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Existence and convergence of sign-changing solutions for Kirchhoff-type p-Laplacian problems involving critical exponent in $\mathbb{R}^N$

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Abstract. We investigate the existence of sign-changing solutions for Kirchhoff-type problems with p-Laplacian involving critical exponent:

$$-\left(1+b|\nabla v|_p^p\right)\Delta_p v + a(x)|v|^{p-2}v = |v|^{p^*-2}v + \lambda f(v), \quad x \in \mathbb{R}^N,$$

where b and λ are positive parameters, $\Delta_p v = \operatorname{div}(|\nabla v|^{p-2}\nabla v)$, $p^* = \frac{Np}{N-p}$, $1 < p < N$, and $|\cdot|_p$ is the Lebesgue L^p-norm. For sufficiently large λ, employing minimization techniques, quantitative deformation lemma and the constrained variational method, we demonstrate the existence of a least-energy sign-changing solution, whose energy is greater than twice that of the ground state solution. Additionally, we show the convergence behavior of the solution as the parameter $b \searrow 0$. Our findings generalize and extend upon recent results in the literature.

Keywords: sign-changing solution, variational methods, p-Laplacian, Kirchhoff-type problem.

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1 Introduction and main result

We are interested in the existence of sign-changing solutions to the following class of Kirchhoff-type problem with the p-Laplacian involving a critical exponent:

$$-\left(1+b|\nabla v|_p^p\right)\Delta_p v + a(x)|v|^{p-2}v = |v|^{p^*-2}v + \lambda f(v), \quad x \in \mathbb{R}^N, \tag{$\mathcal{QKP}$}$$

where b and λ are positive parameters, $\Delta_p v = \operatorname{div}(|\nabla v|^{p-2}\nabla v)$ is the p-Laplacian, $p^* = \frac{Np}{N-p}$, $1 < p < N$, and $f \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$. The term p-Laplacian has emerged as a crucial concept in various physics backgrounds and nonlinear analysis. This functional has found applications in describing the movement of sandpiles [2], modeling Game Theory [11], and is employed in

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image inpainting [6]. A variety of intriguing results have been obtained for problem ($\mathcal{QKP}$) and similar problems, which has been investigated widely, see [7,8] and the reference therein.

Problem ($\mathcal{QKP}$) is derived from the following Kirchhoff equation

$$\begin{cases} -\left(a+b\int_\Omega |\nabla v|^2\right)\Delta v = f(v), & x\in\Omega,\\ v=0, & x\in\partial\Omega. \end{cases} \tag{1.1}$$

Problem (1.1) corresponds to the following stationary form of a Kirchhoff type equation

$$v_{tt} - \left(a+b\int_\Omega |\nabla v|^2 dx\right)\Delta v = f(v), \tag{1.2}$$

where a and b are positive constants. In one and two dimensions, equation (1.2) has been employed to model various phenomena in physics, engineering, and other scientific disciplines, effectively approximating the nonlinear vibrations of beams or plates. In [13], Kirchhoff originally introduced (1.2) as an extension of the classical D'Alembert wave equation, which models the free vibrations of elastic strings

$$\rho\frac{\partial^2 v}{\partial t^2} - \left(\frac{p_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial v}{\partial x}\right|^2 dx\right)\frac{\partial^2 v}{\partial x^2} = f(v),$$

with constants ρ, p_0, h, E, and L.

We introduce a mathematical model that characterizes the behavior of a compressible fluid within a homogeneous, isotropic, rigid porous medium. In this context, $\vec{a}=\vec{a}(x,t)$ represents the seepage velocity, φ is the volumetric moisture content, and $v=v(x,t)$ is the density. The continuity equation is given by:

$$\varphi\frac{\partial v}{\partial t} + \operatorname{div}(v\vec{a}) = 0. \tag{1.3}$$

In the case of laminar flow regime through the porous medium, the relationship between the pressure $\pi=\pi(x,t)$ and the momentum velocity $\rho\vec{a}$ is described by the following Darcy's law:

$$\rho\vec{a} = -\lambda\,\operatorname{grad}\pi. \tag{1.4}$$

The flow rate is not the same in turbulent flow regimes, leading various authors to propose a nonlinear alternative to (1.4). A commonly considered form of the nonlinear Darcy law can be found in works such as Wu et al. [21]:

$$\rho\vec{a} = -\lambda|\operatorname{grad}\pi|^{p-2}\operatorname{grad}\pi, \tag{1.5}$$

where $p>1$. While considering the polytropic gas equation of state

$$\pi = cv, \tag{1.6}$$

where $c>0$ is a positive constant of proportionality. By (1.3), (1.5) and (1.6), we obtain

$$\varphi\frac{\partial v}{\partial t} = c^{p-1}\lambda\,\operatorname{div}\left(|\operatorname{grad}v|^{p-2}\operatorname{grad}v\right). \tag{1.7}$$

Following the variable substitution, we obtain

$$\frac{\partial v}{\partial t} = \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right), \tag{1.8}$$

with p being a real number greater than 1.

$$\Delta_p v := \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|\nabla v|^{p-2}\frac{\partial v}{\partial x_i}\right) = \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right),$$

and

$$v \longmapsto \Delta_p v, \tag{1.9}$$

it is commonly called the p-Laplace operator or the p-Laplacian. In the case of one dimension, one has

$$\Delta_p v = \left(|v'|^{p-2}v'\right)'.$$

Definition 1.1. A solution v is nontrivial provided that $v \neq 0$. A nontrivial solution is called a ground state (or least energy) solution if its energy is the lowest among all nontrivial solutions. Additionally, if v is a nontrivial solution and both v^+ and v^- are nonzero, then v is referred to as sign-changing solution, where $v^+ := \max\{v, 0\}$, $v^- := \min\{v, 0\}$.

Baldelli and Filippucci [4] studied the existence and multiplicity of nontrivial solutions to the following generalized quasilinear Schrödinger equation:

$$-\Delta_p v - \frac{\alpha}{2}\Delta_p\left(|v|^{\alpha}\right)|v|^{\alpha-2}v = \lambda V(x)|v|^{k-2}v + \beta K(x)|v|^{\alpha p^*-2}v \quad \text{in } \mathbb{R}^N,$$

where $\alpha > 1$, β, $\lambda > 0$, and $\alpha < k < \alpha p^*$. The weights are nontrivial and satisfy $0 \leqslant V \in L^{\alpha p^*/(\alpha p^*-k)}(\mathbb{R}^N) \cap \mathcal{C}(\mathbb{R}^N)$, $K \in L^{\infty}(\mathbb{R}^N) \cap \mathcal{C}(\mathbb{R}^N)$. For $N \geqslant 3$ and $1 < p < N$, using the concentration compactness principle, the symmetric Mountain Pass Theorem, the truncation of the energy functional, and the theory of Krasnosel'skii genus, they established the existence of infinitely many nontrivial solutions $v_n \in \mathcal{D}^{1,p}(\mathbb{R}^N) = \{u \in L^{p^*}(\mathbb{R}^N) : \nabla u \in (L^p(\mathbb{R}^N))^N\}$, provided that $1 < \alpha < k < p$.

Li [15] addressed the existence of solutions for the following equation:

$$\begin{aligned}-\operatorname{div}\left(g^p(v)|\nabla v|^{p-2}\nabla v\right) + g^{p-1}(v)g'(u)|\nabla v|^p + V(x)|v|^{p-2}v \\ = K(x)f(v) + Q(x)g(v)|G(v)|^{p^*-2}G(v),\end{aligned}$$

where $1 < p \leqslant N$, $g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^+)$, $V(x)$ and $K(x)$ are positive continuous functions, $Q(x)$ is a bounded continuous function, and $G(v) = \int_0^v g(t)dt$. The author obtained the existence of positive solutions when $N \geqslant 3$. Their proof is based on the truncation of the energy functional and the Mountain Pass Theorem.

In [14], the p-Laplacian Kirchhoff-type equation with logarithmic nonlinearity was investigated:

$$\begin{cases} -\left(a + b|\nabla v|_p^p\right)\Delta_p v = |v|^{q-2}v\ln v^2, & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases} \tag{1.10}$$

Using the constraint variation method, topological degree theory, and the quantitative deformation lemma, the authors established the existence of ground state sign-changing solutions for problem (1.10). In problem (1.10), they assumed that $N > p$, a and b are positive, and $4 \leqslant 2p < q < p^*$ for some $q > p$. Chen et al. [10] investigated a modified version of problem ($\mathcal{QKP}$) with $\lambda = 1$, and the nonlinearity $f(v)$ replaced by $R(x)g(v)$ without the critical term. The modified problem is given by

$$-\left(1 + b|\nabla v|_p^p\right)\Delta_p v + a(x)|v|^{p-2}v = R(x)g(v),$$

in which $R(x)$ and $a(x)$ represent continuous functions that are positive and $v \in \mathcal{D}^{1,p}(\mathbb{R}^N)$. The authors establish the existence of a ground state sign-changing solution by employing the non-Nehari manifold approach (Unlike the Nahari manifold method, the main idea of the non-Nehari manifold approach lies on finding a minimizing sequence for the energy functional outside the manifold). Motivated by the studies mentioned above, in this paper, we suppose that f belongs to $\mathcal{C}^1(\mathbb{R}, \mathbb{R})$ and a belongs to $\mathcal{C}\left(\mathbb{R}^N, \mathbb{R}\right)$ fulfills the following assumptions:

$(\mathcal{A})$ $\inf_{\mathbb{R}^N} a(x) > 0$ and there is R, such that

$$\lim_{|y|\to\infty} m\left\{x \in \mathbb{R}^N :\ x \in B_R(y),\ a(x) \leqslant A\right\} = 0, \quad \forall A > 0,$$

here m denotes the Lebesgue measure.

$(\mathcal{P}_1)$ There holds $\lim_{t\to 0} \frac{f(t)}{|t|^{p-1}} = 0$;

$(\mathcal{P}_2)$ There is $\vartheta \in (2p, p^*)$ such that

$$\lim_{s\to\infty} \frac{f(t)}{|t|^{\vartheta-1}} = 0;$$

$(\mathcal{P}_3)$ There holds $\lim_{t\to\infty} \frac{F(t)}{|t|^{2p}} = +\infty$;

$(\mathcal{P}_4)$ The map $t \mapsto \frac{f(t)}{|t|^{2p-1}}$ is strictly increasing, for all $t \in \mathbb{R}\backslash\{0\}$,

From the assumptions $(\mathcal{P}_3)$ and $(\mathcal{P}_4)$, one obtains

$$f(t)t > 0, \quad t \neq 0; \quad F(t) \geqslant 0, \quad t \in \mathbb{R}. \tag{1.11}$$

Furthermore, from the assumptions $(\mathcal{P}_3)$ and $(\mathcal{P}_4)$, one gets

$$f'(t)t - (2p-1)f(t) < (>)0, \quad \forall t < (>)0. \tag{1.12}$$

For simplicity, we will denote distinct positive constants by c_i, C_i, and C, and $\int$ always stand for $\int_{\mathbb{R}^N}$. Define

$$\mathcal{BC}(\mathbb{R}^N) := \left\{ v \in \mathcal{C}(\mathbb{R}^N) :\ |v|_\infty := \sup_{x\in\mathbb{R}^N} |v(x)| < \infty \right\}.$$

The closure of $\mathcal{C}_c(\mathbb{R}^N)$ in $\mathcal{BC}(\mathbb{R}^N)$ with respect to the uniform norm is the space $\mathcal{C}_0(\mathbb{R}^N)$, where $\mathcal{C}_c(\mathbb{R}^N)$ is space of continuous functions with compact support in $\mathbb{R}^N$. The space of finite measures on $\mathbb{R}^N$ is referred to as $\mathcal{M}(\mathbb{R}^N)$. A sequence $\{\nu_n\}$ converges weakly* to ν in $\mathcal{M}(\mathbb{R}^N)$, written

$$\nu_n \rightharpoonup^* \nu,$$

provided

$$\langle \nu_n, \varphi \rangle \to \langle \nu, \varphi \rangle,$$

$\forall \varphi \in \mathcal{C}_0(\mathbb{R}^N)$. For $1 \leqslant s \leqslant p^*$, we introduce the following norms:

$$\|v\|_{L^s(\mathbb{R}^N)} = |v|_s = |v|_{L^s(\mathbb{R}^N)} := \left(\int |v(x)|^s \,\mathrm{d}x \right)^{\frac{1}{s}}.$$

In order to state our principal results, let

$$W = \left\{ v \in W^{1,p} : \int a(x)|v|^p \,\mathrm{d}x + |\nabla v|_p^p < +\infty \right\},$$

endowed with the following norm

$$\|v\| = \left[\int a(x)|v|^p \,\mathrm{d}x + |\nabla v|_p^p \right]^{\frac{1}{p}}.$$

It is clear that W is continuously embedded into $W^{1,p}(\mathbb{R}^N)$ and consequently the embedding into $L^s(\mathbb{R}^N)$ is also continuous for $s \in [p, p^*]$. This implies that there is a $\gamma_s > 0$ such that

$$|v|_s \leqslant \gamma_s \|v\|, \quad \forall v \in W. \tag{1.13}$$

Lemma 1.2 ([12]). *Assuming that $(\mathcal{A})$ holds. For $p \leqslant s < p^*$, W is compactly embedded into $L^s(\mathbb{R}^N)$.*

Let η_1 denote the first eigenvalue of $-\Delta_p v$. Thus, η_1 represents the smallest value of η such that the problem

$$\begin{cases} -\Delta_p v = \eta |v|^{p-2} v, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.14}$$

possesses a nontrivial solution. Let the functional $\mathcal{K} : W_0^{1,p} \to \mathbb{R}$ be defined as follows:

$$\mathcal{K}(v) = \frac{1}{p} \int_\Omega |\nabla v|^p \,\mathrm{d}x - \frac{\eta}{p} \int_\Omega |v|^p \,\mathrm{d}x.$$

For problem ($\mathcal{QKP}$), the energy functional $\mathcal{K}_b^\lambda : W \to \mathbb{R}$ is given by:

$$\mathcal{K}_b^\lambda = \frac{1}{p}\|v\|^p + \frac{b}{2p}|\nabla v|_p^{2p} - \frac{1}{p^*}|v|_{p^*}^{p^*} \,\mathrm{d}x - \lambda \int F(v) \,\mathrm{d}x, \tag{1.15}$$

for every $v \in W$. Furthermore, $\mathcal{K}_b^\lambda \in \mathcal{C}^1(W, \mathbb{R})$, and its Fréchet derivative is defined as follows:

$$\begin{aligned} \left\langle (\mathcal{K}_b^\lambda)'(v), \varphi \right\rangle = & \left(1 + b \int |\nabla v|^p \,\mathrm{d}x\right) \left(\int |\nabla v|^{p-2} \nabla v \cdot \nabla \varphi \,\mathrm{d}x \right) \\ & + \int a(x)|v|^{p-2} v\varphi \,\mathrm{d}x - \int |v|^{p^*-2} v\varphi \,\mathrm{d}x - \lambda \int f(v)\varphi \,\mathrm{d}x, \end{aligned} \tag{1.16}$$

for every $v, \varphi \in W$. It is widely acknowledged that there exist several intriguing studies concerning the existence of sign-changing solution to problem (1.14). Nevertheless, the approaches employed to seek such solutions are greatly reliant on the subsequent decompositions:

$$\langle \mathcal{K}'(v), v^- \rangle = \langle \mathcal{K}'(v^-), v^- \rangle, \quad \langle \mathcal{K}'(v), v^+ \rangle = \langle \mathcal{K}'\left(v^+\right), v^+ \rangle, \quad \mathcal{K}(v) = \mathcal{K}(v^+) + \mathcal{K}(v^-). \tag{1.17}$$

