Electronic Journal of Qualitative Theory of Differential Equations
2025, No. 25, 1–22; https://doi.org/10.14232/ejqtde.2025.1.25 www.math.u-szeged.hu/ejqtde/

Existence and nonexistence of solutions for generalized quasilinear Kirchhoff–Schrödinger–Poisson system

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Received 21 November 2024, appeared 27 May 2025
Communicated by Gabriele Bonanno

Abstract. In this paper, we consider the existence and nonexistence of solutions for a class of modified Schrödinger–Poisson system with Kirchhoff-type perturbation by use of variational methods. When nonlinear term $h(u) = |u|^{p-2}u$, $1 \leq p < \infty$, the nonexistence of nontrivial solutions of system is demonstrated through Pohožaev identity. When nonlinear term $h(u)$ satisfies appropriate assumptions, taking advantage of critical point theorem, we obtain a positive radial solution and a nontrivial one of system when $g(u)$ satisfies different conditions. Moreover, some convergence properties are established as the parameter $b \to 0$. What is more, the nonexistence of nontrivial solutions in critical case is also proved by use of Pohožaev identity.

Keywords: variational methods, Pohožaev identity, critical point theorem, nonexistence, radial solution.

2020 Mathematics Subject Classification: 35A01, 35A15, 35J10.

1 Introduction

In this paper, we studied the following generalized quasilinear Schrödinger–Poisson system with a Kirchhoff-type perturbation, which is an innovative research topic.

$$\begin{cases} \left(1 + b\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2 dx\right)[-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2] & \\ +V(x)u + \phi u = h(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $b \geq 0$, $V : \mathbb{R}^3 \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are continuous functions, $g \in C^1(\mathbb{R}, \mathbb{R}^+)$ satisfies the following assumption:

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(g_1) g is even with $g'(t) \le 0$ for all $t \ge 0$ and $g(0) = 1$, $\lim_{t\to+\infty} g(t) = a$, $a \in (0, 1)$.

When $\phi = 0$, the system (1.1) has become the following Kirchhoff–Schrödinger equation:

$$\left(1 + b\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2 dx\right)[-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2] + V(x)u = h(x, u), \quad x \in \mathbb{R}^N. \tag{1.2}$$

Problem (1.2) is related to the stationary analogue of the Kirchhoff–Schrödinger type equation, which was proposed by Kirchhoff as an extension of classical D'Alembert's wave equation for free vibrations of elastic strings [13, 18]. Since the Kirchhoff-type problems arise in various models of physical and biological systems, numerous scholars have conducted research on problem (1.2). References [3] and [20] considered the existence and nonexistence of solutions for problem (1.2). For $h(x, u) = \lambda f(u) + g(u)G^5(u)$ as considered in [3], where $G(u) = \int_0^u g(t)dt$. Under some suitable assumptions on $f(u)$, problem (1.2) admits at least one positive ground state solution for $\lambda > \lambda^* > 0$. If $\lambda = 0$, the corresponding equation had no nontrivial solution. For $h(x, u) = f(u)$ and $f(u)$ satisfied appropriate conditions, then problem (1.2) had at least one radial ground state solution. In [20], when $h(x, u)$ satisfied critical or supercritical growth at infinity, the nonexistence result for (1.2) was proved via Pohožaev identity. If $h(x, u)$ showed asymptotically cubic growth at infinity, the existence of positive radial solutions for (1.2) was obtained by use of variational methods. Moreover, some properties were established as the parameter $b \to 0$. Both of the existence of ground state and sign-changing ground state solutions about (1.2) were testified in [23]. Chen et al. [6] applied some new analytical techniques and non-Nehari manifold method to obtain one ground state sign-changing solution $v_b = G^{-1}(u_b)$. Moreover, they illustrated that the energy of $v_b = G^{-1}(u_b)$ is strictly larger than twice of the Nehari type ground state solution. They also established the convergence properties of $v_b = G^{-1}(u_b)$ as the parameter $b \to 0$.

For $g(u) = 1$, problem (1.2) transforms to the following classical Kirchhoff equation:

$$-\left(1 + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + V(x)u = h(x, u), \quad x \in \mathbb{R}^3,$$

which takes into account the changes in length of the string produced by transverse vibrations, hence the nonlocal term appeared. If $b = 0$, problem (1.2) takes the following form:

$$-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = h(x, u), \quad x \in \mathbb{R}^3. \tag{1.3}$$

This equation is related to the existence of solitary wave solutions for quasilinear Schrödinger equations:

$$i\partial_t z = -\Delta z + W(x)z - k(x, |z|) - \Delta l(|z|^2)l'(|z|^2)z, \tag{1.4}$$

where $z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, $l : \mathbb{R} \to \mathbb{R}$ and $k : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are suitable functions. Equation (1.4) appeared in plasma physics and fluid mechanics [25], dissipative quantum mechanics [12] and condensed matter theory [24]. The existing results for equation (1.3) such as nontrivial solutions [15], ground state solutions [4, 31, 33], positive solutions [9,10,29], nonexistence of solutions [14], high-energy solutions [22], multiple solutions [21] and infinitely many solutions [30] were obtained respectively.

System (1.1) is the so called quasilinear Kirchhoff–Schrödinger–Poisson system. If $b = 0$, (1.1) reduces to the following quasilinear Schrödinger–Poisson system:

$$\begin{cases} -\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u + \phi u = h(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.5}$$

In [27], the authors proved the following problem admits at least a ground state solution

$$\begin{cases} -\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u + \phi G(u)g(u) = h(x, u), & x \in \mathbb{R}^3, \\ -\Delta\phi = G^2(u), & x \in \mathbb{R}^3, \end{cases}$$

for $h(x, u) = b(x)|G(u)|^{p-2}G(u)g(u) - c(x)|G(u)|^{q-2}G(u)g(u)$, $2 < q < 4 < p < 6$. Recently, Zhang and Liu obtained a nontrivial ground state solution for $h(x, u) = \lambda f(u) + g(u)G^5(u)$ in reference [38]. By setting $g^2(u) = 1 + 2u^2$ in (1.5), we get the following quasilinear Schrödinger–Poisson system:

$$\begin{cases} -\Delta u + V(x)u + \phi u - ku\Delta(u^2) = h(u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.6}$$

There are a variety of excellent results for system (1.6). References [11, 34, 35] studied the existence of nontrivial solutions respectively for $k = \frac{1}{2}$ or $k = 1$. The authors in [32] obtained the existence of ground state solution and infinitely many geometrically distinct solutions. The sign-changing solutions can be referred to the references [2, 37].

By setting $g^2(u) = 1 + 2u^2$ in (1.1), we get the following modified quasilinear Kirchhoff–Schrödinger–Poisson system:

$$\begin{cases} -(1 + b\int_{\mathbb{R}^3} |\nabla u|^2 dx)\Delta u + V(x)u - \frac{1}{2}u\Delta(u^2) + \phi u = h(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.7}$$

System (1.7) was introduced in [1] very recently, the authors proved that problem (1.7) has at least three solutions: one is positive, one is negative, and one changes its sign. Furthermore, if h is odd with respect to u, they also obtained unbounded sequence of sign-changing solutions. Under appropriate conditions, the authors in [5] discussed the existence of nontrivial nonpositive and nonnegative solutions, a sequence of high-energy solutions via perturbation method. Combining perturbation method with discontinuous finite element method, a series of weak solutions of system (1.7) were gained in [7] for $h(x, u) = K(x)u^{p-2}u$, where $K(x) \in L^{\frac{2}{2-p}}(\mathbb{R}^3)$, $1 < p < 2$ and $K(x) > 0$ for $x \in \mathbb{R}^3$. When system (1.7) involving a nonlocal term and an integral constraint, infinitely many sign-changing solutions were obtained in [8] according to the method of invariant sets of the descending flow combined with the genus theory. The authors in [36] proved that the system (1.7) has a sign-changing solution u_0, which has precisely two nodal domains. The same conclusion was gained in [17] as in [36] for critical case that $h(u) = \lambda|u|^{q-2}u\ln|u|^2 + |u|^4u$.

Nevertheless, there are relatively few achievements on the generalized quasilinear Kirchhoff–Schrödinger–Poisson system (1.1), so the discussion in this paper makes innovations of the pioneering work. Based on the existing results, we extend the results of references [3] and [20] to the generalized quasilinear Kirchhoff–Schrödinger–Poisson system, then we obtain the existence and nonexistence of solutions for system (1.1).

We make some assumptions on $V(x)$ and $h(u)$:

(V_1) $V \in C(\mathbb{R}^3, \mathbb{R})$, $V(x) = V(|x|)$, $0 < V_0 \le V(x) \le V_\infty := \lim_{|x|\to\infty} V(x) < \infty$;

(h_1) $h \in C(\mathbb{R}, \mathbb{R})$, $h(t) = 0$, $\forall t \le 0$ and $\lim_{t\to 0} \frac{h(t)}{t} = 0$;

(h_2) $|h(t)| \le C(1 + |t|^{q-1})$ for some $C > 0$ and $q \in (4, 6)$;

(h_3) there exists $\tau > 4$ such that

$$0 < \tau g(t)H(t) \leq h(t)G(t), \quad \forall t > 0,$$

where $H(t) = \int_0^t h(s)ds$, $G(t) = \int_0^t g(s)ds$.

