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Strong fast invertibility and Lyapunov exponents for linear systems

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Abstract. With the goal of deriving the existence of a dominated splitting, Quas, Thieullen and Zarrabi introduced the concept of strong fast invertibility for linear cocycles in 2019. Here, we take a closer look at strongly fast invertible systems with bounded coefficients. By linking the dimensions at which a system admits strong fast invertibility to the multiplicities of Lyapunov exponents, we are able to give a full characterization of regular strongly fast invertible systems similar to that of systems with stable Lyapunov exponents. In particular, we show that the stability of Lyapunov exponents implies strong fast invertibility (even in the absence of regularity). Central to our arguments are certain induced systems on spaces of exterior products that represent the evolution of volumes.

Finally, we derive convergence results for the computation of Lyapunov exponents via Benettin's algorithm using perturbation theory. While the stronger assumption of stable Lyapunov exponents clearly leaves more freedom on how to choose stepsizes, we derive conditions for the stepsizes with which convergence can be ensured even if a system is only strongly fast invertible.

Keywords: strong fast invertibility, Lyapunov exponents, linear dynamical systems, Benettin's algorithm, convergence analysis.

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1 Introduction

Strong fast invertibility is a new property for dynamical systems introduced by Quas, Thieullen and Zarrabi in 2019 [21]. While it originally served as an ingredient to ensure uniform invertibility of a cocycle along its fastest growing direction (hence the name), strong fast invertibility can be best describes using the evolution of volumes: A linear system is L-dim. strongly fast invertible according to [21] if and only if there is a constant $c > 0$ such that

$$\frac{\| \wedge^L X(t,\tau)\|}{\| \wedge^L X(s,\tau)\|} \leq \| \wedge^L X(t,s)\| \leq c\frac{\| \wedge^L X(t,\tau)\|}{\| \wedge^L X(s,\tau)\|}$$

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for all $t \geq s \geq \tau$, where $\wedge^L X(t,s)$ denotes L-fold exterior product of the Cauchy matrix of the system (see Lemma 3.29). In other words, the maximal growth of L-volumes over a time interval can be computed (up to a constant independent of the interval) by bisecting the interval and computing the maximal growths of L-volumes on the subintervals.

By combining strong fast invertibility with a uniform singular value gap, Quas et al. prove the existence of a *dominated equivariant uniform splitting* of the dynamics, i.e., a splitting into equivariant fast and slow subspaces such that the angle between them is bounded from below and solutions corresponding to the fast subspace grow uniformly exponentially faster than solutions corresponding to the slow subspace. Using an equivalent notion, one may also call the splitting *integrally separated* [22, Definition 2.3].

The latter terminology is usually applied in studies of Lyapunov exponents. In particular, the existence of an integrally separated splitting is necessary to ensure the stability of Lyapunov exponents. Essentially, given an integrally separated splitting, the stability of Lyapunov exponents boils down to the stability on the subspaces of the splitting:

Theorem ([1, Theorem 5.4.9], [3]). *Assume a linear system with bounded coefficients and Lyapunov exponents $\lambda_1 > \cdots > \lambda_p$ with multiplicities $d_1 + \cdots + d_p = d$. The Lyapunov exponents are stable if and only if there exists a Lyapunov transformation reducing the system to block diagonal form*

$$\dot{y} = \operatorname{diag}(B_1(t), \ldots, B_p(t))y,$$

where $B_i(t) \in \mathbb{R}^{d_i \times d_i}$ is upper triangular, such that the following hold:

(i) all non-trivial solutions of $\dot{y}_i = B_i(t)y_i$ have characteristic exponent λ_i,

(ii) λ_i is stable for $\dot{y}_i = B_i(t)y_i$,

(iii) there are constants $a, b > 0$ such that

$$\|Y_i(t,s)^{-1}\|^{-1} \geq b e^{a(t-s)} \|Y_{i+1}(t,s)\|$$

for all $t \geq s$, where $Y_i(t,s)$ denotes the Cauchy matrix of $\dot{y}_i = B_i(t)y_i$.

One of our two main goals is to provide a characterization of strong fast invertibility that allows a direct comparison to stability of Lyapunov exponents via the above theorem. To connect strong fast invertibility, a property that concerns the evolution of volumes, to Lyapunov exponents, we work with certain induced systems on spaces of exterior products. These systems allow us to link Lyapunov exponents to volume growth assuming the original system is regular. Our characterization theorem is the following:

Theorem. *Assume a linear system with bounded coefficients and Lyapunov exponents $\lambda_1 > \cdots > \lambda_p$ with multiplicities $d_1 + \cdots + d_p = d$. If the system is regular and strongly fast invertible at dimensions $d_1 + \cdots + d_l$ for $l = 1, \ldots, p$, then there exists a Lyapunov transformation reducing the system to block diagonal form*

$$\dot{y} = \operatorname{diag}(B_1(t), \ldots, B_p(t))y,$$

where $B_i(t) \in \mathbb{R}^{d_i \times d_i}$ is upper triangular, such that the following hold:

(i) all non-trivial solutions of $\dot{y}_i = B_i(t)y_i$ have characteristic exponent λ_i,

(ii) there is a constant $b > 0$ such that

$$\|Y_i(t,s)^{-1}\|^{-1} \geq b \|Y_{i+1}(t,s)\|$$

for all $t \geq s$, where $Y_i(t,s)$ denotes the Cauchy matrix of $\dot{y}_i = B_i(t)y_i$.

Conversely, any block diagonal system $\dot{y} = B(t)y$ satisfying (i) and (ii) is strongly fast invertible at dimensions $d_1 + \cdots + d_l$ for $l = 1, \ldots, p$.

In particular, our theorem provides an equivalent characterization of strong fast invertibility for systems that are regular and have bounded coefficients. Moreover, it shows that the stability of Lyapunov exponents implies strong fast invertibility at the respective dimensions. More aspects, such as another characterization in the case of simple Lyapunov spectra, can be found in our article. However, we note that there are still interesting aspects to explore that we did not pursue here.

Our second main goal is to derive convergence results for the computation of Lyapunov exponents. More precisely, we focus on *Benettin's algorithm* [4,5] as it is the most fundamental and common algorithm to compute Lyapunov exponents. Its underlying idea is to propagate a set of linear perturbations that are reorthonormalized periodically. The Lyapunov exponents are then computed as averages of volume expansion via the rescaling factors from the orthonormalization procedure.

While it is not difficult to prove convergence of Benettin's algorithm in the absence of numerical errors, integration errors can accumulate and persist. This happens especially if the stepsizes are kept constant. In practice, it is hard to quantify these error, since the exact Lyapunov exponents are usually unknown.

Major efforts have been made by Dieci, Van Vleck and co-authors. They advocate the use of adaptive stepsizes to bound the local integration error and were able to prove error estimates for computed Lyapunov exponents of linear systems that are regular and have stable Lyapunov exponents [10, 11]. While Dieci and Van Vleck proved that the asymptotic limits of the computed exponents can be made arbitrarily close to the true Lyapunov exponents by decreasing the error tolerance or the fixed stepsize, true convergence requires to simultaneously increase the integration time and decrease the stepsizes. This was already conjectured by Mc Donald and Higham in their error analysis for autonomous linear systems in 2001 [17, Section 5].

By tackling both limits simultaneously, we derive new convergence results that differentiate between systems with stable Lyapunov exponents and systems that are only strongly fast invertible. The main difference between the respective convergence results are the requirements for stepsizes. While stepsizes h_n such that

$$\sum_{n=1}^{\infty} h_n = \infty \quad \text{and} \quad h_n \to 0$$

are enough to achieve convergence for systems with stable Lyapunov exponents, we need stricter assumptions to compute Lyapunov exponents for strongly fast invertible systems:

$$\sum_{n=1}^{\infty} h_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} h_n^{p+1} < \infty,$$

where $p > 0$ is the order of consistency of the numerical integrator.