However, when $b > 0$, the functional $\mathcal{K}_b^\lambda$ lacks the same decompositions as in equation (1.17). Specifically, if $v^\pm \neq 0$, it can be seen that

$$\begin{aligned} \mathcal{K}_b^\lambda(v) &= \mathcal{K}_b^\lambda(v^+) + \mathcal{K}_b^\lambda(v^-) + \frac{b}{p}|\nabla v^+|_p^p |\nabla v^-|_p^p > \mathcal{K}_b^\lambda(v^+) + \mathcal{K}_b^\lambda(v^-), \\ \langle (\mathcal{K}_b^\lambda)'(v), v^+ \rangle &= \langle (\mathcal{K}_b^\lambda)'(v^+), v^+ \rangle + b|\nabla v^+|_p^p |\nabla v^-|_p^p > \langle (\mathcal{K}_b^\lambda)'(v^+), v^+ \rangle, \\ \langle (\mathcal{K}_b^\lambda)'(v), v^- \rangle &= \langle (\mathcal{K}_b^\lambda)'(v^-), v^- \rangle + b|\nabla v^+|_p^p |\nabla v^-|_p^p > \langle (\mathcal{K}_b^\lambda)'(v^-), v^- \rangle. \end{aligned}$$

So, the conventional techniques cannot be employed to obtain a sign-changing solutions for problem ($\mathcal{QKP}$). Rather, we use the strategy proposed in [5], which requires defining a constrained set as follows:

$$\mathcal{M}_b^\lambda = \Big\{ v \in W : \ v^\pm \neq 0 \text{ and } \langle (\mathcal{K}_b^\lambda)'(v), v^\pm \rangle = 0 \Big\}, \tag{1.18}$$

and investigate the following minimization problem

$$c_b^\lambda = \inf_{v \in \mathcal{M}_b^\lambda} \mathcal{K}_b^\lambda(v).$$

The inclusion of the nonlocal term in problem ($\mathcal{QKP}$) creates numerous obstacles. Specifically, when compared to the general problem (1.14), in equation (1.17), the decompositions for $\mathcal{K}_b^\lambda$ are much more intricate, which results in some technical challenges while demonstrating the nonemptiness of $\mathcal{M}_b^\lambda$. Furthermore, observe that implicit theorem and the parametric method cannot be employed to address problem ($\mathcal{QKP}$) because of the complex nature of the involved nonlocal term. Consequently, our demonstration follows an alternative approach inspired by [1], specifically by utilizing a modified version of Miranda's theorem (refer to [17]). Additionally, by utilizing degree theory and the quantitative deformation lemma, the minimizer of the constrained problem is shown to be a sign-changing solution.

Now, we proceed to outline our primary results.

Theorem 1.3. *Assume that conditions $(\mathcal{P}_1)$–$(\mathcal{P}_4)$ and $(\mathcal{A})$ are satisfied. Then, there is a positive constant $\lambda^\star$ such that for every λ with $\lambda^\star \leqslant \lambda < \infty$, problem ($\mathcal{QKP}$) possesses a ground state sign-changing solution v_b.*

One purpose of this study is to demonstrate the energy doubling property, which ensures that the energy of any sign-changing solution to problem ($\mathcal{QKP}$) is strictly greater than two times the ground state energy. Although the result is straightforward for equation (1.14), we seek to determine whether it also holds for problem ($\mathcal{QKP}$). The theorem presented below affirms this result for our problem.

Theorem 1.4. *Assume that conditions $(\mathcal{P}_1)$–$(\mathcal{P}_4)$ and $(\mathcal{A})$ are satisfied. Then there is $\lambda^{\star\star} > 0$ such that for all $\lambda^{\star\star} \leqslant \lambda < \infty$, the infimum $c^*_{b,\lambda} > 0$ is attained and the following inequality holds*

$$\mathcal{K}_b^\lambda(v) > 2c^*_{b,\lambda},$$

where $c^*_{b,\lambda} = \inf_{v \in \mathcal{N}_\lambda} \mathcal{K}_b^\lambda(v)$, $\mathcal{N}_b^\lambda = \{v \in W : \ v \neq 0 \text{ and } \langle (\mathcal{K}_b^\lambda)'(v), v\rangle = 0\}$, and v is the solution obtained in Theorem 1.3.

Finally, we demonstrate the convergence behavior of the solution as $b \searrow 0$.

Theorem 1.5. *Assume that conditions $(\mathcal{P}_1)$–$(\mathcal{P}_4)$ and $(\mathcal{A})$ are satisfied. Then, for any sequence $\{b_n\}$ satisfies $b_n \searrow 0$, there is a subsequence of $\{b_n\}$, such that $\{v_{b_n}\}$ converges strongly to v_0 in W, where v_0 is a least energy nodal solution for the problem below*

$$-\Delta_p v + a(x)|v|^{p-2}v = |v|^{p^*-2}v + \lambda f(v), \quad x \in \mathbb{R}^N. \tag{$\mathcal{QKP}_0$}$$

Remark 1.6. We would like to highlight that Theorem 1.3 represents the first result demonstrating the existence of sign-changing solutions for p-Laplacian Kirchhoff-type problems involving critical exponent in $\mathbb{R}^N$.

Remark 1.7. The primary challenges stem from dealing with a degenerate quasi-linear elliptic operator and the increased intricacy of the calculations involved. To surmount these challenges, we employ quantitative deformation lemma and degree theory to derive our findings.

2 Some preliminary results

Let $v \in W$ be fixed with $v^{\pm} \neq 0$, let $g_v : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $G_v : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^2$ defined as follows

$$g_v(\alpha, \beta) = \mathcal{K}_b^{\lambda}(\alpha v^+ + \beta v^-), \tag{2.1}$$

$$G_v(\alpha, \beta) = \Big(\langle (\mathcal{K}_b^{\lambda})'(\alpha v^+ + \beta v^-), \alpha v^+ \rangle, \langle (\mathcal{K}_b^{\lambda})'(\alpha v^+ + \beta v^-), \beta v^- \rangle \Big). \tag{2.2}$$

Lemma 2.1. *Let f and a satisfy $(\mathcal{A})$ and $(\mathcal{P}_1)$–$(\mathcal{P}_4)$. Suppose that $v \in W$ with $v^{\pm} \neq 0$, the following properties hold:*

(A_1) *$\alpha v^+ + \beta v^- \in \mathcal{M}_b^{\lambda}$ if and only if g_v has a critical point $(\alpha, \beta) \in \mathbb{R}_*^+ \times \mathbb{R}_*^+$;*

(A_2) *The pair (α_v, β_v) is the unique critical point of g_v on $\mathbb{R}_*^+ \times \mathbb{R}_*^+$, as well as it is the unique maximum of g_v on $\mathbb{R}^+ \times \mathbb{R}^+$;*

(A_3) *If $\langle (\mathcal{K}_b^{\lambda})'(v), v^{\pm} \rangle \leqslant 0$ then $0 < \alpha_v, \beta_v \leqslant 1$.*

Proof. Proof of (A_1): Note that if $v^{\pm} \neq 0$, one has

$$\begin{aligned}
\left|\nabla(\alpha v^+ + \beta v^-)\right|^p &= \left|\nabla(\alpha v^+ + \beta v^-)\right|^2 \left|\nabla(\alpha v^+ + \beta v^-)\right|^{p-2} \\
&= \left|\nabla(\alpha v^+ + \beta v^-)\right|^{p-2} \nabla(\alpha v^+ + \beta v^-)\nabla(\alpha v^+) \\
&\quad + \left|\nabla(\alpha v^+ + \beta v^-)\right|^{p-2} \nabla(\alpha v^+ + \beta v^-)\nabla(\beta v^-) \\
&= |\nabla(\alpha v^+)|^{p-2}\nabla(\alpha v^+)\nabla(\alpha u^+) + \left|\nabla(\beta v^-)\right|^{p-2} \nabla(\beta v^-)\nabla(\beta u^-) \\
&= |\alpha|^p \left|\nabla v^+\right|^p + |\beta|^p \left|\nabla v^-\right|^p.
\end{aligned}$$

Therefore, a direct computation yields

$$\begin{aligned}
\nabla g_v(\alpha, \beta) &= \left(\frac{\partial g_v}{\partial \alpha}(\alpha, \beta), \frac{\partial g_v}{\partial \beta}(\alpha, \beta) \right) \\
&= \Big(\langle (\mathcal{K}_b^{\lambda})'(\alpha v^+ + \beta v^-), v^+ \rangle, \langle (\mathcal{K}_b^{\lambda})'(\alpha v^+ + \beta v^-), v^- \rangle \Big) \\
&= \left(\frac{1}{\alpha} \langle (\mathcal{K}_b^{\lambda})'(\alpha v^+ + \beta v^-), \alpha v^+ \rangle, \frac{1}{\beta} \langle (\mathcal{K}_b^{\lambda})'(\alpha v^+ + \beta v^-), \beta v^- \rangle \right),
\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
\langle (\mathcal{K}_b^{\lambda})'(\alpha v^+ + \beta v^-), \alpha v^+ \rangle &= b\alpha^p \beta^p |\nabla v^+|_p^p |\nabla v^-|_p^p - \lambda \int f(\alpha v^+)\alpha v^+ \, \mathrm{d}x \\
&\quad + \alpha^p \|v^+\|^p + b\alpha^{2p} |\nabla v^+|_p^{2p} - \alpha^{p^*} |v^+|_{p^*}^{p^*}, \\
\langle (\mathcal{K}_b^{\lambda})'(\alpha v^+ + \beta v^-), \beta v^- \rangle &= b\beta^p \alpha^p |\nabla v^-|_p^p |\nabla v^+|_p^p - \lambda \int f(\beta v^-)\beta v^- \, \mathrm{d}x \\
&\quad + \beta^p \|v^-\|^p + b\beta^{2p} |\nabla v^-|_p^{2p} - \beta^{p^*} |v^-|_{p^*}^{p^*}.
\end{aligned}$$

From (2.3) and the definition of $\mathcal{M}_b^{\lambda}$, it's straightforward to show that (A_1) is satisfied.

Proof of (A_2): From $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$, for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$|f(t)| \leqslant \varepsilon |t|^{p-1} + C_\varepsilon |t|^{\vartheta - 1}, \quad \forall t \in \mathbb{R}. \tag{2.4}$$

From (1.13) and (2.4), we obtain

$$\begin{aligned}\langle (\mathcal{K}_b^\lambda)'(\alpha v^+ + \beta v^-), \alpha v^+\rangle &\geqslant \alpha^p \|v^+\|^p + b\alpha^{2p}|\nabla v^+|_p^{2p} + b\alpha^p\beta^p|\nabla v^+|_p^p|\nabla v^-|_p^p - \alpha^{p*}|v^+|_{p^*}^{p^*}\\ &\quad - \lambda\varepsilon\alpha^p|v^+|_p^p - \lambda C_\varepsilon \alpha^\vartheta |v^+|_\vartheta^\vartheta\\ &\geqslant \alpha^p\left(1-\lambda\varepsilon\gamma_p\right)\|v^+\|^p + b\alpha^{2p}|\nabla v^+|_p^{2p} - \lambda C_\varepsilon\alpha^\vartheta\gamma_\vartheta\|v^+\|^\vartheta\\ &\quad - \alpha^{p*}\gamma_{p*}\|v^+\|^{p^*},\end{aligned}$$

and

$$\begin{aligned}\langle (\mathcal{K}_b^\lambda)'(\alpha v^+ + \beta v^-), \beta v^-\rangle &\geqslant \beta^p \|v^-\|^p + b\beta^{2p}|\nabla v^-|_p^{2p} + b\beta^p\alpha^p|\nabla v^-|_p^p|\nabla v^+|_p^p - \beta^{p*}|v^-|_{p^*}^{p^*}\\ &\quad - \lambda\varepsilon\beta^p|v^-|_p^p - \lambda C_\varepsilon \beta^\vartheta |v^-|_\vartheta^\vartheta\\ &\geqslant \beta^p\left(1-\lambda\varepsilon\gamma_p\right)\|v^-\|^p + b\beta^{2p}|\nabla v^-|_p^{2p} - \lambda C_\varepsilon\beta^\vartheta\gamma_\vartheta\|v^-\|^\vartheta\\ &\quad - \beta^{p*}\gamma_{p*}\|v^-\|^{p^*}.\end{aligned}$$

Select ε in such a manner that $\left(1-\lambda\varepsilon\gamma_p\right) > 0$. Given that $\vartheta \in (2p, p^*)$, one has

$$\langle (\mathcal{K}_b^\lambda)'(\alpha v^+ + \beta v^-), \alpha v^+\rangle > 0, \quad \text{for } \alpha \text{ small enough and all } \beta \geqslant 0, \tag{2.5}$$

and

$$\langle (\mathcal{K}_b^\lambda)'(\alpha v^+ + \beta v^-), \beta v^-\rangle > 0, \quad \text{for } \beta \text{ small enough and all } \alpha \geqslant 0. \tag{2.6}$$

Hence, there is a positive constant τ_1 such that

$$\langle (\mathcal{K}_b^\lambda)'(\tau_1 v^+ + \beta v^-), \tau_1 v^+\rangle > 0, \quad \langle (\mathcal{K}_b^\lambda)'(\alpha v^+ + \tau_1 v^-), \tau_1 v^-\rangle > 0, \tag{2.7}$$

for all $\alpha, \beta \geqslant 0$. Conversely, choosing $\alpha = \tau_2' > \tau_1$. If $\beta \in [\tau_1, \tau_2']$ and τ_2' is large enough, using (1.11), one has that

$$\begin{aligned}\langle (\mathcal{K}_b^\lambda)'(\tau_2' v^+ + \beta v^-), \tau_2' v^+\rangle &\leqslant (\tau_2')^p\|v^+\|^p + b(\tau_2')^{2p}|\nabla v^+|_p^{2p} - (\tau_2')^{p^*}|v^+|_{p*}^{p*}\\ &\quad + b(\tau_2')^{2p}|\nabla v^+|_p^p|\nabla v^-|_p^p\\ &\leqslant 0,\end{aligned}$$

and

$$\begin{aligned}\langle (\mathcal{K}_b^\lambda)'(\alpha v^+ + \tau_2' v^-), \tau_2' v^-\rangle &\leqslant (\tau_2')^p\|v^-\|^p + b(\tau_2')^{2p}|\nabla v^-|_p^{2p} - (\tau_2')^{p^*}|v^-|_{p*}^{p*}\\ &\quad + b(\tau_2')^{2p}|\nabla v^-|_p^p|\nabla v^+|_p^p\\ &\leqslant 0.\end{aligned}$$

Let $\tau_2 > \tau_2'$ be large enough, we get

$$\langle (\mathcal{K}_b^\lambda)'(\alpha v^+ + \tau_2 v^-), \tau_2 v^-\rangle < 0, \qquad \langle (\mathcal{K}_b^\lambda)'(\alpha v^+ + \tau_2 v^-), \tau_2 v^-\rangle < 0, \tag{2.8}$$

for all $\alpha, \beta \in [\tau_1, \tau_2]$. Combining equations (2.7) and (2.8) with Miranda's Theorem [17], we find that there is $(\alpha_v, \beta_v) \in \mathbb{R}_*^+ \times \mathbb{R}_*^+$ such that $G_v(\alpha, \beta) = (0, 0)$, that is, $\alpha v^+ + \beta v^- \in \mathcal{M}_b^\lambda$.