Clearly, (h_1) and (h_2) show that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|h(u)| \leq \varepsilon|u| + C_\varepsilon|u|^{q-1}, \quad \forall u \in \mathbb{R}. \tag{1.8}$$

Now we state our main results.

Theorem 1.1. *For $h(u) = |u|^{p-2}u$ in (1.1), assume that (g_1) holds. If $V(x) + 2\langle\nabla V(x), x\rangle \leq 0$, problem (1.1) has no nontrivial solutions when $p \leq \frac{12a}{5}$. If $\frac{2}{5} \leq a < 1$ and $V(x) + 2\langle\nabla V(x), x\rangle \geq 0$, problem (1.1) has no nontrivial solutions when $p \geq 6$.*

Theorem 1.2. *Assume that (V_1), (g_1) and (h_1)–(h_3) are satisfied, then problem (1.1) has a positive radial solution.*

Theorem 1.3. *Assume that (V_1), (g_1) and (h_1)–(h_3) are satisfied, $\{u_{b_n}\} \subset \mathcal{H}$ are the positive radial solutions obtained in Theorem 1.2 for each $n \in \mathbb{N}$. Then, $u_{b_n} \to u_0$ in $\mathcal{H}$ as $b_n \to 0$, $n \to \infty$, where u_0 is a positive radial solution for problem (1.5).*

The following condition of g is necessary to obtain the next two important results.

(g') $g \in C^1(\mathbb{R}, \mathbb{R}^+)$ is even with $g'(t) \geq 0$ for all $t \geq 0$ and $g(0) = 1$, $tg'(t) < g(t)$ for all $t \in \mathbb{R}$.

Theorem 1.4. *Assume that (V_1), (g') and (h_1)–(h_3) are satisfied, then problem (1.1) has a nontrivial radial solution.*

Theorem 1.5. *Assume that (V_1), (g') and (h_1)–(h_3) are satisfied, $\{u'_{b_n}\} \subset \mathcal{H}$ are the nontrivial radial solutions obtained in Theorem 1.4 for each $n \in \mathbb{N}$. Then, $u'_{b_n} \to u'_0$ in $\mathcal{H}$ as $b_n \to 0$, $n \to \infty$, where u'_0 is a nontrivial radial solution for problem (1.5).*

Finally, we consider the following generalized quasilinear Schrödinger–Poisson system involving a Kirchhoff-type perturbation and critical Sobolev exponent

$$\begin{cases} \left(1 + b\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2\right)[-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2] \\ +V(x)u + \phi u = g(u)G^{p-1}(u), & \text{in} \quad \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in} \quad \mathbb{R}^3. \end{cases} \tag{1.9}$$

Theorem 1.6. *Suppose that (g') holds. If*

$$2V(x) + \langle\nabla V(x), x\rangle \geq 0, \quad \forall x \in \mathbb{R}^3,$$

then problem (1.9) has no nontrivial solutions when $p \geq 6$.

Remark 1.7. Throughout the paper we denote by C, C_i $(i = 1, 2, \dots) > 0$ various positive constants which may vary from line to line and are not essential to the problem.

2 Preliminary results

In this section, we introduce the variational framework associated with problem (1.1). Let $L^p(\mathbb{R}^3)$ be the usual Lebesgue space with the norm $|u|_p = \left(\int_{\mathbb{R}^3} |u|^p dx\right)^{\frac{1}{p}}$.

Define the space H given by

$$H^1 := H^1(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\},$$

with the norm:

$$\|u\| = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx \right)^{\frac{1}{2}},$$

and the corresponding inner is

$$\langle u, v \rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv) dx.$$

Let

$$D^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}.$$

According to the assumption (V_1), we use the space

$$\mathcal{H} := \left\{ u \in H^1(\mathbb{R}^3) : u(x) = u(|x|), \int_{\mathbb{R}^N} V(x) u^2 dx < \infty \right\},$$

with the norm:

$$\|u\|_{\mathcal{H}} = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x) u^2) dx \right)^{\frac{1}{2}},$$

then the embedding $\mathcal{H} \hookrightarrow L^p(\mathbb{R}^3)$ is compact for $2 < p < 6$.

According to [3], the energy functional associated with (1.1) is

$$\begin{aligned} I_b(u) = \frac{1}{2} \int_{\mathbb{R}^3} g^2(u) |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} V(x) |u|^2 dx + \frac{b}{4} \left(\int_{\mathbb{R}^3} g^2(u) |\nabla u|^2 dx \right)^2 \\ + \frac{1}{4} \int_{\mathbb{R}^3} \phi |u|^2 dx - \int_{\mathbb{R}^3} H(u) dx, \end{aligned} \tag{2.1}$$

where $H(t) = \int_0^t h(s) ds$. Since the term $\int_{\mathbb{R}^3} g^2(u) |\nabla u|^2 dx$ is not well defined in $\mathcal{H}$, to overcome this difficulty, a change of variable constructed in [28] is very helpful to us. For any $v \in \mathcal{H}$, let

$$u = G^{-1}(v) \quad \text{and} \quad G(u) = \int_0^u g(t) dt,$$

then

$$\int_{\mathbb{R}^3} g^2(u) |\nabla u|^2 dx = \int_{\mathbb{R}^3} g^2(G^{-1}(v)) |\nabla G^{-1}(v)|^2 dx =: |\nabla v|_2^2 < \infty,$$

and $I_b(u)$ can be reduced to

$$\begin{aligned} J_b(v) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} V(x) |G^{-1}(v)|^2 dx + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla v|^2 dx \right)^2 \\ + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{G^{-1}(v)} |G^{-1}(v)|^2 dx - \int_{\mathbb{R}^3} H(G^{-1}(v)) dx. \end{aligned} \tag{2.2}$$

Then $v \in \mathcal{H}$ is a solution of (1.1) if

$$\langle J_b'(v), \eta \rangle = \int_{\mathbb{R}^3} \nabla v \cdot \nabla \eta dx + \int_{\mathbb{R}^3} V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \eta dx + b \int_{\mathbb{R}^3} |\nabla v|^2 dx \int_{\mathbb{R}^3} \nabla v \cdot \nabla \eta dx$$
$$+ \int_{\mathbb{R}^3} \phi_{G^{-1}(v)} \frac{G^{-1}(v)}{g(G^{-1}(v))} \eta dx - \int_{\mathbb{R}^3} \frac{h(G^{-1}(v))}{g(G^{-1}(v))} \eta dx = 0,$$

for all $\eta \in H^1(\mathbb{R}^3)$.

We observe that by the Lax–Milgram theorem, for given $u \in H^1(\mathbb{R}^3)$, there exists a unique solution $\phi = \phi_u \in D^{1,2}(\mathbb{R}^3)$ satisfying $-\Delta \phi_u = u^2$ in a weak sense. The function ϕ_u is represented by

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy,$$

and it has the following properties.

Lemma 2.1 ([11,26])**.** *The following properties hold:*

(i) *there exists $C > 0$ such that for any $u \in H^1(\mathbb{R}^3)$,*

$$\|\phi_u\|_{D^{1,2}} \le C|u|^2_{\frac{12}{5}}, \quad \int_{\mathbb{R}^3} |\nabla \phi_u|^2 dx = \int_{\mathbb{R}^3} \phi_u u^2 dx \le C\|u\|^4_{\mathcal{H}};$$

(ii) *$\phi_u \ge 0$ for all $u \in H^1(\mathbb{R}^3)$;*

(iii) *$\phi_{tu} = t^2 \phi_u$ for all $t > 0$ and $u \in H^1(\mathbb{R}^3)$;*

(iv) *if $u_j \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3)$, then, up to a subsequence, $\phi_{u_j} \to \phi_u$ in $D^{1,2}(\mathbb{R}^3)$ and*

$$\int_{\mathbb{R}^3} \phi_{u_j} u_j^2 dx \to \int_{\mathbb{R}^3} \phi_u u^2 dx.$$

Lemma 2.2 ([19,22])**.** *For the function g, G, and G^{-1}, the following properties hold under the condition (g_1):*

(1) $\frac{t}{g(t)} g'(t) \le 0$ *for all* $t \ge 0$;

(2) $|t| \le |G^{-1}(t)| \le \frac{|t|}{a}$ *for all* $t \in \mathbb{R}$;

(3) $t^2 \le \frac{t}{g(t)} G(t) \le \frac{t^2}{a}$ *for all* $t \in \mathbb{R}$.

