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Boundedness in a chemotaxis-consumption model with singular sensitivity

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Abstract. This paper deals with the chemotaxis system with singular sensitivity and nonlocal logistic source

$$\begin{cases} u_t = u_{xx} - \left(\frac{u}{v} v_x\right)_x + u^\alpha\left(\gamma - \mu \int_\Omega u^\beta\right), & x \in \Omega, t > 0, \\ v_t = v_{xx} - uv, & x \in \Omega, t > 0, \end{cases}$$

under Neumann boundary conditions in a bounded open interval $\Omega \subset \mathbb{R}$, where $\alpha, \beta, \gamma, \mu$ are positive constants. It is shown that for any $\beta > 1 + \frac{(\alpha-3)_+}{2}$ and $\alpha \geq 1$, the system possesses a global and bounded classical solution. Moreover, we establish uniform-in-time boundedness of $\frac{v_x}{v}$.

Keywords: chemotaxis, nonlocal source, boundedness, singular sensitivity.

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1 Introduction

In this paper, we consider the one-dimensional chemotaxis model with singular sensitivity and nonlocal term

$$\begin{cases} u_t = u_{xx} - \left(\frac{u}{v} v_x\right)_x + u^\alpha\left(\gamma - \mu \int_\Omega u^\beta\right), & x \in \Omega, t > 0, \\ v_t = v_{xx} - uv, & x \in \Omega, t > 0, \\ u_x = v_x = 0, & x \in \partial\Omega, t > 0, \\ (u,v)(x,0) = \big(u_0(x), v_0(x)\big), & x \in \Omega, \end{cases} \tag{1.1}$$

in a bounded open interval $\Omega \subseteq \mathbb{R}$, where α, β, γ, μ are positive constants and where $u = u(x,t)$ denotes the cell density and where $v = v(x,t)$ represents the oxygen concentration.

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The initial data in (1.1) satisfy

$$\begin{cases} u_0 \in C^0\left(\bar{\Omega}\right), \ u_0 \geq 0 \text{ and } u_0 \not\equiv 0, x \in \bar{\Omega}, \\ v_0 \in W^{1,q}\left(\Omega\right)(q>2), \ v_0 > 0 \text{ in } \bar{\Omega} \text{ and } \left(v_0\right)_x = 0\,, x \in \partial\Omega. \end{cases} \tag{1.2}$$

Chemotaxis processes are known to play an important role in various biological contexts. There are also many works on various central aspects like global existence, lager time behaviors, finite time blow-up, and so on (see [4,6,7,11,28,32] and the references therein). When placed at one end of a capillary tube that contains oxygen and an energy source, bacteria of the species E. coli that have a gradient of nutrient concentration form bands that are visible to the unaided eye. They migrate at a constant speed due to a chemotactic mechanism [1]. In order to describe the consumption of the critical substrate and the change in bacterial density by random motion and chemotaxis, Keller and Segel proposed a phenomenological model of wave-like solution behavior without any type of cell kinetics [20], a prototypical version of which is given by:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot \left(\dfrac{u}{v}\nabla v\right), & x \in \Omega, t>0, \\ v_t = \Delta v - uv, & x \in \Omega, t > 0, \end{cases} \tag{1.3}$$

where the second equation models consumption of the signal upon contact with cells, and where in the first equation it is assumed that the chemotactic stimulus is perceived following the Weber–Fechner law, thus requiring the chemotactic sensitivity $\chi \frac{u}{v}$ to be chosen proportional to the reciprocal signal density. When $n=2$, there exists a global generalized solution to (1.3) with $v \to 0$ in $L^p(\Omega)$ as $t \to \infty$ [33], and the solution becomes eventually smooth and converges to the homogeneous steady state if the initial mass $\int_\Omega u_0$ is small [35]. In particular, under an explicit smallness condition on $u_0 \ln u_0 \in L^1(\Omega)$ and $v_0^{-1}\nabla v_0 \in L^2(\Omega)$, the system (1.3) possesses a global classical solution [35]. Moreover, when $n \geq 3$, for all suitably regular and radially symmetric initial data (u_0, v_0), the existence of a globally defined pair (u,v) of radially symmetric functions in an appropriate generalized sense was established in [36]. When uv in (1.3) is replaced by $f(u)v$, $f \in C^1(\mathbb{R})$ essentially behaving like u^β, $\beta \in (0,1)$, $\chi \in (0,1)$, Lankeit and Viglialoro [25] showed that system has a global classical solution with any sufficiently regular initial data. Moreover, if additionally $\int_\Omega u(x,0)$ is sufficiently small, then also their boundedness is achieved. Replacing the first equation in (1.3) by $u_t = \Delta u - \chi \nabla\left(\frac{u}{v}\nabla v\right) + \gamma u - \mu u^k$, the authors in [23] showed that system (1.3) has a generalized global solution for any $\chi, \gamma, \mu > 0$ in the case $k=2$. It is proved that as $\frac{1}{v}$ is replaced by $\phi(v) \in C^1(0,\infty)$ satisfying $\phi(v) \to \infty$ as $v \to 0$, this system possesses a global classical solution if $k>1$ for $n=1$ or $k > 1+n/2$ for $n \geq 2$ [45]. The asymptotic behavior of solutions with $n=2$ is determined [45]. For the local reaction term, a sequence of recent results on the existence of global-in-time solution, long-time behavior, vanishing coefficient limit and optimal time decay rates of the solution in the case $n=1$ were obtained by Zeng and Zhao [41,42]. Recently, many variants of (1.3) have been proposed for applications under different frameworks, see for instance [2].

In those Keller–Segel models (cf. [21]) where v does not stand for a nutrient to be consumed but a signaling substance produced by the bacteria themselves, i.e., the second equation of (1.3) is replaced by $\tau v_t = \epsilon \Delta v - v + u$. For this system, the global solutions are known to exist if χ is sufficiently small, where the precise condition depends on the dimension as well as on $\tau = 0$ [15,29] or $\tau = 1$ [13,38,39]. As to the corresponding case with logistic source $\gamma u - \mu u^k$, the global existence of classical solutions and convergence to constant states were established in [9,16,18,22,27,40,43,44,47,48]. For a generalized solution concept, we refer to [12,26].