Finally, we provide three numerical examples to highlight the effects of different types of stepsizes (adaptive, constant and varying) on the computation of Lyapunov exponents. While the first example is linear and in accordance with our convergence theory, the other two examples are nonlinear and serve as an outlook suggesting that similar convergence properties may be valid for more general classes of systems.

2 Exterior products and powers

Exterior products are a handy tool when it comes to studying the evolution of volumes and hence also Lyapunov exponents. In this section we briefly introduce them and some of their properties. Our main reference is [2, Section 3.2.3].

For $1 \leq L \leq d$, the L-fold *exterior power* of $\mathbb{R}^d$ is the space $\wedge^L \mathbb{R}^d$ consisting of alternating L-linear forms on the dual space $(\mathbb{R}^d)^* \cong \mathbb{R}^d$. A basis can be obtained by taking *exterior products* of basis elements of $\mathbb{R}^d$. For example, the set

$$\{e_I := e_{i_1} \wedge \cdots \wedge e_{i_L} \mid I = (i_1, \ldots, i_L) \text{ with } 1 \leq i_1 < \cdots < i_L \leq d\},$$

where e_i is the i-th unit vector of $\mathbb{R}^d$, defines a natural basis of $\wedge^L \mathbb{R}^d$. In particular, $\wedge^L \mathbb{R}^d$ has dimension $\binom{d}{L}$.

Not all elements of $\wedge^L \mathbb{R}^d$ are *decomposable*, i.e., of the form $u_1 \wedge \cdots \wedge u_L$. Some elements are *indecomposable* and can only be expressed as linear combinations of decomposable elements.

Given subspaces $U, V \subset \mathbb{R}^d$, we define induced subspaces

$$(\wedge^k U) \wedge (\wedge^{L-k} V) := \operatorname{span}\{u_1 \wedge \cdots \wedge u_k \wedge v_{k+1} \wedge \cdots \wedge v_L \mid u_i \in U, v_i \in V\} \subset \wedge^L \mathbb{R}^d$$

as spans of the corresponding induced decomposable elements.

By bilinear extension from the set of decomposable elements to $\wedge^L \mathbb{R}^d$, the following defines a scalar product on $\wedge^L \mathbb{R}^d$:

$$\langle u_1 \wedge \cdots \wedge u_L, v_1 \wedge \cdots \wedge v_L \rangle := \det(\langle u_i, v_j \rangle)_{ij}.$$

In particular, the induced norm of a decomposable element $u_1 \wedge \cdots \wedge u_L$ is the L-volume of the parallelepiped spanned by $u_1, \ldots, u_L$:

$$\|u_1 \wedge \cdots \wedge u_L\| = \sqrt{\det(\langle u_i, u_j \rangle)_{ij}}.$$

Throughout this article norms without annotation always denote euclidean norms or associated matrix norms.

Lemma 2.1. *We have*

(i) $\|u_1 \wedge \cdots \wedge u_L\| \leq \|u_1 \wedge \cdots \wedge u_k\| \cdot \|u_{k+1} \wedge \cdots \wedge u_L\|$

for $u_1, \ldots, u_L \in \mathbb{R}^d$ *and*

(ii) $\langle \hat{u} \wedge \hat{v}, \hat{u}' \wedge \hat{v}' \rangle = \langle \hat{u}, \hat{u}' \rangle \langle \hat{v}, \hat{v}' \rangle$,

(iii) $\|\hat{u} \wedge \hat{v}\| = \|\hat{u}\| \, \|\hat{v}\|$

for $\hat{u}, \hat{u}' \in \wedge^k U$ *and* $\hat{v}, \hat{v}' \in \wedge^{L-k} V$ *with* $U, V \subset \mathbb{R}^d$ *orthogonal.*

Proof. (i) can be found in [2, Subsection 3.2.3].

Let $U, V \subset \mathbb{R}^d$ with $U \perp V$. Choose bases $(u_i)_i$ of U and $(v_j)_j$ of V. We denote the elements of the induced bases of $\wedge^k U$ and $\wedge^{L-k} V$ by u_I and v_J respectively. Since

$$\langle u_I \wedge v_J, u_{I'} \wedge v_{J'} \rangle$$

is the determinant of a block diagonal matrix, it is equal to the product of the determinants of both blocks, which is

$$\langle u_I, u_{I'} \rangle \langle v_J, v_{J'} \rangle.$$

Thus, (ii) holds on basis elements and by bilinearity of the inner product on arbitrary elements.

Assertion (iii) follows immediately from (ii) by setting $\hat{u} = \hat{u}'$ and $\hat{v} = \hat{v}'$. ☐

Next, we discuss several constructions for linear maps. The L-fold exterior power of $A \in \mathbb{R}^{d\times d}$ is defined via

$$(\wedge^L A)(u_1 \wedge \cdots \wedge u_L) := Au_1 \wedge \cdots \wedge Au_L.$$

Similarly, one may define

$$((\wedge^k A) \wedge (\wedge^{L-k} B))(u_1 \wedge \cdots \wedge u_L) := Au_1 \wedge \cdots \wedge Au_k \wedge Bu_{k+1} \wedge \cdots \wedge Bu_L.$$

for $A, B \in \mathbb{R}^{d\times d}$. Another helpful construction is

$$\hat{A}^L(u_1 \wedge \cdots \wedge u_L) := \sum_{k=1}^{L} u_1 \wedge \cdots \wedge u_{k-1} \wedge Au_k \wedge u_{k+1} \wedge \cdots \wedge u_L.$$

Since we will make extensive use of these induced maps, we state and derive a list of basic properties for them.

Lemma 2.2. *The following are true for $A, B \in \mathbb{R}^{d\times d}$:*

(i) $\wedge^L I_{\mathbb{R}^d} = I_{\wedge^L \mathbb{R}^d}$,

(ii) $\wedge^L (AB) = (\wedge^L A)(\wedge^L B)$,

(iii) $(\wedge^L A)^{-1} = \wedge^L A^{-1}$ *if* $A \in GL(d, \mathbb{R})$,

(iv) $\widehat{\alpha A + \beta B}^L = \alpha \hat{A}^L + \beta \hat{B}^L$,

(v) $\| \wedge^L A\| = \sigma_1(A) \ldots \sigma_L(A)$, *where* $\sigma_1 \geq \cdots \geq \sigma_d$ *denote the singular values,*

(vi) $\| \wedge^L A\| \leq \|A\|^L$,

(vii) $\|\hat{A}^L\| \leq L\|A\|$,

(viii) $\|(\wedge^k A) \wedge (\wedge^{L-k} B)\| \leq \binom{d}{L}^{\frac{1}{2}} \| \wedge^k A\| \, \| \wedge^{L-k} B\|$,

(ix) if $A[u_1, \ldots, u_L] = QR$, *then* $\|(\wedge^L A)(u_1 \wedge \cdots \wedge u_L)\| = r_{11} \ldots r_{LL}$, *where* $r_{ii} \geq 0$ *denote the diagonal elements of* R *ordered in decreasing size,*

(x) $\det(\wedge^L A) = \det(A)^{\binom{d-1}{L-1}}$.

Proof. (i)–(vii) and (x) can be found in [2, Subsection 3.2.3].