Under condition (g'), the following properties hold:

(4) $|G^{-1}(t)| \le \frac{1}{g(0)} |t| = |t|$ *for all* $t \in \mathbb{R}$;

(5) $\frac{G^{-1}(t)t}{g(G^{-1}(t))} \le |G^{-1}(t)|^2$ *for all* $t \in \mathbb{R}$;

(6) $\lim_{|t| \to 0} \frac{G^{-1}(t)}{t} = \frac{1}{g(0)} = 1$ *and*

$$\lim_{|t| \to \infty} \frac{G^{-1}(t)}{t} = \begin{cases} \frac{1}{g(\infty)}, & \text{if } g \text{ is bounded}, \\ 0, & \text{if } g \text{ is unbounded}. \end{cases}$$

3 Proof of the main results

3.1 Proof of Theorem 1.1

In this section, we will prove the nonexistence of nontrivial solutions for the following system:

$$\begin{cases} \left(1+b\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2dx\right)[-\mathrm{div}(g^2(u)\nabla u)+g(u)g'(u)|\nabla u|^2] \\ +V(x)u+\phi u=|u|^{p-2}u, & x\in\mathbb{R}^3, \\ -\Delta\phi=u^2, & x\in\mathbb{R}^3. \end{cases} \tag{3.1}$$

By a standard argument in [20,35], we can obtain the following Pohožaev identity.

Lemma 3.1 (Pohožaev identity). *If $v\in\mathcal{H}$ is a weak solution of (3.1), then v satisfies*

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2dx+\frac{3}{2}\int_{\mathbb{R}^3}V(x)|G^{-1}(v)|^2dx+\frac{1}{2}\int_{\mathbb{R}^3}\langle\nabla V(x),x\rangle|G^{-1}(v)|^2dx \\ &\quad+\frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla v|^2dx\right)^2+\frac{5}{4}\int_{\mathbb{R}^3}\phi_{G^{-1}(v)}|G^{-1}(v)|^2dx=\frac{3}{p}\int_{\mathbb{R}^3}|G^{-1}(v)|^pdx. \end{aligned} \tag{3.2}$$

Proof of Theorem 1.1. Indeed, because $(v,\phi_{G^{-1}(v)})\in\mathcal{H}\times D^{1,2}(\mathbb{R}^3)$ is a solution of (3.1), we have the following equation:

$$\begin{aligned} &\int_{\mathbb{R}^3}|\nabla v|^2dx+\int_{\mathbb{R}^3}V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}vdx+b\left(\int_{\mathbb{R}^3}|\nabla v|^2dx\right)^2 \\ &\quad+\int_{\mathbb{R}^3}\phi_{G^{-1}(v)}\frac{G^{-1}(v)}{g(G^{-1}(v))}vdx=\int_{\mathbb{R}^3}\frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))}vdx. \end{aligned} \tag{3.3}$$

Multiply the equation of the (3.3) by $\frac{5}{4}$ and combined with (3.2), we have

$$\begin{aligned} &\frac{3}{4}\int_{\mathbb{R}^3}|\nabla v|^2dx+\int_{\mathbb{R}^3}V(x)\left(\frac{5}{4}\frac{G^{-1}(v)v}{g(G^{-1}(v))}-\frac{3}{2}|G^{-1}(v)|^2\right)dx+\frac{3}{4}b\left(\int_{\mathbb{R}^3}|\nabla v|^2dx\right)^2 \\ &\quad-\frac{1}{2}\int_{\mathbb{R}^3}\langle\nabla V(x),x\rangle|G^{-1}(v)|^2dx+\frac{5}{4}\int_{\mathbb{R}^3}\phi_{G^{-1}(v)}\left(\frac{G^{-1}(v)v}{g(G^{-1}(v))}-|G^{-1}(v)|^2\right)dx \\ &=\int_{\mathbb{R}^3}|G^{-1}(v)|^{p-2}\left(\frac{5}{4}\frac{G^{-1}(v)v}{g(G^{-1}(v))}-\frac{3}{p}|G^{-1}(v)|^2\right)dx. \end{aligned} \tag{3.4}$$

If $V(x)+2\langle\nabla V(x),x\rangle\leq 0$, then by Lemma 2.2-(3) that the left hand of (3.4) is nonnegative, so that the equation (3.4) has no nontrivial solutions when $p\leq\frac{12a}{5}$.

Combining (3.2) and (3.3), it can be concluded that

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^3}V(x)\left(\frac{G^{-1}(v)v}{g(G^{-1}(v))}-3|G^{-1}(v)|^2\right)dx-\frac{1}{2}\int_{\mathbb{R}^3}\langle\nabla V(x),x\rangle|G^{-1}(v)|^2dx \\ &\quad+\frac{1}{2}\int_{\mathbb{R}^3}\phi_{G^{-1}(v)}\left(\frac{G^{-1}(v)v}{g(G^{-1}(v))}-\frac{5}{2}|G^{-1}(v)|^2\right)dx \\ &=\int_{\mathbb{R}^3}|G^{-1}(v)|^{p-2}\left(\frac{G^{-1}(v)v}{2g(G^{-1}(v))}-\frac{3}{p}|G^{-1}(v)|^2\right)dx. \end{aligned} \tag{3.5}$$

Under conditions $\frac{2}{5}\leq a<1$, Lemma 2.2-(3) and $V(x)+2\langle\nabla V(x),x\rangle\geq 0$, for $p\geq 6$, the equation (3.5) has no nontrivial solutions. □

3.2 Proof of Theorem 1.2

This section provides the proof of Theorem 1.2. Clearly, as mentioned previously, we will devote to studying the functional J_b. It is hard to prove the boundedness of the (PS) sequence of J_b, so that finding a special bounded (PS) sequence of $J_{b,\mu}$ may provides great help,

$$\begin{aligned} J_{b,\mu}(v) := &\frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2dx + \frac{1}{2}\int_{\mathbb{R}^3}V(x)|G^{-1}(v)|^2dx + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2dx\right)^2 \\ &+ \frac{1}{4}\int_{\mathbb{R}^3}\phi_{G^{-1}(v)}|G^{-1}(v)|^2dx - \mu\int_{\mathbb{R}^3}H(G^{-1}(v))dx, \end{aligned} \tag{3.6}$$

where $\mu \in [1, 2]$.

Next, we list a useful critical point theorem proposed by Jeanjean [3, 16], which is crucial to obtain our main result.

Theorem 3.2. *Let $(X, \|\cdot\|)$ be a Banach space and $\mathbb{I} \subset \mathbb{R}^+$ an interval. Consider the following family of C^1-functionals on X:*

$$J_{b,\mu}(v) = A(v) - \mu B(v), \quad \forall \mu \in \mathbb{I},$$

where $B(v) \geq 0$, $\forall v \in X$, and either $A(v) \to +\infty$ or $B(v) \to +\infty$ as $\|v\| \to \infty$. Assume that there are two points v_1, v_2 in X, such that

$$c_\mu := \inf_{\gamma\in\Gamma_\mu}\max_{t\in[0,1]} J_{b,\mu}(\gamma(t)) > \max\{J_{b,\mu}(v_1), J_{b,\mu}(v_2)\}, \quad \forall \mu \in \mathbb{I},$$

where

$$\Gamma_\mu = \{\gamma \in C([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2\}.$$

Then, for almost every $\mu \in \mathbb{I}$, there is a sequence $\{v_n\} \subset X$, such that

(i) $\{v_n\}$ is bounded;

(ii) $J_{b,\mu}(v_n) \to c_\mu$;

(iii) $J'_{b,\mu}(v_n) \to 0$ in the dual X^{-1} of X.

Moreover, the map $\mu \mapsto c_\mu$ is continuous from the left.

Lemma 3.3. *Assume that (V_1), (g_1), (h_1)–(h_3) are satisfied, then*

(i) for $\mu \in [1, 2]$, there exists $v \in \mathcal{H}\setminus\{0\}$ such that $J_{b,\mu}(v) < 0$;

(ii) there exists ρ, $\alpha > 0$ such that $J_{b,\mu}(v) \geq \alpha$ for $\|v\|_{\mathcal{H}} = \rho$.

Proof. (i) According to (h_1) and (h_3), there exists constant $C > 0$ such that

$$H(G^{-1}(v)) \geq C|v|^\tau \quad \text{for all } v \geq 0 \quad \text{and} \quad H(G^{-1}(v)) \equiv 0 \quad \text{for all } v \leq 0,$$

where τ is defined in (h_3).

For any fixed $\psi \in \mathcal{H}$ with $\psi > 0$, by Lemma 2.2-(2), we have for $\tau > 4$

$$\begin{aligned} J_{b,\mu}(t\psi) = &\frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla t\psi|^2 + V(x)|G^{-1}(t\psi)|^2\right)dx + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla t\psi|^2dx\right)^2 \\ &+ \frac{1}{4}\int_{\mathbb{R}^3}\phi_{G^{-1}(t\psi)}|G^{-1}(t\psi)|^2dx - \mu\int_{\mathbb{R}^3}H(G^{-1}(t\psi))dx \\ \leq\ & Ct^2\|\psi\|^2_{\mathcal{H}} + \frac{bt^4}{4}\left(\int_{\mathbb{R}^3}|\nabla\psi|^2dx\right)^2 + \frac{t^4}{4}C\|\psi\|^4_{\mathcal{H}} - \mu Ct^\tau\int_{\mathbb{R}^3}|\psi|^\tau dx \\ \to\ & -\infty, \end{aligned}$$

as $t \to \infty$, which implies that (i) holds if we take $v = t\psi$ with t sufficiently large.