Nonlocal term in mathematical models can forecast how a disassociated sticky cell population will aggregate and react to the adhesive pressures produced by binding during cell-cell adhesion. See [3,17,30,31] for the reference. Bian et al. were devoted to the analysis of nonnegative solutions for the chemotaxis model with nonlocal nonlinear source in bounded domain in [5]

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + u^\alpha \left(1 - \int_\Omega u^\beta\right), & x \in \Omega, t > 0, \\ 0 = \Delta v - v + u, & x \in \Omega, t > 0, \end{cases} \tag{1.4}$$

where $\alpha \geq 1$, $\beta > 1$. They proved that if spatial dimension $n \geq 3$, either $2 \leq \alpha < 1 + \frac{2\beta}{n}$ or $\alpha < 2$ and $\frac{(2+n)(2-\alpha)}{n} < 1 + \frac{2\beta}{n} - \alpha$, the model admits a classical solution which is uniformly bounded. The same conclusion is established for the full-parabolic version in [8]. Recently, when the second equation of (1.4) is replaced by $v_t = \epsilon \Delta v - v + u$ and $-\chi \nabla \cdot (u \nabla v)$ is replaced by $-\chi \nabla \cdot (\frac{u}{v} \nabla v)$, the authors in [10] provide corresponding conditions to ensure the existence and boundedness of solutions, respectively.

In this paper, we consider the chemotaxis-consumption system with singular sensitivity and nonlocal logistic source. It is the purpose of this work to investigate the question of global existence and boundedness to (1.1) when $n = 1$. We first consider a non-singular chemotaxis system for (u, w) based on the transformation $w = -\ln \frac{v}{\|v_0\|_{L^\infty(\Omega)}}$ (see [33]). We obtain the key estimate on $\|w_x\|_{L^2(\Omega)}$ by deriving a subtle estimate for $\int_\Omega w_x^2 + \int_\Omega u^p$ with $1 < p < 2$. This is crucial to imply any further useful global regularity information on the quantity u itself. Before stating our results, we state that $x_+ = \max\{x, 0\}$.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}$ be a bounded open interval. Suppose that $\gamma, \mu > 0$ and that*

$$\beta > 1 + \frac{(\alpha - 3)_+}{2}, \quad \alpha \geq 1. \tag{1.5}$$

Then for any choice of u_0 and v_0 complying with (1.2), the problem (1.1) possesses a global bounded classical solution. Moreover, there exists some $\bar{M}$ such that

$$\left\| \frac{v_x}{v} \right\|_{L^\infty(\Omega)} \leq \bar{M}, \quad t > 0. \tag{1.6}$$

This paper is structured as follows. In Section 2, we present some preliminaries. In Section 3, we establish the global boundedness for $\frac{v_x}{v}$ and u in L^∞-norm.

2 Preliminaries

We need the Neumann heat semigroup in $\{e^{t\Delta}\}_{t \geq 0}$ estimates in $\Omega \subset \mathbb{R}^n$ to prove the local existence and the global existence of solutions. The following lemma below could be found in the vast existing literature, for instance, [14,34].

Lemma 2.1 ([14,34])**.** *Let $n \geq 1$, $\{e^{t\Delta}\}_{t \geq 0}$ be the Neumann heat semigroup in Ω, and $\lambda_1 > 0$ denotes the first nonzero eigenvalue of $-\Delta$ in Ω with respect to the Neumann boundary condition. Then there exist $K_1, \ldots, K_4 > 0$ depending on Ω only such that the following properties:*

(i) If $1 \leq p_2 \leq p_1 \leq \infty$, then

$$\|e^{t\Delta} z\|_{L^{p_1}(\Omega)} \leq K_1 \left(1 + t^{-\frac{n}{2}(\frac{1}{p_2} - \frac{1}{p_1})}\right) e^{-\lambda_1 t} \|z\|_{L^{p_2}(\Omega)}, \quad t \in (0, T) \tag{2.1}$$

for all $z \in L^{p_2}(\Omega)$ satisfying $\int_\Omega z = 0$.

(ii) *If* $1 \le p_2 \le p_1 \le \infty$, *then*

$$\|\nabla e^{t\Delta} z\|_{L^{p_1}(\Omega)} \le K_2 \left(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p_2}-\frac{1}{p_1})}\right) e^{-\lambda_1 t} \|z\|_{L^{p_2}(\Omega)}, \quad t \in (0,T) \tag{2.2}$$

for all $z \in L^{p_2}(\Omega)$.

(iii) *If* $2 \le p_2 \le p_1 \le \infty$, *then*

$$\|\nabla e^{t\Delta} z\|_{L^{p_1}(\Omega)} \le K_3 \left(1 + t^{-\frac{n}{2}(\frac{1}{p_2}-\frac{1}{p_1})}\right) e^{-\lambda_1 t} \|\nabla z\|_{L^{p_2}(\Omega)}, \quad t \in (0,T) \tag{2.3}$$

for all $z \in W^{1,p_2}(\Omega)$.

(iv) *If* $1 < p_2 \le p_1 < \infty$ *or* $1 < p_2 < \infty$ *and* $p_1 = \infty$, *then*

$$\|e^{t\Delta} \nabla z\|_{L^{p_1}(\Omega)} \le K_4 \left(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p_2}-\frac{1}{p_1})}\right) e^{-\lambda_1 t} \|z\|_{L^{p_2}(\Omega)}, \quad t \in (0,T) \tag{2.4}$$

for all $z \in (L^{p_2}(\Omega))^n$.

Let us state a basic result on local existence and extensibility of classical solutions, which can be proved by well-established fixed point arguments (see [19,37,46]). However, we could not find a precise reference that covers our model, therefore, we show a short proof here.

Lemma 2.2. *Let* $\alpha, \beta \ge 1$ *and* $\gamma, \mu > 0$. *Suppose that* u_0 *and* v_0 *satisfy* (1.2). *Then there exist a maximal* $T \in (0, \infty]$ *and a nonnegative classical solution* (u, v) *of functions*

$$\begin{cases} u \in C^0\Big(\bar{\Omega} \times [0,T)\Big) \cap C^{2,1}\Big(\bar{\Omega} \times (0,T)\Big), \\ v \in C^0\Big(\bar{\Omega} \times [0,T)\Big) \cap C^{2,1}\Big(\bar{\Omega} \times (0,T)\Big) \cap L^\infty_{loc}\Big([0,T), W^{1,q}(\Omega)\Big) \end{cases}$$

satisfying (1.1) *in the classical sense in* $\Omega \times (0,T)$. *Moreover, we have the following alternative:*

$$\text{Either} \quad T = \infty, \quad \text{or} \quad \|u\|_{L^\infty(\Omega)} + \|v\|_{W^{1,q}(\Omega)} \to \infty \quad \text{as } t \to T.$$