To show $(viii)$, write

$$I_1 = (i_1, \ldots, i_k) \text{ and } I_2 = (i_{k+1}, \ldots, i_L)$$

for a given tuple $I = (i_1, \ldots, i_L)$ and estimate

$$\begin{aligned}\|(\wedge^k A) \wedge (\wedge^{L-k} B)\| &= \sup_{\sum \alpha_I^2 = 1} \left\| ((\wedge^k A) \wedge (\wedge^{L-k} B)) \left(\sum \alpha_I e_I\right) \right\| \\ &\leq \sup_{\sum \alpha_I^2 = 1} \sum |\alpha_I| \, \|((\wedge^k A) e_{I_1}) \wedge ((\wedge^{L-k} B) e_{I_2})\| \leq \sup_{\sum \alpha_I^2 = 1} \sum |\alpha_I| \, \|(\wedge^k A) e_{I_1}\| \, \|(\wedge^{L-k} B) e_{I_2}\| \\ &\leq \binom{d}{L}^{\frac{1}{2}} \| \wedge^k A\| \, \| \wedge^{L-k} B\|.\end{aligned}$$

Finally, we prove (ix):

$$\|(\wedge^L A)(u_1 \wedge \cdots \wedge u_L)\|^2 = \det(\langle Au_i, Au_j \rangle)_{ij} = \det((QR)^T (QR)) = \det(R^T R) = r_{11}^2 \ldots r_{LL}^2.$$

□

Lemma 2.3. *Let* $A = \operatorname{diag}(A_1, A_2) \in \mathbb{R}^{d\times d}$ *and set* $B := \operatorname{diag}(A_1, 0)$ *and* $C := \operatorname{diag}(0, A_2)$. *The following are true:*

(i) $\wedge^L A = \sum_{k=0}^{L} (\wedge^k B) \wedge (\wedge^{L-k} C)$,

(ii) $\| \wedge^L A\| = \max_k \|(\wedge^k B) \wedge (\wedge^{L-k} C)\|$,

(iii) $\|(\wedge^k B) \wedge (\wedge^{L-k} C)\| \geq \| \wedge^k B\| \, \| \wedge^{L-k} C\|$.

Proof. Let $A_1 \in \mathbb{R}^{d_1 \times d_1}$. Given a basis element

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_L},$$

choose j such that $i_j \leq d_1 < i_{j+1}$. It holds

$$(\wedge^L A) e_I = ((\wedge^j B) \wedge (\wedge^{L-j} C)) e_I.$$

Moreover, all summands in

$$\sum_{k=0}^{L} ((\wedge^k B) \wedge (\wedge^{L-k} C)) e_I$$

vanish except for $k = j$. Hence, the two maps in (i) coincide on basis elements e_I.

The subspaces

$$W_k := \left(\wedge^k \left(\mathbb{R}^{d_1} \times \{0\}\right)\right) \wedge \left(\wedge^{L-k} \left(\{0\} \times \mathbb{R}^{d-d_1}\right)\right)$$

for $k = 0, \ldots, L$ form an orthogonal decomposition of $\wedge^L \mathbb{R}^d$. Since

$$\operatorname{im}\left((\wedge^k B) \wedge (\wedge^{L-k} C)\right) \subset W_k$$

and

$$W_i \subset \ker\left((\wedge^k B) \wedge (\wedge^{L-k} C)\right)$$

for $i \neq k$, (ii) easily follows by means of this decomposition.

To prove (iii), we use (iii) of Lemma 2.1. It follows that

$$\begin{aligned}\|(\wedge^k A) \wedge (\wedge^{L-k} B)\| &\geq \max_{\|\hat{u} \wedge \hat{v}\| = 1} \|(\wedge^k A)\hat{u} \wedge (\wedge^{L-k} B)\hat{v}\| = \max_{\|\hat{u}\|=1} \|(\wedge^k A)\hat{u}\| \max_{\|\hat{v}\|=1} \|(\wedge^{L-k} B)\hat{v}\| \\ &= \| \wedge^k A\| \, \| \wedge^{L-k} B\|,\end{aligned}$$

where maxima are with respect to $\hat{u} \wedge \hat{v} \in W_k$. □

Next, we relate the principle angles between two complementary subspaces of $\mathbb{R}^d$ to a principle angle on $\wedge^L \mathbb{R}^d$.

Proposition 2.4. *Let* $\mathbb{R}^d = U \oplus V$ *with* $\dim U = L$. *Set* $\hat{U} := \wedge^L U$ *and* $\hat{V} := \left(\wedge^L V^\perp\right)^\perp$. *Then*

$$\prod_{i=1}^{\min(L, d-L)} \sin \alpha_i(U, V) = \sin \alpha_1(\hat{U}, \hat{V}), \tag{2.1}$$

where $0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \frac{\pi}{2}$ *are the principle angles between the respective subspaces.*

Proof. First we pass to the orthogonal complement of V in order to work with two subspaces of the same dimension. According to [25] it holds

$$\prod_{i=1}^{\min(L,d-L)} \sin \alpha_i(U,V) = \prod_{i=1}^{L} \sin\left(\frac{\pi}{2} - \alpha_i(U,V^\perp)\right) = \prod_{i=1}^{L} \cos \alpha_i(U,V^\perp).$$

The concept of "higher dimensional angle" coined in [13] helps us to transition to $\wedge^L \mathbb{R}^d$. In fact, the higher dimensional angle $\theta(U,V^\perp)$ between two subspaces U and $V^\perp$ of the same dimension satisfies

$$\cos \theta(U,V^\perp) = \prod_{i=1}^{L} \cos \alpha_i(U,V^\perp)$$

and is defined via

$$\cos \theta(U,V^\perp) := \frac{\langle \alpha, \beta \rangle}{\|\alpha\| \, \|\beta\|},$$

where $\alpha, \beta \in \wedge^L \mathbb{R}^d$ are decomposable elements representing U and $V^\perp$, i.e., $\alpha = u_1 \wedge \cdots \wedge u_L$ with $\operatorname{span}(u_1,\ldots,u_L) = U$ and $\beta = v_1' \wedge \cdots \wedge v_L'$ with $\operatorname{span}(v_1',\ldots,v_L') = V^\perp$. The latter is nothing else than the angle between α and β or the principle angle between their corresponding 1-dimensional subspaces:

$$\cos \theta(U,V^\perp) = \cos \alpha_1(\wedge^L U, \wedge^L V^\perp) = \cos \alpha_1(\hat{U}, \hat{V}^\perp) = \sin \alpha_1(\hat{U}, \hat{V}). \qquad \square$$

Corollary 2.5. *In Proposition 2.4 it holds*

$$\left(\wedge^L V^\perp\right)^\perp = \oplus_{k=0}^{L-1} \left((\wedge^k U) \wedge (\wedge^{L-k} V) \right).$$

In particular, equation (2.1) is true for

$$\begin{aligned} U &:= \operatorname{span}\{Ae_1, \ldots, Ae_L\}, \\ V &:= \operatorname{span}\{Ae_{L+1}, \ldots, Ae_d\}, \\ \hat{U} &= \operatorname{span}\{(\wedge^L A)(e_1 \wedge \cdots \wedge e_L)\}, \\ \hat{V} &= \operatorname{span}\{(\wedge^L A)(e_{i_1} \wedge \cdots \wedge e_{i_L}) \mid \{i_1, \ldots, i_L\} \neq \{1, \ldots, L\}\} \end{aligned}$$

for any given $A \in GL(d, \mathbb{R})$.

Proof. The identity can be checked by using that both spaces have codimension 1 and by showing that the inner product between elements of the form

$$v_1' \wedge \cdots \wedge v_L' \in \wedge^L V^\perp$$

and

$$u_1 \wedge \cdots \wedge u_k \wedge v_{k+1} \wedge \cdots \wedge v_L$$

with $u_i \in U$ and $v_i \in V$ vanishes for $k < L$. $\square$

3 Asymptotic properties of linear systems

Consider the linear differential equation

$$\dot{x} = A(t)x, \quad A \in C(\mathbb{R}_{\geq 0}, \mathbb{R}^{d \times d}) \tag{3.1}$$

with bounded coefficients, i.e.,

$$M := \sup_t \|A(t)\| < \infty.$$

We write $X(t)$ for a fundamental matrix of equation (3.1) and $X(t,s) := X(t)X(s)^{-1}$, $t \geq s$, for the Cauchy matrix. By requiring that A is bounded, solutions grow or decay at most exponentially fast:

$$e^{-M(t-s)} \leq \|X(t,s)^{\pm 1}\| \leq e^{M(t-s)}$$

for all $t \geq s$.