Next, we demonstrate the pair (α_v, β_v) is unique. To finalize the proof, we will investigate two separate cases.

Case 1) $v \in \mathcal{M}_b^\lambda$.

For any $v \in \mathcal{M}_b^\lambda$, one has that

$$\|v^+\|^p + b|\nabla v^+|_p^p|\nabla v^-|_p^p + b|\nabla v^+|_p^{2p} = |v^+|_{p*}^{p^*} + \lambda\int f(v^+)v^+\,\mathrm{d}x, \tag{2.9}$$

and

$$\|v^-\|^p + b|\nabla v^-|_p^p|\nabla v^+|_p^p + b|\nabla v^-|_p^{2p} = |v^-|_{p*}^{p^*} + \lambda \int f(v^-)v^- \, dx. \tag{2.10}$$

Consider a pair of numbers (α_0, β_0) satisfying $\alpha_0 v^+ + \beta_0 v^- \in \mathcal{M}_b^\lambda$ and $0 < \alpha_0 \leqslant \beta_0$. So, we have that

$$\alpha_0^p\|v^+\|^p + b\alpha_0^p\beta_0^p|\nabla v^+|_p^p|\nabla v^-|_p^p + b\alpha_0^{2p}|\nabla v^+|_p^{2p} = \alpha_0^{p^*}|v^+|_{p^*}^{p^*} + \lambda \int f(\alpha_0 v^+)\alpha_0 v^+ \, dx,$$

and

$$\beta_0^p\|v^-\|^p + b\beta_0^p\alpha_0^p|\nabla v^-|_p^p|\nabla v^+|_p^p + b\beta_0^{2p}|\nabla v^-|_p^{2p} = \beta_0^{p^*}|v^-|_{p^*}^{p^*} + \lambda \int f(\beta_0 v^-)\beta_0 v^- \, dx.$$

Hence, thanks to $0 < \alpha_0 \leqslant \beta_0$, one has that

$$\frac{1}{\alpha_0^p}\|v^+\|^p + b|\nabla v^+|_p^p|\nabla v^-|_p^p + b|\nabla v^+|_p^{2p} \leqslant \alpha_0^{p^*-2p}|v^+|_{p^*}^{p^*} + \lambda \int \frac{f(\alpha_0 v^+)}{(\alpha_0 v^+)^{2p-1}}(v^+)^{2p} \, dx, \tag{2.11}$$

and

$$\frac{1}{\beta_0^p}\|v^-\|^p + b|\nabla v^-|_p^p|\nabla v^+|_p^p + b|\nabla v^-|_p^{2p} \geqslant \beta_0^{p^*-2p}|v^-|_p^{p^*} + \lambda \int \frac{f(\beta_0 v^-)}{(\beta_0 v^-)^{2p-1}}(v^-)^{2p} \, dx. \tag{2.12}$$

Combining (2.10) with (2.12), one has that

$$\left(\frac{1}{\beta_0^p} - 1\right)\|v^-\|^p \geqslant \left(\beta_0^{p^*-2p} - 1\right)|v^-|_{p^*}^{p^*} + \lambda \int \left[\frac{f(\beta_0 v^-)}{(\beta_0 v^-)^{2p-1}} - \frac{f(v^-)}{(v^-)^{2p-1}}\right](v^-)^{2p} \, dx.$$

If $\beta_0 > 1$, the left-hand side of the aforementioned inequality becomes less than zero, thereby contradicting condition $(\mathcal{P}_4)$ that guarantees the positivity of the right-hand side. Consequently, if follows that $0 < \alpha_0 \leqslant \beta_0 \leqslant 1$. Similarly, combining (2.9) with (2.11), thanks to $0 < \alpha_0 \leqslant \beta_0$, we obtain

$$\left(\frac{1}{\alpha_0^p} - 1\right)\|v^+\|^p \leqslant \left(\alpha_0^{p^*-2p} - 1\right)|v^+|_{p^*}^{p^*} + \lambda \int \left[\frac{f(\alpha_0 v^+)}{(\alpha_0 v^+)^{2p-1}} - \frac{f(v^+)}{(v^+)^{2p-1}}\right](v^+)^{2p} \, dx.$$

In view of $(\mathcal{P}_4)$, we have $\alpha_0 \geqslant 1$. Due to this, $\alpha_0 = \beta_0 = 1$. So, we conclude that if $v \in \mathcal{M}_b^\lambda$, the unique pair such that $\alpha_v v^+ + \beta_v v^- \in \mathcal{M}_b^\lambda$ is $(\alpha_v, \beta_v) = (1, 1)$.

Case 2) $v \notin \mathcal{M}_b^\lambda$.

Assume the existence of two pairs $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$, satisfy the following condition

$$u_1 = \alpha_1 v^+ + \beta_1 v^- \in \mathcal{M}_b^\lambda, \qquad u_2 = \alpha_2 v^+ + \beta_2 v^- \in \mathcal{M}_b^\lambda.$$

Therefore, one has

$$u_2 = \left(\frac{\alpha_2}{\alpha_1}\right)\alpha_1 v^+ + \left(\frac{\beta_2}{\beta_1}\right)\beta_1 v^- = \left(\frac{\alpha_2}{\alpha_1}\right)u_1^+ + \left(\frac{\beta_2}{\beta_1}\right)u_1^- \in \mathcal{M}_b^\lambda.$$

By $u_1 \in \mathcal{M}_b^\lambda$, we have that

$$\frac{\beta_2}{\beta_1} = \frac{\alpha_2}{\alpha_1} = 1.$$

Hence, $\alpha_1 = \alpha_2, \beta_1 = \beta_2$.

Lastly, we show that the unique maximum of g_v on $\mathbb{R}^+ \times \mathbb{R}^+$ is (α_v, β_v).
Indeed, using (1.11), it can be seen that

$$\begin{aligned}
g_v(\alpha,\beta) &= \mathcal{K}_b^\lambda(\alpha v^+ + \beta v^-) \\
&= \frac{1}{p}\|\alpha v^+ + \beta v^-\|^p + \frac{b}{2p}|\nabla \alpha v^+ + \beta v^-|_p^{2p} - \frac{1}{p^*}|\alpha v^+ + \beta v^-|_{p^*}^{p^*} \\
&\quad - \lambda \int F(\alpha v^+ + \beta v^-)\,\mathrm{d}x \\
&\leqslant \frac{\alpha^p}{p}\|v^+\|^p + \frac{\beta^p}{p}\|v^-\|^p - \frac{\beta^{p^*}}{p^*}|v^-|_{p^*}^{p^*} - \frac{\alpha^{p^*}}{p^*}|v^+|_{p^*}^{p^*} + \frac{b\alpha^{2p}}{2p}|\nabla v^+|_p^{2p} \\
&\quad + \frac{b\beta^{2p}}{2p}|\nabla v^-|_p^{2p} + \frac{b\alpha^p\beta^p}{p}|\nabla v^+|_p^p|\nabla v^-|_p^p.
\end{aligned} \tag{2.13}$$

Thus, for any $v \in W$ with $v^\pm \neq 0$, in light of equation (2.13) and the condition $p^* > 2p$, one has

$$\lim_{|(\alpha,\beta)|\to\infty} g_v(\alpha,\beta) = -\infty.$$

Thus, (α_v, β_v) represents the unique critical point of g_v within the domain $\mathbb{R}^+ \times \mathbb{R}^+$. Hence, our task is to establish that it is impossible to reach a maximum point on the boundary $(0, \beta_0)$ and $(\alpha_0, 0)$. For the sake of contradiction, let us assume that $(0, \beta_0)$ is a maximum of g_v with $\beta_0 \geqslant 0$. Consequently, it follows that

$$\begin{aligned}
g_v(\alpha,\beta_0) &= \frac{\alpha^p}{p}\|v^+\|^p + \frac{\beta_0^p}{p}\|v^-\|^p - \frac{\beta_0^{p^*}}{p^*}|v^-|_{p^*}^{p^*} - \frac{\alpha^{p^*}}{p^*}|v^+|_{p^*}^{p^*} + \frac{b\beta_0^{2p}}{2p}|\nabla v^-|_p^{2p} \\
&\quad + \frac{b\alpha^{2p}}{2p}|\nabla v^+|_p^{2p} + \frac{b\alpha^p\beta_0^p}{p}|\nabla v^+|_p^p|\nabla v^-|_p^p - \lambda\int F(\alpha v^+) - \lambda \int F(\beta_0 v^-).
\end{aligned}$$

Thus, it can be seen that

$$\begin{aligned}
\frac{\partial g_v}{\partial \alpha}(\alpha,\beta_0) &= \alpha^{p-1}\|v^+\|^p - \alpha^{p^*-1}|v^+|_{p^*}^{p^*} + b\alpha^{2p-1}|\nabla v^+|_p^{2p} + b\alpha^{p-1}\beta_0^p|\nabla v^+|_p^p|\nabla v^-|_p^p \\
&\quad - \lambda \int f(\alpha v^+)v^+\,\mathrm{d}x \\
&> 0,
\end{aligned}$$

if α is sufficiently small. This implies that g_v is increasing with respect to α if α is sufficiently small. This leads to a contradiction. Likewise, g_v can not attain its global maximum on $(\alpha_0, 0)$ with $\alpha_0 \geqslant 0$.

Proof of (A_3): Suppose $\alpha_v \geqslant \beta_v > 0$. By $\alpha_v v^+ + \beta_v v^- \in \mathcal{M}_b^\lambda$, we have that

$$\alpha_v^p\|v^+\|^p + b\alpha_v^{2p}|\nabla v^+|_p^{2p} + b\alpha_v^{2p}|\nabla v^+|_p^p|\nabla v^-|_p^p \geqslant \lambda \int f(\alpha_v v^+)\alpha_v v^+\,\mathrm{d}x + \alpha_v^{p^*}|v^+|_{p^*}^{p^*}. \tag{2.14}$$

Conversely, from the inequality $\langle (\mathcal{K}_b^\lambda)'(v), v^+\rangle \leqslant 0$, we deduce that

$$\lambda \int f(v^+)v^+\,\mathrm{d}x + |v^+|_{p^*}^{p^*} \geqslant \|v^+\|^p + b|\nabla v^+|_p^p|\nabla v^-|_p^p + b|\nabla v^+|_p^{2p}. \tag{2.15}$$

Combining (2.14) and (2.15), we obtain

$$\left(\frac{1}{\alpha_v^p} - 1\right)\|v^+\|^p \geqslant \left(\alpha_v^{p^*-2p} - 1\right)|v^+|_{p^*}^{p^*} + \lambda \int \left[\frac{f(\alpha_v v^+)}{(\alpha_v v^+)^{2p-1}} - \frac{f(v^+)}{(v^+)^{2p-1}}\right](v^+)^{2p}\,\mathrm{d}x.$$

Hence, due to condition $(\mathcal{P}_4)$, we can infer that $\alpha_v \leqslant 1$. Therefore, we establish that $0 < \alpha_v, \beta_v \leqslant 1$. Consequently, Lemma 2.1 has been proved. $\square$

The following result plays a crucial role to prove that c_b^λ is achieved.

Lemma 2.2. *Let $c_b^\lambda = \inf_{v\in\mathcal{M}_b^\lambda} \mathcal{K}_b^\lambda(v)$, then we have that*

$$\lim_{\lambda\to\infty} c_b^\lambda = 0.$$

Proof. For every $v \in \mathcal{M}_\lambda$, it holds that

$$\|v^+\|^p + b|\nabla v^+|_p^{2p} + b|\nabla v^+|_p^p|\nabla v^-|_p^p = |v^+|_{p^*}^{p^*} + \lambda\int f(v^+)v^+ \,\mathrm{d}x,$$

and

$$\|v^-\|^p + b|\nabla v^+|_p^{2p} + b|\nabla v^-|_p^p|\nabla v^+|_p^p = |v^-|_{p^*}^{p^*} + \lambda\int f(v^-)v^- \,\mathrm{d}x.$$

Then by (1.13) and (2.4), we get

$$\left(1-\lambda\varepsilon\gamma_p\right)\|v^+\|^p \leqslant \lambda C_\varepsilon\gamma_\vartheta\|v^+\|^\vartheta + \gamma_{p*}\|v^+\|^{p^*},$$

and

$$\left(1-\lambda\varepsilon\gamma_p\right)\|v^-\|^p \leqslant \lambda C_\varepsilon\gamma_\vartheta\|v^-\|^\vartheta + \gamma_{p*}\|v^-\|^{p^*}.$$

Taking $\varepsilon < \frac{1}{\lambda\gamma_p}$ such that $1-\lambda\varepsilon\gamma_p > 0$. Given that $\vartheta \in (2p, p^*)$, there is $\rho > 0$ such that

$$\|v^+\| \geqslant \rho, \quad \|v^-\| \geqslant \rho, \quad \forall v \in \mathcal{M}_b^\lambda. \tag{2.16}$$

Conversely, for any $v \in \mathcal{M}_b^\lambda$, it becomes evident that

$$\begin{aligned}\langle(\mathcal{K}_b^\lambda)'(v), v\rangle &= \langle(\mathcal{K}_b^\lambda)'(v), v^+\rangle + \langle(\mathcal{K}_b^\lambda)'(v), v^-\rangle \\ &= 0.\end{aligned}$$

From (1.12), we can deduce that

$$f(t)t - 2pF(t) \geqslant 0, \tag{2.17}$$

is increasing when $t > 0$ and decreasing when $t < 0$. Thus, we get

$$\begin{aligned}\mathcal{K}_b^\lambda(v) &= \mathcal{K}_b^\lambda(v) - \frac{1}{2p}\langle(\mathcal{K}_b^\lambda)'(v), v\rangle \\ &= \frac{(p-1)}{2p}\|v\|^p + \frac{p^*-2p}{2pp^*}|v|_{p^*}^{p^*} + \frac{\lambda}{2p}\int\left[f(v)v - 2pF(v)\right]\,\mathrm{d}x \\ &\geqslant \frac{(p-1)}{2p}\|v\|^p \\ &> \frac{(p-1)}{2p}\rho^p.\end{aligned}$$

Therefore, $\mathcal{K}_b^\lambda$ is bounded below on $\mathcal{M}_b^\lambda$. Hence, $c_b^\lambda = \inf_{v\in\mathcal{M}_b^\lambda}\mathcal{K}_b^\lambda(v)$ is well defined.