Proof. We first fix

$$K > 3 \max\{\|u_0\|_{L^\infty(\Omega)}, \|v_0\|_{L^q(\Omega)}, 6K_3\|v_0'\|_{L^q(\Omega)}\}, \tag{2.5}$$

where K_3 is as in (2.3). For small $T_* \in (0,1)$ to be specified below, in the Banach space

$$X =: C^0([0,T_*]; C^0(\bar{\Omega})) \times C^0([0,T_*]; W^{1,q}(\Omega))$$

we consider the closed set

$$\begin{aligned} S := \{(u,v) \in X \mid \|u\|_{L^\infty((0,T_*);L^\infty(\Omega))} \le K,\ \|v\|_{L^\infty((0,T_*);W^{1,q}(\Omega))} \le K, \\ \inf_{x\in\Omega} v(\cdot,t) \ge \frac{1}{e} \inf_{x\in\Omega} v_0(x),\ for\ a.e.\ t \in (0,T_*)\}. \end{aligned} \tag{2.6}$$

For $(u,v) \in S$ and $t \in (0,T_*)$, we let

$$\begin{aligned} \Phi(u,v)(t) &:= \begin{pmatrix} \phi_1(u,v)(t) \\ \phi_2(u,v)(t) \end{pmatrix} \\ &:= \begin{pmatrix} e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}(\frac{u}{v}v_x)_x + \gamma\int_0^t e^{(t-s)\Delta}u^\alpha - \mu\int_0^t e^{(t-s)\Delta}\left(u^\alpha \int_\Omega u^\beta\right) \\ e^{t\Delta}v_0 - \int_0^t e^{(t-s)\Delta}uv \end{pmatrix}. \end{aligned}$$

On dropping a nonpositive term, we use (2.4) and (2.6) to estimate

$$
\begin{aligned}
\|\phi_1(u,v)(t)\|_{L^\infty(\Omega)} \leq\ & \|u_0\|_{L^\infty(\Omega)} + \frac{K_4 e}{\inf\limits_{x\in\Omega} v_0(x)} \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2q}} e^{-\lambda_1(t-s)} \|u\|_{L^\infty(\Omega)} \|v_x\|_{L^q(\Omega)} \\
& + \gamma \int_0^t \sup_{t\in(0,T)} \|u\|^\alpha_{L^\infty(\Omega)} \\
\leq\ & \frac{K}{3} + CT_*^{\frac{1}{2}-\frac{1}{2q}} + \gamma K^\alpha T_*, \quad t \in (0,T_*),
\end{aligned}
\tag{2.7}
$$

where we use C to denote a generic positive constant independent of T^* here and below. Here when we deal with the first and third term we have used that $\|e^{\sigma\Delta} z\|_{L^{r_3}(\Omega)} \leq \|z\|_{L^{r_3}(\Omega)}$, for all $z \in L^{r_3}(\Omega)$ with $1 \leq r_3 \leq \infty$.

We next derive the $W^{1,q}$-estimate on $\phi_2(u,v)$. On the one hand, we drop a nonpositive term to get

$$
\|\phi_2(u,v)(t)\|_{L^q(\Omega)} = \left\| e^{t\Delta} v_0 - \int_0^t e^{(t-s)\Delta} uv \right\|_{L^q(\Omega)} \leq \|v_0\|_{L^q(\Omega)} \leq \frac{K}{3} \tag{2.8}
$$

for all $t \in (0,T_*)$. On the other hand, it follows from (2.2), (2.3), (2.5) and (2.6) that

$$
\begin{aligned}
\|(\phi_2(u,v)(t))_x\|_{L^q(\Omega)} &= \left\| (e^{t\Delta} v_0)_x - \int_0^t (e^{(t-s)\Delta} uv)_x \right\|_{L^q(\Omega)} \\
&\leq 2K_3 \|(v_0)_x\|_{L^q(\Omega)} + K_2 \int_0^t (t-s)^{-\frac{1}{2}} e^{-\lambda_1(t-s)} \|uv\|_{L^q(\Omega)} \\
&\leq \frac{K}{3} + CT_*^{\frac{1}{2}}, \quad t \in (0,T_*).
\end{aligned}
\tag{2.9}
$$

A combination of (2.8) and (2.9) entails

$$
\|\phi_2(u,v)(t)\|_{W^{1,q}(\Omega)} \leq \frac{2K}{3} + CT_*^{\frac{1}{2}}, \quad t \in (0,T_*). \tag{2.10}
$$

The comparison principle to the second equation of (1.1) leads to $\|v\|_{L^\infty(\Omega)} \leq \|v_0\|_{L^\infty(\Omega)}$ for all $t \in (0,T_*)$. By the order preserving of the Neumann heat semigroup, we find

$$
\begin{aligned}
\phi_2(u,v)(t) \geq e^{t\Delta} v_0 - \int_0^t e^{(t-s)\Delta} uv &\geq \inf_{x\in\Omega} v_0(x) - \int_0^t \|e^{(t-s)\Delta} uv\|_{L^\infty(\Omega)} \\
&\geq \inf_{x\in\Omega} v_0(x) - K\|v_0\|_{L^\infty(\Omega)} T_*, \quad t \in (0,T_*).
\end{aligned}
\tag{2.11}
$$

If we take T_* small enough, then it follows from (2.7), (2.10) and (2.11) that $\Phi S \subseteq S$.

Moreover, with T_* still at our disposal, we proceed to check that for all $(u_1,v_1),(u_2,v_2)$ belonging to S,

$$
\begin{aligned}
\|\phi_1(u_1,v_1)(t) - \phi_1(u_2,v_2)(t)\|_{L^\infty(\Omega)} \leq\ & \int_0^t \left\| e^{(t-s)\Delta} \left(\frac{u_1 (v_1)_x}{v_1} - \frac{u_2 v_2'}{v_2} \right)_x \right\|_{L^\infty(\Omega)} \\
& + \gamma \int_0^t \|e^{(t-s)\Delta}(u_1^\alpha - u_2^\alpha)\|_{L^\infty(\Omega)} \\
& + \mu \int_0^t \left\| e^{(t-s)\Delta} u_2^\alpha \int_\Omega (u_2^\beta - u_1^\beta) \right\|_{L^\infty(\Omega)} \\
& + \mu \int_0^t \left\| e^{(t-s)\Delta} (u_2^\alpha - u_1^\alpha) \int_\Omega u_1^\beta \right\|_{L^\infty(\Omega)}
\end{aligned}
\tag{2.12}
$$

for all $t \in (0, T_*)$. For the first term of the right-hand side of (2.12), in accordance with (2.6) and (2.4), we can estimate

$$\begin{aligned} &\int_0^t \left\| e^{(t-s)\Delta} \left(\frac{u_1(v_1)_x}{v_1} - \frac{u_2(v_2)_x}{v_2} \right)_x \right\|_{L^\infty(\Omega)} \\ &\quad \le K_2 \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2q}} e^{-\lambda_1(t-s)} \left\| \frac{u_1(v_1)_x}{v_1} - \frac{u_2(v_2)_x}{v_2} \right\|_{L^q(\Omega)} \\ &\quad \le K_2 T_*^{\frac{1}{2}-\frac{1}{2q}} \|(u_1,v_1)-(u_2,v_2)\|_X. \end{aligned} \tag{2.13}$$