In our article, equation (3.1) will be changed by so-called *Lyapunov transformations*. These transformations leave asymptotic properties[1] like boundedness of the system matrix, the Lyapunov spectrum, regularity, the stability of Lyapunov exponents, or strong fast invertibility invariant.

Definition 3.1 ([1, Definition 3.1.1]). $x = L(t)y$ is called *Lyapunov transformation* if L is continuously differentiable, everywhere invertible and $L, L^{-1}, \dot{L}$ are bounded. The transformed system $\dot{y} = B(t)y$ is given by

$$B(t) = L(t)^{-1}A(t)L(t) - L(t)^{-1}\dot{L}(t).$$

It is easy to see that Lyapunov transformations form a group with respect to multiplication. A particular Lyapunov transformation is obtained via the Gram–Schmidt procedure.

Proposition 3.2 ([1, Lemma 3.3.1 and Theorem 3.3.1]). *There is an orthogonal Lyapunov transformation such that the fundamental matrix $Y(t)$ of the transformed system is upper triangular.*

3.1 Characteristic exponents

In this subsection, we define characteristic exponents and Lyapunov exponents.

Definition 3.3 ([1, Definition 2.1.1]). The *characteristic exponent* of $f : [0, \infty) \to \mathbb{R}$ is given by

$$\chi[f] := \limsup_{t \to \infty} \frac{1}{t} \log |f(t)| \in \mathbb{R} \cup \{\pm\infty\}.$$

A few handy properties follow easily from the definition (see [1, Section 2.1] and [2, Lemma 3.2.1]):

(i) $\chi[c] = 0$ for $c \neq 0$

(ii) $\chi[cf] = \chi[f]$ for $c \neq 0$,

(iii) $\chi[|f|^c] = c\chi[f]$ for $c \in \mathbb{R}$ (set $0(\pm\infty) = 0$),

(iv) $\chi[f] \leq \chi[g]$ if $|f| \leq |g|$,

(v) $\chi[f+g] \leq \max(\chi[f], \chi[g])$ with equality if $\chi[f] \neq \chi[g]$,

(vi) $\chi[fg] \leq \chi[f] + \chi[g]$ (if the right-hand side makes sense).

[1] We will define all mentioned properties throughout our article.

In the context of dynamical systems, we apply characteristic exponents to measure the (upper) exponential growth of solutions $x(t)$:

$$\chi[x] := \chi[\|x\|] = \limsup_{t\to\infty} \frac{1}{t} \log \|x(t)\|.$$

Note that $\chi[x]$ is independent of the chosen norm because all norms on $\mathbb{R}^d$ are equivalent.[2]

Since solutions with distinct characteristic exponents are linearly independent, the characteristic exponents can take at most d values

$$\infty > M \geq \lambda_1 > \cdots > \lambda_p \geq -M > -\infty.$$

In particular, there is a filtration of subspaces of the space of solutions called *Lyapunov filtration*

$$\mathbb{R}^d \cong V_1 \supset \cdots \supset V_p \supset V_{p+1} := \{0\}$$

with $V_i := \{x \,|\, \chi[x] \leq \lambda_i\}$ satisfying

$$\chi[x] = \lambda_i \qquad \Longleftrightarrow \qquad x \in V_i \setminus V_{i+1}.$$

We set $d_i := \dim V_i - \dim V_{i+1}$.

Remark 3.4. Sometimes the Lyapunov filtration is defined on the space of initial vectors. In fact, both versions of filtration spaces are naturally isomorphic:

$$V_i' := \{v \mid \chi[X(t,0)v] \leq \lambda_i\} = \{x(0) \mid x \in V_i\}.$$

One may also define the filtration spaces using a given fundamental matrix $X(t)$:

$$V_{i,X} := \{v \mid \chi[X(t)v] \leq \lambda_i\}.$$

Definition 3.5. The *(forward) Lyapunov spectrum* $(\lambda_i, d_i)_{i=1,\ldots,p}$ of a system consists of its *Lyapunov exponents* λ_i together with their *multiplicities* d_i. Since we sometimes count Lyapunov exponents according to their multiplicities, we define $\Lambda_1, \ldots, \Lambda_d$ via

$$\Lambda_{d_1+\cdots+d_{i-1}+j} := \lambda_i \quad \text{for } j = 1, \ldots, d_i.$$

A Lyapunov exponent λ_i is called *simple* or *nondegenerate* if $d_i = 1$ and otherwise *degenerate*. If all Lyapunov exponents are simple, we call the Lyapunov spectrum *simple*.

Example 3.6. Let $\epsilon > 0$. We set $a_n := n^2 - \log n$ and $b_n := n^2$ and define $f_n \in C(\mathbb{R}_{\geq 0}, \mathbb{R})$ through

$$f_n = \begin{cases} 0, & t \in [0, a_n - \epsilon] \\ \frac{t-(a_n-\epsilon)}{\epsilon}, & t \in (a_n - \epsilon, a_n] \\ 1, & t \in (a_n, b_n] \\ \frac{b_n+\epsilon-t}{\epsilon}, & t \in (b_n, b_n + \epsilon] \\ 0. & t \in (b_n + \epsilon, \infty) \end{cases}$$

[2] In infinite dimensions the characteristic exponents generally depend on the chosen norm, although independence can be achieved on certain scales of Banach spaces [6].

For small ϵ the functions f_n have disjoint supports. Thus, the system matrix

$$A(t) := \begin{pmatrix} 1 & 0 \\ 0 & a_{22}(t) \end{pmatrix}$$

with $a_{22}(t) := \sum_{n\in\mathbb{N}} f_n(t)$ is bounded and continuous. A fundamental matrix is given by

$$X(t) := \begin{pmatrix} e^t & 0 \\ 0 & e^{\int_0^t a_{22}(\tau)\, d\tau} \end{pmatrix}.$$

Since

$$1 \le e^{\int_0^{b_n+\epsilon} a_{22}(\tau)\, d\tau} \le e^{\sum_{k=1}^n (\log(k)+2\epsilon)} = n! e^{2\epsilon n},$$

we have

$$0 \le \frac{1}{t} \log e^{\int_0^t a_{22}(\tau)\, d\tau} \le \frac{2\epsilon(n+1)}{n^2} + \frac{\log((n+1)!)}{n^2}$$

for $t \in [b_n, b_{n+1} + \epsilon]$. In particular, the system exhibits solutions with characteristic exponents:

$$\chi[X(t)e_1] = 1 \quad \text{and} \quad \chi[X(t)e_2] = 0.$$

Since the characteristic exponents are distinct, the columns of $X(t)$ realize the whole Lyapunov spectrum ($\lambda_1 = 1$ and $\lambda_2 = 0$) and can be used to compute the Lyapunov filtration ($V_{1,X} = \mathbb{R}^2$ and $V_{2,X} = \{0\} \times \mathbb{R}$).

We may always choose[3] a basis of solutions that realizes the whole Lyapunov spectrum. In that case

$$\sum_{i=1}^{d} \chi[x_i] = \sum_{i=1}^{p} d_i \lambda_i. \tag{3.2}$$

Definition 3.7 ([1, Definition 2.4.2])**.** A basis $x_1, \ldots, x_d$ of solutions is called *normal* if $\sum_i \chi[x_i]$ is minimal, i.e., if equation (3.2) holds. Moreover, we call a fundamental matrix *normal* if its columns form a normal basis.

If we order a normal basis such that $\chi[x_i]$ decreases with i, then $\chi[x_i] = \Lambda_i$.