(ii) Let $\varepsilon \in \left(0, \frac{a^2 V_0}{2\mu}\right)$, by Lemma 2.2-(2) and (1.8), we obtain

$$\begin{aligned} J_{b,\mu}(v) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|G^{-1}(v)|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla v|^2 dx\right)^2 \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{G^{-1}(v)}|G^{-1}(v)|^2 dx - \mu\int_{\mathbb{R}^3} H(G^{-1}(v))dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3}\left(V_0 - \frac{\mu\varepsilon}{a^2}\right)|v|^2 dx - \frac{C_\varepsilon \mu}{q a^q}\int_{\mathbb{R}^3} |v|^q dx. \\ &\geq C_1\|v\|_{\mathcal{H}}^2 - C_2\|v\|_{\mathcal{H}}^q. \end{aligned}$$

Hence, for $q > 4$ we can choose $\|v\|_{\mathcal{H}} = \rho > 0$ small enough such that $J_{b,\mu}(v) \geq \alpha > 0$. □

Define

$$A(v) := \frac{1}{2}\int_{\mathbb{R}^3}\left[|\nabla v|^2 + V(x)|G^{-1}(v)|^2\right]dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla v|^2 dx\right)^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{G^{-1}(v)}|G^{-1}(v)|^2 dx,$$

$$B(v) := \int_{\mathbb{R}^3} H(G^{-1}(v))dx.$$

It is deduced from (V_1) and Lemma 2.2-(2) that

$$A(v) > \frac{1}{2}\|v\|_{\mathcal{H}}^2 \to +\infty, \quad \text{as } \|v\|_{\mathcal{H}} \to +\infty.$$

Moreover, from (h_1), it can be observed that $B(v) = \int_{\mathbb{R}^3} H(G^{-1}(v))dx \geq 0$, $\forall v \in \mathcal{H}$.

Based on the above facts and Lemma 3.3, the conclusion of Theorem 3.2 holds. It shows that for a.e. $\mu \in [1, 2]$, there is a bounded $(PS)_{c_\mu}$ sequence $\{v_n\} \subset \mathcal{H}$, satisfying $J_{b,\mu}(v_n) \to c_\mu$ and $J'_{b,\mu}(v_n) \to 0$, where c_μ is the mountain pass level.

Lemma 3.4. *Suppose (V_1), (g_1), (h_1)–(h_3) hold, $\{v_n\}$ is the sequence obtained above, going if necessary to a subsequence, $v_n \to v$ in $\mathcal{H}$.*

Proof. Since $\{v_n\} \subset \mathcal{H}$ is bounded, up to a subsequence, there exists $v \in \mathcal{H}$ such that

$$\begin{aligned} v_n &\rightharpoonup v \quad \text{in } \mathcal{H}, \\ v_n &\to v \quad \text{in } L^r(\mathbb{R}^3),\ 2 < r < 6, \\ v_n(x) &\to v(x) \quad \text{a.e. on } x \in \mathbb{R}^3. \end{aligned}$$

Define $\varphi : \mathbb{R} \to \mathbb{R}$ by $\varphi(t) = \frac{G^{-1}(t)}{g(G^{-1}(t))}$. According to (g_1), then $a < g(t) \leq 1$ for $t \in \mathbb{R}$, jointly with Lemma 2.2-(1), we have

$$\varphi'(t) = \frac{1}{g^2(G^{-1}(t))}\left[1 - \frac{G^{-1}(t)g'(G^{-1}(t))}{g(G^{-1}(t))}\right] \geq \frac{1}{g^2(G^{-1}(t))} \geq 1.$$

According to the mean value theorem, for any $x \in \mathbb{R}^3$, there exists a function $\xi(x)$ between $v(x)$ and $v_n(x)$ such that

$$\begin{aligned} &\int_{\mathbb{R}^3} V(x)\left[\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{G^{-1}(v)}{g(G^{-1}(v))}\right](v_n - v)dx \\ &= \int_{\mathbb{R}^3} V(x)\varphi'(\xi)|v_n - v|^2 dx \\ &\geq \int_{\mathbb{R}^3} V(x)|v_n - v|^2 dx, \end{aligned}$$

and with the help of Lemma 2.1, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\left[\phi_{G^{-1}(v_n)}\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}-\phi_{G^{-1}(v)}\frac{G^{-1}(v)}{g(G^{-1}(v))}\right](v_n-v)dx\\
&=\int_{\mathbb{R}^3}\phi_{G^{-1}(v_n)}\left[\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}-\frac{G^{-1}(v)}{g(G^{-1}(v))}\right](v_n-v)dx\\
&\quad+\int_{\mathbb{R}^3}\left(\phi_{G^{-1}(v_n)}-\phi_{G^{-1}(v)}\right)\frac{G^{-1}(v)}{g(G^{-1}(v))}(v_n-v)dx\\
&\geq\int_{\mathbb{R}^3}\phi_{G^{-1}(v_n)}\varphi'(\xi)|v_n-v|^2dx\\
&\geq\int_{\mathbb{R}^3}\phi_{G^{-1}(v_n)}|v_n-v|^2dx\\
&\to 0.
\end{aligned}
$$

Choose $A_n=\int_{\mathbb{R}^3}|\nabla v_n|^2dx$ and $A=\int_{\mathbb{R}^3}|\nabla v|^2dx$. By $v_n\rightharpoonup v$ in $\mathcal{H}$, thus $\{A_n-A\}$ is bounded. By the weak convergence of v_n in $\mathcal{H}$, we get $b(A_n-A)\int_{\mathbb{R}^3}\nabla v\nabla(v_n-v)dx=o_n(1)$. So that

$$
\begin{aligned}
&b\int_{\mathbb{R}^3}|\nabla v_n|^2dx\int_{\mathbb{R}^3}\nabla v_n\nabla(v_n-v)dx-b\int_{\mathbb{R}^3}|\nabla v|^2dx\int_{\mathbb{R}^3}\nabla v\nabla(v_n-v)dx\\
&\geq b(A_n-A)\int_{\mathbb{R}^3}\nabla v\nabla(v_n-v)dx\\
&\to\ 0,
\end{aligned}
$$

as $n\to\infty$. Noting that $a<g(t)\leq 1$, then by Lemma 2.2-(2), (1.8) and Hölder inequality, $v_n\to v$ in $L^r(\mathbb{R}^3)$, $2<r<6$, we obtain

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3}\left[\frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))}-\frac{h(G^{-1}(v))}{g(G^{-1}(v))}\right](v_n-v)dx\right|\\
&\leq C\int_{\mathbb{R}^3}\left[\varepsilon(|v_n|+|v|)+C_\varepsilon(|v_n|^{q-1}+|v|^{q-1})\right]|v_n-v|dx\\
&\leq C\varepsilon(|v_n|_2+|v|_2)|v_n-v|_2+CC_\varepsilon(|v_n|_q^{q-1}+|v|_q^{q-1})|v_n-v|_q\\
&\to 0,
\end{aligned}
$$

as $n\to\infty$. Then

$$
\begin{aligned}
o_n(1)&=\langle J'_{b,\mu}(v_n)-J'_{b,\mu}(v),v_n-v\rangle\\
&=\int_{\mathbb{R}^3}|\nabla(v_n-v)|^2dx+\int_{\mathbb{R}^3}V(x)\left[\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}-\frac{G^{-1}(v)}{g(G^{-1}(v))}\right](v_n-v)dx\\
&\quad+b\left[\int_{\mathbb{R}^3}|\nabla v_n|^2dx\int_{\mathbb{R}^3}\nabla v_n\cdot\nabla(v_n-v)dx-\int_{\mathbb{R}^3}|\nabla v|^2dx\int_{\mathbb{R}^3}\nabla v\cdot\nabla(v_n-v)dx\right]\\
&\quad+\int_{\mathbb{R}^3}\left[\phi_{G^{-1}(v_n)}\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}-\phi_{G^{-1}(v)}\frac{G^{-1}(v)}{g(G^{-1}(v))}\right](v_n-v)dx\\
&\quad-\mu\int_{\mathbb{R}^3}\left[\frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))}-\frac{h(G^{-1}(v))}{g(G^{-1}(v))}\right](v_n-v)dx\\
&\geq\int_{\mathbb{R}^3}|\nabla(v_n-v)|^2dx+\int_{\mathbb{R}^3}V(x)|v_n-v|^2dx\\
&=\|v_n-v\|^2_{\mathcal{H}}.
\end{aligned}
$$

Therefore, $v_n\to v$ in $\mathcal{H}$. □

According to Theorem 3.2, there is a bounded sequence $\{v_n\} \subset \mathcal{H}$ satisfying $J_{b,\mu}(v_n) \to c_\mu$ and $J'_{b,\mu}(v_n) \to 0$. From Lemma 3.4, for every $\mu \in [1,2]$, we get that $J_{b,\mu}$ has a nontrivial critical point v satisfying $J_{b,\mu}(v) = c_\mu$ and $J'_{b,\mu}(v) = 0$. Therefore, there exists $\{\mu_n\} \subset [1,2]$ such that $\lim_{n\to\infty} \mu_n = 1$, $\{v_{\mu_n}\} \subset \mathcal{H}$ such that $J_{b,\mu_n}(v_{\mu_n}) = c_{\mu_n} > 0$ and $J'_{b,\mu_n}(v_{\mu_n}) = 0$.

