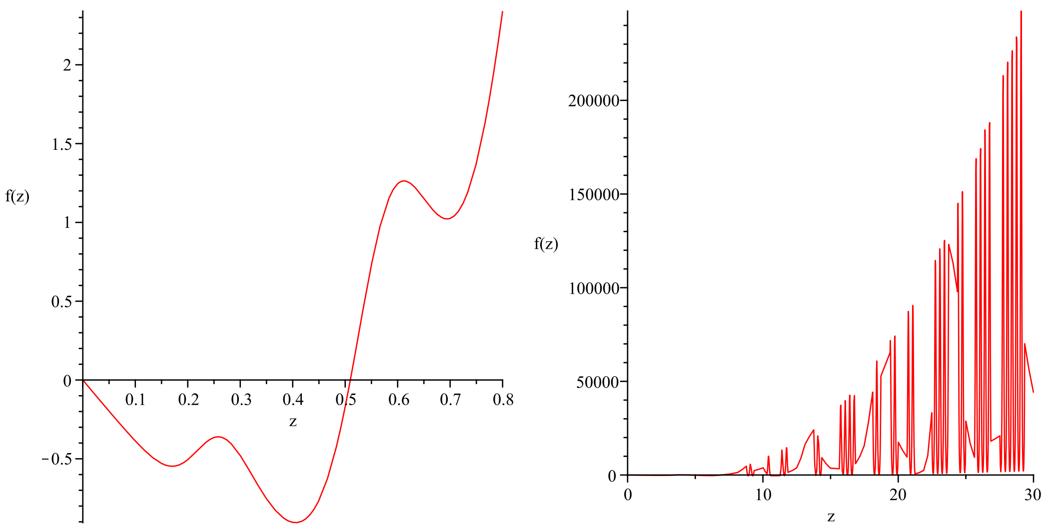

Figure 4.2: For the parameters stated in (a), we highlight the existence of both a local minimum and a global minimum near the origin. For the parameters stated in (b), we highlight the existence of a global minimum point and the coercivity of the function f.

4.4 Proof of Theorem 3.7

We now proceed to prove our first existence result for the case (1.3). Due to the fact that relation (1.3) holds, we have that $1 < p^- < 2 < p^+$ and $2 < p^*(x) = \frac{Np(x)}{N-p(x)}$. We will consider $W := W_0^{1,p(x)}(\Omega)$ as the appropriate functional framework.

In order to identify the critical points of our energy functional, we first aim to prove that $T_{\lambda,\varepsilon}$ is coercive. To this end, we have that

$$
\begin{aligned}
T_{\lambda,\varepsilon}(u) &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{c\varepsilon}{p^+}\min\left\{\|u\|_W^{p^+}, \|u\|_W^{p^-}\right\} - \frac{\lambda}{2}\int_\Omega u^2 dx - \lambda\varepsilon\int_\Omega u dx \\
&\geq \frac{C_\lambda}{2}\int_\Omega |\nabla u|^2 dx + \frac{c\varepsilon}{p^+}\min\left\{\|u\|_W^{p^+}, \|u\|_W^{p^-}\right\} - \lambda\varepsilon C_{1p}\|u\|_W,
\end{aligned}
$$

where $C_{1p} > 0$ is a constant as in previous proofs (we also have used of relation (4.2)). Now, letting $\|u\|_W \to \infty$, since $1 < p^-$ we get that

$$
\lim_{\|u\|_W \to \infty} T_{\lambda,\varepsilon}(u) = +\infty
$$

so the fact that $T_{\lambda,\varepsilon}$ is coercive holds true. In what follows the existence of a nontrivial critical point is obtained in the same fashion as in the proof of Theorem 3.4. □

4.5 Proof of Theorem 3.8

Since, similarly to the proof of Theorem 3.5, the proof of this theorem is longer, we will split it into several steps. Also, as in the proof of Theorem 3.7, we will consider $W := W_0^{1,p(x)}(\Omega)$ as the appropriate functional framework. Furthermore, since relation (1.3) implies that $2 < p^+ < p^*(x)$ for all $x \in \overline{\Omega}$, throughout this proof, we can use the same embeddings for the functional spaces as in the proof of Theorem 3.5.