Since $\alpha, \beta \ge 1$, from the mean value theorem and (2.6), we know

$$\|u_2^\alpha - u_1^\alpha\|_{L^\infty(\Omega)} \le \alpha K^{\alpha-1} \|u_2 - u_1\|_{L^\infty(\Omega)}, \qquad \|u_2^\beta - u_1^\beta\|_{L^\infty(\Omega)} \le \beta K^{\beta-1} \|u_2 - u_1\|_{L^\infty(\Omega)}.$$

We thus gain that

$$\begin{aligned} &\gamma \int_0^t \|e^{(t-s)\Delta}(u_1^\alpha - u_2^\alpha)\|_{L^\infty(\Omega)} + \mu \int_0^t \left\| e^{(t-s)\Delta} u_2^\alpha \int_\Omega (u_2^\beta - u_1^\beta) \right\|_{L^\infty(\Omega)} \\ &\quad + \mu \int_0^t \left\| e^{(t-s)\Delta} (u_2^\alpha - u_1^\alpha) \int_\Omega u_1^\beta \right\|_{L^\infty(\Omega)} \\ &\le C T_* \|u_1 - u_2\|_{L^\infty(\Omega)}, \quad t \in (0, T_*). \end{aligned} \tag{2.14}$$

Thus (2.12), (2.13), (2.14) provide

$$\|\phi_1(u_1,v_1)(t) - \phi_1(u_2,v_2)(t)\|_{L^\infty(\Omega)} \le C T_*^{\frac{1}{2}-\frac{1}{2q}} \|(u_1,v_1)-(u_2,v_2)\|_X, \quad t \in (0, T_*). \tag{2.15}$$

Similar process to (2.8)-(2.10), we can conclude that

$$\begin{aligned} &\|\phi_2(u_1,v_1)(t) - \phi_2(u_2,v_2)(t)\|_{W^{1,q}(\Omega)} \\ &\le \int_0^t \|e^{(t-s)\Delta} u_1(v_1 - v_2)\|_{W^{1,q}(\Omega)} + \int_0^t \|e^{(t-s)\Delta} v_2(u_1 - u_2)\|_{W^{1,q}(\Omega)} \\ &\le C T_*^{\frac{1}{2}} \|(u_1,v_1)-(u_2,v_2)\|_X, \quad t \in (0, T_*), \end{aligned} \tag{2.16}$$

so that Φ is shown to be a contraction if T_* is sufficiently small. Accordingly, the Banach fixed point theorem asserts the existence of some $(u, v) \in S$ such that $\Phi(u, v) = (u, v)$. Relying on straightforward regularity arguments including standard semigroup techniques, it can easily be checked that in fact (u, v) lies in the asserted regularity class and is a classical solution of (1.1) in $\Omega \times (0, T_*)$.

Finally, the nonnegative of (u, v) can be obtained by the strong maximum principle, which can be derived as that in [24]. It is sufficient to justify the conditions of Theorem B.1 in [24] in the following. Firstly, the equation

$$v_t = v_{xx} - uv, \quad x \in \Omega, t \in (0, T)$$

is considered. We interpret u as given function from above, i.e. $u \in C^0(\bar{\Omega} \times [0, T_*)) \cap C^{2,1}(\bar{\Omega} \times (0, T_*))$. For given $T_1 < T_*$ one can fine $C > 0$ such that

$$u \le C, \quad \bar{\Omega} \times [0, T_1]. \tag{2.17}$$

With $b \equiv 0 \in C^1(\bar{\Omega} \times (0, T_1), \mathbb{R}) \cap L^\infty_{loc}(\Omega \times [0, T_1), \mathbb{R})$ fulfilling $b_x = 0$ on $\partial\Omega \times (0, T_1)$ and $f(x, t, u) := -u(x, t)v \in C^0(\Omega \times [0, T_1) \times \mathbb{R})$ satisfying

$$|f(x,t,v_1) - f(x,t,v_2)| = |u(v_1 - v_2)| \le C|v_1 - v_2|, \quad x \in \bar{\Omega}, t \in [0, T), v_1, v_2 \in K$$

with compact K. Define $\underline{v} = \inf_{x\in\Omega} v_0(x)e^{-Ct}$. A direct calculation shows that

$$\begin{cases} \underline{v}_t = -C\underline{v} = \underline{v}_{xx} - C\underline{v} \le \underline{v}_{xx} - u\underline{v}, & x \in \Omega,\ t \in (0, T_1), \\ v_x = 0, & x \in \partial\Omega,\ t \in (0, T_1), \\ \underline{v}(\cdot, 0) \le v_0(x), & x \in \Omega. \end{cases}$$

which by comparison implies

$$v(x,t) \ge \inf_{x\in\Omega} v_0(x)e^{-Ct} > 0 \quad \text{on } \Omega \times (0, T_1). \tag{2.18}$$

For given function $v \in C^0\big(\bar{\Omega} \times [0, T_*)\big) \cap C^{2,1}\big(\bar{\Omega} \times (0, T_*)\big)$ satisfying (2.18), we can study the first equation of (1.1)

$$u_t = u_{xx} - \Big(\frac{u}{v}v_x\Big)_x + u^\alpha\Big(\gamma - \mu\int_\Omega u^\beta\Big).$$

In order to apply Theorem B.1 in [24] to $b(x,t) = \frac{v_x}{v}$, $f(x,t,u) = u^\alpha(\gamma - \mu\int_\Omega u^\beta)$, we now verify the conditions of the comparison theorem. To avoid repetition, we just point out the differences, which are the local Lipschitz estimation of $f(x,t,u)$. For every compact K,

$$\begin{aligned} |f(x,t,u_1) - f(x,t,u_2)| &= \left| u_1^\alpha\Big(\gamma - \mu\int_\Omega u_1^\beta\Big) - u_2^\alpha\Big(\gamma - \mu\int_\Omega u_2^\beta\Big)\right| \\ &\le \gamma|u_1^\alpha - u_2^\alpha| + \mu\left|u_2^\alpha\int_\Omega(u_2^\beta - u_1^\beta)\right| + \mu\left|(u_2^\alpha - u_1^\alpha)\int_\Omega u_1^\beta\right| \\ &\le L(K,\alpha,\beta)|u_1 - u_2|, \quad x \in \bar{\Omega}, t \in [0, T_1), u_1, u_2 \in K, \end{aligned}$$

where $L(K,\alpha,\beta)$ is a positive constant. Therefore, the comparision theorem becomes applicable to suitable subsolutions and supersolutions of this equation. Moreover, since 0 is a subsolution of the equation, we have $u \ge 0$. $\square$

A first basic property of this solution is immediate.