Proposition 3.8 ([1, Theorem 2.5.1, Corollary 2.5.1 and Remark 2.5.2])**.** *Any fundamental matrix $X(t) = [x_1, \ldots, x_d]$ satisfies*

$$\sum_{i=1}^{d} \chi[x_i] \ge \sum_{i=1}^{p} d_i \lambda_i \ge \chi[\det X(t)].$$

If equality holds, then $X(t)$ is normal. However, $X(t)$ being normal does not imply equality in general.

The largest Lyapunov exponent can be expressed as the characteristic exponent of $\|X(t)\|$.

Proposition 3.9. *The largest Lyapunov exponent satisfies*

$$\lambda_1 = \chi[\|X(t)\|].$$

Proof. One inequality is trivial and the other follows from

$$\|X(t)\| \le \sqrt{d} \max_i \|X(t)e_i\|_1,$$

where e_i is the i-th unit vector of $\mathbb{R}^d$. □

[3] Construct a basis by iteratively choosing d_i solutions in $V_i \setminus V_{i+1}$ for $i = p, \ldots, 1$.

3.2 Regularity

We now introduce the notion of regularity, which implies that characteristic exponents of solutions can be obtained as limits instead of limes superiors.

Definition 3.10 ([2, p. 115])**.** We call a system *(forward) regular*[4] if

$$\sum_{i=1}^{p} d_i\lambda_i = \liminf_{t\to\infty} \frac{1}{t} \log|\det X(t)|.$$

In case of triangular systems regularity can be checked via the diagonal elements of the system matrix[5]. Moreover, one may check regularity of general systems through Perron's regularity test, which compares the Lyapunov spectra of the system and its adjoint system $\dot{y} = -A(t)^T y$ [1, Theorem 3.6.1].

Proposition 3.11 ([1, Theorem 3.9.1])**.** *Regular systems have sharp characteristic exponents, i.e., it holds*

$$\chi[x] = \lim_{t\to\infty} \frac{1}{t} \log\|x(t)\|.$$

Example 3.12. The system from Example 3.6 is regular: Indeed, the fundamental matrix

$$X(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{\int_0^t a_{22}(\tau)\,d\tau} \end{pmatrix}$$

is diagonal and our previous estimates show that the diagonal elements have sharp characteristic exponents:

$$1 + 0 = \lambda_1 + \lambda_2 = \lim_{t\to\infty} \frac{1}{t} \log e^t + \lim_{t\to\infty} \frac{1}{t} \log e^{\int_0^t a_{22}(\tau)\,d\tau} = \lim_{t\to\infty} \frac{1}{t} \log|\det X(t)|.$$

Alternatively, one may show that diagonal elements of the system matrix $A(t)$ have finite mean values.

All Lyapunov exponents of a regular system can be obtained via characteristic exponents of singular values of $X(t)$.

Proposition 3.13. *If equation* (3.1) *is regular, then*

$$\Lambda_i = \lim_{t\to\infty} \frac{1}{t} \log \sigma_i(X(t))$$

for all $i = 1, \dots, d$, where $\sigma_1 \geq \cdots \geq \sigma_d$ denote the singular values.

---

[4]The definition from Arnold's book is equivalent to [1, Definition 3.5.1]. This can be checked via [1, Lemma 3.5.1] and the Liouville–Ostrogradski formula

$$\det X(t) = \det X(t_0) e^{\int_{t_0}^t \operatorname{tr}(A(\tau))\,d\tau}.$$

[5][1, Corollary 3.8.1] states that a lower triangular system is regular if and only if its diagonal elements $a_{ii}(t)$ have finite mean values:

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t a_{ii}(s)\,ds < \infty.$$

Since adjoint systems of regular systems are regular ([1, Corollary 3.6.1]), the same holds true for upper triangular systems.

We prove Proposition 3.13 in the next subsection using induced systems on the space of exterior products. Next, let us state a result on singular vectors of $X(t)$.

Proposition 3.14. *If equation* (3.1) *is regular and $u_1(t), \dots, u_d(t)$ denote the right singular vectors of $X(t)$, then*

$$U_{i,X}(t) := \operatorname{span}(u_{d_1+\cdots+d_{i-1}+1}(t), \dots, u_{d_1+\cdots+d_i}(t))$$

converges to $U_{i,X} := V_{i+1,X}^{\perp} \cap V_{i,X}$ exponentially fast. More precisely, it holds

$$\limsup_{t\to\infty} \frac{1}{t} \log \max_{\substack{u\in U_{i,X}(t),\\ u'\in U_{j,X},\\ \|u\|=\|u'\|=1}} |\langle u, u'\rangle| \leq -|\lambda_i - \lambda_j|$$

for $i \neq j$.

Proof. Since equation (3.1) has bounded coefficients and its induced systems are regular (see Lemma 3.19), the deterministic version of the multiplicative ergodic theorem [2, Proposition 3.4.2] applies. In its proof, Arnold shows that the filtration $F_X(t)$ given by the spaces

$$V_{i,X}(t) := U_{p,X}(t) \oplus \cdots \oplus U_{i,X}(t)$$

converges exponentially fast to the Lyapunov filtration F_X given by the spaces

$$V_{i,X} = U_{p,X} \oplus \cdots \oplus U_{i,X}$$

using the metric δ (see [2, Equation 3.4.10]) on the manifold of filtrations. In particular, he shows that

$$\limsup_{n\to\infty} \frac{1}{n} \log \delta(F_X(n), F_X) \leq -h,$$

where $h > 0$ is a parameter also appearing in the definition of δ. Disentangling the metric yields the desired convergence statement for discrete time.

The version for continuous time follows as described in the proof of [2, Theorem 3.4.1] since

$$\limsup_{n\to\infty} \frac{1}{n} \sup_{0\leq t\leq 1} \log \|X(n+t, n)^{\pm 1}\| \leq 0. \qquad \square$$

If at least two singular values coincide, the singular value decomposition is not unique. However, according to Proposition 3.13 the singular values corresponding to different Lyapunov exponents are distinct for large t. In particular, the spaces $U_{i,X}(t)$ are uniquely defined for large t.

Corollary 3.15. *If equation* (3.1) *is regular and λ_1 is simple, then*

$$\lim_{t\to\infty} \frac{\|X(t)v\|}{\|X(t)\|} = \|v - P_{V_{2,X}} v\|$$

for $v \in \mathbb{R}^d$, where $P_{V_{2,X}}$ denotes the orthogonal projection onto $V_{2,X}$.

Proof. To prove the corollary, we use the two orthogonal decompositions $\mathbb{R}^d = U_{1,X} \oplus V_{2,X}$ and $\mathbb{R}^d = U_{1,X}(t) \oplus V_{2,X}(t)$ from Proposition 3.14 and its proof. Since $d_1 = 1$, there are unit vectors u_1 and $u_1(t)$ spanning $U_{1,X}$ and $U_{1,X}(t)$. Moreover, Proposition 3.14 implies

$$\lim_{t\to\infty} \|P_{V_{2,X}(t)} u_1\| = 0,$$

where $P_{V_{2,X(t)}}$ denotes the orthogonal projection onto $V_{2,X}(t)$. We estimate $\|X(t)u_1\| \leq \|X(t)\|$ and

$$\begin{aligned}\|X(t)u_1\| &= \|X(t)(\langle u_1, u_1(t)\rangle u_1(t) + P_{V_{2,X(t)}}u_1)\| \geq |\langle u_1, u_1(t)\rangle|\,\|X(t)\| - \|X(t)\|\,\|P_{V_{2,X(t)}}u_1\| \\ &= \left(\sqrt{1-\|P_{V_{2,X(t)}}u_1\|^2} - \|P_{V_{2,X(t)}}u_1\|\right)\|X(t)\|.\end{aligned}$$