Step 1. We will prove the existence of a "mountain" near the origin, that is the existence of some constants $r > 0$ and $\alpha > 0$ such that

$$
T_{\lambda,\varepsilon}(u) \geq \alpha > 0,
$$

for any $u \in W$ with $\|u\|_W = r$. Without losing generality we may suppose that $\|u\|_W < 1$, then we have that

$$
\begin{aligned}
T_{\lambda,\varepsilon}(u) &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{c\varepsilon}{p^+}\|u\|_W^{p^+} - \frac{\lambda}{2}\int_\Omega u^2 dx - \lambda\varepsilon\int_\Omega u dx \\
&\geq \frac{C_{2p}}{2}\|u\|_W^2 + \frac{c\varepsilon}{p^+}\|u\|_W^{p^+} - \frac{\lambda C'_{2p}}{2}\|u\|_W^2 - \lambda\varepsilon C_{1p}\|u\|_W \\
&\geq r\left[\left(\frac{C_{2p}}{2} + \frac{c\varepsilon}{p^+}\right) r^{p^+-1} - \lambda\left(\frac{C'_{2p}}{2} + \varepsilon C_{1p}\right)\right],
\end{aligned}
$$

where the constants $C_{2p} > 0$, $C'_{2p} > 0$, $C_{1p} > 0$ are the same as in the previous proofs.

So, by the above inequality if we take $\lambda \in (0, \lambda^*)$, with λ^* defined as it follows:

$$
\lambda^* = \frac{C_{2p}p^+ + 2c\varepsilon}{C'_{2p} + 2\varepsilon C_{1p}} \cdot r^{p^+-1} \tag{4.15}
$$

we can find a constant $\alpha = \alpha\big(\frac{C_{2p}p^{+}+2c\varepsilon}{C'_{2p}+2\varepsilon C_{1p}} \cdot r^{p^{+}-1}\big) > 0$ such that

$$T_{\lambda,\varepsilon}(u) \geq \alpha > 0.$$

Step 2. We now proceed to prove the existence of a "valley" near the origin. To this end, we aim to show that there exists $e \in W$, with $e > 0$, such that

$$T_{\lambda,\varepsilon}(te) < 0,$$

provided that $t > 0$ is sufficiently small.

In this regard, we approximate our energy functional using similar techniques as in Step 2 of the proof of Theorem 3.5. We obtain

$$T_{\lambda,\varepsilon}(te) \leq \frac{t^2}{2}\int_\Omega |\nabla e|^2 dx + t^{p^-} \cdot C_e\varepsilon - \frac{\lambda t^2}{2}\int_\Omega e^2 dx - \lambda\varepsilon t\int_\Omega e dx,$$

where $C_e = 2C_\phi|\alpha|_{p'(x)}\|e\|_W^{p^-} + \frac{\beta}{p^-}\|e\|_W^{p^-}$, and $C_\phi > 0$ is a constant depending on the potential ϕ.

Since $t \in (0,1)$, the above relation implies that

$$T_{\lambda,\varepsilon}(te) \leq t^{p^-}\left(\frac{1}{2}\int_\Omega |\nabla e|^2 dx + C_e\varepsilon\right) - \lambda\varepsilon t\int_\Omega e dx.$$

Now, since $1 < p^-$, it follows that

$$T_{\lambda,\varepsilon}(te) < 0,$$

provided that

$$t < t_0 = \frac{\lambda\varepsilon\int_\Omega e dx}{\frac{1}{2}\int_\Omega |\nabla e|^2 dx + C_e\varepsilon}.$$

Step 3. To prove that the geometrical properties established in Step 1 and Step 2 lead to the existence of a nontrivial critical point for our energy functional, we proceed as in the proof of Theorem 3.5. Therefore, our goal is to find a Palais–Smale sequence whose limit is the corresponding critical point.

Let $\lambda^* > 0$ be as defined in (4.15). Keeping in mind the results established in Step 1, we have that

$$\inf_{\partial B(0,r)} T_{\lambda,\varepsilon} > 0, \tag{4.16}$$

where $B(0,r)$ denotes the ball centered at 0 with radius r in W.