Lemma 2.3. *Let $\alpha, \beta \ge 1$ and $\gamma, \mu > 0$. Then*

$$\int_\Omega u \le M_0 := \max\left\{\int_\Omega u_0, |\Omega|^{1-\frac{1}{\beta}}\left(\frac{\gamma}{\mu}\right)^{\frac{1}{\beta}}\right\}, \quad t \in (0, T). \tag{2.19}$$

Proof. Integrating the first equation of (1.1), using the Hölder inequality, we obtain

$$\frac{d}{dt}\int_\Omega u \le \int_\Omega u^\alpha\left[\gamma - |\Omega|^{1-\beta}\mu\left(\int_\Omega u\right)^\beta\right], \quad t \in (0, T),$$

which implies (2.19) by a straightforward ODE analysis. $\square$

Now following a standard procedure of changing variables in (1.1) we substitute

$$w = -\ln \frac{v}{\|v_0\|_{L^\infty(\Omega)}}$$

and thus infer that $w \geq 0$ in $\Omega \times (0, T)$ and

$$(u, w) \in \Big(C^0\big(\bar{\Omega} \times [0, T)\big) \cap C^{2,1}\big(\bar{\Omega} \times (0, T)\big)\Big)^2$$

solves

$$\begin{cases} u_t = u_{xx} + (uw_x)_x + u^\alpha\Big(\gamma - \mu \int_\Omega u^\beta\Big), & x \in \Omega, t \in (0, T), \\ w_t = w_{xx} - w_x^2 + u, & x \in \Omega, t \in (0, T), \\ u_x = w_x = 0, & x \in \partial\Omega, t \in (0, T), \\ u(x, 0) = u_0,\ w(x, 0) = -\ln \dfrac{v_0}{\|v_0\|_{L^\infty(\Omega)}}, & x \in \Omega. \end{cases} \tag{2.20}$$

We now prepare an estimate on $\int_\Omega u^k$ for all $k > 1$.

Lemma 2.4. *Let $\gamma, \mu > 0$. Suppose that u_0 and v_0 satisfy (1.2). Assume that α, β satisfy (1.5). Then for all $k > 1$, there exists $M_1(k) > 0$ such that*

$$\frac{d}{dt}\int_\Omega u^k + \int_\Omega u^k \leq -\frac{2(k-1)}{k}\int_\Omega \Big|\Big(u^{\frac{k}{2}}\Big)_x\Big|^2 - k(k-1)\int_\Omega u^{k-1}u_x w_x + M_1, \quad t \in (0, T). \tag{2.21}$$

Proof. Testing the first equation of (1.1) by ku^{k-1} and using Young's inequality, we obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^k + \int_\Omega u^k = & -\frac{4(k-1)}{k}\int_\Omega \Big|\Big(u^{\frac{k}{2}}\Big)_x\Big|^2 - k(k-1)\int_\Omega u^{k-1}u_x w_x + \int_\Omega u^k \\ & + k\gamma\int_\Omega u^{k+\alpha-1} - k\mu\int_\Omega u^{k+\alpha-1}\int_\Omega u^\beta, \quad t \in (0, T). \end{aligned} \tag{2.22}$$

We apply the Gagliardo–Nirenberg inequality and Young's inequality to find positive constants $C_0(\Omega), C_1(k, \Omega)$ such that

$$\begin{aligned} \int_\Omega u^k & \leq C_0\left(\int_\Omega \Big|\Big(u^{\frac{k}{2}}\Big)_x\Big|^2\right)^{\frac{k-1}{k+1}} \|u\|_{L^1(\Omega)}^{\frac{2k}{k+1}} + C_0\|u\|_{L^1(\Omega)}^k \\ & \leq \frac{k-1}{k}\int_\Omega \Big|\Big(u^{\frac{k}{2}}\Big)_x\Big|^2 + C_1\|u\|_{L^1(\Omega)}^k, \quad t \in (0, T). \end{aligned} \tag{2.23}$$

Let us fix β_0 satisfying

$$\beta_0 < \beta \tag{2.24}$$

and

$$\frac{\alpha - 1}{2} < \beta_0 < k + \alpha - 1, \tag{2.25}$$

which is possible because $\beta > \frac{\alpha-1}{2}$. By virtue of Hölder's inequality and (2.24) we see that

$$-k\mu\int_\Omega u^{k+\alpha-1}\int_\Omega u^\beta \leq -k\mu|\Omega|^{1-\frac{\beta}{\beta_0}}\int_\Omega u^{k+\alpha-1}\left(\int_\Omega u^{\beta_0}\right)^{\frac{\beta}{\beta_0}}. \tag{2.26}$$

Due to the condition (2.25), we may invoke the Gagliardo–Nirenberg inequality along with the Young inequality, which says that there exist positive constants $C_2(\alpha,\gamma,k,\Omega)$ and $C_3(\alpha,\beta,\gamma,k,\Omega)$ satisfying

$$\begin{aligned} k\gamma\int_\Omega u^{k+\alpha-1}dx &\le C_2\left(\int_\Omega |(u^{\frac{k}{2}})_x|^2dx\right)^{\frac{a(k+\alpha-1)}{k}} \|u\|_{L^{\frac{Mk}{2}}(\Omega)}^{(1-a)(k+\alpha-1)} + C_2\|u\|_{L^1(\Omega)}^{k+\alpha-1} \\ &\le \frac{k-1}{k}\int_\Omega |(u^{\frac{k}{2}})_x|^2 + C_3\|u\|_{L^{\frac{Mk}{2}}(\Omega)}^{\frac{(1-a)k(k+\alpha-1)}{k-a(k+\alpha-1)}} + C_2\|u\|_{L^1(\Omega)}^{k+\alpha-1}, \end{aligned} \tag{2.27}$$

where

$$M = 2\beta_0(k+\alpha-1+\beta)k^{-1}(\beta+\beta_0)^{-1}$$

and where

$$a = \beta k(k+\alpha-1-\beta_0)(k+\alpha-1)^{-1}[(\beta+\beta_0)k+(k+\alpha-1+\beta)\beta_0]^{-1} \in (0,1),$$

thanks to (2.25). Moreover, making full use of (2.25) we see that actually

$$\beta_0 < \frac{Mk}{2} < k+\alpha-1, \tag{2.28}$$

and that

$$\frac{k(k+\alpha-1)(1-a)}{[k-a(k+\alpha-1)](k+\alpha-1+\beta)} < 1. \tag{2.29}$$