Thus, it holds

$$\lim_{t\to\infty} \frac{\|X(t)u_1\|}{\|X(t)\|} = 1.$$

Now, we decompose $v \in \mathbb{R}^d$ into $v = \langle v, u_1\rangle u_1 + v_2$ according to $\mathbb{R}^d = U_{1,X} \oplus V_{2,X}$. Since the system is regular and the characteristic exponents satisfy $\chi[X(t)v_2] \leq \lambda_2$ and $\chi[\|X(t)\|] = \lambda_1$, we have

$$\lim_{t\to\infty} \frac{\|X(t)v_2\|}{\|X(t)\|} = 0.$$

The claim of the corollary follows from

$$\frac{\|X(t)v\|}{\|X(t)\|} \leq |\langle v, u_1\rangle| \frac{\|X(t)u_1\|}{\|X(t)\|} + \frac{\|X(t)v_2\|}{\|X(t)\|}$$

and

$$\frac{\|X(t)v\|}{\|X(t)\|} \geq |\langle v, u_1\rangle| \frac{\|X(t)u_1\|}{\|X(t)\|} - \frac{\|X(t)v_2\|}{\|X(t)\|}. \qquad \square$$

3.3 Induced systems

Equation (3.1) induces a system on $\wedge^L \mathbb{R}^d$ via

$$\dot{\hat{x}} = \widehat{A(t)}^L \hat{x}. \tag{3.3}$$

In the following, we refer to equation (3.3) using the term *induced system*.

A fundamental matrix of the induced system can be obtained by taking the exterior power of a fundamental matrix of the original system:

Proposition 3.16. *It holds*

$$\frac{d}{dt}(\wedge^L X) = \hat{A}^L(\wedge^L X).$$

Proof. Since the determinant is multilinear, we have

$$\begin{aligned}\left\langle \frac{d}{dt}(\wedge^L X)e_I, e_J\right\rangle &= \frac{d}{dt}\langle Xe_{i_1}\wedge\cdots\wedge Xe_{i_L}, e_J\rangle = \sum_{k=1}^{L}\left\langle Xe_{i_1}\wedge\cdots\wedge\frac{d}{dt}(Xe_{i_k})\wedge\cdots\wedge Xe_{i_L}, e_J\right\rangle \\ &= \left\langle \sum_{k=1}^{L} Xe_{i_1}\wedge\cdots\wedge AXe_{i_k}\wedge\cdots\wedge Xe_{i_L}, e_J\right\rangle = \left\langle \hat{A}^L(\wedge^L X)e_I, e_J\right\rangle. \qquad \square\end{aligned}$$

Several properties carry over from the original system to the induced system. For instance, the Lyapunov spectrum $(\lambda_{i,L}, d_{i,L})_{i=1,\ldots,p_L}$ of the induced system can be related to the spectrum of the original system if the system is regular (see also [2, Theorem 5.3.1]).

Theorem 3.17. *Let equation (3.1) be regular. The Lyapunov exponents $\lambda_{i,L}$ of the induced system are the different values given by*

$$\Lambda_{i_1} + \cdots + \Lambda_{i_L}$$

for indices $i_1 < \cdots < i_L$. The corresponding multiplicity $d_{i,L}$ is the number of combinations of indices for which $\lambda_{i,L}$ can be achieved. In particular,

$$\lambda_{1,L} = \Lambda_1 + \cdots + \Lambda_L.$$

If $L = d_1 + \cdots + d_l$, then $\lambda_{1,L}$ is simple and the second space of the Lyapunov filtration $V_{2,L}$ is given by

$$\operatorname{span}\{x_1 \wedge \cdots \wedge x_L \mid \operatorname{span}(x_1, \ldots, x_L) \cap V_{l+1} \neq \{0\}\}.$$

Remark 3.18. If equation (3.1) is not regular, the spectrum of the induced system can differ from what is described in Theorem 3.17.

Proof. In the absence of regularity, the fastest growing solutions do not necessarily span the fastest growing subspace. This can be the case if the characteristic exponents of the fastest growing solutions are only obtainable along distinct subsequences.

Indeed, one readily checks that

$$X(t) := \begin{pmatrix} e^{t\sin(\log(1+t))} & 0 & 0 \\ 0 & e^{t\cos(\log(1+t))} & 0 \\ 0 & 0 & e^{\frac{1}{2}t} \end{pmatrix}$$

is a normal fundamental matrix corresponding to a bounded, continuous system with Lyapunov exponents $\Lambda_1 = \Lambda_2 = 1$ and $\Lambda_3 = 1/2$. Since the characteristic exponents of the first two columns of $X = [x_1, x_2, x_3]$ are not sharp, the system is not regular. Moreover, one computes

$$\begin{aligned} \chi[x_1 \wedge x_2] &= \chi[e^{t(\sin(\log(1+t))+\cos(\log(1+t)))}] = \sqrt{2} \\ \chi[x_1 \wedge x_3] &= \chi[e^{t(\sin(\log(1+t))+\frac{1}{2})}] = \frac{3}{2} \\ \chi[x_2 \wedge x_3] &= \chi[e^{t(\cos(\log(1+t))+\frac{1}{2})}] = \frac{3}{2}. \end{aligned}$$

Since $x_1(t) \wedge x_3(t)$ and $x_2(t) \wedge x_3(t)$ are orthogonal for each t, any nontrivial linear combination of them has characteristic exponent $3/2$. In particular, $\Lambda_{1,2} = \Lambda_{2,2} = 3/2$ and $\Lambda_{3,2} = \sqrt{2}$. Thus, even though x_1 and x_2 have the highest characteristic exponents, their associated volume element has the lowest characteristic exponent. □

We prove Theorem 3.17 using the following lemma:

Lemma 3.19. *If equation (3.1) is regular, then the induced basis of a normal basis is normal and the induced system is regular.*

Proof. Let $x_1, \ldots, x_d$ be a normal basis such that $\chi[x_i] = \Lambda_i$. Since

$$\chi[x_{i_1} \wedge \cdots \wedge x_{i_L}] \leq \chi[x_{i_1}] + \cdots + \chi[x_{i_L}] = \Lambda_{i_1} + \cdots + \Lambda_{i_L},$$

we have

$$\sum_{i_1<\cdots<i_L} \Lambda_{i_1} + \cdots + \Lambda_{i_L} \geq \sum_{i_1<\cdots<i_L} \chi[x_{i_1} \wedge \cdots \wedge x_{i_L}] \geq \chi[\det(\wedge^L X(t))]$$
$$\geq \liminf_{t\to\infty} \frac{1}{t} \log|\det(\wedge^L X(t))| = \binom{d-1}{L-1} \liminf_{t\to\infty} \frac{1}{t} \log|\det X(t)| = \binom{d-1}{L-1} \sum_{i=1}^{d} \Lambda_i.$$

Each index i appears in precisely $\binom{d-1}{L-1}$ combinations of indices $i_1 < \cdots < i_L$. Hence, the above inequalities are actually equalities, proving that the induced basis is a normal basis and that the induced system is regular. □

The beginning of the proof of Lemma 3.19 implies the following:

Proposition 3.20. *It holds*

$$\lambda_{1,L} \leq \Lambda_1 + \cdots + \Lambda_L$$

independent of the regularity of equation (3.1).