Lemma 3.5. *Suppose that (g_1), (h_1) and (h_3) are satisfied, then sequence $\{v_{\mu_n}\}$ is bounded in $\mathcal{H}$.*

Proof. Since sequence $\{v_{\mu_n}\} \subset \mathcal{H}$ satisfies

$$J_{b,\mu_n}(v_{\mu_n}) = c_{\mu_n} \quad \text{and} \quad J'_{b,\mu_n}(v_{\mu_n}) = 0.$$

Then we have

$$\begin{aligned} J_{b,\mu_n}(v_{\mu_n}) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v_{\mu_n}|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|G^{-1}(v_{\mu_n})|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla v_{\mu_n}|^2 dx\right)^2 \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{G^{-1}(v_{\mu_n})}|G^{-1}(v_{\mu_n})|^2 dx - \mu_n \int_{\mathbb{R}^3} H(G^{-1}(v_{\mu_n}))dx \\ &= c_{\mu_n} > 0. \end{aligned} \tag{3.7}$$

We also deduce that

$$\begin{aligned} \langle J'_{b,\mu_n}(v_{\mu_n}), v_{\mu_n}\rangle &= \int_{\mathbb{R}^3} |\nabla v_{\mu_n}|^2 dx + \int_{\mathbb{R}^3} V(x)\frac{G^{-1}(v_{\mu_n})}{g(G^{-1}(v_{\mu_n}))} v_{\mu_n} dx \\ &\quad + b\left(\int_{\mathbb{R}^3} |\nabla v_{\mu_n}|^2 dx\right)^2 + \int_{\mathbb{R}^3} \phi_{G^{-1}(v_{\mu_n})}\frac{G^{-1}(v_{\mu_n})}{g(G^{-1}(v_{\mu_n}))} v_{\mu_n} dx \\ &\quad - \mu_n \int_{\mathbb{R}^3} \frac{h(G^{-1}(v_{\mu_n}))}{g(G^{-1}(v_{\mu_n}))} v_{\mu_n} dx \\ &= o(1)\|v_{\mu_n}\|, \end{aligned} \tag{3.8}$$

as $n \to \infty$. It follows from (3.7), (3.8), (h_3) and Lemma 2.2-(2), (3) that

$$\begin{aligned} &c_{\mu_n} + 1 + \|v_{\mu_n}\| \\ &\quad \geq J_{b,\mu_n}(v_{\mu_n}) - \frac{a}{4}\langle J'_{b,\mu_n}(v_{\mu_n}), v_{\mu_n}\rangle \\ &\quad \geq \left(\frac{1}{2} - \frac{a}{4}\right)\int_{\mathbb{R}^3} |\nabla v_{\mu_n}|^2 dx + \int_{\mathbb{R}^3} V(x)G^{-1}(v_{\mu_n})\left[\frac{1}{2}G^{-1}(v_{\mu_n}) - \frac{a v_{\mu_n}}{4g(G^{-1}(v_{\mu_n}))}\right]dx \\ &\qquad + \left(\frac{1}{4} - \frac{a}{4}\right) b \left(\int_{\mathbb{R}^3} |\nabla v_{\mu_n}|^2 dx\right)^2 + \int_{\mathbb{R}^3} \phi_{G^{-1}(v_{\mu_n})} G^{-1}(v_{\mu_n})\left[\frac{1}{4}G^{-1}(v_{\mu_n}) - \frac{a v_{\mu_n}}{4g(G^{-1}(v_{\mu_n}))}\right]dx \\ &\qquad - \mu_n \int_{\mathbb{R}^3}\left[H(G^{-1}(v_{\mu_n})) - \frac{a}{4}\frac{h(G^{-1}(v_{\mu_n}))}{g(G^{-1}(v_{\mu_n}))} v_{\mu_n}\right]dx \\ &\quad \geq \left(\frac{1}{2} - \frac{a}{4}\right)\int_{\mathbb{R}^3} |\nabla v_{\mu_n}|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} V(x)|G^{-1}(v_{\mu_n})|^2 dx \\ &\quad \geq \frac{1}{4}\|v_{\mu_n}\|^2_{\mathcal{H}}, \end{aligned}$$

which means that $\{v_{\mu_n}\}$ is bounded in $\mathcal{H}$. □

Proof of Theorem 1.2. Taking a subsequence of $\{v_{\mu_n}\}$ still represented by $\{v_{\mu_n}\}$, because $\{v_{\mu_n}\}$ is bounded in $\mathcal{H}$, similar to the proof of Lemma 3.4, we obtain $v_{\mu_n} \to v_\mu$ in $\mathcal{H}$. From Theorem 3.2, we know that $\mu \mapsto c_\mu$ is continuous from the left. So

$$\begin{aligned}\lim_{n\to\infty} J_b(v_{\mu_n}) &= \lim_{n\to\infty}\left[J_{b,\mu_n}(v_{\mu_n}) + (\mu_n - 1)\int_{\mathbb{R}^3} H(G^{-1}(v_{\mu_n}))dx\right]\\ &= \lim_{n\to\infty} c_{\mu_n} = \tilde{c}.\end{aligned}$$

In addition,

$$\lim_{n\to\infty}\langle J_b'(v_{\mu_n}), \eta\rangle = \lim_{n\to\infty}\left[\langle J_{b,\mu_n}'(v_{\mu_n}), \eta\rangle + (\mu_n - 1)\int_{\mathbb{R}^3}\frac{h(G^{-1}(v_{\mu_n}))}{g(G^{-1}(v_{\mu_n}))}\eta dx\right] = 0,$$

for any $\eta \in C_0^\infty(\mathbb{R}^3)$, which means that $J_b'(v) = 0$ satisfies $J_b(v) = \tilde{c} > 0$.

Let $v^- = \min\{v, 0\}$. Using Lemma 2.2-(2), (3) and with help of the assumption (h_1) and Lemma 2.1, we have

$$\begin{aligned}0 &= \langle J_b'(v), v^-\rangle\\ &= \int_{\mathbb{R}^3}\left(|\nabla v^-|^2 + V(x)\frac{G^{-1}(v^-)}{g(G^{-1}(v^-))}v^-\right)dx\\ &\geq \int_{\mathbb{R}^3}|\nabla v^-|^2 + V(x)|v^-|^2 dx\\ &\geq \|v^-\|_{\mathcal{H}}^2.\end{aligned}$$

It shows that $v^- \equiv 0$. Applying the strong maximum principle, we obtain $v > 0$. □

4 Asymptotic properties

Now, we are in a situation to give the proof of convergence properties.

Proof of Theorem 1.3. If v_{b_n} is a critical point of J_{b_n}, which is obtained in Theorem 1.2 for each $n \in \mathbb{N}$. Similar to the proof of Lemma 3.3, for $b_n \to 0$, $n \to \infty$, $\{v_{b_n}\}$ is a (PS) sequence, which is bounded in $\mathcal{H}$. There exists a subsequence of $\{b_n\}$, still denoted by $\{b_n\}$, such that $v_{b_n} \rightharpoonup v_0$ in $\mathcal{H}$. It is easy to obtain

$$\begin{aligned}\|v_{b_n} - v_0\|_{\mathcal{H}}^2 \leq{}& \langle J_{b_n}'(v_{b_n}) - J_0'(v_0), v_{b_n} - v_0\rangle - b_n\int_{\mathbb{R}^3}|\nabla v_{b_n}|^2dx\int_{\mathbb{R}^3}\nabla v_{b_n}\nabla(v_{b_n} - v_0)dx\\ &+ \int_{\mathbb{R}^3}\left[\phi_{G^{-1}(v_0)}\frac{G^{-1}(v_0)}{g(G^{-1}(v_0))} - \phi_{G^{-1}(v_{b_n})}\frac{G^{-1}(v_{b_n})}{g(G^{-1}(v_{b_n}))}\right](v_{b_n} - v_0)dx\\ &+ \int_{\mathbb{R}^3}\left[\frac{h(G^{-1}(v_{b_n}))}{g(G^{-1}(v_{b_n}))} - \frac{h(G^{-1}(v_0))}{g(G^{-1}(v_0))}\right](v_{b_n} - v_0)dx\\ ={}& o_n(1).\end{aligned}$$

On the one hand, for all $\eta \in \mathcal{H}\backslash\{0\}$, in view of Lemma 2.2-(2), we can use the Lebesgue dominated convergence theorem to obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}V(x)\frac{G^{-1}(v_{b_n})\eta}{g(G^{-1}(v_{b_n}))}dx = \int_{\mathbb{R}^3}V(x)\frac{G^{-1}(v_0)\eta}{g(G^{-1}(v_0))}dx,$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\frac{h(G^{-1}(v_{b_n}))\eta}{g(G^{-1}(v_{b_n}))}dx = \int_{\mathbb{R}^3}\frac{h(G^{-1}(v_0))\eta}{g(G^{-1}(v_0))}dx.$$