Indeed, (2.29) is possible since the definition of a ensures that

$$\begin{aligned} &\frac{k(k+\alpha-1)(1-a)}{[k-a(k+\alpha-1)](k+\alpha-1+\beta)} \\ &\quad= \frac{(k+\alpha-1)[\beta_0 k+(k+\alpha-1+\beta)\beta_0]+\beta\beta_0 k}{(k+\alpha-1)[\beta_0 k+(k+\alpha-1+\beta)\beta_0]+\beta\beta_0 k+\beta(k+\alpha-1+\beta)[2\beta_0-(\alpha-1)]}. \end{aligned}$$

Therefore, in view of (2.28) and (2.29), one further application of Young's inequality and the interpolation inequality provides $C_4(\alpha,\beta,\gamma,\mu,k,\Omega)$ such that

$$\begin{aligned} C_3\|u\|_{L^{\frac{Mk}{2}}(\Omega)}^{\frac{(1-a)k(k+\alpha-1)}{k-a(k+\alpha-1)}} &\le C_3\left[\int_\Omega u^{k+\alpha-1}\left(\int_\Omega u^{\beta_0}\right)^{\frac{\beta}{\beta_0}}\right]^{\frac{k(k+\alpha-1)(1-a)}{[k-a(k+\alpha-1)](k+\alpha-1+\beta)}} \\ &\le k\mu|\Omega|^{1-\frac{\beta}{\beta_0}}\int_\Omega u^{k+\alpha-1}\left(\int_\Omega u^{\beta_0}\right)^{\frac{\beta}{\beta_0}} + C_4. \end{aligned} \tag{2.30}$$

From (2.22), (2.23), (2.26), (2.27) and (2.30), we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^k + \int_\Omega u^k \le &-\frac{2(k-1)}{k}\int_\Omega \left|\left(u^{\frac{k}{2}}\right)_x\right|^2 - k(k-1)\int_\Omega u^{k-1}u_x w_x + C_1\|u\|_{L^1(\Omega)}^k \\ &+ C_2\|u\|_{L^1(\Omega)}^{k+\alpha-1} + C_4, \quad t\in(0,T). \end{aligned} \tag{2.31}$$

Therefore, (2.21) results from (2.31) by taking $M_1 = C_1M_0^k + C_2M_0^{k+\alpha-1} + C_4$. □

3 Global boundedness of solutions

This section is devoted to the derivation of estimates for w, so as to finally obtain the global boundedness of solutions. For this purpose, we establish L^2 estimate on w_x and L^4 estimate on w_x for our present setting.

Lemma 3.1. *Let $\gamma,\mu > 0$, and suppose that u_0 and v_0 satisfy (1.2). Assume that α,β satisfy (1.5). Then there exists constant $M_2 > 0$ depend on $\alpha,\beta,\gamma,\mu,\Omega$ such that*

$$\int_\Omega w_x^2 \le M_2, \quad t \in (0,T). \tag{3.1}$$

Proof. Recalling $(w_0)_x = 0$ on $\partial\Omega$, we multiply the second equation in (2.20) by $-w_{xx}$ and integrate by parts to see, using Young's inequality, that

$$\frac{d}{dt}\int_\Omega w_x^2 = -2\int_\Omega w_{xx}^2 - 2\int_\Omega u w_{xx} \le -\int_\Omega w_{xx}^2 + \int_\Omega u^2, \quad t \in (0,T). \tag{3.2}$$

For fixed $1 < p < 2$, applying $k = p$ to (2.21) takes the form

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p \le & -\frac{2(p-1)}{p}\int_\Omega \left|\left(u^{\frac{p}{2}}\right)_x\right|^2 - p(p-1)\int_\Omega u^{p-1}u_x w_x + M_1 \\ = & -\frac{2(p-1)}{p}\int_\Omega \left|\left(u^{\frac{p}{2}}\right)_x\right|^2 + (p-1)\int_\Omega u^p w_{xx} + M_1, \quad t \in (0,T),\end{aligned}$$

which along with Young's inequality derives

$$\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p \le -\frac{2(p-1)}{p}\int_\Omega |(u^{\frac{p}{2}})_x|^2 + \frac{1}{2}\int_\Omega w_{xx}^2 + \frac{(p-1)^2}{2}\int_\Omega u^{2p} + M_1 \tag{3.3}$$

for all $t \in (0,T)$. According to Poincaré's inequality and the fact $(w_0)_x = 0$ on $\partial\Omega$, we know there is constant $C_5(\Omega)$ such that

$$-\frac{1}{2}\int_\Omega w_{xx}^2 \le -\frac{1}{2C_5}\int_\Omega w_x^2. \tag{3.4}$$

Recalling $p > 1$, Young's inequality provide $C_7(p,\Omega) > 0$ such that

$$\int_\Omega u^2 \le \int_\Omega u^{2p} + C_7. \tag{3.5}$$

Collecting (3.2)–(3.5) we see that

$$\begin{aligned}&\frac{d}{dt}\left(\int_\Omega w_x^2 + \int_\Omega u^p\right) + C_6\left(\int_\Omega w_x^2 + \int_\Omega u^p\right) \\ &\le -\frac{2(p-1)}{p}\int_\Omega |(u^{\frac{p}{2}})_x|^2 + \left[1 + \frac{(p-1)^2}{2}\right]\int_\Omega u^{2p} + C_8, \quad t \in (0,T),\end{aligned} \tag{3.6}$$

where $C_6 = \min\{1, \frac{1}{2C_5}\}$ and $C_8 = C_7 + M_1$. Here our assumption for $1 < p < 2$ warrants that

$$0 < \frac{2p-1}{p+1} < 1. \tag{3.7}$$

We may once more rely on Young's inequality together with Gagliardo–Nirenberg inequality to see that there exist constants $C_9(p,\Omega), C_{10}(p,\Omega) > 0$ such that