Now, by the results established in Step 2, we have the existence of $e \in W$ such that $T_{\lambda,\varepsilon}(te) < 0$, provided that $t > 0$ is sufficiently small. Moreover, since $2 < p^*(x)$, it follows that $W \hookrightarrow L^2(\Omega) \hookrightarrow L^1(\Omega)$, and

$$T_{\lambda,\varepsilon}(u) \geq \left(\frac{C_{2p}}{2} + \frac{c\varepsilon}{p^+}\right)\|u\|_W^{p^+} - \left(\frac{\lambda C'_{2p}}{2} + \lambda\varepsilon\right)\|u\|_W,$$

for any $u \in B(0,r)$.

Therefore, we can say that there exists a constant $c_* < 0$ such that

$$-\infty < c_* := \inf_{\overline{B(0,r)}} T_{\lambda,\varepsilon} < 0.$$

Now, by the above relations, let ϑ be a constant such that:

$$0 < \vartheta < \inf_{\partial B(0,r)} T_{\lambda,\varepsilon} - \inf_{B(0,r)} T_{\lambda,\varepsilon}.$$

Using Ekeland's variational principle (see [17]) for the energy functional $T_{\lambda,\varepsilon} : \overline{B(0,r)} \to \mathbb{R}$ we may find $u_\vartheta \in \overline{B(0,r)}$ such that

$$\begin{aligned} T_{\lambda,\varepsilon}(u_\vartheta) &< \inf_{\overline{B(0,r)}} T_{\lambda,\varepsilon} + \vartheta \\ T_{\lambda,\varepsilon}(u_\vartheta) &< T_{\lambda,\varepsilon}(u) + \vartheta \|u - u_\vartheta\|_W, u \neq u_\vartheta. \end{aligned}$$

Since

$$T_{\lambda,\varepsilon}(u_\vartheta) \leq \inf_{\overline{B(0,r)}} T_{\lambda,\varepsilon} + \vartheta \leq \inf_{B(0,r)} T_{\lambda,\varepsilon} + \vartheta < \inf_{\partial B(0,r)} T_{\lambda,\varepsilon},$$

we obtain that $u_\vartheta \in B(0,r)$.

Now, let us define $J_{\lambda,\varepsilon} : \overline{B(0,r)} \to \mathbb{R}$ as

$$J_{\lambda,\varepsilon}(u) = T_{\lambda,\varepsilon}(u) + \vartheta \|u - u_\vartheta\|.$$

Hence, by the same arguments as in the previous proof (Theorem 3.5, see also [5, 21, 28]), we conclude that u_ϑ is a minimum point of $J_{\lambda,\varepsilon}$, and we have

$$\frac{J_{\lambda,\varepsilon}(u_\vartheta + tw) - J_{\lambda,\varepsilon}(u_\vartheta)}{t} \geq 0, \tag{4.17}$$

for small $t > 0$ and any $w \in B(0,1)$.

Thus, relation (4.17) implies that

$$\frac{T_{\lambda,\varepsilon}(u_\vartheta - tw) - T_{\lambda,\varepsilon}(u_\vartheta)}{t} + \vartheta \|w\|_W \geq 0.$$

Letting $t \to 0$, it follows that

$$\langle T'_{\lambda,\varepsilon}(u_\vartheta), w \rangle + \vartheta \|w\|_W > 0,$$

which implies that $\|T'_{\lambda,\varepsilon}(u_\vartheta)\|_W \leq \vartheta$.

From the above relations, we obtain the existence of a sequence $(u_n)_{n\geq 1} \subseteq B(0,r)$ such that

$$\begin{aligned} T_{\lambda,\varepsilon}(u_n) &\to c_*, \\ T'_{\lambda,\varepsilon}(u_n) &\to 0. \end{aligned}$$

Now, since relation (1.3) implies that $2 < p^*(x)$, using the same arguments as in Step 3 of the proof of Theorem 3.5, we obtain that there exists some $u_* \in W$ such that $u_n \to u_*$ in W and

$$\begin{aligned} T_{\lambda,\varepsilon}(u_*) &= c_* < 0, \\ T'_{\lambda,\varepsilon}(u_*) &= 0, \end{aligned}$$

therefore, u_* is a critical point of the energy functional $T_{\lambda,\varepsilon}$, and thus every $\lambda \in (0, \lambda^*)$ is an eigenvalue for problem ($P_{\lambda,\varepsilon}$). □

