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Coexisting steady-state solutions of a class of reaction-diffusion systems with different boundary conditions

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Abstract. In this work, we investigate a class of reaction-diffusion system in which both species are influenced by self-diffusion. By introducing two particular functions, we provide a complete characterization of the parameter ranges such that coexisting steady-state solutions of the system do not exist under three boundary conditions. Then based on the maximum principle, a sufficient condition for the existence of constant coexisting solutions of the system under Neumann boundary conditions was derived.

Keywords: reaction-diffusion system, steady-state, existence, boundary condition.

2020 Mathematics Subject Classification: 35B50, 35J57.

1 Introduction

In [14], Shigesada, Kawasaki and Teramoto introduced the following system with cross-diffusions and self-diffusions, when they took a nonlinear dispersive force and an environmental potential function into consideration,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta[(d_1 + a_{11}u + a_{12}v)u] + u(1 - u - a_1 v), & x \in \Omega, t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta[(d_2 + a_{21}u + a_{22}v)v] + v(1 - a_2 u - v), & x \in \Omega, t > 0, \\ \alpha_1 u + \beta_1 \dfrac{\partial u}{\partial \nu} = \alpha_2 v + \beta_2 \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) \geq 0,\ v(x,0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary, and satisfies the interior ball condition at any $x \in \partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$. u and v are the densities of two competing species, α_i, β_i and a_{ij} $(i,j = 1,2)$ are nonnegative constants, $\alpha_i + \beta_i > 0$ $(i = 1,2)$, a_i and d_i $(i = 1,2)$ are all positive constants, a_{11} and a_{22} stand for the

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self-diffusion pressures, while a_{12} and a_{21} are the cross-diffusion pressures, a_1, a_2 describe the inter-specific competitions, and d_1, d_2 are their diffusion rates [19].

In order to analyze and describe the above reaction-diffusion model, it is necessary to clarify the boundary conditions of the region. When $\alpha_i = 0$ $(i = 1, 2)$, we have $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0$, which is called Neumann boundary condition. At this point, individuals who reach the boundary will be reflected back into the region without leaving, meaning that the species is in an isolated environment. When $\beta_i = 0$ $(i = 1, 2)$, we have $u = v = 0$. These indicate that individuals who encounter the boundary cross it immediately and thereby maintain the density on the boundary at zero [1]. This means that the boundary $\partial\Omega$ can effectively absorb all individuals encountering it. Therefore, this boundary is called absorbed, also known as Dirichlet boundary condition. When $\alpha_i, \beta_i > 0$ $(i = 1, 2)$, it is called Robin boundary condition in the mathematical literature.

Since the model, which was abbreviated as SKT model, was proposed, numerous experts and scholars have conducted extensive and in-depth research on it. These results mainly include the existence, boundedness and global convergence of classical solutions, the local and global existence of weak solutions, the existence, nonexistence and stability of steady-state solutions, the existence of traveling wave solutions, and so on.

In fact, the study of standard SKT models is quite difficult. More experts and scholars are turning to the study of certain special forms of SKT model; see [2, 4–8, 12, 15, 16]. Among them, when the spatial dimension is 1 and $d_1 = d_2$, [6] proved the global existence of smooth solutions. Under the same conditions, [15] obtained the uniform boundedness and convergence of smooth solutions. In 2015, Lou and Winkler [12] used the comparison principle and Sobolev regularity theory to obtain the global existence and uniform boundedness of smooth solutions on bounded convex domain when the spatial dimension is less than 3 and $d_1 = d_2$. Taking self-diffusion into consideration, [5] proved the global existence of the unique smooth solution in any spatial dimension; and [8] obtained the global existence of the unique classical solution by using Sobolev embedding theory under the condition $d_1 = d_2$.

For research on steady-state solutions, one can refer to [9–11]. In 1996, Lou and Ni [9] used the maximum principle and Lyapunov functional theory to prove that in the weak competition case, if self-diffusion and/or cross-diffusion are relatively weaker than diffusion, then there is still no nonconstant steady-state solution. At the same time, they proved that in the weak competition case, with one of the cross-diffusion pressures arbitrarily given but fixed, it is expect to find non-constant steady-state solutions if the other cross-diffusion pressure is large enough [9]. Without considering the influence of self-diffusion, [10] obtained a sufficient condition such that the SKT model has no nonconstant steady-state solutions. When $a_{21} = a_{22} = 0$, Lou et al. [11] provided the parameter ranges such that the system has no nonconstant positive solutions for $a_{11} = 0$ and $a_{11} \neq 0$, respectively.

It is obvious to see that the above studies were conducted under Neumann boundary conditions, which is also the most extensively studied scenario. In addition, some scholars have also studied the SKT model under Dirichlet boundary conditions, which can be found in [3, 13, 17, 20] and references therein. For example, the sufficient conditions for the existence of positive steady-state solutions of the system are given in [13] using the fixed point theory in the case of fixed or sufficiently large cross-diffusion coefficients, respectively. The existence of steady-state solutions for a one-dimensional system was studied using the singular perturbation method [17].