$$\begin{aligned}\left[1+\frac{(p-1)^2}{2}\right]\int_\Omega u^{2p} &= \frac{(p-1)^2}{2}\|u^{\frac{p}{2}}\|^4_{L^4(\Omega)} \\ &\le C_9\left(\int_\Omega \left|(u^{\frac{p}{2}})_x\right|^2\right)^{\frac{2p-1}{p+1}}\|u\|^{\frac{3p}{p+1}}_{L^1(\Omega)} + C_9\|u\|^{2p}_{L^1(\Omega)} \\ &\le \frac{p-1}{p}\int_\Omega |(u^{\frac{p}{2}})_x|^2 + C_{10}\|u\|^{\frac{3p}{2-p}}_{L^1(\Omega)} + C_9\|u\|^{2p}_{L^1(\Omega)}\end{aligned} \tag{3.8}$$

for all $t \in (0,T)$, which together with (3.6) and (3.8) asserts

$$\frac{d}{dt}\left(\int_\Omega w_x^2 + \int_\Omega u^p\right) + C_6\left(\int_\Omega w_x^2 + \int_\Omega u^p\right) \le C_8 + C_9\|u\|^{2p}_{L^1(\Omega)} + C_{10}\|u\|^{\frac{3p}{2-p}}_{L^1(\Omega)} \tag{3.9}$$

for all $t \in (0,T)$. Thereupon, (3.1) can be derived by (3.9) and (2.19). □

Lemma 3.2. *Let $\gamma,\mu > 0$, and suppose that u_0 and v_0 satisfy (1.2). Assume that α,β satisfy (1.5). Then there exists $M_3(\alpha,\beta,\gamma,\mu,\Omega) > 0$ such that*

$$\int_\Omega w_x^4 + \int_\Omega u^2 \le M_3, \quad t \in (0,T). \tag{3.10}$$

Proof. By (2.21) with $k = 2$, we get

$$\frac{d}{dt}\int_\Omega u^2 + \int_\Omega u^2 \le -\int_\Omega u_x^2 - 2\int_\Omega u u_x w_x + M_1.$$

Fixing $\theta > 1$, we use Young's inequality and Hölder's inequality to obtain

$$\frac{d}{dt}\int_\Omega u^2 + \int_\Omega u^2 \le -\frac{1}{2}\int_\Omega u_x^2 + 2\|u\|^2_{L^{2\theta}(\Omega)}\|w_x^2\|_{L^{\frac{\theta}{\theta-1}}(\Omega)} + M_1, \quad t \in (0,T). \tag{3.11}$$

Now another application of Young's inequality, followed by Hölder's inequality, reveals that

$$\begin{aligned}\frac{d}{dt}\int_\Omega w_x^4 &= -12\int_\Omega w_x^2 w_{xx}^2 - 12\int_\Omega w_x^2 w_{xx} u \\ &\le -12\int_\Omega w_x^2 w_{xx}^2 + 8\int_\Omega w_x^2 w_{xx}^2 + \frac{9}{2}\int_\Omega w_x^2 u^2 \\ &\le -\int_\Omega |(w_x^2)_x|^2 + \frac{9}{2}\|u\|^2_{L^{2\theta}(\Omega)}\|w_x^2\|_{L^{\frac{\theta}{\theta-1}}(\Omega)}\end{aligned} \tag{3.12}$$

for all $t \in (0,T)$. Adding (3.11) and (3.12), we write

$$\begin{aligned}&\frac{d}{dt}\left(\int_\Omega w_x^4 + \int_\Omega u^2\right) + \left(\int_\Omega w_x^4 + \int_\Omega u^2\right) \\ &\le -\int_\Omega |(w_x^2)_x|^2 - \frac{1}{2}\int_\Omega u_x^2 + \|w_x^2\|^2_{L^2(\Omega)} + \frac{13}{2}\|u\|^2_{L^{2\theta}(\Omega)}\|w_x^2\|_{L^{\frac{\theta}{\theta-1}}(\Omega)} + M_1\end{aligned} \tag{3.13}$$

for all $t \in (0,T)$. Using the Gagliardo–Nirenberg inequality, we deduce $C_{11}(\Omega) > 0$ such that

$$\|u\|^2_{L^{2\theta}(\Omega)} \le 4C_{11}^2\left(\int_\Omega u_x^2\right)^{\frac{2\theta-1}{3\theta}}\|u\|^{\frac{2(\theta+1)}{3\theta}}_{L^1(\Omega)} + 4C_{11}^2\|u\|^2_{L^1(\Omega)}, \quad t \in (0,T) \tag{3.14}$$

and

$$\|w_x^2\|_{L^{\frac{\theta}{\theta-1}}(\Omega)} \le C_{11}\left(\int_\Omega |(w_x^2)_x|^2\right)^{\frac{1}{3\theta}} \|w_x^2\|_{L^1(\Omega)}^{\frac{3\theta-2}{3\theta}} + C_{11}\|w_x^2\|_{L^1(\Omega)}, \quad t\in(0,T) \tag{3.15}$$

as well as

$$\|w_x^2\|_{L^2(\Omega)}^2 \le 4C_{11}^2\left(\int_\Omega |(w_x^2)_x|^2\right)^{\frac{1}{3}} \|w_x^2\|_{L^1(\Omega)}^{\frac{4}{3}} + 4C_{11}^2\|w_x^2\|_{L^1(\Omega)}^2, \quad t\in(0,T). \tag{3.16}$$

Therefore, in view of $\frac{2\theta-1}{3\theta}+\frac{1}{3\theta}=\frac{2}{3}<1$, we apply Young's inequality and recall (3.14)–(3.16) to estimate

$$\begin{aligned}
&\|w_x^2\|_{L^2(\Omega)}^2 + \frac{13}{2}\|u\|_{L^{2\theta}(\Omega)}^2\|w_x^2\|_{L^{\frac{\theta}{\theta-1}}(\Omega)} \\
&\le \frac{1}{2}\int_\Omega |u_x|^2 + \frac{3}{4}\int_\Omega |(w_x^2)_x|^2 + C_{11}\|u\|_{L^1(\Omega)}^{\frac{2(\theta+1)}{\theta}}\|w_x^2\|_{L^1(\Omega)}^{\frac{3\theta-2}{\theta}} + C_{11}\|u\|_{L^1(\Omega)}^2\|w_x^2\|_{L^1(\Omega)}^{\frac{3\theta}{\theta+1}} \\
&\quad + C_{11}\|u\|_{L^1(\Omega)}^{\frac{6\theta}{3\theta-1}}\|w_x^2\|_{L^1(\Omega)}^{\frac{3\theta-2}{3\theta-1}} + C_{11}\|u\|_{L^1(\Omega)}^2\|w_x^2\|_{L^1(\Omega)} + C_{11}\|w_x^2\|_{L^1(\Omega)}^2
\end{aligned} \tag{3.17}$$

for all $t\in(0,T)$. Here C_{11} is a constant that depends on Ω. In view of (2.19), (3.1), (3.13) and (3.17), this immediately leads to (3.10), upon an ODE comparison principle. □

With the estimate (3.10) at hand, we now turn to establish uniform-in-time boundedness of u and w_x.