Proof of Theorem 3.17. Let $x_1, \ldots, x_d$ be a normal basis such that $\chi[x_i] = \Lambda_i$. Since the induced basis is normal, it realizes the whole Lyapunov spectrum of the induced system. Our claims about the spectrum in Theorem 3.17 now follow from

$$\chi[x_{i_1} \wedge \cdots \wedge x_{i_L}] = \Lambda_{i_1} \wedge \cdots \wedge \Lambda_{i_L}.$$

Moreover, $V_{2,L}$ is spanned by solutions

$$x_{i_1} \wedge \cdots \wedge x_{i_L}$$

such that

$$\Lambda_{i_1} + \cdots + \Lambda_{i_L} < \Lambda_1 + \cdots + \Lambda_L.$$

If $L = d_1 + \cdots + d_l$, the latter is equivalent to $x_{i_L} \in V_{l+1}$, which proves that $V_{2,L}$ is a subset of the set defined in Theorem 3.17. On the other hand, their dimensions must coincide since the codimension of $V_{2,L}$ is $d_{1,L} = 1$ and neither set contains the solution $x_1 \wedge \cdots \wedge x_L$. □

As a direct consequence, we get Proposition 3.13.

Proof of Proposition 3.13. The previous lemma and Proposition 3.9 imply

$$\Lambda_L = (\Lambda_1 + \cdots + \Lambda_L) - (\Lambda_1 + \cdots + \Lambda_{L-1}) = \lim_{t\to\infty} \frac{1}{t} \log \| \wedge^L X(t)\| - \lim_{t\to\infty} \frac{1}{t} \log \| \wedge^{L-1} X(t)\|$$
$$= \lim_{t\to\infty} \frac{1}{t} \log \frac{\sigma_1(X(t)) \ldots \sigma_L(X(t))}{\sigma_1(X(t)) \ldots \sigma_{L-1}(X(t))} = \lim_{t\to\infty} \frac{1}{t} \log \sigma_L(X(t)).$$

□

Corollary 3.21. *If equation* (3.1) *is regular and* $L = d_1 + \cdots + d_l$, *then*

$$\lim_{t\to\infty} \frac{\|(\wedge^L X(t))\hat{v}\|}{\| \wedge^L X(t)\|} > 0$$

for every $\hat{v} \notin V_{2,L,X}$.

Proof. Since the induced system on the space of L-fold exterior products is regular and $\lambda_{1,L}$ is simple, the claim follows from Corollary 3.15. □

3.4 Stability of Lyapunov exponents

The stability of Lyapunov exponents has been widely studied. Here, we mainly state results listed in the overview from [1, Chapter V] and refer to [7] for the relation to integral separation.

Definition 3.22 ([1, Definition 5.2.1]). The Lyapunov exponents of equation (3.1) are called *stable* if for any $\epsilon > 0$ there exist a $\delta > 0$ such that the Lyapunov exponents of the continuously perturbed system $\dot{\tilde{x}} = (A(t) + Q(t))\tilde{x}$ with $\sup_{t\geq 0} \|Q(t)\| < \delta$ satisfy

$$|\Lambda_i - \tilde{\Lambda}_i| < \epsilon$$

for $i = 1, \dots, d$.

The stability of Lyapunov exponents can be characterized by a uniform gap between the exponential growth rates of solutions corresponding to different Lyapunov exponents (this property is called *integral separation*; see also [7]).

Theorem 3.23 ([1, Theorem 5.4.8]). *The Lyapunov exponents of a system with simple spectrum are stable if and only if there exists a fundamental matrix $X(t) = [x_1(t), \dots, x_d(t)]$ and constants $a, b > 0$ such that*

$$\frac{\|x_i(t)\|}{\|x_i(s)\|} \geq b e^{a(t-s)} \frac{\|x_{i+1}(t)\|}{\|x_{i+1}(s)\|}$$

for all $t \geq s$.

In fact, one may use such a fundamental matrix to construct a Lyapunov transformation that reduces equation (3.1) to a diagonal system [1, Theorem 5.3.1].

The stability of Lyapunov exponents for systems with degenerate spectra can be characterized in a similar fashion by transforming the system to block diagonal form.

Theorem 3.24 ([1, Theorem 5.4.9], [3]). *The Lyapunov exponents of equation (3.1) are stable if and only if there exists a Lyapunov transformation reducing the system to block diagonal form*

$$\dot{y} = \operatorname{diag}(B_1(t), \dots, B_p(t))y,$$

where $B_i(t) \in \mathbb{R}^{d_i \times d_i}$ is upper triangular, such that the following hold:

(i) all non-trivial solutions of $\dot{y}_i = B_i(t)y_i$ have characteristic exponent λ_i,

(ii) λ_i is stable[6] *for $\dot{y}_i = B_i(t)y_i$,*

(iii) there are constants $a, b > 0$ such that

$$\|Y_i(t,s)^{-1}\|^{-1} \geq b e^{a(t-s)} \|Y_{i+1}(t,s)\| \tag{3.4}$$

for all $t \geq s$, where $Y_i(t,s)$ denotes the Cauchy matrix of $\dot{y}_i = B_i(t)y_i$.

Example 3.25. Let us return to Example 3.6. The system

$$A(t) = \begin{pmatrix} 1 & 0 \\ 0 & a_{22}(t) \end{pmatrix}$$

[6][1, Theorem 5.4.9] uses the condition $\overline{\omega}_i = \lambda_i = \Omega_i$, which is equivalent to the stability of λ_i on the subsystem (this follows from [1, Theorem 5.4.9] applied to the subsystem). The quantities $\overline{\omega}_i$ and Ω_i can be computed in terms of *lower* and *upper functions* for the subsystem (see [1, Definition 5.1.2, Definition 5.1.3 and p. 171]).

is already in block diagonal form such that the first block corresponds to $\lambda_1 = 1$ and the second block corresponds to $\lambda_2 = 0$. However, it does not satisfy (iii) of Theorem 3.24 since there are no constants $a, b > 0$ such that

$$e^{(b_n - a_n) - \int_{a_n}^{b_n} a_{22}(\tau) d\tau} = e^{\log n - \log n} = 1$$

is bounded from below by $be^{a(b_n - a_n)} = bn^a$ for all $n \in \mathbb{N}$.

Now, any Lyapunov transformation that maps to a system of the same block diagonal form satisfying (i) must be a diagonal transformation

$$L(t) = \begin{pmatrix} l_{11}(t) & 0 \\ 0 & l_{22}(t) \end{pmatrix}.$$

Since $l_{11}(t)$ and $l_{22}(t)$ are bounded from above and below by positive constants, the Lyapunov transformation does not change whether (iii) is satisfied or not. In particular, this proves that the Lyapunov exponents of the system cannot be stable.

Systems with stable Lyapunov exponents retain their spectra under perturbations that tend to zero.

Theorem 3.26 ([3])**.** *Assume equation* (3.1) *has stable Lyapunov exponents. If $Q(t)$ is a bounded, piecewise continuous perturbation such that*

$$\|Q(t)\| \to 0 \text{ as } t \to \infty,$$

then the perturbed system $\dot{\tilde{x}} = (A(t) + Q(t))\tilde{x}$ has the same Lyapunov spectrum.

While L^∞-perturbations of regular systems do not retain regularity in general, we have the following result:

Proposition 3.27. *If equation* (3.1) *is regular, has stable Lyapunov exponents, and $Q(t)$ is a bounded, piecewise continuous perturbation such that*

$$\|Q(t)\| \to 0 \text{ as } t \to \infty,$$

then the perturbed system is regular.

Proof. Theorem 3.26 implies that

$$\Lambda_1 + \cdots + \Lambda_d = \tilde{\Lambda}_1 + \cdots + \tilde{\Lambda}_d.$$

Moreover, due to the Liouville–Ostrogradski formula, we have

$$\det \tilde{X}(t) = \det \tilde{X}(0)\, e^{\int_0^t \operatorname{tr}(A(\tau)+Q(\tau))d\tau} = \det(\tilde{X}(0)X(0)^{-1}) \det X(t)\, e^{\int_0^t \operatorname{tr}(Q(\tau))d\tau}.$$

Since

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{tr}(Q(\tau))d\tau = 0,$$

regularity of the original system implies regularity of the perturbed system:

$$\begin{aligned}\Lambda_1 + \cdots + \Lambda_d = \tilde{\Lambda}_1 + \cdots + \tilde{\Lambda}_d &\geq \liminf_{t\to\infty} \frac{1}{t} \log |\det \tilde{X}(t)| = \liminf_{t\to\infty} \frac{1}{t} \log |\det X(t)| \\ &= \Lambda_1 + \cdots + \Lambda_d.\end{aligned}$$

□

3.5 Strong fast invertibility

In this subsection, we introduce *strong fast invertibility*, which is a weaker concept than the stability of Lyapunov exponents but still sufficient for their computation. Our main objectives are a characterization result that allows us to compare strong fast invertibility to the stability of Lyapunov exponents and perturbation results in preparation for the analysis of Benettin's algorithm.