Based on the previous work [19], where one species has cross-diffusion and another species has self-diffusion, now we consider the following steady-state model, which indicates that there are self-diffusions in both competing species and there is no cross-diffusion in either,

$$\begin{cases} \Delta[(d_1+a_{11}u)u]+u(1-u-a_1v)=0, & x\in\Omega,\\ \Delta[(d_2+a_{22}v)v]+v(1-a_2u-v)=0, & x\in\Omega,\\ \alpha_1u+\beta_1\dfrac{\partial u}{\partial\nu}=\alpha_2v+\beta_2\dfrac{\partial v}{\partial\nu}=0, & x\in\partial\Omega. \end{cases} \tag{1.2}$$

We aim to obtain sufficient conditions such that the system (1.2) has no coexisting solutions under three different boundary conditions, and establish the parameter ranges for the existence of constant coexisting solutions under Neumann boundary conditions. Considering that u and v represent species densities, we focus on the nonnegative classical solution (u,v) of (1.2), which means that $(u,v)\in(C^1(\overline{\Omega})\cap C^2(\Omega))^2$, $u,v\geq 0$ in $\overline{\Omega}$, and satisfies (1.2) in the pointwise sense [19].

The remainder of this work is organized as follows. Section 2 gives some basic preliminaries which imply the strict positivity of the nontrivial solutions of system (1.2). In Section 3, based on the boundedness of solutions, we obtain two different parameter ranges for nonexistence of coexisting solutions under three boundary conditions. In Section 4, we establish the sufficient conditions for the existence of constant coexisting solutions under Neumann boundary conditions.

2 Preliminaries

First of all, we can obtain the positivity of the classical solutions of (1.2) in Ω, which is crucial in subsequent section.

Proposition 2.1. *Let (u,v) be a nonnegative classical solution of (1.2). Then if $u\not\equiv 0$, we have $u>0$ in Ω, and if $v\not\equiv 0$, we have $v>0$ in Ω.*

Proof. We only prove $u>0$ in Ω whenever $u\not\equiv 0$, since the positivity of v in Ω can be proved in a similar way. Otherwise, there is $x_0\in\Omega$ such that $u(x_0)=\min_{x\in\overline{\Omega}}u(x)=0$.

It follows from the first equation of (1.2) that

$$(d_1+2a_{11}u)\Delta u+2a_{11}|\nabla u|^2+u(1-u-a_1v)=0.$$

Let

$$Lu=-(d_1+2a_{11}u)\Delta u-2a_{11}|\nabla u|^2+cu \quad \text{with } c=u+a_1v.$$

Then

$$c\geq 0 \quad \text{and} \quad Lu=u\geq 0 \quad \text{in } \Omega.$$

So, an application of the strong maximum principle shows that u is a constant in Ω, and thus $u=0$, a contradiction to $u\not\equiv 0$. This completes the proof. □

Remark 2.2. In the case of Neumann or Robin boundary conditions, we can get further that $u,v>0$ in $\overline{\Omega}$ by Hopf's boundary lemma. In fact, for example, considering the case of Robin boundary conditions, suppose that there is $x_0\in\overline{\Omega}$ such that $u(x_0)=\min_{x\in\overline{\Omega}}u(x)=0$. If $x_0\in\Omega$, we can directly derive a contradiction by Proposition 2.1. If $x_0\in\partial\Omega$, then $u(x_0)<u(x)$ for all $x\in\Omega$. Since $Lu\geq 0$ in Ω and Ω satisfies the interior ball condition at $x_0\in\partial\Omega$, it follows from Hopf's boundary lemma that $\frac{\partial u}{\partial\nu}(x_0)<0$. Hence

$$\alpha_1u(x_0)+\beta_1\frac{\partial u}{\partial\nu}(x_0)<0,$$

a condiction. Therefore, $u>0$ in $\overline{\Omega}$.

3 Nonexistence of coexisting steady-state solutions

In this section, we will discuss the nonexistence of coexisting solutions for system (1.2) under three different boundary conditions.

3.1 Neumann boundary condition

In this subsection, assume that $\alpha_1 = \alpha_2 = 0$, that is, we consider the following system,

$$\begin{cases} \Delta[(d_1 + a_{11}u)u] + u(1 - u - a_1 v) = 0, & x \in \Omega, \\ \Delta[(d_2 + a_{22}v)v] + v(1 - a_2 u - v) = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

Firstly, we give the following lemma, which indicates that u and v are both bounded.

Lemma 3.1. *Suppose that (u, v) is a nonnegative classical solution of (3.1). If $u \not\equiv 0$, then $u \le 1$ in $\overline{\Omega}$. Similarly, if $v \not\equiv 0$, then $v \le 1$ in $\overline{\Omega}$.*

Proof. We only prove the boundedness of u, while the boundedness of v can be similarly obtained, leaving it for interested readers. Rewrite the first equation of the system (3.1) as follows

$$\begin{cases} (d_1 + 2a_{11}u)\Delta u + 2a_{11}|\nabla u|^2 + u(1 - u - a_1 v) = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.2}$$

Let $f(x) = u(x)(1 - u(x) - a_1 v(x))$, $x \in \Omega$. Proposition 2.2 in [9] implies that there exist $x_1, x_2 \in \overline{\Omega}$ such that

$$u(x_1) = \max_{x \in \overline{\Omega}} u(x) \triangleq M, \qquad u(x_2) = \min_{x \in \overline{\Omega}} u(x) \triangleq m,$$

and

$$f(x_1) \ge 0, \qquad f(x_2) \le 0,$$

that is

$$M(1 - M - a_1 v(x_1)) \ge 0, \qquad m(1 - m - a_1 v(x_2)) \le 0.$$

We thus have

$$1 - a_1 v(x_2) \le m \le u(x) \le M \le 1 - a_1 v(x_1) \quad \text{for all } x \in \overline{\Omega}.$$

Combining this with the nonnegativity of v, we obtain $u \le 1$ in $\overline{\Omega}$. □

Next, we provide a sufficient condition for the nonexistence of coexisting solutions under Neumann boundary conditions.