By Lemma 2.1, we can obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\phi_{G^{-1}(v_{b_n})}\frac{G^{-1}(v_{b_n})\eta}{g(G^{-1}(v_{b_n}))}dx=\int_{\mathbb{R}^3}\phi_{G^{-1}(v_0)}\frac{G^{-1}(v_0)\eta}{g(G^{-1}(v_0))}dx.$$

On the other hand, we have $\langle J'_{b_n}(v_{b_n}),\eta\rangle=o_n(1)$ and $\langle J'_0(v_0),\eta\rangle=o_n(1)$. Moreover,

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\nabla v_{b_n}\nabla\eta dx=\int_{\mathbb{R}^3}\nabla v_0\nabla\eta dx,$$

$$\lim_{n\to\infty}b_n\int_{\mathbb{R}^3}|\nabla v_{b_n}|^2dx\int_{\mathbb{R}^3}\nabla v_{b_n}\nabla\eta=0.$$

Thus,

$$\int_{\mathbb{R}^3}\nabla v_0\nabla\eta dx+\int_{\mathbb{R}^3}V(x)\frac{G^{-1}(v_0)}{g(G^{-1}(v_0)}\eta dx+\int_{\mathbb{R}^3}\phi_{G^{-1}(v_0)}\frac{G^{-1}(v_0)}{g(G^{-1}(v_0))}\eta dx=\int_{\mathbb{R}^3}\frac{h(G^{-1}(v_0))}{g(G^{-1}(v_0))}\eta dx.$$

It shows that v_0 is a positive solution of (1.5). □

5 Proof of Theorem 1.4 and Theorem 1.5

In this section, we will prove the existence of nontrivial radial solution for system (1.1) under the condition (g').

Lemma 5.1. *Assume that (V_1), (g'), (h_1) and (h_3) are satisfied, then*

(i) for $\mu\in[1,2]$, there exists $v\in\mathcal{H}\backslash\{0\}$ such that $J_{b,\mu}(v)<0$;

(ii) there exists $\rho_1,\alpha_1>0$ such that $J_{b,\mu}(v)\geq\alpha_1$ and $\|v\|_{\mathcal{H}}=\rho_1$.

Proof. (i) According to (h_1) and (h_3), there exists constant $C>0$ such that

$$H(G^{-1}(v))\geq C|v|^\tau\quad\text{for all }\ v\geq 0\quad\text{and}\quad H(G^{-1}(v))\equiv 0\quad\text{for all }\ v\leq 0,$$

where τ is defined in (h_3).

For any fixed $\psi_1\in\mathcal{H}$ with $\psi_1>0$, by Lemma 2.2-(4), we have

$$\begin{aligned}J_{b,\mu}(t\psi_1)&=\frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla t\psi_1|^2+V(x)|G^{-1}(t\psi_1)|^2\right)dx+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla t\psi_1|^2dx\right)^2\\&\quad+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{G^{-1}(t\psi_1)}|G^{-1}(t\psi_1)|^2dx-\mu\int_{\mathbb{R}^3}H(G^{-1}(t\psi_1))dx\\&\leq\frac{t^2}{2}\|\psi_1\|^2_{\mathcal{H}}+\frac{b}{4}t^4\left(\int_{\mathbb{R}^3}|\nabla\psi_1|^2dx\right)^2+\frac{t^4}{4}C\|\psi_1\|^4_{\mathcal{H}}-\mu Ct^\tau\int_{\mathbb{R}^3}|\psi_1|^\tau dx\\&\to-\infty,\end{aligned}$$

as $t\to\infty$ for $\tau>4$, which implies that (i) holds if we take $v=t\psi_1$ with t sufficiently large.

(ii) It follows from (3.6) that

$$\begin{aligned}J_{b,\mu}(v)&=\frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla v|^2+V(x)|G^{-1}(v)|^2\right)dx+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2dx\right)^2\\&\quad+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{G^{-1}(v)}|G^{-1}(v)|^2dx-\mu\int_{\mathbb{R}^3}H(G^{-1}(v))dx\\&=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2dx-\int_{\mathbb{R}^3}\left(-\frac{1}{2}V(x)|G^{-1}(v)|^2+\mu H(G^{-1}(v))\right)dx\\&\quad+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2dx\right)^2+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{G^{-1}(v)}|G^{-1}(v)|^2dx.\end{aligned}\tag{5.1}$$

Let $A(x,s) := -\frac{1}{2}V(x)|G^{-1}(s)|^2 + \mu H(G^{-1}(s))$, then by Lemma 2.2-(6), we have

$$\lim_{s\to 0}\frac{A(x,s)}{|s|^2} = \lim_{s\to 0}\left[-\frac{1}{2}V(x)\left|\frac{G^{-1}(s)}{s}\right|^2 + \mu\frac{H(G^{-1}(s))}{|s|^2}\right] = -\frac{1}{2}V(x), \tag{5.2}$$

and

$$\lim_{s\to+\infty}\frac{A(x,s)}{|s|^6} = \lim_{s\to+\infty}\left[-\frac{1}{2}V(x)\left|\frac{G^{-1}(s)}{s}\right|^2\left(\frac{1}{|s|^4}\right) + \mu\frac{H(G^{-1}(s))}{|s|^6}\right] = 0. \tag{5.3}$$

Thus, by (5.2) and (5.3), for $\epsilon > 0$ sufficiently small, there exists a constant $C_\epsilon > 0$ such that

$$A(x,s) \le \left(-\frac{1}{2}V(x) + \epsilon\right)|s|^2 + C_\epsilon|s|^6. \tag{5.4}$$

Then, according to (V_1), (5.1), (5.4), Lemma 2.1 and Sobolev embedding theorem, we have

$$\begin{aligned}
J_{b,\mu}(v) &\ge \frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2dx - \int_{\mathbb{R}^3}\left(-\frac{1}{2}V(x)+\epsilon\right)|v|^2dx - \int_{\mathbb{R}^3}C_\epsilon|v|^6dx \\
&\quad + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2dx\right)^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{G^{-1}(v)}|G^{-1}(v)|^2dx \\
&\ge \frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2dx + \frac{1}{2}\int_{\mathbb{R}^3}V(x)v^2dx - \epsilon\int_{\mathbb{R}^3}|v|^2dx - C_\epsilon\int_{\mathbb{R}^3}|v|^6dx \\
&\ge \frac{1}{2}\|v\|^2_{\mathcal{H}} - \epsilon C\|v\|^2_{\mathcal{H}} - C_\epsilon\|v\|^6_{\mathcal{H}}.
\end{aligned}$$

It follows that

$$J_{b,\mu}(v) \ge C\|v\|^2_{\mathcal{H}} - C\|v\|^6_{\mathcal{H}},$$

if we choose sufficiently small $\rho_1 > 0$, which implies that

$$J_{b,\mu}(v) \ge C\rho_1^2 - C\rho_1^6 := \alpha_1 > 0. \qquad \square$$

Obviously, as defined in Section 3.2, $A(v)$ and $B(v)$ are the same, we can still obtain that $B(v) \ge 0$ for $v \in \mathcal{H}$ and $A(v) \to +\infty$ as $\|v\| \to \infty$. Also by Lemma 5.1 that the conclusion of Theorem 3.2 holds, then there is a bounded $(PS)_{c'_\mu}$ sequence $\{v'_n\} \subset \mathcal{H}$, satisfying $J_{b,\mu}(v'_n) \to c'_\mu$ and $J'_{b,\mu}(v'_n) \to 0$, where c'_μ is the mountain pass level.

Lemma 5.2. *Suppose (V_1), (g'), (h_1)–(h_3) hold, $\{v'_n\}$ is the sequence obtained above, going if necessary to a subsequence, $v'_n \to v'$ in $\mathcal{H}$.*

Proof. Since $\{v'_n\} \subset \mathcal{H}$ is bounded, up to a subsequence, there exists $v' \in \mathcal{H}$ such that

$$\begin{aligned}
v'_n &\rightharpoonup v' \quad \text{in } \mathcal{H}, \\
v'_n &\to v' \quad \text{in } L^r(\mathbb{R}^3),\ \text{for } 2 < r < 6, \\
v'_n(x) &\to v'(x) \quad \text{a.e. on } x \in \mathbb{R}^3.
\end{aligned}$$

Firstly, we claim that there exists $C > 0$ such that

$$\int_{\mathbb{R}^3}\left[|\nabla(v'_n - v')|^2 + V(x)\left(\frac{G^{-1}(v'_n)}{g(G^{-1}(v'_n))} - \frac{G^{-1}(v')}{g(G^{-1}(v'))}\right)(v'_n - v')\right]dx \ge C\|v'_n - v'\|^2_{\mathcal{H}}. \tag{5.5}$$

Indeed, we may assume $v_n' \neq v'$ (otherwise the conclusion is trivial). Set

$$\omega_n' = \frac{v_n' - v'}{\|v_n' - v'\|}$$

and

$$l_n = \frac{1}{v_n' - v'}\left(\frac{G^{-1}(v_n')}{g(G^{-1}(v_n'))} - \frac{G^{-1}(v')}{g(G^{-1}(v'))}\right).$$