Fix $v \in W$ with $v^\pm \neq 0$. Lemma 2.1 guarantees that for every $\lambda > 0$, there is $\alpha_\lambda, \beta_\lambda > 0$ such that $\alpha_\lambda v^+ + \beta_\lambda v^- \in \mathcal{M}_b^\lambda$. From Lemma 2.1, one has

$$\begin{aligned}0 \leqslant c_b^\lambda &\leqslant \mathcal{K}_b^\lambda(\alpha_\lambda v^+ + \beta_\lambda v^-) \\ &\leqslant \frac{1}{p}\|\alpha_\lambda v^+ + \beta_\lambda v^-\|^p + \frac{b}{2p}|\nabla(\alpha_\lambda v^+ + \beta_\lambda v^-)|_p^{2p} \\ &= \alpha_\lambda^p\frac{1}{p}\|v^+\|^p + \beta_\lambda^p\frac{1}{p}\|v^-\|^p + \alpha_\lambda^{2p}\frac{b}{2p}|\nabla v^+|_p^{2p} + \beta_\lambda^{2p}\frac{b}{2p}|\nabla v^-|_p^{2p} \\ &\quad + \alpha_\lambda^p\beta_\lambda^p\frac{b}{p}|\nabla v^+|_p^p|\nabla v^-|_p^p.\end{aligned}$$

We need only verify that $\alpha_\lambda \to 0$ and $\beta_\lambda \to 0$, as $\lambda \to \infty$.
Define

$$\mathcal{G}_v = \left\{(\alpha_\lambda, \beta_\lambda) \in \mathbb{R}^+ \times \mathbb{R}^+ : G_v\left(\alpha_\lambda, \beta_\lambda\right) = (0,0), \lambda > 0\right\},$$

In light of (2.4), we have that

$$\begin{aligned}\alpha_\lambda^{p^*}|v^+|_{p^*}^{p^*} + \beta_\lambda^{p^*}|v^-|_{p^*}^{p^*} &\leqslant \lambda \int f(\alpha_\lambda v^+)\alpha_\lambda v^+ \, dx + \lambda \int f(\beta_\lambda v^-)\beta_\lambda v^- \, dx \\ &\quad + \alpha_\lambda^{p^*}|v^+|_{p^*}^{p^*} + \beta_\lambda^{p^*}|v^-|_{p^*}^{p^*} \\ &= \|\alpha_\lambda v^+ + \beta_\lambda v^-\|^p + b|\nabla(\alpha_\lambda v^+ + \beta_\lambda v^-)|_p^{2p}.\end{aligned}$$

Therefore, $\mathcal{G}_v$ is bounded, since $p^* > 2p$. Assume that $\{\lambda_n\}$ is a sequence in $(0,\infty)$ with $\lambda_n \to \infty$, as n tends to infinity. Then there is α_0 and β_0 such that $(\alpha_{\lambda_n}, \beta_{\lambda_n}) \to (\alpha_0, \beta_0)$, as $n \to \infty$. We assert that $\alpha_0 = \beta_0 = 0$. Suppose, the contrary $\alpha_0 > 0$ or $\beta_0 > 0$. Given that $\alpha_{\lambda_n}v^+ + \beta_{\lambda_n}v^- \in \mathcal{M}_b^{\lambda_n}$, then for any $n \in \mathbb{N}$, we have that

$$\begin{aligned}&\|\alpha_{\lambda_n}v^+ + \beta_{\lambda_n}v^-\|^p + b|\nabla(\alpha_{\lambda_n}v^+ + \beta_{\lambda_n}v^-)|_p^{2p} \\ &\quad = \lambda_n \int f(\alpha_{\lambda_n}v^+ + \beta_{\lambda_n}v^-)(\alpha_{\lambda_n}v^+ + \beta_{\lambda_n}v^-) \, dx + |\alpha_{\lambda_n}v^+ + \beta_{\lambda_n}v^-|_{p^*}^{p^*}.\end{aligned} \tag{2.18}$$

Then, invoking $\alpha_{\lambda_n}v^+ \to \alpha_0 v^+, \beta_{\lambda_n}v^- \to \beta_0 v^-$in W along with the Lebesgue dominated convergence theorem, we obtain that

$$\int f(\alpha_{\lambda_n}v^+ + \beta_{\lambda_n}v^-)(\alpha_{\lambda_n}v^+ + \beta_{\lambda_n}v^-) \to \int f(\alpha_0 v^+ + \beta_0 v^-)(\alpha_0 v^+ + \beta_0 v^-) > 0,$$

as $n \to \infty$. This leads to a contradiction with equation (2.18), because $\lambda_n \to \infty$, as $n \to \infty$ and and the sequence $\{\alpha_{\lambda_n}v^+ + \beta_{\lambda_n}v^-\}$ is bounded in W. Hence, we conclude that $\alpha_0 = \beta_0 = 0$, from which we infer that $\lim_{\lambda\to\infty} c_b^\lambda = 0$. □

Let ϕ be a standard cut off function, namely ϕ belongs to the space $C_c^\infty\left(\mathbb{R}^N\right)$, satisfying $0 \leqslant \phi \leqslant 1$, ϕ being equal to 1 on $B_{1/2}(0)$ and supp $\phi \subset B_1(0)$. For any $\epsilon > 0$, let $\phi_\epsilon(x) = \phi\left(\frac{x}{\epsilon}\right)$.

Lemma 2.3. *Let $1 < p < N$, for every $x_i \in \mathbb{R}^N$ and for $u \in L^p(\mathbb{R}^N)$, we have*

$$|v(x)\nabla\phi_\epsilon(x - x_i)|_{L^p(\mathbb{R}^N)}^p \leqslant |\nabla\phi|^p_{L^{\frac{p^*p}{p^*-p}}(\mathbb{R}^N)} |v|^p_{L^{p^*}(B_\epsilon(x_i))}.$$

Proof. Using the Hölder inequality, one has

$$\begin{aligned}\left|v(x)\nabla\phi_\epsilon(x - x_i)\right|_{L^p(\mathbb{R}^N)}^p &\leqslant \int \epsilon^{-p}\left|\nabla\phi\left(\frac{x - x_i}{\epsilon}\right)\right|^p |v(x)|^p \, dx \\ &\leqslant \epsilon^{-p}\left(\int_{B_\epsilon(x_i)}\left|\nabla\phi\left(\frac{x - x_i}{\epsilon}\right)\right|^{\frac{p^*p}{p^*-p}} dx\right)^{\frac{p^*-p}{p^*}} |v|^p_{L^{p^*}(B_\epsilon(x_i))} \\ &\leqslant \epsilon^{-p}\epsilon^{N\left(\frac{p^*-p}{p^*}\right)}|\nabla\phi|^p_{L^{\frac{p^*p}{p^*-p}}(\mathbb{R}^N)} |v|^p_{L^{p^*}(B_\epsilon(x_i))}.\end{aligned}$$

The result follows by noting that $p = \left(\frac{p^*-p}{p^*}\right)N$. □

With the above results, we will now undertake a three-step process to obtain that $c_b^\lambda = \inf_{v\in\mathcal{M}_b^\lambda}\mathcal{K}_b^\lambda(v)$ is achieved.

Lemma 2.4. *There is $\lambda^\star > 0$ such that for all $\lambda \in [\lambda^\star, \infty)$, the infimum c_b^λ is attained.*

Proof. As defined by c_b^λ, there is a sequence $\{v_n\} \subset \mathcal{M}_b^\lambda$ satisfies

$$\lim_{n\to\infty} \mathcal{K}_b^\lambda(v_n) = c_b^\lambda.$$

On the other hand, by using (2.17) and $\langle (\mathcal{K}_b^\lambda)'(v_n), v_n\rangle = 0$, we get

$$\begin{aligned} c_b^\lambda + o(1) &= \mathcal{K}_b^\lambda(v_n) - \frac{1}{2p}\langle (\mathcal{K}_b^\lambda)'(v_n), u_n\rangle \\ &= \frac{(p-1)}{2p}\|v\|^p + \frac{p^*-2p}{2pp^*}|v|_{p^*}^{p^*} + \frac{\lambda}{2p}\int [f(v)v - 2pF(v)] \\ &\geqslant \frac{(p-1)}{2p}\|v\|^p. \end{aligned} \tag{2.19}$$

Therefore, $\{v_n\}$ is bounded in W, i.e., there is a constant $M > 0$ such that

$$\|v_n\| \leqslant M, \quad \forall n \in \mathbb{N}. \tag{2.20}$$

From (2.16), (2.19), and Ekeland's variational principle [20], $\{v_n\}$ represents a $(\mathrm{PS})_{c_b^\lambda}$ sequence for $\mathcal{K}_b^\lambda|_{\mathcal{M}_b^\lambda}$, i.e.

$$\mathcal{K}_b^\lambda(v_n) \to c_b^\lambda, \quad (\mathcal{K}_b^\lambda)'(v_n) \to 0. \tag{2.21}$$

Since v_n is bounded in W, there existence a $v \in W$ and a subsequence $\{v_n\}$ where

$$\begin{aligned} v_n &\rightharpoonup v && \text{in } W, \\ v_n &\to v && \text{in } L^s \text{ for } p \leqslant s < p^*, \\ v_n &\to v && \text{a.e. in } \mathbb{R}^N, \\ v_n &\rightharpoonup v && \text{in } L^{p^*}, \\ f(v_n) &\rightharpoonup f(v) && \text{in } (L^{p^*})', \\ |v_n|^{p^*-2}v_n &\rightharpoonup |v|^{p^*-2}v && \text{in } (L^{p^*})'. \end{aligned} \tag{2.22}$$

We migh suppose that

$$\begin{aligned} |\nabla v_n|^p &\rightharpoonup^* \mu \text{ in } \mathcal{M}(\mathbb{R}^N), \\ |v_n|^{p^*} &\rightharpoonup^* \nu \text{ in } \mathcal{M}(\mathbb{R}^N). \end{aligned} \tag{2.23}$$

for some measures μ and ν. Then, according to Concentration-Compactness principle [16], there exists an at most countable set $\mathfrak{J}$ and $\{x_i\}_{i\in\mathfrak{J}} \subset \mathbb{R}^N$, $\{\mu_i\}_{i\in\mathfrak{J}} \subset (0, +\infty)$, $\{\nu_i\}_{i\in\mathfrak{J}} \subset (0, +\infty)$ such that

$$\begin{aligned} \nu &= \sum_{i\in\mathfrak{J}} \nu_i \delta_{x_i} + |v|^{p^*}, \\ \mu &\geqslant \sum_{i\in\mathfrak{J}} \mu_i \delta_{x_i} + |\nabla v|^p, \end{aligned} \tag{2.24}$$

with $S\nu_i^{\frac{p}{p^*}} \leqslant \mu_i$, where

$$S := \inf_{\substack{v\in W\setminus\{0\},\\ |v|_{p^*}=1}} \frac{\|v\|^p}{|v|_{p^*}^p}.$$

Our claim is that $\nabla v_n^\pm \to \nabla v^\pm$ almost everywhere in $\mathbb{R}^N$. Following the approach in [3,9], we will carry out the proof in two steps.

Step a) The set $\mathfrak{J}$ is either finite or empty. Let's take a fixed $i \in \mathfrak{J}$ and let $\epsilon > 0$ be chosen small enough. Utilizing Lemma 2.3, we can conclude that

$$|v_n\phi_\epsilon(x-x_i)|_p^p \leqslant |v_n|_p^p,$$

and

$$\begin{aligned}|\nabla(v_n\phi_\epsilon(x-x_i))|_p^p &\leqslant 2^{p-1}\Big||\nabla\phi_\epsilon(x-x_i)|^p\,|v_n|^p\Big|_1 + 2^{p-1}\Big||\phi_\epsilon(x-x_i)|^p|\nabla v_n|^p\Big|_1\\ &\leqslant 2^{p-1}|v_n|_{p^*}^p|\nabla\phi|^p_{L^{\frac{p^*p}{p^*-p}}(\mathbb{R}^N)} + 2^{p-1}\|v_n\|^p.\end{aligned}$$

Consequently, a constant C_1 exists such that, for any $\epsilon > 0$ and $i \in \mathfrak{J}$, we have

$$\|v_n\phi_\epsilon(x-x_i)\| \leqslant C_1.$$

It is clear that $\{v_n\phi_\epsilon(x-x_i)\}_{n=1}^{+\infty} \subset W$. Therefore, $\{v_n\phi_\epsilon(x-x_i)\}$ is bounded in W. For any fixed $x_i \in \mathfrak{J}$, from (2.21), we obtain

$$\langle(\mathcal{K}_b^\lambda)'(v_n), v_n\phi_\epsilon(x-x_i)\rangle = o(1).$$

Thus, one gets

$$\begin{aligned}&(1+b\|v_n\|^p)\int|\nabla v_n|^p\,\phi_\epsilon(x-x_i)\,\mathrm{d}x + (1+b\|v_n\|^p)\int|\nabla v_n|^{p-2}\,\nabla v_n\cdot v_n\nabla\phi_\epsilon(x-x_i)\,\mathrm{d}x\\ &\quad= \int|v_n|^{p^*}\phi_\epsilon(x-x_i)\,\mathrm{d}x + \lambda\int f(v_n)v_n\phi_\epsilon(x-x_i)\,\mathrm{d}x + o(1).\end{aligned} \tag{2.25}$$

By means of the Young inequality, for any $\delta > 0$, there is C_δ such that

$$\left|\int|\nabla v_n|^{p-2}\nabla v_n\cdot v_n\nabla\phi_\epsilon(x-x_i)\,\mathrm{d}x\right| \leqslant C_\delta|v_n\nabla\phi_\epsilon(x-x_i)|_p^p + \delta\|v_n\|_p^p.$$

Applying the Strauss Lemma [19], we obtain

$$\lim_{n\to\infty}|v_n\nabla\phi_\epsilon(x-x_i)|_p^p = |v\nabla\phi_\epsilon(x-x_i)|_p^p.$$

Therefore, utilizing Lemma 2.3, we derive

$$\begin{aligned}\limsup_{n\to\infty}\left|\int|\nabla v_n|^{p-2}\nabla v_n\cdot v_n\nabla\phi_\epsilon(x-x_i)\,\mathrm{d}x\right| &\leqslant C_\delta|v\nabla\phi_\epsilon(x-x_i)|_p^p + \delta\limsup_{n\to\infty}\|v_n\|^p\\ &\leqslant \delta C_2 + C_\delta C_3|v|^p_{L^{p^*}(B_\epsilon(x_i))},\end{aligned} \tag{2.26}$$

where $C_3 = |\nabla\phi|^p_{\frac{p^*p}{p^*-p}}$ is constant and independent of ϵ and δ. Using (2.4) along with the Strauss Lemma [19] and (2.22), we deduce

$$\lim_{n\to\infty}\int f(v_n)v_n\phi_\epsilon(x-x_i)\,\mathrm{d}x = \int f(v)v\phi_\epsilon(x-x_i)\,\mathrm{d}x. \tag{2.27}$$

Combining (2.25)–(2.27) and using (2.20), we obtain

$$\begin{aligned}\lim_{n\to\infty}\int|\nabla v_n|^p\,\phi_\epsilon(x-x_i)\,\mathrm{d}x &\leqslant \lambda\int f(v)v\phi_\epsilon(x-x_i)\,\mathrm{d}x + \lim_{n\to\infty}\int|v_n|^{p^*}\phi_\epsilon(x-x_i)\,\mathrm{d}x\\ &\quad+ \delta(1+bM^p)\limsup_{n\to\infty}\|v_n\|^p + C_\delta C_3(1+bM^p)|v|^p_{L^{p^*}(B_\epsilon(x_i))}.\end{aligned}$$

Therefore, since $\{v_n\}$ is bounded in W and by (2.23), we can conclude that

$$\int \phi_\epsilon(x-x_i)\,\mathrm{d}\mu \leqslant \lambda \int f(v)v\phi_\epsilon(x-x_i)\,\mathrm{d}x + \int \phi_\epsilon(x-x_i)\,\mathrm{d}\nu + \delta C_4 + C_\delta C_5 |v|^p_{L^{p^*}(B_\epsilon(x_i))},$$

where C_4 and C_5 are independent of ϵ and δ. Thus, by the conditions $0 \leqslant \phi \leqslant 1$ and $\phi \equiv 1$ on $B_{1/2}(0)$, one gets

$$\begin{aligned} \mu\left(B_{\epsilon/2}(x_i)\right) &\leqslant \int \phi_\epsilon(x-x_i)\,\mathrm{d}\nu \\ &\leqslant \nu\left(B_\epsilon(x_i)\right) + \lambda \int\limits_{B_\epsilon(x_i)} f(v)v\,\mathrm{d}x + \delta C_4 + C_\delta C_5 |v|^p_{L^{p^*}(B_\epsilon(x_i))}. \end{aligned} \tag{2.28}$$

From (2.24), we have that

$$\begin{aligned} \nu_i^{\frac{p}{p^*}} S &\leqslant \mu_i \\ &\leqslant \mu\left(B_{\epsilon/2}(x_i)\right), \end{aligned}$$

and

$$\lim_{\epsilon \to 0^+} \nu\left(B_\epsilon(x_i)\right) = \nu_i.$$

Taking the limit as $\epsilon \to 0^+$ in (2.28) yields

$$a\nu_i^{\frac{p}{p^*}} S \leqslant \nu_i + \delta C_4.$$

As $\delta > 0$ is arbitrary, we conclude that

$$a\nu_i^{\frac{p}{p^*}} S \leqslant \nu_i,$$

i.e.,

$$\nu_i \geqslant (aS)^{\frac{p^*}{p^*-p}} > 0.$$

Since ν is a bounded positive measure, it follows that $\mathfrak{J}$ must be either a empty set or finite. The proof of step a is now finished.