Theorem 3.2. *Suppose that (u, v) is a nonnegative classical solution of (3.1). If*

(i) $$a_1 > 1 > a_2, \qquad d_1 > d_2 \quad \textit{and} \quad d_1 \ge d_2 + 2a_{22}$$

or

(ii) $$a_1 < 1 < a_2, \qquad d_1 < d_2 \quad \textit{and} \quad d_2 \ge d_1 + 2a_{11},$$

then system (3.1) *has no coexisting solutions, that is, at least one species is extinct.*

Proof. (i) We argue by contradiction. Suppose that $u \not\equiv 0$ and $v \not\equiv 0$. It follows from Proposition 2.1 that $u, v > 0$ in Ω, which allows us to rewrite system (3.1) as follows,

$$\begin{cases} \dfrac{\Delta[(d_1 + a_{11}u)u]}{u} = -1 + u + a_1 v, & x \in \Omega, \\ \dfrac{\Delta[(d_2 + a_{22}v)v]}{v} = -1 + a_2 u + v, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.3}$$

Let

$$w_1 = d_1 + a_{11}u \quad \text{and} \quad w_2 = d_2 + a_{22}v. \tag{3.4}$$

Since $a_1 > 1 > a_2$, we can obtain that $\frac{\Delta(w_1 u)}{u} > \frac{\Delta(w_2 v)}{v}$, that is,

$$\frac{u\Delta w_1 + 2\nabla u \cdot \nabla w_1 + w_1 \Delta u}{u} > \frac{v\Delta w_2 + 2\nabla v \cdot \nabla w_2 + w_2 \Delta v}{v} \quad \text{in } \Omega. \tag{3.5}$$

We procedure the following calculation based on the inequality above,

$$\begin{aligned}
&\operatorname{div}\Big[(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v)\frac{u}{v}\Big] \\
&= \big(uv\Delta w_1 + v\nabla u \cdot \nabla w_1 + u\nabla v \cdot \nabla w_1 - uv\Delta w_2 - v\nabla u \cdot \nabla w_2 - u\nabla v \cdot \nabla w_2 \\
&\quad + vw_1\Delta u + v\nabla u \cdot \nabla w_1 + w_1\nabla u \cdot \nabla v - uw_2\Delta v - u\nabla v \cdot \nabla w_2 - w_2\nabla u \cdot \nabla v\big)\frac{u}{v} \\
&\quad + \big(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v\big) \cdot \Big(\frac{1}{v}\nabla u - \frac{u}{v^2}\nabla v\Big) \\
&> \big(u\nabla v \cdot \nabla w_1 - v\nabla u \cdot \nabla w_2 + w_1\nabla u \cdot \nabla v - w_2\nabla u \cdot \nabla v\big)\frac{u}{v} \\
&\quad + \big(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v\big) \cdot \Big(\frac{1}{v}\nabla u - \frac{u}{v^2}\nabla v\Big) \\
&= |\nabla u|^2(a_{11}u + w_1) + |\nabla v|^2(a_{22}v + w_2)\frac{u^2}{v^2} \\
&\quad + \nabla u \cdot \nabla v\Big[(a_{11}u + w_1)\frac{u}{v} + (-a_{22}v - w_2)\frac{u}{v} + (-a_{11}u - w_1 - a_{22}v - w_2)\frac{u}{v}\Big] \\
&= |\nabla u|^2(d_1 + 2a_{11}u) + |\nabla v|^2(d_2 + 2a_{22}v)\frac{u^2}{v^2} - 2(d_2 + 2a_{22}v)\frac{u}{v}\nabla u \cdot \nabla v \\
&= |\nabla u|^2(d_1 + 2a_{11}u - d_2 - 2a_{22}v) + \Big(\sqrt{d_2 + 2a_{22}v}\nabla u - \frac{u}{v}\sqrt{d_2 + 2a_{22}v}\nabla v\Big)^2.
\end{aligned}$$

As $d_1 > d_2$, $d_1 \geq d_2 + 2a_{22}$ and $v \leq 1$, we have

$$d_1 + 2a_{11}u - d_2 - 2a_{22}v \geq d_1 + 2a_{11}u - d_2 - 2a_{22} \geq 0 \quad \text{in } \Omega.$$

Thus,

$$\operatorname{div}\Big[(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v)\frac{u}{v}\Big] > 0. \tag{3.6}$$