Argue by contradiction and assume that

$$\int_{\mathbb{R}^3} (|\nabla \omega_n'|^2 + V(x) l_n(x) \omega_n'^2) dx \to 0.$$

Since

$$\frac{d}{ds}\left(\frac{G^{-1}(s)}{g(G^{-1}(s))}\right) = \frac{1}{g^2(G^{-1}(s))}\left(1 - \frac{G^{-1}(s) g'(G^{-1}(s))}{g(G^{-1}(s))}\right) > 0,$$

$\frac{G^{-1}(s)}{g(G^{-1}(s))}$ is strictly increasing, and for each $C > 0$ there exists $\delta' > 0$ such that

$$\frac{d}{ds}\left(\frac{G^{-1}(s)}{g(G^{-1}(s))}\right) > \delta',$$

when $|s| \leq C$. Hence, we see that $l_n(x)$ is positive. Hence

$$\int_{\mathbb{R}^3} |\nabla \omega_n'|^2 dx \to 0, \quad \int_{\mathbb{R}^3} V(x) l_n(x) \omega_n'^2 dx \to 0, \quad \text{and} \quad \int_{\mathbb{R}^3} V(x) \omega_n'^2 dx \to 1.$$

Similar to Lemma 2.5 in Reference [30], we assert that for each $\varepsilon' > 0$, there exists $C_1 > 0$ independent of n such that $\text{meas}(\Omega_n) < \varepsilon'$, where $\Omega_n := \{x \in \mathbb{R}^3 : |v_n'(x) - v'(x)| \geq C_1\}$, it can be inferred that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^3 \setminus \Omega_n} V(x) \omega_n'^2 dx \leq C \int_{\mathbb{R}^3 \setminus \Omega_n} V(x) \frac{|G^{-1}(v_n' - v')|^2}{\|v_n' - v'\|^2} dx \leq C \int_{\mathbb{R}^3} V(x) l_n(x) dx \to 0. \tag{5.6}$$

On the other hand, by the integral absolutely continuity, there exists $\varepsilon' > 0$ such that whenever $\Omega \subset \mathbb{R}^3$ and $\text{meas}(\Omega) < \varepsilon'$,

$$\int_{\Omega} V(x) \omega_n'^2 dx \leq \frac{1}{2}. \tag{5.7}$$

Combining (5.6) with (5.7), we have

$$\int_{\mathbb{R}^3} V(x) \omega_n'^2 dx = \int_{\mathbb{R}^3 \setminus \Omega_n} V(x) \omega_n'^2 dx + \int_{\Omega_n} V(x) \omega_n'^2 dx \leq \frac{1}{2} + o_n(1),$$

which implies $1 \leq \frac{1}{2}$, a contradiction. This implies that (5.5) holds. What's more, by Lemma 2.1, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left[\phi_{G^{-1}(v_n')} \frac{G^{-1}(v_n')}{g(G^{-1}(v_n'))} - \phi_{G^{-1}(v')} \frac{G^{-1}(v')}{g(G^{-1}(v'))}\right] (v_n' - v') dx \\
&\geq \int_{\mathbb{R}^3} \phi_{G^{-1}(v_n')} \left[\frac{G^{-1}(v_n')}{g(G^{-1}(v_n'))} - \frac{G^{-1}(v')}{g(G^{-1}(v'))}\right] (v_n' - v') dx \\
&\quad + \int_{\mathbb{R}^3} \left(\phi_{G^{-1}(v_n')} - \phi_{G^{-1}(v')}\right) \frac{G^{-1}(v')}{g(G^{-1}(v'))} (v_n' - v') dx \\
&\to 0.
\end{aligned} \tag{5.8}$$

Similar to Lemma 3.4, we can easily obtain

$$\begin{aligned} & b\int_{\mathbb{R}^3}|\nabla v_n'|^2dx\int_{\mathbb{R}^3}\nabla v_n'\nabla(v_n'-v')dx - b\int_{\mathbb{R}^3}|\nabla v'|^2dx\int_{\mathbb{R}^3}\nabla v'\nabla(v_n'-v')dx \\ & \geq b\int_{\mathbb{R}^3}\left(|\nabla v_n'|^2-|\nabla v'|^2\right)dx\int_{\mathbb{R}^3}\nabla v'\nabla(v_n'-v')dx \\ & \to 0, \end{aligned} \tag{5.9}$$

as $n\to\infty$. From (g'), there is $g(t)\geq 1$, $\forall t\geq 0$. In addition, by Lemma 2.2-(4), (1.8) and Hölder inequality, $v_n\to v$ in $L^r(\mathbb{R}^3)$, $2<r<6$, we obtain

$$\begin{aligned} & \left|\int_{\mathbb{R}^3}\left[\frac{h(G^{-1}(v_n'))}{g(G^{-1}(v_n'))}-\frac{h(G^{-1}(v'))}{g(G^{-1}(v'))}\right](v_n'-v')dx\right| \\ & \leq \int_{\mathbb{R}^3}\left[\varepsilon(|v_n'|+|v'|)+C_\varepsilon(|v_n'|^{q-1}+|v'|^{q-1})\right]|v_n'-v'|dx \\ & \leq \varepsilon(|v_n'|_2+|v'|_2)|v_n'-v'|_2+C_\varepsilon(|v_n'|_q^{q-1}+|v'|_q^{q-1})|v_n'-v'|_q \\ & \to 0. \end{aligned} \tag{5.10}$$

By virtue of (5.5), (5.8)–(5.10), we have

$$\begin{aligned} o_n(1) &= \langle J_{b,\mu}'(v_n')-J_{b,\mu}'(v'), v_n'-v'\rangle \\ &= \int_{\mathbb{R}^3}|\nabla(v_n'-v')|^2dx+\int_{\mathbb{R}^3}V(x)\left[\frac{G^{-1}(v_n')}{g(G^{-1}(v_n'))}-\frac{G^{-1}(v')}{g(G^{-1}(v'))}\right](v_n'-v')dx \\ &\quad + b\left[\int_{\mathbb{R}^3}|\nabla v_n'|^2dx\int_{\mathbb{R}^3}\nabla v_n'\cdot\nabla(v_n'-v')dx-\int_{\mathbb{R}^3}|\nabla v'|^2dx\int_{\mathbb{R}^3}\nabla v'\cdot\nabla(v_n'-v')dx\right] \\ &\quad + \int_{\mathbb{R}^3}\left[\phi_{G^{-1}(v_n')}\frac{G^{-1}(v_n')}{g(G^{-1}(v_n'))}-\phi_{G^{-1}(v')}\frac{G^{-1}(v')}{g(G^{-1}(v'))}\right](v_n'-v')dx \\ &\quad - \mu\int_{\mathbb{R}^3}\left[\frac{h(G^{-1}(v_n'))}{g(G^{-1}(v_n'))}-\frac{h(G^{-1}(v'))}{g(G^{-1}(v'))}\right](v_n'-v')dx \\ &\geq C\|v_n'-v'\|_{\mathcal{H}}^2+o_n(1), \end{aligned}$$

which implies that $v_n'\to v'$ in $\mathcal{H}$. □

From Lemma 5.2, for every $\mu\in[1,2]$, we get that $J_{b,\mu}$ has a nontrivial critical point v' satisfying $J_{b,\mu}(v')=c_\mu'$ and $J_{b,\mu}'(v')=0$. Therefore, there exists $\{\mu_n\}\subset[1,2]$ such that $\lim_{n\to\infty}\mu_n=1$, $\{v_{\mu_n}'\}\subset\mathcal{H}$ such that $J_{b,\mu_n}(v_{\mu_n}')=c_{\mu_n}'>0$ and $J_{b,\mu_n}'(v_{\mu_n}')=0$.

Lemma 5.3. *Suppose that (V_1), (g'), (h_1) and (h_3) are satisfied, then any sequence $\{v_{\mu_n}'\}$ is bounded in $\mathcal{H}$.*

Proof. Since $\{v_{\mu_n}'\}\subset\mathcal{H}$ satisfies $J_{b,\mu_n}(v_{\mu_n}')=c_{\mu_n}'>0$, we have

$$\begin{aligned} J_{b,\mu_n}(v_{\mu_n}') &= \frac{1}{2}\int_{\mathbb{R}^3}|\nabla v_{\mu_n}'|^2dx+\frac{1}{2}\int_{\mathbb{R}^3}V(x)|G^{-1}(v_{\mu_n}')|^2dx+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla v_{\mu_n}'|^2dx\right)^2 \\ &\quad +\frac{1}{4}\int_{\mathbb{R}^3}\phi_{G^{-1}(v_{\mu_n}')}|G^{-1}(v_{\mu_n}')|^2dx-\mu_n\int_{\mathbb{R}^3}H(G^{-1}(v_{\mu_n}'))dx \\ &= c_{\mu_n}'>0. \end{aligned} \tag{5.11}$$