Step b) The convergence $\nabla v_n \to \nabla v$ holds a.e. in $\mathbb{R}^N$. First, let's assume that $\mathfrak{J}$ is a finite nonempty set, namely $\mathfrak{J} = \{1, 2, \ldots, m\}$. For any $\epsilon > 0$, we define an open set $\Omega_\epsilon \subset \mathbb{R}^N$ as follows

$$\Omega_\epsilon = B_{1/2\epsilon}(0) \setminus \bigcup_{i=1}^m B_\epsilon(x_i).$$

There is $\epsilon_0 > 0$ such that

$$B_{\epsilon_0}(x_{i_1}) \cap B_{\epsilon_0}(x_{i_2}) = \emptyset,$$

for any $i_1, i_2 \in \mathfrak{J}$ with $i_1 \neq i_2$, which are possible since $\mathfrak{J}$ is a finite set. Now, let $0 < \zeta < \epsilon_0$. Our claim is

$$\int_{\Omega_\zeta} \left(|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v\right) \cdot \nabla(v_n - v)\,\mathrm{d}x \to 0. \tag{2.29}$$

Indeed, Define, for $x \in \mathbb{R}^N$ with $0 < \epsilon < \zeta$,

$$\psi_\epsilon(x) = \phi_{1/\epsilon}(x) - \sum_{i=1}^m \phi_\epsilon\left(x - x_i\right),$$

where ϕ was given in Lemma (2.3). Therefore, we have

$$\psi_\epsilon(x) \in C_c^\infty\left(\mathbb{R}^N\right),$$
$$\psi_\epsilon(x) = 0 \text{ on } \bigcup_{i=1}^{m} B_{\epsilon/2}(x_i),$$
$$\psi_\epsilon(x) = 1 \text{ on } B_{1/2\epsilon}(0) \setminus \bigcup_{i=1}^{m} B_\epsilon(x_i).$$

From Lemma 2.3 and the fact that $\{v_n\}$ is bounded in W for all $n \in \mathbb{N}$, It is straightforward to verify that $\{\psi_\epsilon v_n\}$ is bounded in W for all n. Using (2.21), we deduce that as $n \to \infty$, the following holds

$$(1+b\|v_n\|^p)\left(\int |\nabla v_n|^p \psi_\epsilon \, dx + \int |\nabla v_n|^{p-2}\nabla v_n \cdot v_n \nabla \psi_\epsilon \, dx\right) = \lambda \int f(v_n) v_n \psi_\epsilon \, dx + \int |v_n|^{p^*} \psi_\epsilon \, dx + o(1).$$

As in step a, we conclude that

$$\lim_{n\to\infty}(1+b\|v_n\|^p)\int |\nabla v_n|^p \, \psi_\epsilon \, dx = \lim_{n\to\infty}\left(-(1+b\|v_n\|^p)\int |\nabla v_n|^{p-2}\nabla v_n \cdot v_n \nabla \psi_\epsilon \, dx\right) + \lim_{n\to\infty}\int |v_n|^{p^*}\psi_\epsilon \, dx + \lambda \int f(v) v \psi_\epsilon \, dx.$$

Given that $\psi_\epsilon \equiv 0$ on $\bigcup_{i=1}^m B_{\epsilon/2}(x_i)$ and $\nu = |v|^{p^*} + \sum_{i=1}^m \nu_i \delta_{x_i}$, we obtain

$$\lim_{n\to\infty}\int |v_n|^{p^*} \, \psi_\epsilon \, dx = \int \psi_\epsilon \, d\nu = \int |v|^{p^*} \psi_\epsilon \, dx.$$

Thus, one has

$$\lim_{n\to\infty}(1+b\|v_n\|^p)\int |\nabla v_n|^p \, \psi_\epsilon \, dx = \lim_{n\to\infty}\left(-(1+b\|v_n\|^p)\int |\nabla v_n|^{p-2}\nabla v_n \cdot u_n \nabla \psi_\epsilon \, dx\right) + \lambda \int f(v) v \psi_\epsilon \, dx + \int |v|^{p^*}\psi_\epsilon \, dx. \tag{2.30}$$

Similarly, in light of (2.21), we conclude that as $n \to \infty$, the following holds

$$(1+b\|v_n\|^p)\int |\nabla v_n|^{p-2} \, \nabla v_n \cdot \psi_\epsilon \nabla v \, dx = -(1+b\|v_n\|^p)\int |\nabla v_n|^{p-2}\nabla v_n \cdot v \nabla \psi_\epsilon \, dx + \int |v_n|^{p^*-2} v_n v \psi_\epsilon \, dx + \lambda \int f(v_n) v \psi_\epsilon \, dx + o(1). \tag{2.31}$$

From Lemma 2.3 and the Young inequality, one has

$$\limsup_{n\to\infty}\left|\int |\nabla v_n|^{p-2}\nabla v_n \cdot v \nabla \psi_\epsilon \, dx\right| \leqslant \delta \limsup_{n\to\infty}\|v_n\|^p + C_\delta |v \nabla \psi_\epsilon|_p^p \leqslant \delta C_2 + C_\delta C_3 |v|^p_{L^{p^*}(B_\epsilon(x_i))}. \tag{2.32}$$

Utilizing the fact that $h : \mathbb{R}^N \to \mathbb{R}$ defined by $h(x) = |x|^p$ is strictly convex, one has

$$0 \leqslant \left(|\nabla v_n|^{p-2} \nabla v_n - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla \left(v_n - v \right), \tag{2.33}$$

On the other hand, since $\nabla v_n \rightharpoonup \nabla v$ in $L^p(\mathbb{R}^N)$, one has

$$\int \psi_\epsilon |\nabla v|^{p-2} \nabla v \cdot (\nabla v_n - \nabla v) \, \mathrm{d}x \to 0. \tag{2.34}$$

Then, taking into account that $\psi_\epsilon \equiv 1$ on Ω_ϵ, and combining (2.16), (2.22), (2.30)-(2.31), (2.33)-(2.34), along with the boundedness of $\{v_n\}$, it follows that as n tends to infinity, we deduce that

$$\begin{aligned}
&\int_{\Omega_\zeta} \left(|\nabla v_n|^{p-2} \nabla v_n - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla (v_n - v) \, \mathrm{d}x \\
&\quad \leqslant (1 + b\|v_n\|^p) \int \psi_\epsilon \left(|\nabla v_n|^{p-2} \nabla v_n - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla (v_n - v) \, \mathrm{d}x \\
&\quad = (1 + b\|v_n\|^p) \int \Big(\psi_\epsilon |\nabla v_n|^{p-2} \nabla v_n \cdot \nabla v_n - \psi_\epsilon |\nabla v_n|^{p-2} \nabla v_n \cdot \nabla v \\
&\qquad - \psi_\epsilon |\nabla v|^{p-2} \nabla v \cdot \nabla v_n + \psi_\epsilon |\nabla v|^{p-2} \nabla v \cdot \nabla v \Big) \, \mathrm{d}x \\
&\quad = \lim_{n \to \infty} (1 + b\|v_n\|^p) \int |\nabla v_n|^{p-2} \nabla v_n \cdot v \nabla \psi_\epsilon \, \mathrm{d}x \\
&\qquad - \lim_{n \to \infty} (1 + b\|v_n\|^p) \int |\nabla v_n|^{p-2} \nabla v_n \cdot v_n \nabla \psi_\epsilon \, \mathrm{d}x + o(1).
\end{aligned}$$

Given that $0 \leqslant \psi_\varepsilon \leqslant 1$, and $\psi_\epsilon = 1$ on Ω_ϵ for $0 < \epsilon < \epsilon_0$, by (2.26) and (2.32), as n tends to infinity, we conclude that

$$\begin{aligned}
\int\limits_{\Omega_\zeta} \left(|\nabla v_n|^{p-2} \nabla v_n - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla (v_n - v) \, \mathrm{d}x &\leqslant 2(1 + bM^p) C_\delta C_3 |v|^p_{L^{p^*}(B_\epsilon(x_i))} \\
&\quad + 2(1 + bM^p) \delta C_2 + o(1),
\end{aligned} \tag{2.35}$$

where constants C_2, C_3 are independent of δ and ϵ. Therefore, in (2.35), first let $n \to \infty$, then let $\epsilon \to 0^+$, and finally let $\delta \to 0^+$, yielding

$$\lim_{n \to \infty} \int\limits_{\Omega_\zeta} \left(|\nabla v_n|^{p-2} \nabla v_n - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla (v_n - v) \, \mathrm{d}x = 0. \tag{2.36}$$

Using an elementary inequality (see [18])

$$\left(a - b, |a|^{p-2} a - |b|^{p-2} b \right) \geqslant \begin{cases} C_p |a - b|^p & \text{if } p \geqslant 2, \\ C_p \frac{|a-b|^2}{(|a| + |b|)^{2-p}} & \text{if } 1 < p < 2, \end{cases}$$

where $a, b \in \mathbb{R}^N$. If $p \geqslant 2$, the convergence of (2.36) to zero as $n \to \infty$ implies that

$$\lim_{n \to \infty} C_p \int\limits_{\Omega_\zeta} |\nabla v_n - \nabla v|^p \, \mathrm{d}x = 0. \tag{2.37}$$

Moreover, when $1 < p < 2$, one has

$$\lim_{n \to \infty} C_p \int\limits_{\Omega_\zeta} \frac{|\nabla v_n - \nabla v|^2}{(|\nabla v| + |\nabla v_n|)^{2-p}} \, \mathrm{d}x = 0.$$

Therefore, applying the Hölder inequality, one gets

$$\begin{aligned}\int\limits_{\Omega_\zeta} |\nabla (v_n - v)|^p \, \mathrm{d}x &= \int\limits_{\Omega_\zeta} \frac{|\nabla (v_n - v)|^p (|\nabla v_n| + |\nabla v|)^{p(p-2)/2}}{(|\nabla v_n| + |\nabla v|)^{p(p-2)/2}} \, \mathrm{d}x \\ &\leqslant \left(\int\limits_{\Omega_\zeta} \frac{|\nabla (v_n - v)|^2}{(|\nabla v_n| + |\nabla v|)^{2-p}} \, \mathrm{d}x \right)^{p/2} \\ &\quad \times \left(\int\limits_{\Omega_\zeta} (|\nabla v_n| + |\nabla v|)^p \, \mathrm{d}x \right)^{(2-p)/2} .\end{aligned} \tag{2.38}$$

Hence, from (2.37) and (2.38), one can conclude

$$\nabla v_n \to \nabla v \quad \text{a.e. in } \Omega_\zeta.$$

Because ζ is arbitrary, by the Cantor diagonal argument, a subsequence $\{v_n\}$ can be chosen such that

$$\nabla v_n \to \nabla v \quad \text{a.e. on } \mathbb{R}^N, \tag{2.39}$$

as $n \to \infty$. If $\mathfrak{J}$ is empty, then $v_n \to v$ in L^{p^*}, it is easy to see that (2.36) is true for $\mathbb{R}^N$ in place of Ω_ζ. Hence (2.39) holds. From the convergences in (2.22) and (2.39), we have

$$\begin{aligned} v_n^\pm &\rightharpoonup v^\pm && \text{in } W, \\ v_n^\pm &\to v^\pm && \text{in } L^s \text{ for } p \leqslant s < p^*), \\ v_n^\pm &\to v^\pm && \text{a.e. in } \mathbb{R}^N, \\ \nabla v_n^\pm &\to \nabla v^\pm && \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

Using Fatou's lemma, the weak lower semicontinuity of norm, and the Brézis–Lieb lemma, one gets

$$\begin{aligned}
&\liminf_{n\to\infty} \mathcal{K}_b^\lambda(\alpha v_n^+ + \beta v_n^-) \\
&\geqslant \frac{\alpha^p}{p} \lim_{n\to\infty} \left(\|v^+\|^p + \|v_n^+ - v^+\|^p\right) + \frac{\beta^p}{p} \lim_{n\to\infty} \left(\|v^-\|^p + \|v_n^- - v^-\|^p\right) \\
&\quad + \frac{b\alpha^{2p}}{2p} \left[|\nabla v^+|_p^p + \lim_{n\to\infty} |\nabla v_n^+ - \nabla v^+|_p^p\right]^2 + \frac{b\beta^{2p}}{2p} \left[|\nabla v^-|_p^p + \lim_{n\to\infty} |\nabla v_n^- - \nabla v^-|_p^p\right]^2 \\
&\quad + \frac{b\alpha^p\beta^p}{p} \liminf_{n\to\infty} |\nabla v_n^+|_p^p |\nabla v_n^-|_p^p - \lambda \int F(\alpha v^+) \, \mathrm{d}x - \lambda \int F(\beta v^-) \, \mathrm{d}x \\
&\quad - \frac{\alpha^{p^*}}{p^*} \left[|v^+|_{p^*}^{p^*} + \lim_{n\to\infty} |v_n^+ - v^+|_{p^*}^{p^*}\right] - \frac{\beta^{p^*}}{p^*} \left[|v^-|_{p^*}^{p^*} + \lim_{n\to\infty} |v_n^- - v^-|_{p^*}^{p^*}\right] \\
&\geqslant \mathcal{K}_b^\lambda(\alpha v^+ + \beta v^-) + \frac{\alpha^p}{p} A_1 + \frac{b\alpha^{2p}}{2p} A_3^2 + \frac{b\alpha^{2p}}{p} A_3 |\nabla v^+|_p^p - \frac{\alpha^{p^*}}{p^*} B_1 + \frac{\beta^p}{p} A_2 \\
&\quad + \frac{b\beta^{2p}}{2p} A_4^2 + \frac{b\beta^{2p}}{p} A_4 |\nabla v^-|_p^p - \frac{\beta^{p^*}}{p^*} B_2,
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \lim_{n\to\infty} \|v_n^+ - v^+\|^p, & A_2 &= \lim_{n\to\infty} \|v_n^- - v^-\|^p, & A_3 &= \lim_{n\to\infty} |\nabla v_n^+ - \nabla v^+|_p^p, \\
A_4 &= \lim_{n\to\infty} |\nabla v_n^- - \nabla v^-|_p^p, & B_1 &= \lim_{n\to\infty} |v_n^+ - v^+|_{p^*}^{p^*}, & B_2 &= \lim_{n\to\infty} |v_n^- - v^-|_{p^*}^{p^*}.
\end{aligned}$$

In view of Lemma 2.1 (A_2), one obtains

$$\mathcal{K}_b^\lambda(v_n) \geqslant \mathcal{K}_b^\lambda(\alpha v_n^+ + \beta v_n^-),$$

for all $\alpha, \beta \geqslant 0$. That is,

$$\begin{aligned} c_b^\lambda \geqslant \mathcal{K}_b^\lambda(\alpha v^+ + \beta v^-) + \frac{\alpha^p}{p} A_1 + \frac{b\alpha^{2p}}{2p} A_3^2 + \frac{b\alpha^{2p}}{p} A_3 |\nabla v^+|_p^p - \frac{\alpha^{p^*}}{p^*} B_1 + \frac{\beta^p}{p} A_2 \\ + \frac{b\beta^{2p}}{2p} A_4^2 + \frac{b\beta^{2p}}{p} A_4 |\nabla v^-|_p^p - \frac{\beta^{p^*}}{p^*} B_2, \end{aligned} \tag{2.40}$$

for all $\alpha, \beta \geqslant 0$.