On the other hand, we can see from Neumann boundary conditions that

$$\begin{aligned}
&\int_\Omega \mathrm{div}\left[(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v)\frac{u}{v}\right]\mathrm{d}x \\
&= \int_{\partial\Omega}\left(uv\frac{\partial w_1}{\partial \nu} - uv\frac{\partial w_2}{\partial \nu} + vw_1\frac{\partial u}{\partial \nu} - uw_2\frac{\partial v}{\partial \nu}\right)\frac{u}{v}\mathrm{d}S \\
&= \int_{\partial\Omega}\left(a_{11}uv\frac{\partial u}{\partial \nu} - a_{22}uv\frac{\partial v}{\partial \nu} + vw_1\frac{\partial u}{\partial \nu} - uw_2\frac{\partial v}{\partial \nu}\right)\frac{u}{v}\mathrm{d}S \\
&= 0,
\end{aligned}$$

which is contradict to (3.6). Therefore, $u \equiv 0$ or $v \equiv 0$.

(ii) In this case, due to $a_1 < 1 < a_2$, we can see that $\frac{\Delta(w_1 u)}{u} < \frac{\Delta(w_2 v)}{v}$, which can be expanded into the following form

$$\frac{u\Delta w_1 + 2\nabla u\cdot\nabla w_1 + w_1\Delta u}{u} < \frac{v\Delta w_2 + 2\nabla v\cdot\nabla w_2 + w_2\Delta v}{v} \quad \text{in } \Omega, \tag{3.7}$$

where w_1, w_2 are defined in (3.4). Similar to the calculations of (i), we consider the process as follows

$$\begin{aligned}
&\mathrm{div}\Big[(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v)\frac{v}{u}\Big] \\
&< \big(u\nabla v\cdot\nabla w_1 - v\nabla u\cdot\nabla w_2 + w_1\nabla u\cdot\nabla v - w_2\nabla u\cdot\nabla v\big)\frac{v}{u} \\
&\quad + \big(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v\big)\cdot\Big(-\frac{v}{u^2}\nabla u + \frac{1}{u}\nabla v\Big) \\
&= |\nabla u|^2\big(-a_{11}u - w_1\big)\frac{v^2}{u^2} + |\nabla v|^2\big(-a_{22}v - w_2\big) \\
&\quad + \nabla u\cdot\nabla v\Big[(a_{11}u + w_1)\frac{v}{u} + (-a_{22}v - w_2)\frac{v}{u} + (a_{11}u + w_1 + a_{22}v + w_2)\frac{v}{u}\Big] \\
&= -|\nabla u|^2\big(d_1 + 2a_{11}u\big)\frac{v^2}{u^2} - |\nabla v|^2\big(d_2 + 2a_{22}v\big) + 2(d_1 + 2a_{11}u)\frac{v}{u}\nabla u\cdot\nabla v \\
&= |\nabla v|^2\big(-d_2 - 2a_{22}v + d_1 + 2a_{11}u\big) - \Big(\frac{v}{u}\sqrt{d_1 + 2a_{11}u}\nabla u - \sqrt{d_1 + 2a_{11}u}\nabla v\Big)^2.
\end{aligned}$$

Since $d_1 < d_2$, $d_2 \geq d_1 + 2a_{11}$ and $u \leq 1$, we see

$$-d_2 - 2a_{22}v + d_1 + 2a_{11}u \leq -d_2 - 2a_{22}v + d_1 + 2a_{11} \leq 0 \quad \text{in } \Omega.$$

It is shown that

$$\mathrm{div}\Big[(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v)\frac{v}{u}\Big] < 0. \tag{3.8}$$

Now, based on the boundary conditions, we can still obtain the following result

$$\int_\Omega \mathrm{div}\left[(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v)\frac{v}{u}\right]\mathrm{d}x = 0,$$

which leads to a contradiction. This completes the proof. □

Remark 3.3. Based on Proposition 2.1, we constructed two subtle auxiliary functions and proved the conclusion by contradiction. From the conclusion, it can be seen that the theorem provides a specific characterization that when the inter-specific competition rate and diffusion rate of some species are relatively high, it will lead to some species extinction.

Furthermore, under the conditions (i) and (ii) of Theorem 3.2, we selected two sets of values and conducted numerical simulations for the following system respectively,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta[(d_1 + a_{11}u)u] + u(1 - u - a_1 v), & x \in (0,1), t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta[(d_2 + a_{22}v)v] + v(1 - a_2 u - v), & x \in (0,1), t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x = 0,1, t > 0, \\ u(x,0) = u_0(x) \geq 0,\ v(x,0) = v_0(x) \geq 0, & x \in (0,1), \end{cases} \tag{3.9}$$

as shown in Figure 3.1 and 3.2. The simulation results show that, species u becomes extinct under condition (i) while species v becomes extinct under condition (ii). Therefore, it is nature to ask whether species u necessarily extinct under condition (i) and species v necessarily extinct under condition (ii). Our results have not provided an answer, and this is our future work.