We also deduce that

$$\begin{aligned}\langle J'_{b,\mu_n}(v'_{\mu_n}), v'_{\mu_n}\rangle &= \int_{\mathbb{R}^3} |\nabla v'_{\mu_n}|^2 dx + \int_{\mathbb{R}^3} V(x) \frac{G^{-1}(v'_{\mu_n})}{g(G^{-1}(v'_{\mu_n}))} v'_{\mu_n} dx \\ &\quad + b\left(\int_{\mathbb{R}^3} |\nabla v'_{\mu_n}|^2 dx\right)^2 + \int_{\mathbb{R}^3} \phi_{G^{-1}(v'_{\mu_n})} \frac{G^{-1}(v'_{\mu_n})}{g(G^{-1}(v'_{\mu_n}))} v'_{\mu_n} dx \\ &\quad - \mu_n \int_{\mathbb{R}^3} \frac{h(G^{-1}(v'_{\mu_n}))}{g(G^{-1}(v'_{\mu_n}))} v'_{\mu_n} dx.\end{aligned} \tag{5.12}$$

It follows from (5.11), (5.12), (h_3) and Lemma 2.2-(5) that

$$\begin{aligned}&2c'_{\mu_n} + 1 + \|v'_{\mu_n}\| \\ &\geq J_{b,\mu_n}(v'_{\mu_n}) - \frac{1}{4}\langle J'_{b,\mu_n}(v'_{\mu_n}), v'_{\mu_n}\rangle \\ &= \frac{1}{4}\int_{\mathbb{R}^3} |\nabla v'_{\mu_n}|^2 dx + \int_{\mathbb{R}^3} V(x) G^{-1}(v'_{\mu_n}) \left[\frac{1}{2} G^{-1}(v'_{\mu_n}) - \frac{v'_{\mu_n}}{4g(G^{-1}(v'_{\mu_n}))}\right] dx \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{G^{-1}(v'_{\mu_n})} G^{-1}(v'_{\mu_n}) \left[G^{-1}(v'_{\mu_n}) - \frac{v'_{\mu_n}}{g(G^{-1}(v'_{\mu_n}))}\right] dx \\ &\quad - \mu_n \int_{\mathbb{R}^3} \left[H(G^{-1}(v'_{\mu_n})) - \frac{1}{4}\frac{h(G^{-1}(v'_{\mu_n}))}{g(G^{-1}(v'_{\mu_n}))} v'_{\mu_n}\right] dx \\ &\geq \frac{1}{4}\int_{\mathbb{R}^3} |\nabla v'_{\mu_n}|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} V(x) |G^{-1}(v'_{\mu_n})|^2 dx.\end{aligned} \tag{5.13}$$

According to the assumption (h_3) and (V_1), we have $H(s) \geq CG(s)^\tau \geq CG(s)^2$ for all $s \geq 1$. Then by Lemma 2.2-(4)

$$\begin{aligned}&\int_{\{x:\ |G^{-1}(v'_{\mu_n})|>1\}} V(x) v'^2_{\mu_n} dx \\ &\leq C \int_{\{x:\ |G^{-1}(v'_{\mu_n})|>1\}} H(G^{-1}(v'_{\mu_n})) dx \\ &\leq C \int_{\mathbb{R}^3} H(G^{-1}(v'_{\mu_n})) dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla v'_{\mu_n}|^2 dx\right)^2 + C\int_{\mathbb{R}^3} \phi_{G^{-1}(v'_{\mu_n})} v'^2_{\mu_n} dx \\ &\leq C\left[\frac{1}{2}\int_{\mathbb{R}^3} |\nabla v'_{\mu_n}|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x) |G^{-1}(v'_{\mu_n})|^2 dx - c_1 + o_n(1)\right].\end{aligned} \tag{5.14}$$

On the other hand, for $x \in \{x :\ |G^{-1}(v'_n)| \leq 1\}$ we know that

$$\begin{aligned}\frac{1}{g^2(1)} \int_{\{x:\ |G^{-1}(v'_{\mu_n})|\leq 1\}} V(x) v'^2_{\mu_n} dx &\leq C \int_{\{x:\ |G^{-1}(v'_{\mu_n})|\leq 1\}} V(x) |G^{-1}(v'_{\mu_n})|^2 dx \\ &\leq C \int_{\mathbb{R}^3} V(x) |G^{-1}(v'_{\mu_n})|^2 dx.\end{aligned} \tag{5.15}$$

Since (g'), we know that $g(s)$ is nondecreasing. Combining (5.13) and (5.14) with (5.15), we deduce that $\{v'_{\mu_n}\}$ is bounded in $\mathcal{H}$. □

Proof of Theorem 1.4. Similar to the proof of Theorem 1.2, we take a subsequence of $\{v'_{\mu_n}\}$, and still represented by $\{v'_{\mu_n}\}$, because $\{v'_{\mu_n}\}$ is bounded in $\mathcal{H}$, similar to the proof of Lemma 5.2, we obtain $v'_{\mu_n} \to v'_\mu$ in $\mathcal{H}$. From Theorem 3.2, we known that $\mu \mapsto c'_\mu$ is continuous from the left. So

$$\begin{aligned}\lim_{n\to\infty} J_b(v'_{\mu_n}) &= \lim_{n\to\infty}\left[J_{b,\mu_n}(v'_{\mu_n}) + (\mu_n - 1)\int_{\mathbb{R}^3} H(G^{-1}(v'_{\mu_n}))dx\right] \\ &= \lim_{n\to\infty} c'_{\mu_n} = \tilde{c}'.\end{aligned} \tag{5.16}$$

In addition,

$$\lim_{n\to\infty}\langle J'_b(v'_{\mu_n}), \eta\rangle = \lim_{n\to\infty}\left[\langle J'_{b,\mu_n}(v'_{\mu_n}), \eta\rangle + (\mu_n - 1)\int_{\mathbb{R}^3}\frac{h(G^{-1}(v'_{\mu_n}))}{g(G^{-1}(v'_{\mu_n}))}\eta dx\right] = 0, \tag{5.17}$$

for any $\eta \in C_0^\infty(\mathbb{R}^3)$, which means that $J'_b(v') = 0$ satisfies $J_b(v') = \tilde{c}' > 0$. □

Proof of Theorem 1.5. The proof process is similar to Theorem 1.3, and specific steps are omitted here. □

6 Nonexistence in critical case

In this section, we prove the nonexistence of nontrivial solutions for system (1.9) by Pohožaev identity.

Lemma 6.1. *(Pohožaev identity) If $v \in \mathcal{H}$ is a weak solution of problem (1.9), then we have the following Pohožaev identity:*

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2dx + \frac{3}{2}\int_{\mathbb{R}^3}V(x)|G^{-1}(v)|^2dx + \frac{1}{2}\int_{\mathbb{R}^3}\langle\nabla V(x), x\rangle|G^{-1}(v)|^2dx \\ &+ \frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla v|^2dx\right)^2 + \frac{5}{4}\int_{\mathbb{R}^3}\phi_{G^{-1}(v)}|G^{-1}(v)|^2dx - \frac{3}{p}\int_{\mathbb{R}^3}|v|^pdx = 0.\end{aligned} \tag{6.1}$$

Proof of Theorem 1.6. Suppose that $v \in \mathcal{H}$ is a solution to system (1.9), then v satisfies

$$\begin{aligned}&\int_{\mathbb{R}^3}|\nabla v|^2dx + \int_{\mathbb{R}^3}V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}vdx + b\left(\int_{\mathbb{R}^3}|\nabla v|^2dx\right)^2 \\ &+ \int_{\mathbb{R}^3}\phi_{G^{-1}(v)}\frac{G^{-1}(v)}{g(G^{-1}(v))}vdx - \int_{\mathbb{R}^3}|v|^pdx = 0.\end{aligned} \tag{6.2}$$

Combining (6.1) and (6.2), we can get

$$\begin{aligned}&\int_{\mathbb{R}^3}V(x)\left(\frac{1}{2}\frac{G^{-1}(v)v}{g(G^{-1}(v))} - \frac{3}{2}|G^{-1}(v)|^2\right)dx - \frac{1}{2}\int_{\mathbb{R}^3}\langle\nabla V(x), x\rangle|G^{-1}(v)|^2dx \\ &+ \int_{\mathbb{R}^3}\phi_{G^{-1}(v)}\left(\frac{1}{2}\frac{G^{-1}(v)v}{g(G^{-1}(v))} - \frac{5}{4}|G^{-1}(v)|^2\right)dx = \left(\frac{1}{2} - \frac{3}{p}\right)\int_{\mathbb{R}^3}|v|^pdx.\end{aligned} \tag{6.3}$$

Since Lemma 2.2-(5) that $\frac{G^{-1}(v)}{g(G^{-1}(v))}v \le |G^{-1}(v)|^2$, if $2V(x) + \langle\nabla V(x), x\rangle \ge 0$, then (6.3) implies that $v = 0$ for $p \ge 6$. So that $0 = u = G^{-1}(v)$. Then (1.9) has no nontrivial solutions. □

Declarations

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors declare that they have equal contributions.

Funding

Jing Zhang was supported by the Natural Science Foundation of Inner Mongolia Autonomous Region (No. 2022MS01001, 2023LHMS01005), Research Program of science and technology at Universities of Inner Mongolia Autonomous Region (No. NJYT23100).

Availability of data and materials

Not applicable.

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