Step 1: We claim that $v^\pm \neq 0$. Suppose, for the sake of contradiction, that $v^+ = 0$. Letting $\beta = 0$ in (2.40), one has

$$c_b^\lambda \geqslant \frac{\alpha^p}{p} A_1 + \frac{b\alpha^{2p}}{2p} A_3^2 - \frac{\alpha^{p^*}}{p^*} B_1, \tag{2.41}$$

for all $\alpha \geqslant 0$.

Firstly, we consider the case where $B_1 = 0$. If $A_1 = 0$, then $v_n^+ \to v^+$ in W. From (2.16), it follows that $\|v^+\| > 0$. This contradicts the fact that we assume $v^+ = 0$. If $A_1 > 0$, one has

$$c_b^\lambda \geqslant \frac{\alpha^p}{p} A_1 > 0,$$

for all $\alpha \geqslant 0$, which gives a contradiction with Lemma 2.2.

Finally, we consider the case where $B_1 > 0$. Lemma 2.2 ensures the existence of a $\lambda^\star > 0$ suth that the following inequality holds

$$c_b^\lambda < \frac{p}{N} S^{N/p^2}, \quad \forall \lambda \geqslant \lambda^\star. \tag{2.42}$$

From (1.13) and $B_1 > 0$, we conclude $A_1 > 0$. Using (2.41), one has

$$\begin{aligned} \frac{p}{N} S^{N/p^2} &\leqslant \frac{p}{N} \left[\frac{A_1}{B_1^{p/p^*}} \right]^{N/p^2} \\ &\leqslant \max_{\alpha \geqslant 0} \left\{ \frac{\alpha^p}{p} A_1 - \frac{\alpha^{p^*}}{p^*} B_1 \right\} \\ &\leqslant \max_{\alpha \geqslant 0} \left\{ \frac{\alpha^p}{p} A_1 + \frac{b\alpha^{2p}}{2p} A_3^2 - \frac{\alpha^{p^*}}{p^*} B_1 \right\} \\ &\leqslant c_b^\lambda. \end{aligned}$$

Thanks to (2.42), by (2.41), ones has

$$\begin{aligned} c_b^\lambda &< \frac{p}{N} S^{N/p^2} \\ &\leqslant \max_{\alpha \geqslant 0} \left\{ \frac{\alpha^p}{p} A_1 + \frac{b\alpha^{2p}}{2p} A_3^2 - \frac{\alpha^{p^*}}{p^*} B_1 \right\} \\ &\leqslant c_b^\lambda, \end{aligned}$$

which is a contradiction. Consequently, $v^+ \neq 0$. Likewise, we deduce that $v^- \neq 0$ in a similar manner.

Step 2: We will prove that $B_1 = B_2 = 0$. We just prove $B_1 = 0$ (in an analogous manner, we demonstrate that $B_2 = 0$). Suppose, for the sake of contradiction, that $B_1 > 0$.

First case $B_2 > 0$. As both $B_1 > 0$, and $B_2 > 0$, it follows A_1, and $A_2 > 0$. Let $\widetilde{\alpha}$ and $\widetilde{\beta}$ satisfy

$$\frac{\widetilde{\alpha}^p}{p}A_1 + \frac{b\widetilde{\alpha}^{2p}}{2p}A_3^2 - \frac{\widetilde{\alpha}^{p^*}}{p^*}B_1 = \max_{\alpha\geqslant 0}\left\{\frac{\alpha^p}{p}A_1 + \frac{b\alpha^{2p}}{2p}A_3^2 - \frac{\alpha^{p^*}}{p^*}B_1\right\},$$

and

$$\frac{\widetilde{\beta}^p}{p}A_2 + \frac{b\widetilde{\beta}^{2p}}{2p}A_4^2 - \frac{\widetilde{\beta}^{p^*}}{p^*}B_2 = \max_{\beta\geqslant 0}\left\{\frac{a\beta^p}{p}A_2 + \frac{b\beta^{2p}}{2p}A_4^2 - \frac{\beta^{p^*}}{p^*}B_2\right\}.$$

Due to the compactness of the set $[0,\widetilde{\alpha}] \times [0,\widetilde{\beta}]$ is compact and the continuity of g_v, there is a pair$(\alpha_v, \beta_v) \in [0,\widetilde{\alpha}] \times [0,\widetilde{\beta}]$ satisfying

$$g_v(\alpha_v, \beta_v) = \max_{(\alpha,\beta)\in[0,\widetilde{\alpha}]\times[0,\widetilde{\beta}]} g_v(\alpha, \beta).$$

Therefore, it suffices to verify that a maximum cannot be attained on the boundary, i.e., $(\alpha_v, \beta_v) \in (0,\widetilde{\alpha}) \times (0,\widetilde{\beta})$. Observe that if β is sufficiently small, one has

$$\begin{aligned} g_v(\alpha, 0) &= \mathcal{K}_b^\lambda(\alpha v^+) \\ &< \mathcal{K}_b^\lambda(\alpha v^+) + \mathcal{K}_b^\lambda(\beta v^-) \\ &\leqslant \mathcal{K}_b^\lambda(\alpha v^+ + \beta v^-) \\ &= g_v(\alpha, \beta), \end{aligned}$$

for all $\alpha \in [0,\widetilde{\alpha}]$. Thus, there is $\beta_0 \in [0,\widetilde{\beta}]$ such that $g_v(\alpha, 0) \leqslant g_v(\alpha, \beta_0)$, for all $\alpha \in [0,\widetilde{\alpha}]$, i.e., $(\alpha_v, \beta_v) \notin [0,\widetilde{\alpha}] \times \{0\}$. Using a similar approach, we can demonstrate that $(\alpha_v, \beta_v) \notin \{0\} \times [0,\widetilde{\beta}]$.

It is straightforward to observe that

$$\frac{\alpha^p}{p}A_1 + \frac{b\alpha^{2p}}{2p}A_3^2 + \frac{b\alpha^{2p}}{p}A_3|\nabla v^+|_p^p - \frac{\alpha^{p^*}}{p^*}B_1 > 0, \quad \alpha \in (0,\widetilde{\alpha}], \tag{2.43}$$

and

$$\frac{\beta^p}{p}A_2 + \frac{b\beta^{2p}}{2p}A_4^2 + \frac{b\beta^{2p}}{p}A_4|\nabla v^-|_p^p - \frac{\beta^{p^*}}{p^*}B_2 > 0, \quad \beta \in (0,\widetilde{\beta}]. \tag{2.44}$$

Therefore, one gets

$$\begin{aligned} \frac{p}{N}S^{N/p^2} \leqslant{}& \frac{\widetilde{\alpha}^p}{p}A_1 + \frac{b\widetilde{\alpha}^{2p}}{2p}A_3^2 + \frac{b\widetilde{\alpha}^{2p}}{p}A_3\int |\nabla v^+|^p\,\mathrm{d}x - \frac{\widetilde{\alpha}^{p^*}}{p^*}B_1 + \frac{\beta^p}{p}A_2 \\ &+ \frac{b\beta^{2p}}{2p}A_4^2 + \frac{b\beta^{2p}}{p}A_4|\nabla v^-|_p^p - \frac{\beta^{p^*}}{p^*}B_2, \end{aligned}$$

and

$$\begin{aligned} \frac{p}{N}S^{N/p^2} \leqslant{}& \frac{\alpha^p}{p}A_1 + \frac{b\alpha^{2p}}{2p}A_3^2 + \frac{b\alpha^{2p}}{p}A_3|\nabla v^+|_p^p - \frac{\alpha^{p^*}}{p^*}B_1 + \frac{\widetilde{\beta}^p}{p}A_2 \\ &+ \frac{b\widetilde{\beta}^{2p}}{2p}A_4^2 + \frac{b\widetilde{\beta}^{2p}}{p}A_4|\nabla v^-|_p^p - \frac{\widetilde{\beta}^{p^*}}{p^*}B_2, \end{aligned}$$

for all $\alpha \in [0, \widetilde{\alpha}], \beta \in [0, \widetilde{\beta}]$. Combining this with (2.40), one obtains

$$g_v(\alpha, \widetilde{\beta}) \leqslant 0, \quad \forall \alpha \in [0, \widetilde{\alpha}],$$

and, one also has

$$g_v(\widetilde{\alpha}, \beta) \leqslant 0, \quad \forall \beta \in [0, \widetilde{\beta}].$$

That is, $(\alpha_v, \beta_v) \notin \{\widetilde{\alpha}\} \times [0, \widetilde{\beta}]$ and $(\alpha_v, \beta_v) \notin \times [0, \widetilde{\alpha}] \times \{\widetilde{\beta}\}$.

In conclusion, it follows that $(\alpha_v, \beta_v) \in (0, \widetilde{\alpha}) \times (0, \widetilde{\beta})$. Thus, the pair (α_v, β_v) is a critical point of g_v. Hence, $\alpha_v v^+ + \beta_v v^- \in \mathcal{M}_b^\lambda$. So, combining (2.40), (2.43) and (2.44), one has that

$$\begin{aligned} c_b^\lambda &\geqslant \mathcal{K}_b^\lambda(\alpha_v v^+ + \beta_v v^-) + \frac{\alpha_v^p}{p} A_1 + \frac{b\alpha_v^{2p}}{2p} A_3^2 + \frac{b\alpha_v^{2p}}{p} A_3 |\nabla v^+|_p^p - \frac{\alpha_v^{p^*}}{p^*} B_1 + \frac{\beta_v^p}{p} A_2 \\ &\quad + \frac{b\beta_v^{2p}}{2p} A_4^2 + \frac{b\beta_v^{2p}}{p} A_4 |\nabla v^-|_p^p - \frac{\beta_v^{p^*}}{p^*} B_2 \\ &> \mathcal{K}_b^\lambda(\alpha_v v^+ + \beta_v v^-) \\ &\geqslant c_b^\lambda. \end{aligned}$$

This leads to a contradiction. Hence $B_1 = 0$.

Second case $B_2 = 0$. In this situation, maximization can be carried out in $[0, \widetilde{\alpha}] \times [0, \infty)$. In fact, it can be shown that there is $\beta_0 \in [0, \infty)$ satisfying

$$\mathcal{K}_b^\lambda(\alpha v^+ + \beta v^-) \leqslant 0,$$

for all $(\alpha, \beta) \in [0, \widetilde{\alpha}] \times [\beta_0, \infty)$. Hence, there is $(\alpha_v, \beta_v) \in [0, \widetilde{\alpha}] \times [0, \infty)$ that satisfies

$$g_v(\alpha_v, \beta_v) = \max_{(\alpha,\beta) \in [0,\widetilde{\alpha}] \times [0,\infty)} g_v(\alpha, \beta).$$

Next, we show that $(\alpha_v, \beta_v) \in (0, \widetilde{\alpha}) \times (0, \infty)$. One has

$$g_v(\alpha, 0) < g_v(\alpha, \beta), \quad \forall \alpha \in [0, \widetilde{\alpha}] \text{ and } \beta \text{ sufficiently small},$$

thus, $(\alpha_v, \beta_v) \notin [0, \widetilde{\alpha}] \times \{0\}$. At the same time, one has

$$g_v(0, \beta) < g_v(\alpha, \beta), \quad \forall \beta \in [0, \infty) \text{ and } \alpha \text{ sufficiently small}.$$

Therefore, we get that $(\alpha_v, \beta_v) \notin \{0\} \times [0, \infty)$.

Conversely, for all $\beta \in [0, \infty)$, we have

$$\begin{aligned} \frac{p}{N} S^{N/p^2} &\leqslant \frac{\widetilde{\alpha}^p}{p} A_1 + \frac{b\widetilde{\alpha}^{2p}}{2p} A_3^2 + \frac{b\widetilde{\alpha}^{2p}}{p} A_3 |\nabla v^+|_p^p - \frac{\widetilde{\alpha}^{p^*}}{p^*} B_1 + \frac{\beta^p}{p} A_2 \\ &\quad + \frac{b\beta^{2p}}{2p} A_4^2 + \frac{b\beta^{2p}}{p} A_4 |\nabla v^-|_p^p. \end{aligned}$$

Thus, one has

$$g_v(\widetilde{\alpha}, \beta) \leqslant 0, \quad \forall \beta \in [0, \infty).$$

Therefore, $(\alpha_v, \beta_v) \notin \{\widetilde{\alpha}\} \times [0, \infty)$. Then, $(\alpha_v, \beta_v) \in (0, \widetilde{\alpha}) \times (0, \infty)$. Hence, (α_v, β_v) is an inner maximizer of g_v in $[0, \widetilde{\alpha}) \times [0, \infty)$. And so $\alpha_v v^+ + \beta_v v^- \in \mathcal{M}_b^\lambda$.

Consequently, based on (2.43), we get

$$\begin{aligned}
c_b^\lambda &\geqslant \mathcal{K}_b^\lambda(\alpha_v v^+ + \beta_v v^-) + \frac{\alpha_v^p}{p} A_1 + \frac{b\alpha_v^{2p}}{2p} A_3^2 + \frac{b\alpha_v^{2p}}{p} A_3 |\nabla v^+|_p^p - \frac{\alpha_v^{p^*}}{p^*} B_1 + \frac{\beta_v^p}{p} A_2 \\
&\quad + \frac{b\beta_v^{2p}}{2p} A_4^2 + \frac{b\beta_v^{2p}}{p} A_4 |\nabla v^-|_p^p \\
&> \mathcal{K}_b^\lambda(\alpha_v v^+ + \beta_v v^-) \\
&\geqslant c_b^\lambda,
\end{aligned}$$

which is absurd.

Hence, based on the preceding discussion, it follows that $B_1 = B_2 = 0$.