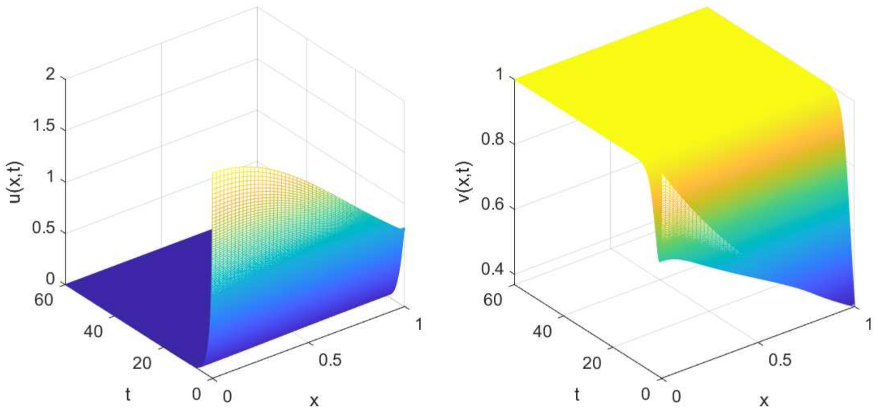

Figure 3.1: Numerical simulations for system (3.9) with $u_0(x) = 2e^{-x}$, $v_0(x) = e^{-x}$, $d_1 = 0.7, d_2 = 0.2, a_{11} = 0.1, a_{22} = 0.2$, $a_1 = 1.8$ and $a_2 = 0.5$.

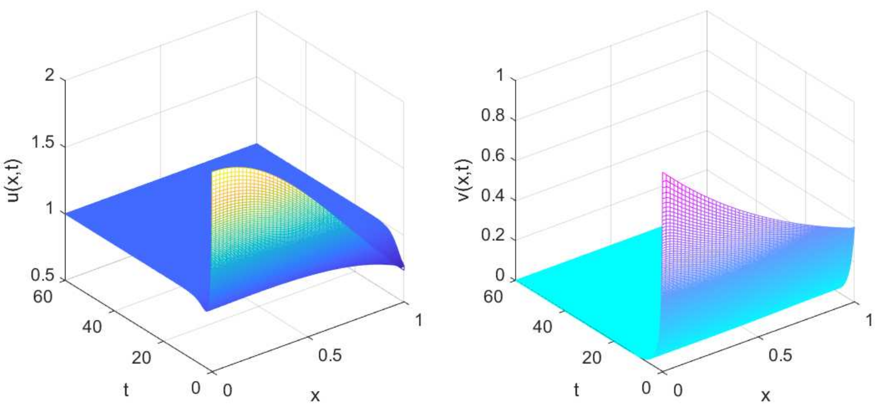

Figure 3.2: Numerical simulations for system (3.9) with $u_0(x) = 2e^{-x}$, $v_0(x) = e^{-x}$, $d_1 = 0.1, d_2 = 0.4, a_{11} = 0.1, a_{22} = 0.2$, $a_1 = 0.8$ and $a_2 = 1.5$.

3.2 Dirichlet boundary condition

In this subsection, we consider the case of $\beta_1 = \beta_2 = 0$, which leads to the following system,

$$\begin{cases} \Delta[(d_1 + a_{11}u)u] + u(1 - u - a_1 v) = 0, & x \in \Omega, \\ \Delta[(d_2 + a_{22}v)v] + v(1 - a_2 u - v) = 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{3.10}$$

Now, we can also deduce that u and v are both bounded.

Lemma 3.4. *Suppose that (u, v) is a nonnegative classical solution of (3.10). If $u \not\equiv 0$, then $u \le 1$ in $\overline{\Omega}$. Similarly, if $v \not\equiv 0$, then $v \le 1$ in $\overline{\Omega}$.*

Proof. We only need to prove the boundedness of v. Proposition 2.1 implies that $v > 0$ in Ω if $v \not\equiv 0$. The second equation of system (3.10) can be transformed into the following form,

$$\begin{cases} (d_2 + 2a_{22}v)\Delta v + 2a_{22}|\nabla v|^2 + v(1 - a_2 u - v) = 0, & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases} \tag{3.11}$$

Suppose on the contrary that there exists a point x_0 such that $v(x_0) = \max_{\overline{\Omega}} v(x) > 1$. Obviously, $x_0 \in \Omega$. So we have

$$\Delta v(x_0) \le 0, \ \nabla v(x_0) = 0 \text{ and } v(x_0)(1 - a_2 u(x_0) - v(x_0)) \ge 0.$$

Since $v > 0$ in Ω, we obtain that $1 - a_2 u(x_0) - v(x_0) \ge 0$, that is, $1 \ge a_2 u(x_0) + v(x_0)$, a contradiction. □

If we restrict attention to the Dirichlet boundary conditions, we can establish the parameter ranges such that the system (3.10) has no coexisting solutions.

Theorem 3.5. *Suppose that (u, v) is a nonnegative classical solution of (3.10). If*

(i) $$a_1 > 1 > a_2, \quad d_1 > d_2 \quad \text{and} \quad d_1 \ge d_2 + 2a_{22}$$

or

(ii) $$a_1 < 1 < a_2, \quad d_1 < d_2 \quad \text{and} \quad d_2 \ge d_1 + 2a_{11},$$

then system (3.10) has no coexisting solutions.