Proof of Theorem 1.1. In view of Lemma 2.2, it is routine to check that for any fixed $t_0\in(0,T)$, $\sup_{0\le t\le t_0}\|u(\cdot,t)\|_{L^\infty(\Omega)}$, $\sup_{0\le t\le t_0}\|w_x(\cdot,t)\|_{L^q(\Omega)}$ are bounded. It is sufficient to justify it for $t\in(t_0,T)$.

Employing the Neumann heat semigroup $\{e^{\Delta t}\}_{t\ge 0}$, we represent w_x according to

$$w_x(t) = \left(e^{t\Delta}w_0\right)_x - \int_0^t \left(e^{(t-s)\Delta}w_x^2\right)_x + \int_0^t \left(e^{(t-s)\Delta}u\right)_x, \quad t\in(t_0,T),$$

recalling (2.2), (2.3) and (3.10), we obtain $C_{12}(\Omega)>0$ such that

$$\begin{aligned}
\|w_x\|_{L^\infty(\Omega)} &\le C_{12}[1+t^{-\frac{1}{2q}}]e^{-\lambda_1 t}\|w_0'\|_{L^q(\Omega)} + C_{12}\int_0^t [1+(t-s)^{-\frac{3}{4}}]e^{-\lambda_1(t-s)} \sup_{t\in(0,T)}\|w_x\|_{L^4(\Omega)}^2 \\
&\quad + C_{12}\int_0^t [1+(t-s)^{-\frac{3}{4}}]e^{-\lambda_1(t-s)} \sup_{t\in(0,T)}\|u\|_{L^2(\Omega)} \\
&\le C_{13}, \quad t\in(t_0,T),
\end{aligned} \tag{3.18}$$

where $C_{13}=C_{12}[1+t_0^{-\frac{1}{2q}}]\|w_0'\|_{L^q(\Omega)} + 2C_{12}M_3^{\frac{1}{2}}\int_0^t[1+(t-s)^{-\frac{3}{4}}]e^{-\lambda_1(t-s)}$ is a bounded constant because $\int_0^t[1+(t-s)^{-\frac{3}{4}}]e^{-\lambda_1(t-s)}$ is finite. In consequence, (3.18) and the definition of w hence establish (1.6). Using (2.21) and Young's inequality along with (3.18), we obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u^k + \int_\Omega u^k &\le -\frac{k-1}{k}\int_\Omega \left|\left(u^{\frac{k}{2}}\right)_x\right|^2 - \frac{k(k-1)}{4}\int_\Omega u^{k-2}u_x^2 \\
&\quad - k(k-1)\int_\Omega u^{k-1}u_x w_x + M_1 \\
&\le -\frac{k-1}{k}\int_\Omega \left|\left(u^{\frac{k}{2}}\right)_x\right|^2 + k(k-1)C_{13}^2\int_\Omega u^k + M_1,
\end{aligned} \tag{3.19}$$

for all $t \in (t_0, T)$ and $k > 2$. Following procedures analogous to (2.23) we can obtain $C_{14}(k, \alpha, \beta, \gamma, \mu, \Omega) > 0$ such that

$$k(k-1)C_{13}^2 \int_\Omega u^k \leq \frac{(k-1)}{k} \int_\Omega \left| \left(u^{\frac{k}{2}}\right)_x \right|^2 + C_{14} \|u\|_{L^1(\Omega)}^k, \quad t \in (t_0, T). \tag{3.20}$$

A combination of the above estimates (3.19) and (3.20) yields that

$$\frac{d}{dt} \int_\Omega u^k + \int_\Omega u^k \leq M_1 + C_{14} \|u\|_{L^1(\Omega)}^k, \quad t \in (t_0, T).$$

It holds true that for any $2 < k < \infty$, there exists $C_{15}(k, \alpha, \beta, \gamma, \mu, \Omega) > 0$ such that

$$\int_\Omega u^k \leq C_{15}, \quad t \in (t_0, T), \tag{3.21}$$

here we use the ordinary differential equations comparison argument together with (2.19). By the variation-of-constants formula, u can be represented as

$$\begin{aligned} u = {} & e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} (u w_x)_x + \int_0^t e^{(t-s)\Delta} \left(\frac{\gamma}{|\Omega|} \int_\Omega u^\alpha \right) \\ & + \gamma \int_0^t e^{(t-s)\Delta} \left(u^\alpha - \frac{1}{|\Omega|} \int_\Omega u^\alpha \right) - \mu \int_0^t e^{(t-s)\Delta} \left(u^\alpha \int_\Omega u^\beta \right), \quad t \in (t_0, T). \end{aligned}$$

On dropping a nonpositive term, we have

$$\begin{aligned} u \leq {} & e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} (u w_x)_x + \int_0^t e^{(t-s)\Delta} \left(\frac{\gamma}{|\Omega|} \int_\Omega u^\alpha \right) \\ & + \gamma \int_0^t e^{(t-s)\Delta} \left(u^\alpha - \frac{1}{|\Omega|} \int_\Omega u^\alpha \right), \quad t \in (t_0, T). \end{aligned}$$

With this representation and standard smoothing estimates (2.1) and (2.4), we arrive at

$$\begin{aligned} \|u\|_{L^\infty(\Omega)} \leq {} & \|u_0\|_{L^\infty(\Omega)} + C_{16} \int_0^t [1 + (t-s)^{-\frac{3}{4}}] e^{-\lambda_1 (t-s)} \sup_{t \in (0,T)} \|u\|_{L^2(\Omega)} \|w_x\|_{L^\infty(\Omega)} \\ & + \left[\frac{\gamma}{|\Omega|} + 2C_{16} \gamma \int_0^t [1 + (t-s)^{-\frac{1}{2}}] e^{-\lambda_1 (t-s)} \right] \left(\sup_{t \in (0,T)} \|u\|_{L^\alpha(\Omega)} \right)^\alpha, \quad t \in (t_0, T) \end{aligned}$$

with $C_{16}(\Omega) > 0$, whence observing (3.18) and (3.21), we therefore obtain the desired global boundedness of u. Consequently, Theorem 1.1 is completed. □

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