Step 3: Lastly, we establish the attainment of c_b^λ. Given $v^\pm \neq 0$, according to Lemma 2.1, there is $\alpha_v, \beta_v > 0$ such that $\widetilde{v} := \alpha_v v^+ + \beta_v v^- \in \mathcal{M}_b^\lambda$. Moreover, we have that $\langle (\mathcal{K}_b^\lambda)'(v), v^\pm \rangle \leqslant 0$. Lemma (2.1) implies that $0 < \alpha_v, \beta_v \leqslant 1$.
Combining $v_n \in \mathcal{M}_b^\lambda$ and Lemma 2.1, one gets

$$\begin{aligned}
\mathcal{K}_b^\lambda(v_n^+ + v_n^-) &= \mathcal{K}_b^\lambda(v_n) \\
&\geqslant \mathcal{K}_b^\lambda(\alpha_v v_n^+ + \beta_v v_n^-).
\end{aligned}$$

Taking into consideration $B_1 = B_2 = 0$ and the semicontinuity of the norm, one has

$$\begin{aligned}
c_b^\lambda &\leqslant \mathcal{K}_b^\lambda(\widetilde{v}) = \mathcal{K}_b^\lambda(\widetilde{v}) - \frac{1}{2p} \left\langle (\mathcal{K}_b^\lambda)'(\widetilde{v}), \widetilde{v} \right\rangle \\
&= \frac{(p-1)}{2p} \|\widetilde{v}\|^p + \frac{p^* - 2p}{2pp^*} |\widetilde{v}|_{p^*}^{p^*} + \frac{\lambda}{2p} \int [f(\widetilde{v})\widetilde{v} - 2pF(\widetilde{v})] \, dx \\
&= \frac{(p-1)}{2p} \left(\|\alpha_v v^+\|^p + \|\beta_v v^-\|^p \right) + \frac{p^* - 2p}{2pp^*} \left[|\alpha_v v^+|_{p^*}^{p^*} + |\beta_v v^-|_{p^*}^{p^*} \right] \\
&\quad + \frac{\lambda}{2p} \left[\int f(\alpha_v v^+)\alpha_v v^+ - 2pF(\alpha_v v^+) + \int f(\beta_v v^-)\beta_v v^- - 2pF(\beta_v v^-) \right] \\
&\leqslant \frac{(p-1)}{2p} \|v\|^p + \frac{p^* - 2p}{2pp^*} |v|_{p^*}^{p^*} + \frac{\lambda}{2p} \int f(v)v - 2pF(v) \\
&\leqslant \liminf_{n \to \infty} \left[\mathcal{K}_b^\lambda(v_n) - \frac{1}{2p} \left\langle (\mathcal{K}_b^\lambda)'(v_n), v_n \right\rangle \right] \\
&= \liminf_{n \to \infty} \mathcal{K}_b^\lambda(v_n) \\
&= c_b^\lambda.
\end{aligned}$$

Therefore, it follows that $\alpha_v = \beta_v = 1$, and c_b^λ is attained by $v_b := v^+ + v^- \in \mathcal{M}_b^\lambda$. □

3 Proof of theorems

We begin by proving Theorem 1.3.

Proof of Theorem 1.3. By applying Lemma 2.4, we can demonstrate that the minimizer v_b for c_b^λ is a sign-changing solution to problem ($\mathcal{QKP}$). As $v_b \in \mathcal{M}_b^\lambda$, one has

$$\langle (\mathcal{K}_b^\lambda)'(v_b), v_b^\pm \rangle = 0.$$

For $(\alpha,\beta)\in(\mathbb{R}^+\times\mathbb{R}^+)\backslash(1,1)$, using Lemma 2.1 and Lemma 2.4, we can conclude

$$\mathcal{K}_b^\lambda(\alpha v_b^+ + \beta v_b^-) < \mathcal{K}_b^\lambda(v_b^+ + v_b^-) = c_b^\lambda. \tag{3.1}$$

At this point, we aim to demonstrate that $(\mathcal{K}_b^\lambda)'(v_b)=0$. Suppose, the contrary $(\mathcal{K}_b^\lambda)'(v_b)\neq 0$. It follows that there is $\delta>0$ and $\theta>0$ such that for all $\|u-v_b\|\geqslant 3\delta$, one has

$$\|(\mathcal{K}_b^\lambda)'(u)\|\geqslant\theta.$$

Select $\sigma\in\left(0,\min\left\{\frac{1}{2^{(p-1)}},\frac{\delta}{2^{(p-1)/p}\|v_b\|}\right\}\right)$. Define $D:=(1-\sigma,1+\sigma)\times(1-\sigma,1+\sigma)$, and

$$k(\alpha,\beta)=\alpha v_b^+ + \beta v_b^-,\quad \forall(\alpha,\beta)\in D.$$

From (3.1), one has that

$$\widetilde{c}_\lambda := \max_{\alpha,\beta\in\partial D}\mathcal{K}_b^\lambda\circ k < c_b^\lambda.$$

Let $S_\delta := B_\delta(v_b)$ and $\varepsilon:=\min\left\{\frac{(c_b^\lambda-\widetilde{c}_\lambda)}{2},\frac{\theta\delta}{8}\right\}$, it follows from [20, Lemma 2.3] that there is a deformation $\eta\in\mathcal{C}([0,1]\times W,W)$ such that

(a) If $u\notin(\mathcal{K}_b^\lambda)^{-1}\left(\left[c_b^\lambda-2\varepsilon,c_b^\lambda+2\varepsilon\right]\cap S_{2\delta}\right)$, then $\eta(1,u)=u$,

(b) $\eta\left(1,(\mathcal{K}_b^\lambda)^{c_b^\lambda+\varepsilon}\cap S_\delta\right)\subset\left(\mathcal{K}_b^\lambda\right)^{c_b^\lambda-\varepsilon}$,

(c) $\mathcal{K}_b^\lambda(\eta(1,u))\leqslant\mathcal{K}_b^\lambda(u),\quad\forall u\in W.$

To begin, we must demonstrate that

$$\max_{(\alpha,\beta)\in\bar{D}}\mathcal{K}_b^\lambda(\eta(1,k(\alpha,\beta)))<c_b^\lambda. \tag{3.2}$$

Indeed, by Lemma 2.1, one has

$$\mathcal{K}_b^\lambda(k(\alpha,\beta))\leqslant c_b^\lambda < c_b^\lambda+\varepsilon,$$

i.e,

$$k(\alpha,\beta)\in\left(\mathcal{K}_b^\lambda\right)^{c_b^\lambda+\varepsilon}.$$

Conversely, one has

$$\begin{aligned}\|k(\alpha,\beta)-v_b\|^p &= \|(\beta-1)v_b^- + (\alpha-1)v_b^+\|^p\\ &\leqslant 2^{p-1}\left((\beta-1)^p\|v_b^-\|^p+(\alpha-1)^p\|v_b^+\|^p\right)\\ &\leqslant 2^{p-1}\sigma^p\|v_b\|^p\\ &<\delta^p.\end{aligned}$$

It follows that

$$k(\alpha,\beta)\in S_\delta\quad\forall(\alpha,\beta)\in\bar{D}.$$

Thus, in light of (b), one gets

$$\mathcal{K}_b^\lambda(\eta(1,k(\alpha,\beta)))<c_b^\lambda-\varepsilon.$$

Hence, we confirm that (3.2) holds.

Our next step is to show that $\eta(1, k(D)) \cap \mathcal{M}_b^\lambda \neq \emptyset$, leading to a contradiction with the definition of c_b^λ. Define $h(\alpha,\beta) := \eta(1, k(\alpha,\beta))$,

$$\begin{aligned}\Psi_0(\alpha,\beta) :=& \left(\langle (\mathcal{K}_b^\lambda)'(k(\alpha,\beta)), v_b^+\rangle, \langle (\mathcal{K}_b^\lambda)'(k(\alpha,\beta)), v_b^-\rangle\right)\\ =& \left(\langle (\mathcal{K}_b^\lambda)'(\alpha v_b^+ + \beta v_b^-), v_b^+\rangle, \langle (\mathcal{K}_b^\lambda)'(\alpha v_b^+ + \beta v_b^-), v_b^-\rangle\right)\\ :=& \left(\varphi^1_{v_b}(\alpha,\beta), \varphi^2_{v_b}(\alpha,\beta)\right),\end{aligned}$$

and

$$\Psi_1(\alpha,\beta) := \left(\frac{1}{\alpha}\langle (\mathcal{K}_b^\lambda)'(h(\alpha,\beta)), (h(\alpha,\beta))^+\rangle, \frac{1}{\beta}\langle (\mathcal{K}_b^\lambda)'(h(\alpha,\beta)), (h(\alpha,\beta))^-\rangle\right).$$

A straightforward computation yields

$$\begin{aligned}\left.\frac{\varphi^1_{v_b}(\alpha,\beta)}{\partial\alpha}\right|_{(1,1)} =& (p-1)\|v_b^+\|^p - (p^*-1)|v_b^+|_{p^*}^{p^*} + b(2p-1)|\nabla v_b^+|_p^{2p}\\ &- \lambda\int \partial_\alpha f(v_b^+)(v_b^+)^2\,\mathrm{d}x + b(p-1)|\nabla v_b^+|_p^p|\nabla v_b^-|_p^p,\\ \left.\frac{\varphi^2_{v_b}(\alpha,\beta)}{\partial\beta}\right|_{(1,1)} =& (p-1)\|v_b^-\|^p - (p^*-1)|v_b^-|_{p^*}^{p^*} + b(2p-1)|\nabla v_b^-|_p^{2p}\\ &- \lambda\int \partial_\alpha f(v_b^-)(v_b^-)^2\,\mathrm{d}x + b(p-1)|\nabla v_b^-|_p^p|\nabla v_b^+|_p^p,\end{aligned}$$

$$\left.\frac{\varphi^1_{v_b}(\alpha,\beta)}{\partial\beta}\right|_{(1,1)} = bp|\nabla v_b^+|_p^p|\nabla v_b^-|_p^p, \qquad \left.\frac{\varphi^2_{v_b}(\alpha,\beta)}{\partial\alpha}\right|_{(1,1)} = bp|\nabla v_b^-|_p^p|\nabla v_b^+|_p^p.$$

Let

$$M = \begin{bmatrix} \left.\frac{\varphi^1_{v_b}(\alpha,\beta)}{\partial\alpha}\right|_{(1,1)} & \left.\frac{\varphi^2_{v_b}(\alpha,\beta)}{\partial\alpha}\right|_{(1,1)} \\ \left.\frac{\varphi^1_{v_b}(\alpha,\beta)}{\partial\beta}\right|_{(1,1)} & \left.\frac{\varphi^2_{v_b}(\alpha,\beta)}{\partial\beta}\right|_{(1,1)} \end{bmatrix}.$$

Employing condition $(\mathcal{P}_4)$ with $t \neq 0$, it follows that

$$\partial_t f(t)t^2 - (2p-1)f(t)t > 0.$$

Thus, since $v_b \in \mathcal{M}_b^\lambda$, one has

$$\begin{aligned}\det M =& \left.\frac{\varphi^1_{v_b}(\alpha,\beta)}{\partial\alpha}\right|_{(1,1)} \times \left.\frac{\varphi^2_{v_b}(\alpha,\beta)}{\partial\beta}\right|_{(1,1)} - \left.\frac{\varphi^1_{v_b}(\alpha,\beta)}{\partial\beta}\right|_{(1,1)} \times \left.\frac{\varphi^2_{v_b}(\alpha,\beta)}{\partial\alpha}\right|_{(1,1)}\\ =& \Big[p\|v_b^+\|^p + bp|\nabla v_b^+|_p^p|\nabla v_b^-|_p^p + (p^*-2p)|v_b^+|_{p^*}^{p^*}\\ &+ \lambda\int (\partial_\alpha f(v_b^+)(v_b^+)^2 - (2p-1)f(v_b^+)(v_b^+)\,\mathrm{d}x\Big]\\ &\times \Big[p\|v_b^-\|^p + bp|\nabla v_b^+|_p^p|\nabla v_b^-|_p^p + (p^*-2p)|v_b^-|_{p^*}^{p^*}\\ &+ \lambda\int (\partial_\beta f(v_b^-)(v_b^-)^2 - (2p-1)f(v_b^-)(v_b^-)\,\mathrm{d}x\Big]\\ &- (bp)^2|\nabla v_b^+|_p^{2p}|\nabla v_b^-|_p^{2p}\\ >& 0.\end{aligned}$$

As $(1,1)$ is the unique isolated zero of the $\mathcal{C}^1$ function $\Psi_0(\alpha,\beta)$, it follows from degree theory that $\deg(\Psi_0, D, 0) = 1$. Consequently, taking both (3.2) and (a) into account, one has

$$h(\alpha,\beta) = k(\alpha,\beta), \quad \text{on } \partial D.$$

Thus, it follows that $\deg(\Psi_1, D, 0) = 1$. Hence, $\Psi_1(\alpha_0,\beta_0) = 0$ for some $(\alpha_0,\beta_0) \in D$ so that

$$\eta(1, k(\alpha_0,\beta_0)) = h(\alpha_0,\beta_0) \in \mathcal{M}_b^\lambda.$$

Thereby contradicting (3.2). Thus, $(\mathcal{K}_b^\lambda)'(v_b) = 0$, which implies v_b is a critical point of $\mathcal{K}_b^\lambda$. Therefore, we conclude that v_b is a sign-changing solution for problem ($\mathcal{QKP}$). To conclude our proof, we show that v has exactly two nodal domains. We proceed by assuming, for contradiction, that

$$v = v_1 + v_2 + v_3, \quad \text{with } v_i \neq 0, v_1 \geqslant 0, v_2 \leqslant 0,$$
$$\operatorname{supp}(v_i) \cap \operatorname{supp}(v_j) = \emptyset, \quad \text{for } i \neq j,\ i,j = 1,2,3,$$

and

$$\langle (\mathcal{K}_b^\lambda)'(v), v_i\rangle = 0, \quad \text{for } i = 1,2,3.$$

Let $u := v_1 + v_2$, we observe that $u^+ = v_1$ and $u^- = v_2$. Thus, $u^\pm \neq 0$. Therefore, there is a unique (α_u,β_u) of positive numbers such that $\alpha_u v_1 + \beta_u v_2 \in \mathcal{M}_b^\lambda$.