Proof. (i) According to the proof of Theorem 3.2-(i), the following inequality still holds

$$\operatorname{div}\Big[(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v)\frac{u}{v}\Big] > 0,$$

where $w_1 = d_1 + a_{11}u$ and $w_2 = d_2 + a_{22}v$.

Now, we consider the following integral

$$\begin{aligned} &\int_\Omega \operatorname{div}\left[\left(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v\right)\frac{u}{v}\right] \mathrm{d}x \\ &= \int_{\partial\Omega} \left(uv\frac{\partial w_1}{\partial \nu} - uv\frac{\partial w_2}{\partial \nu} + vw_1\frac{\partial u}{\partial \nu} - uw_2\frac{\partial v}{\partial \nu}\right)\frac{u}{v}\mathrm{d}S \\ &= \int_{\partial\Omega} \left(a_{11}u^2\frac{\partial u}{\partial \nu} - a_{22}u^2\frac{\partial v}{\partial \nu} + uw_1\frac{\partial u}{\partial \nu} - \left(d_2\frac{u^2}{v} + a_{22}u^2\right)\frac{\partial v}{\partial \nu}\right)\mathrm{d}S. \end{aligned}$$

It is easy to see that the function $\frac{u^2}{v}$ in the last term of the integrand does not make sense on $\partial\Omega$. So in such a case we cannot make calculations directly. Let

$$\Omega_\varepsilon = \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) > \varepsilon\} \quad \text{for any small } \varepsilon > 0.$$

Since $(u, v) \in (C^1(\overline{\Omega}) \cap C^2(\Omega))^2$, we observe that

$$(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v)\frac{u}{v} \in C^1(\overline{\Omega_\varepsilon}).$$

Then divergence theorem implies that

$$\begin{aligned}
&\int_\Omega \mathrm{div}\left[(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v)\frac{u}{v}\right] \mathrm{d}x \\
&= \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \mathrm{div}\left[(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v)\frac{u}{v}\right] \mathrm{d}x \\
&= \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} \left(a_{11}u^2\frac{\partial u}{\partial \nu} - a_{22}u^2\frac{\partial v}{\partial \nu} + uw_1\frac{\partial u}{\partial \nu} - \left(d_2\frac{u^2}{v} + a_{22}u^2\right)\frac{\partial v}{\partial \nu}\right) \mathrm{d}S.
\end{aligned}$$

Let

$$\begin{aligned}
I_1(\varepsilon) &= \int_{\partial\Omega_\varepsilon} \left(a_{11}u^2\frac{\partial u}{\partial \nu} - a_{22}u^2\frac{\partial v}{\partial \nu} + uw_1\frac{\partial u}{\partial \nu} - a_{22}u^2\frac{\partial v}{\partial \nu}\right) \mathrm{d}S, \\
I_2(\varepsilon) &= \int_{\partial\Omega_\varepsilon} d_2\frac{u^2}{v}\frac{\partial v}{\partial \nu} \mathrm{d}S.
\end{aligned}$$

Obviously, $I_1(\varepsilon)$ approaches zero as $\varepsilon \to 0$ in terms of Dirichlet boundary conditions. In order to deal with the term $I_2(\varepsilon)$, we write

$$V = \left\{\varphi(x) \in C^1(\overline{\Omega}) \mid \varphi(x) > 0, x \in \Omega;\ \varphi|_{\partial\Omega} = 0;\ \frac{\partial\varphi}{\partial\nu}\Big|_{\partial\Omega} < 0\right\}.$$

By Hopf's Lemma, we have $\frac{\partial u(x_0)}{\partial \nu} < 0$ and $\frac{\partial v(x_0)}{\partial \nu} < 0$ for any $x_0 \in \partial\Omega$, and thus $u \in V$ and $v \in V$. We now define

$$g(x) := \begin{cases} \dfrac{u(x)}{v(x)}, & x \in \Omega, \\[2mm] \dfrac{\partial u(x)}{\partial \nu} \Big/ \dfrac{\partial v(x)}{\partial \nu}, & x \in \partial\Omega. \end{cases}$$

Then Lemma 2.4 in [18] shows that $g(x) \in C\big(\overline{\Omega}, (0, +\infty)\big)$. Therefore, we conclude that $I_2(\varepsilon)$ also approaches zero as $\varepsilon \to 0$. These considerations motivate that

$$\int_\Omega \mathrm{div}\left[(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v)\frac{u}{v}\right] \mathrm{d}x = 0 \tag{3.12}$$

thanks to Lebesgue dominated convergence theorem and Dirichlet boundary conditions, a contradiction.

(ii) On one hand, it is obvious to see from Theorem 3.2-(ii) that

$$\mathrm{div}\left[(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v)\frac{v}{u}\right] < 0, \tag{3.13}$$

due to $d_1 < d_2$, $d_2 \geq d_1 + 2a_{11}$ and Lemma 3.4.

On the other hand, we can deduce from the discussion on boundary integrals similar to (i) that

$$\int_\Omega \mathrm{div}\left[(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v)\frac{v}{u}\right] \mathrm{d}x = 0. \tag{3.14}$$

Therefore, a contradiction can also be obtained, and the conclusion is valid. □

Remark 3.6. It is easy to see that the conditions of this theorem are the same as those of Theorem 3.2. In fact, since the equations in system (3.1) and (3.10) are the same, the conditions of Theorem 3.2 and 3.5 are given to ensure that the constructed auxiliary function is strictly positive or strictly negative within the region, which is independent of boundary conditions.