Remark 4.7. As in the previous cases in order to enhance clarity and improve the readability of the paper, we present Figure 4.3 illustrating a simple function that exhibits the same geometrical properties as our energy functional. For simplicity, we omit the x-dependence of f and define $f : \mathbb{R} \to \mathbb{R}$ as

$$f(z) = \frac{1}{2}|z|^2 + \varepsilon |z|^{p(z)} - \lambda \left(\frac{1}{2}|z|^2 + \varepsilon \cdot z \right).$$

We consider the following parameters $\varepsilon = 10, \lambda = 0.4, p(x) = 2 + 0.5 \cdot \sin(6\pi x)$ for:

(a) the behavior near the origin;

(b) the behavior away from the origin.

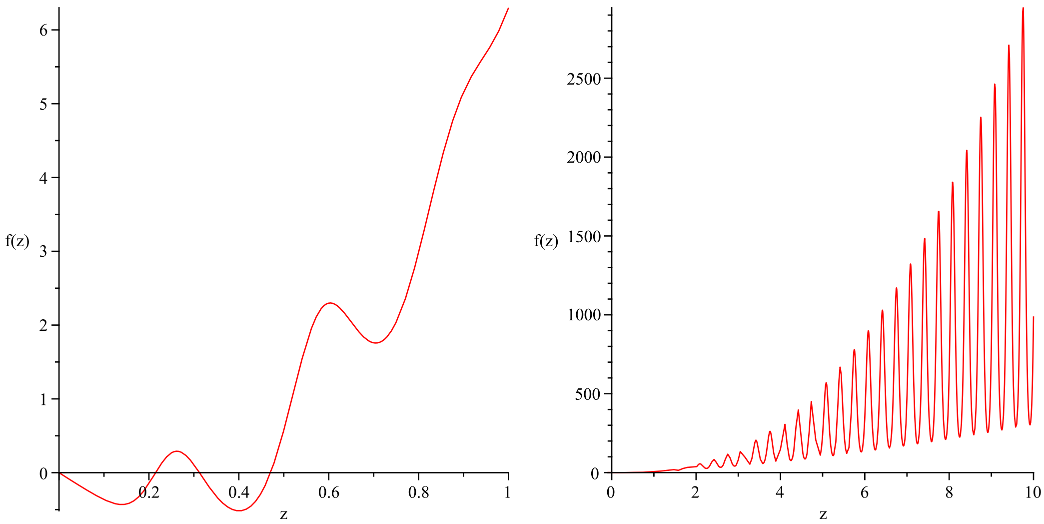

Figure 4.3: For the parameters specified above, the first figure illustrates the existence of a both a local minimum and a global minimum near the origin, while the second figure demonstrates the coercivity of the function f.

5 Final remarks and open questions

Remark 5.1. At this point, we do not know the multiplicity of eigenvalues in any of the cases. For a simpler problem, similar to the problem corresponding to the case (1.1), it has been proved that there exists a unique eigenfunction in the interval $(0, \lambda_1(\Delta))$. For more details, we refer to [14].

Remark 5.2. The only case where we can prove the existence of eigenvalues for large values of λ is in (1.2). Also, in this case, we can obtain a concentration result near the origin. At this point, we do not know whether the solutions given by Theorems 3.5 and 3.6 coincide for $\lambda \in (0, \lambda_0)$.

Remark 5.3. We observe that in (1.3), there is a concentration of the spectrum in the near proximity of the origin. At this point, the order relation between the parameters $\lambda_1(\Delta)$ and λ^* is not known. Furthermore, the multiplicity of eigenvalues in the interval $(0, \lambda_1(\Delta)) \cap (0, \lambda^*)$ is also unknown, as we lack information on whether the solutions given by Theorems 3.7 and 3.8 coincide.

Remark 5.4. Since we only have some information on the qualitative behavior for large values of the parameter λ in (1.2) (we refer here to Theorem 3.6), a more challenging question is to develop an exhaustive analysis for all $\lambda > 0$ in the other two cases.

Remark 5.5. A promising direction for future research emerges when the reaction function is defined as $f(x,z) = |u|^{p^*(x)-1} + \varepsilon$. This formulation enables an investigation into the interplay between the growth behavior of the potential function ϕ and the critical growth behavior of the reaction function f. For related results in this direction, we refer to [10,16].

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