Hence,

$$\mathcal{K}_b^\lambda(\alpha_u v_1 + \beta_u v_2) \geqslant c_b^\lambda.$$

Additionally, by noting that $\langle (\mathcal{K}_b^\lambda)'(v), v_i\rangle = 0$, one gets $\langle (\mathcal{K}_b^\lambda)'(u), u^\pm\rangle < 0$. From Lemma 2.1, it follows that $(\alpha_u,\beta_u) \in (0,1] \times (0,1]$. Conversely, one has

$$\begin{aligned}
0 &= \frac{1}{2p}\langle (\mathcal{K}_b^\lambda)'(v), v_3\rangle \\
&= \frac{1}{2p}\|v_3\|^p + \frac{b}{2p}|\nabla v_3|_p^{2p} + \frac{b}{2p}|\nabla v_3|_p^p|\nabla v_1|_p^p + \frac{b}{2p}|\nabla v_3|_p^p|\nabla v_2|_p^p \\
&\quad - \frac{\lambda}{2p}\int f(v_3)v_3 \, \mathrm{d}x - \frac{1}{2p}|v|_{p^*}^{p^*} \\
&< \mathcal{K}_b^\lambda(v_3) + \frac{b}{2p}|\nabla v_3|_p^p|\nabla v_1|_p^p + \frac{b}{2p}|\nabla v_3|_p^p|\nabla v_2|_p^p.
\end{aligned}$$

Hence, by (2.17) we obtain

$$\begin{aligned}
c_b^\lambda &\leqslant \mathcal{K}_b^\lambda(\alpha_u v_1 + \beta_u v_2) \\
&= \mathcal{K}_b^\lambda(\alpha_u v_1 + \beta_u v_2) - \frac{1}{2p}\langle (\mathcal{K}_b^\lambda)'(\alpha_u v_1 + \beta_u v_2), (\alpha_u v_1 + \beta_u v_2)\rangle \\
&= \frac{1}{2p}\left(\|\alpha_u v_1\|^p + \|\beta_u v_2\|^p\right) + \frac{\lambda}{2p}\int [f(\alpha_u v_1)(\alpha_u v_1) - 2pF(\alpha_u v_1)] \, \mathrm{d}x \\
&\quad + \frac{\lambda}{2p}\int [f(\beta_u v_2)(\beta_u v_2) - 2pF(\beta_u v_2)] \, \mathrm{d}x \\
&\quad + \frac{p^* - 2p}{2pp^*}\alpha_u^{p^*}|v_1|_{p^*}^{p^*} x + \frac{p^* - 2p}{2pp^*}\beta_u^{p^*}|v_2|_{p^*}^{p^*} \\
&\leqslant \mathcal{K}_b^\lambda(v_1 + v_2) - \frac{1}{2p}\langle (\mathcal{K}_b^\lambda)'(v_1 + v_2), (v_1 + v_2)\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{K}_b^\lambda(v_1+v_2) + \frac{1}{2p}\langle (\mathcal{K}_b^\lambda)'(v), v_3\rangle + \frac{b}{2p}|\nabla v_3|_p^p|\nabla v_1|_p^p + \frac{b}{2p}|\nabla v_3|_p^p|\nabla v_2|_p^p \\
&< \mathcal{K}_b^\lambda(v_1) + \mathcal{K}_b^\lambda(v_2) + \mathcal{K}_b^\lambda(v_3) + \frac{b}{2p}\left(|\nabla v_3|_p^p + |\nabla v_2|_p^p\right)|\nabla v_1|_p^p \\
&\quad + \frac{b}{2p}\left(|\nabla v_3|_p^p + |\nabla v_1|_p^p\right)|\nabla v_2|_p^p + \frac{b}{2p}\left(|\nabla v_1|_p^p + |\nabla v_2|_p^p\right)|\nabla v_3|_p^p \\
&= \mathcal{K}_b^\lambda(v) \\
&= c_b^\lambda,
\end{aligned}
$$

this leads to a contradiction, which implies that $v_3 = 0$. Hence, v has precisely two nodal domains. Utilizing Theorem 1.3, we prove the existence of a least energy sign-changing solution v_b to problem ($\mathcal{QKP}$). □

Next, we provide a proof for Theorem 1.4, asserting that the energy of v_b is strictly greater than twice the ground state energy.

Proof of Theorem 1.4. Similarly to the proof of Lemma 2.2, there is $\lambda_1^\star > 0$ such that for all $\lambda \geqslant \lambda_1^\star$, there is $v \in \mathcal{N}_b^\lambda$ such that $\mathcal{K}_b^\lambda(v) = c_{b,\lambda}^* > 0$. Using standard arguments the critical points of the functional $\mathcal{K}_b^\lambda$ on $\mathcal{N}_b^\lambda$ are critical points of $\mathcal{K}_b^\lambda$ in W. As a result, $(\mathcal{K}_b^\lambda)'(v) = 0$, which implies that v is a ground state solution of problem ($\mathcal{QKP}$). Theorem 1.3 indicates that for all $\lambda \geqslant \lambda^\star$, there exists a least energy sign-changing solution v_b to problem ($\mathcal{QKP}$) that changes its sign only once. Define $\lambda^{\star\star} = \max\{\lambda_1^\star, \lambda^\star\}$. Assume that $v_b = v_b^+ + v_b^-$. Following the approach in the proof of Lemma 2.1, there are $\alpha_{v_b^+} > 0$ and $\beta_{v_b^-} > 0$ such that

$$
\alpha_{v_b^+} v_b^+ \in \mathcal{N}_b^\lambda, \qquad \beta_{v_b^-} v_b^- \in \mathcal{N}_b^\lambda.
$$

Moreover, according to Lemma 2.1, $\alpha_{v_b^+}, \beta_{v_b^-} \in (0,1)$. Consequently, by applying Lemma 2.1 again, one obtains

$$
\begin{aligned}
2c_{b,\lambda}^* &\leqslant \mathcal{K}_b^\lambda(\alpha_{v_b^+} v_b^+) + \mathcal{K}_b^\lambda(\beta_{v_b^-} v_b^-) \\
&\leqslant \mathcal{K}_b^\lambda(\alpha_{v_b^+} v_b^+ + \beta_{v_b^-} v_b^-) \\
&< \mathcal{K}_b^\lambda(v_b^+ + v_b^-) \\
&= c_b^\lambda.
\end{aligned}
$$

Hence, $c_{b,\lambda}^* > 0$ cannot be attained by any sign-changing function. □

Lastly, we provide the proof for Theorem 1.5.

Proof of Theorem 1.5. **Step 1:** We prove that for any sequence $\{b_n\}$, $\{v_{b_n}\}$ is bounded in W if $b_n \searrow 0$. Let $\chi \in C_0^\infty\left(\mathbb{R}^N\right)$ be a nonzero function with fixed $\chi^\pm \neq 0$. Similar to the argument presented in Lemma 2.1, for any $b \in [0,1]$, there is (τ_1, τ_2) that are independent of b, such that

$$
\langle (\mathcal{K}_b^\lambda)'(\tau_1\chi^+ + \tau_2\chi^-), \tau_1\chi^+\rangle < 0, \quad \text{and} \quad \langle (\mathcal{K}_b^\lambda)'(\tau_1\chi^+ + \tau_2\chi^-), \tau_2\chi^-\rangle < 0.
$$

As per Lemma 2.1, for any $b \in [0,1]$, there is a unique $(\alpha_\chi(b), \beta_\chi(b)) \in (0,1] \times (0,1]$ such that $\bar{\chi} := \alpha_\chi(b)\tau_1\chi^+ + \beta_\chi(b)\tau_2\chi^- \in \mathcal{M}_b^\lambda$.

Hence, using (2.4), we can deduce that for any $b \in [0,1]$,

$$\begin{aligned}
\mathcal{K}_b^\lambda(v_b) &\leqslant \mathcal{K}_b^\lambda(\bar{\chi}) \\
&= \mathcal{K}_b^\lambda(\bar{\chi}) - \frac{1}{2p}\langle(\mathcal{K}_b^\lambda)'(\bar{\chi}), \bar{\chi}\rangle \\
&= \frac{(p-1)}{2p}\|\bar{\chi}\|^p + \frac{p^*-2p}{2pp^*}|\bar{\chi}|_{p^*}^{p^*} + \frac{\lambda}{2p}\int [f(\bar{\chi})\bar{\chi} - 2pF(\bar{\chi})]\,\mathrm{d}x \\
&\leqslant \frac{(p-1)}{2p}\|\bar{\chi}\|^p + \frac{p^*-2p}{2pp^*}|\bar{\chi}|_{p^*}^{p^*} + \frac{\lambda}{2p}\left(C_\varepsilon|\bar{\chi}|_\vartheta^\vartheta + \varepsilon|\bar{\chi}|_p^p\right) \\
&\leqslant \frac{(p-1)}{2p}\left(\|\tau_1\chi^+\|^p + \|\tau_2\chi^-\|^p\right) + \frac{p^*-2p}{2pp^*}\left[|\tau_1\chi^+|_{p^*}^{p^*} + |\tau_2\chi|_{p^*}^{p^*}\right] \\
&\quad + \frac{\lambda}{2p}\left[\left(\varepsilon|\tau_1\chi^+|_p^p + C_\varepsilon|\tau_1\chi^+|_\vartheta^\vartheta\right) + \left(\varepsilon|\tau_2\chi^-|_p^p + C_\varepsilon|\tau_2\chi^-|_\vartheta^\vartheta\right)\right] \\
&= C_\chi.
\end{aligned}$$

Therefore, as $n \to \infty$, one gets

$$\begin{aligned}
C_\chi + 1 &\geqslant \mathcal{K}_{b_n}^\lambda(v_{b_n}) \\
&= \mathcal{K}_{b_n}^\lambda(v_{b_n}) - \frac{1}{2p}\langle(\mathcal{K}_{b_n}^\lambda)'(v_{b_n}), v_{b_n}\rangle \\
&\geqslant \frac{(p-1)}{2p}\|v_{b_n}\|^p.
\end{aligned}$$

Hence, $\{v_{b_n}\}$ is bounded in W.

Step 2: We demonstrate the existence of a nodal solution v_0 to problem ($\mathcal{QKP}$). As the sequence $\{v_{b_n}\}$ is bounded in W, up to a subsequence, there is $v_0 \in W$ such that

$$\begin{aligned}
v_{b_n} &\rightharpoonup v_0 && \text{in } W, \\
v_{b_n} &\to v_0 && \text{in } L^s(\mathbb{R}^N), \text{ for } s \in [p, p^*), \\
v_{b_n} &\to v_0 && \text{a.e. in } \mathbb{R}^N, \\
\nabla v_{b_n} &\to \nabla v_0 && \text{a.e. in } \mathbb{R}^N.
\end{aligned} \tag{3.3}$$

Since $\{v_{b_n}\}$ is a weak solution of problem ($\mathcal{QKP}$) with $b = b_n$, one obtains

$$\begin{aligned}
\int (|\nabla v_{b_n}|^{p-2}\nabla v_{b_n}\cdot\nabla\varphi + a(x)|v_{b_n}|^{p-2}\varphi)\,\mathrm{d}x + b_n|\nabla v_{b_n}|_p^p \int |v_{b_n}|^{p-2}\nabla v_{b_n}\cdot\nabla\varphi\,\mathrm{d}x \\
= \lambda\int f(v_{b_n})\varphi dx + \int |v_{b_n}|^{p^*-2}v_{b_n}\varphi\,\mathrm{d}x,
\end{aligned} \tag{3.4}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. By combining equations (3.3) and (3.4) with Step 1, we can conclude that

$$\begin{aligned}
\int (|\nabla v_0|^{p-2}\nabla v_0\cdot\nabla\varphi + a(x)|v_0|^{p-2}\varphi)\,\mathrm{d}x + b_n|\nabla v_0|_p^p \int_{\mathbb{R}^N} |v_0|^{p-2}\nabla v_0\cdot\nabla\varphi\,\mathrm{d}x \\
= \lambda\int f(v_0)\varphi\,\mathrm{d}x + \int |v_0|^{p^*-2}v_0\varphi\,\mathrm{d}x,
\end{aligned}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Thus, v_0 is a weak solution to problem ($\mathcal{QKP}_0$). In a manner akin to Lemma 2.2, we can infer that $v_0^+ \neq 0$ and $v_0^- \neq 0$.

Step 3: Here, we demonstrate that problem ($\mathcal{QKP}_0$) has a least energy nodal solution u_0, and that there is a unique pair $(\alpha_{b_n}, \beta_{b_n}) \in \mathbb{R}^+ \times \mathbb{R}^+$ satisfying $\alpha_{b_n} u_0^+ + \beta_{b_n} u_0^- \in \mathcal{M}_{b_n}^\lambda$. Furthermore, we show that as $n \to \infty$, $(\alpha_{b_n}, \beta_{b_n})$ converges to $(1,1)$.

Using an approach similar to the one in the proof of Theorem 1.3, it follows that problem ($\mathcal{QKP}_0$) possesses a nodal solution u_0 with least energy, where $\mathcal{K}_0^\lambda(u_0) = c_0^\lambda$ and $(\mathcal{K}_0^\lambda)'(u_0) = 0$. Then, following Lemma 2.1, we can easily obtain the existence and uniqueness of a pair $(\alpha_{b_n}, \beta_{b_n})$ such that $\alpha_{b_n} u_0^+ + \beta_{b_n} u_0^- \in \mathcal{M}_{b_n}^\lambda$, with $\alpha_{b_n} > 0$ and $\beta_{b_n} > 0$. In order to complete the proof, it is enough to prove that $(\alpha_{b_n}, \beta_{b_n}) \to (1,1)$ as $n \to \infty$. Indeed, as $\alpha_{b_n} u_0^+ + \beta_{b_n} u_0^- \in \mathcal{M}_{b_n}^\lambda$, one has that

$$\begin{aligned} \alpha_{b_n}^p \|u_0^+\|^p + b_n \alpha_{b_n}^{2p} |\nabla u_0^+|_p^{2p} + b_n \alpha_{b_n}^p \beta_{b_n} |\nabla u_0^+|_p^p |\nabla u_0^-|_p^p \\ = \lambda \int f(\alpha_{b_n} u_0^+) \alpha_{b_n} u_0^+ \, dx + \alpha_{b_n}^{p^*} |u_0^+|_{p^*}^{p^*}, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \beta_{b_n}^p \|u_0^-\|^p + b_n \beta_{b_n}^{2p} |\nabla u_0^-|_p^{2p} + b_n \beta_{b_n}^p \alpha_{b_n} |\nabla u_0^-|_p^p |\nabla u_0^+|_p^p \\ = \lambda \int f(\alpha_{b_n} u_0^-) \beta_{b_n} u_0^- \, dx + \beta_{b_n}^{p^*} |u_0^-|_{p^*}^{p^*}. \end{aligned} \tag{3.6}$$

using the fact that $b_n \searrow 0$, it follows that $\{\alpha_{b_n}\}$ and $\{\beta_{b_n}\}$ are bounded. Suppose, up to a subsequence, $\alpha_{b_n} \to \alpha_0$ and $\beta_{b_n} \to \beta_0$. Then by (3.5) and (3.6), we have

$$\alpha_0^p \|u_0^+\|^p = \lambda \int f(\alpha_0 u_0^+) \alpha_0 u_0^+ \, dx + \alpha_0^{p^*} |u_0^+|_{p^*}^{p^*}, \tag{3.7}$$

and

$$\beta_0^p \|u_0^-\|^p = \lambda \int f(\beta_0 u_0^-) \beta_0 u_0^- \, dx + \beta_0^{p^*} |u_0^-|_{p^*}^{p^*}. \tag{3.8}$$

Noticing that v_0 is a nodal solution to problem ($\mathcal{QKP}_0$), one has

$$\|u_0^+\|^p = \lambda \int f(u_0^+) u_0^+ \, dx + |u_0^+|_{p^*}^{p^*}, \tag{3.9}$$

and

$$\|u_0^+\|^p = \lambda \int f\left(u_0^-\right) u_0^- \, dx + |u_0^-|_{p^*}^{p^*}. \tag{3.10}$$

Therefore, from (3.7)–(3.10), we can easily obtain that $(\alpha_0, \beta_0) = (1,1)$. We can now complete the proof of Theorem 1.5. We claim that v_0 obtained in Step 2 is a least energy solution to problem ($\mathcal{QKP}_0$). In fact, according to Step 3 and Lemma 2.1, we see that

$$\begin{aligned} \mathcal{K}_0^\lambda(u_0) \leqslant \mathcal{K}_0^\lambda(v_0) &= \lim_{n\to\infty} \mathcal{K}_{b_n}^\lambda(v_{b_n}) \\ &\leqslant \lim_{n\to\infty} \mathcal{K}_{b_n}^\lambda(\alpha_{b_n} u_0^+ + \beta_{b_n} u_0^-) \\ &= \lim_{n\to\infty} \mathcal{K}_0^\lambda(u_0^+ + u_0^-) \\ &= \mathcal{K}_0^\lambda\left(u_0\right), \end{aligned}$$

which completes the proof of Theorem 1.5. □

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Statements and Declarations

Conflict of interest

The authors declare that they have no conflict of interest.

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