3.3 Robin boundary condition

In this subsection, we consider the following system

$$\begin{cases} \Delta[(d_1 + a_{11}u)u] + u(1 - u - a_1 v) = 0, & x \in \Omega, \\ \Delta[(d_2 + a_{22}v)v] + v(1 - a_2 u - v) = 0, & x \in \Omega, \\ \alpha_1 u + \beta_1 \dfrac{\partial u}{\partial \nu} = \alpha_2 v + \beta_2 \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{3.15}$$

where $\alpha_i > 0$, $\beta_i > 0$, $i = 1, 2$. Under this boundary conditions, the boundedness of u and v is also not difficult to obtain.

Lemma 3.7. *Suppose that (u, v) is a nonnegative classical solution of (3.15). If $u \not\equiv 0$, then $u \leq 1$ in $\overline{\Omega}$. Similarly, if $v \not\equiv 0$, then $v \leq 1$ in $\overline{\Omega}$.*

Proof. We only prove the previous statement. First, we can obtain that

$$\begin{cases} (d_1 + 2a_{11}u)\Delta u + 2a_{11}|\nabla u|^2 + u(1 - u - a_1 v) = 0, & x \in \Omega, \\ \alpha_1 u + \beta_1 \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.16}$$

Suppose that there exists a point $x_0 \in \overline{\Omega}$, such that $u(x_0) = \max_{x \in \overline{\Omega}} u(x) > 1$.

(i) If $x_0 \in \Omega$, we have $\Delta u(x_0) \leq 0$, $\nabla v(x_0) = 0$ and $u(x_0)(1 - u(x_0) - a_1 v(x_0)) \geq 0$. Since $u > 0$ in Ω, we obtain that $1 - u(x_0) - a_1 v(x_0) \geq 0$, that is, $1 \geq u(x_0) + a_1 v(x_0)$, a contradiction.

(ii) If $x_0 \in \partial\Omega$, then $u(x_0) > u(x)$ for all $x \in \Omega$. Thus we can see that $\frac{\partial u}{\partial \nu}(x_0) \geq 0$. Since $u > 0$ in $\overline{\Omega}$ by Remark 2.2, we have $\alpha_1 u(x_0) + \beta_1 \frac{\partial u}{\partial \nu}(x_0) > 0$, a contradiction. □

Theorem 3.8. *Suppose that (u, v) is a nonnegative classical solution of (3.15). If*

(i) $\quad a_1 > 1 > a_2, \quad d_1 > d_2, \quad d_1 \geq d_2 + 2a_{22}, \quad \alpha_1 \geq \beta_1$ *and* $\alpha_2 \leq \beta_2$

or

(ii) $\quad a_1 < 1 < a_2, \quad d_1 < d_2, \quad d_2 \geq d_1 + 2a_{11}, \quad \alpha_1 \leq \beta_1$ *and* $\alpha_2 \geq \beta_2,$

then system (3.15) has no coexisting solutions.

Proof. We only prove the first case. Now, we can still obtain

$$\operatorname{div}\Big[(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v)\frac{u}{v}\Big] > 0,$$

where $w_1 = d_1 + a_{11}u$ and $w_2 = d_2 + a_{22}v$.

Notice that $\frac{\partial u}{\partial \nu} = -\frac{\alpha_1}{\beta_1} u$, $\frac{\partial v}{\partial \nu} = -\frac{\alpha_2}{\beta_2} v$ in $\partial\Omega$. It consequently seems advisable to consider the following integral

$$\begin{aligned}
&\int_\Omega \operatorname{div}\left[(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v)\frac{u}{v}\right] \mathrm{d}x \\
&= \int_{\partial\Omega}\left(a_{11}uv\frac{\partial u}{\partial \nu} - a_{22}uv\frac{\partial v}{\partial \nu} + vw_1\frac{\partial u}{\partial \nu} - uw_2\frac{\partial v}{\partial \nu}\right)\frac{u}{v}\mathrm{d}S \\
&= \int_{\partial\Omega}\left(-a_{11}\frac{\alpha_1}{\beta_1}u^3 + a_{22}\frac{\alpha_2}{\beta_2}u^2 v - \frac{\alpha_1}{\beta_1}u^2 w_1 + \frac{\alpha_2}{\beta_2}u^2 w_2\right)\mathrm{d}S \\
&= \int_{\partial\Omega} u^2\left(-2a_{11}\frac{\alpha_1}{\beta_1}u + 2a_{22}\frac{\alpha_2}{\beta_2}v - d_1\frac{\alpha_1}{\beta_1} + d_2\frac{\alpha_2}{\beta_2}\right)\mathrm{d}S \\
&\le \int_{\partial\Omega} u^2\left(-2a_{11}\frac{\alpha_1}{\beta_1}u + 2a_{22}\frac{\alpha_2}{\beta_2} - d_1\frac{\alpha_1}{\beta_1} + d_2\frac{\alpha_2}{\beta_2}\right)\mathrm{d}S \\
&\le \int_{\partial\Omega} u^2\left(-2a_{11}\frac{\alpha_1}{\beta_1}u + 2a_{22} - d_1 + d_2\right)\mathrm{d}S \\
&\le 0,
\end{aligned}$$

due to Lemma 3.7, $\alpha_1 \ge \beta_1$, $\alpha_2 \le \beta_2$ and $d_1 \ge d_2 + 2a_{22}$, a contradiction. □

4 Existence of coexisting solutions

In this section, we will investigate the existence of coexisting solutions for system (3.1).

Theorem 4.1. *Let $a_1 < 1$ and $a_2 < 1$. Suppose that (u, v) is a nonnegative classical solution of (3.1). If $u \not\equiv 0$ and $v \not\equiv 0$, then*

$$u \equiv \frac{1-a_1}{1-a_1a_2} \quad \textit{and} \quad v \equiv \frac{1-a_2}{1-a_1a_2}. \tag{4.1}$$

Proof. Recall from Lemma 3.1 that there exist $x_1, x_2 \in \overline{\Omega}$ such that

$$u(x_1) = \max_{x\in\overline{\Omega}} u(x) \triangleq M, \qquad u(x_2) = \min_{x\in\overline{\Omega}} u(x) \triangleq m,$$

and

$$1 - a_1 v(x_2) \le m \le u(x) \le M \le 1 - a_1 v(x_1) \quad \text{for all } x \in \overline{\Omega}. \tag{4.2}$$

Similarly, there exist $x_3, x_4 \in \overline{\Omega}$ such that

$$v(x_3) = \max_{x\in\overline{\Omega}} v(x) \triangleq M', \qquad v(x_4) = \min_{x\in\overline{\Omega}} v(x) \triangleq m',$$

and

$$1 - a_2 u(x_4) \le m' \le v(x) \le M' \le 1 - a_2 u(x_3) \quad \text{for all } x \in \overline{\Omega}. \tag{4.3}$$

We first prove $u \equiv \frac{1-a_1}{1-a_1a_2}$. Combining (4.2) with (4.3), we obtain

$$m \ge 1 - a_1 v(x_2) \ge 1 - a_1(1 - a_2 u(x_3)) \ge 1 - a_1 + a_1 a_2 m, \tag{4.4}$$

and

$$M \le 1 - a_1 v(x_1) \le 1 - a_1(1 - a_2 u(x_4)) \le 1 - a_1 + a_1 a_2 M. \tag{4.5}$$

Notice that since $a_1 < 1$ and $a_2 < 1$, we have

$$\frac{1-a_1}{1-a_1a_2} \le m \le u(x) \le M \le \frac{1-a_1}{1-a_1a_2}. \tag{4.6}$$

Hence

$$u \equiv \frac{1-a_1}{1-a_1a_2}.$$

Similarly, it can be proved that

$$\frac{1-a_2}{1-a_1a_2} \le m' \le v(x) \le M' \le \frac{1-a_2}{1-a_1a_2}. \tag{4.7}$$

Consequently,

$$v \equiv \frac{1-a_2}{1-a_1a_2}.$$

This completes the proof. □

Remark 4.2. Here we only give sufficient conditions for the existence of nonzero constant solutions for the system under Neumann boundary conditions. In fact, for systems under the other two boundary conditions, it is easy to see that if there is a constant solution, it can only be the zero solution.

In addition, under the conditions of Theorem 4.1, we also selected a set of values and conducted numerical simulations for system (3.9), as shown in Figure 4.1. The simulation results show that species u and v tend to constant coexisting solutions $\left(\frac{1-a_1}{1-a_1a_2}, \frac{1-a_2}{1-a_1a_2}\right) \approx (0.33, 0.83)$.

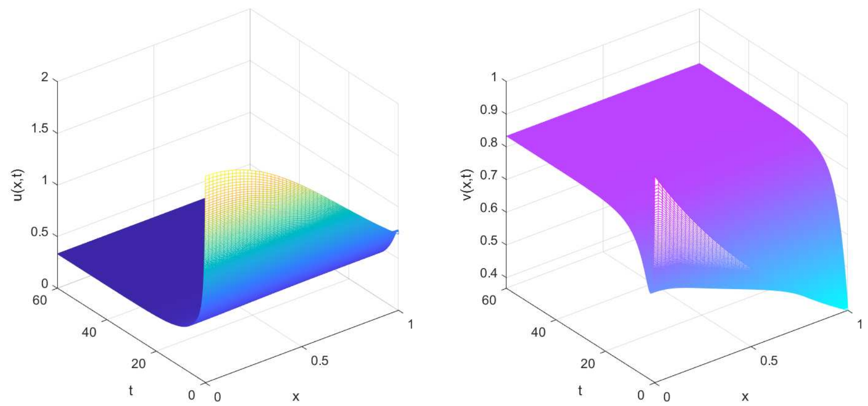

Figure 4.1: Numerical simulations for system (3.9) with $u_0(x) = 2e^{-x}$, $v_0(x) = e^{-x}$, $d_1 = 0.1, d_2 = 0.2, a_{11} = 0.1, a_{22} = 0.2$, $a_1 = 0.8$ and $a_2 = 0.5$.

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