3.5.1 Definition and relation to induced systems

Quas et al. introduce three notions of fast invertibility (weak/standard/strong) to analyze the existence of a dominated splitting for discrete-time systems [21]. Here, we focus solely on the strong version, since weak and standard fast invertibility are trivially satisfied for systems with bounded coefficients.

Definition 3.28 ([21])**.** Equation (3.1) is said to be *L-dimensionally strongly fast invertible*[7] if

$$c_{FI,L} := \sup_{t\geq s\geq \tau} \prod_{k=1}^{L} \frac{\sigma_k(X(t,s))\sigma_k(X(s,\tau))}{\sigma_k(X(t,\tau))} < \infty.$$

There are several equivalent formulations immediately visible from the definition.

Lemma 3.29. *The following are equivalent:*

(i) Equation (3.1) *is L-dim. strongly fast invertible.*

(ii) The induced system on $\wedge^L \mathbb{R}^d$ *is* 1*-dim. strongly fast invertible.*

(iii) There is a constant $c > 0$ *such that*

$$\frac{\| \wedge^L X(t,\tau)\|}{\| \wedge^L X(s,\tau)\|} \leq \| \wedge^L X(t,s)\| \leq c\frac{\| \wedge^L X(t,\tau)\|}{\| \wedge^L X(s,\tau)\|}$$

for all $t \geq s \geq \tau$*.*

(iv) $\sup_{t\geq s} \prod_{k=1}^{L} \frac{\sigma_k(X(t,s))\sigma_k(X(s))}{\sigma_k(X(t))} < \infty$ *for any fundamental matrix* $X(t)$*.*

(v) $\sup_{t\geq s\geq \tau\geq T} \prod_{k=1}^{L} \frac{\sigma_k(X(t,s))\sigma_k(X(s,\tau))}{\sigma_k(X(t,\tau))} < \infty$ *for any* $T > 0$*.*

Proof. The relation between strong fast invertibility and exterior products follows from Lemma 2.2. The third characterization is merely a reformulation of strong fast invertibility for the induced system. Equivalence of strong fast invertibility to the fourth characterization follows from

$$\frac{\| \wedge^L X(t,s)\| \, \| \wedge^L X(s)\|}{\| \wedge^L X(t)\|} \leq \| \wedge^L X(0)\| \, \| \wedge^L X(0)^{-1}\| \frac{\| \wedge^L X(t,s)\| \, \| \wedge^L X(s,0)\|}{\| \wedge^L X(t,0)\|}$$

and

$$\begin{aligned}
&\frac{\| \wedge^L X(t,s)\| \, \| \wedge^L X(s,\tau)\|}{\| \wedge^L X(t,\tau)\|} \\
&\quad = \frac{\| \wedge^L X(t,s)\| \, \| \wedge^L X(s)\|}{\| \wedge^L X(t)\|} \cdot \frac{\| \wedge^L X(s,\tau)\| \, \| \wedge^L X(\tau)\|}{\| \wedge^L X(s)\|} \cdot \frac{\| \wedge^L X(t)\|}{\| \wedge^L X(t,\tau)\| \, \| \wedge^L X(\tau)\|} \\
&\quad \leq \frac{\| \wedge^L X(t,s)\| \, \| \wedge^L X(s)\|}{\| \wedge^L X(t)\|} \cdot \frac{\| \wedge^L X(s,\tau)\| \, \| \wedge^L X(\tau)\|}{\| \wedge^L X(s)\|}.
\end{aligned}$$

[7]The name "fast invertibility" stems from the context of [21] as it ensures that the cocylce is uniformly invertible on the fastest subspace of the respective dimension (without assuming bounded coefficients).

Finally, using

$$\wedge^L X(T+t,t) \wedge^L X(t,s) = \wedge^L X(T+t,T+s) \wedge^L X(T+s,s),$$

and $\sup_t \|\hat{A}^L(t)\| \leq LM$, one may show the final characterization:

$$\frac{\| \wedge^L X(t,s)\| \, \| \wedge^L X(s,\tau)\|}{\| \wedge^L X(t,\tau)\|} \leq e^{6TLM} \frac{\| \wedge^L X(T+t,T+s)\| \, \| \wedge^L X(T+s,T+\tau)\|}{\| \wedge^L X(T+t,T+\tau)\|}. \qquad \square$$

Maybe the most intuitive of the above characterizations of strong fast invertibility is the third. It means that the maximal growth of L-volumes along any interval can be computed (up to a constant independent of the interval) by bisecting the interval and multiplying the maximal growths of L-volumes on the subintervals.

Proposition 3.30. *Every system is d-dim. strongly fast invertible.*

Proof. Since $\dim(\wedge^d \mathbb{R}^d) = 1$, we have $c_{FI,d} = 1$. □

Example 3.31. The system from Example 3.6 is strongly fast invertible at dimensions 1 and 2: Due to the previous proposition, we only need to consider the first dimension. We compute the Cauchy matrix

$$X(t,s) := \begin{pmatrix} e^{t-s} & 0 \\ 0 & e^{\int_s^t a_{22}(\tau)\,d\tau} \end{pmatrix}.$$

Since $0 \leq a_{22}(t) \leq 1$, the first diagonal entry of $X(t,s)$ is always larger than the second for $t \geq s$. In particular, it holds

$$\sigma_1(X(t,s)) = e^{t-s},$$

which implies 1-dim. strong fast invertibility.

3.5.2 Characterization and comparison to stability of Lyapunov exponents

To derive the characterization theorem mentioned in the introduction, we first prove that regularity and strong fast invertibility imply the existence of a Lyapunov transformation that brings the system into block diagonal form.

Proposition 3.32. *Assume equation (3.1) has Lyapunov exponents $\lambda_1 > \cdots > \lambda_p$ with multiplicities $d_1 + \cdots + d_p = d$. If the system is regular and strongly fast invertible at dim. $d_1 + \cdots + d_l$ for $l = 1, \ldots, p$, then there is a Lyapunov transformation reducing the system to block diagonal form:*

$$\dot{y} = \operatorname{diag}(B_1(t), \ldots, B_p(t))y,$$

where $B_i \in \mathbb{R}^{d_i \times d_i}$ is upper triangular, such that all non-trivial solutions of $\dot{y}_i = B_i y_i$ have characteristic exponent λ_i.

We prove the proposition via induced systems using two auxiliary lemmata and the following result:

Theorem 3.33 ([1, Theorems 3.3.3 and 3.3.4])**.** *There exists a Lyapunov transformation reducing equation (3.1) to block diagonal form:*

$$\dot{y} = \operatorname{diag}(B_1(t), \ldots, B_k(t))y,$$

where $B_i \in \mathbb{R}^{n_i \times n_i}$ is upper triangular, if and only if there is a fundamental matrix

$$X = [X_1, \dots, X_k]$$

with $X_i \in \mathbb{R}^{d \times n_i}$ satisfying

$$\inf_t \frac{G(X)}{G(X_1)\dots G(X_k)} > 0, \tag{3.5}$$

where G denotes the Gram determinant.

Remark 3.34. Given a fundamental matrix $X(t)$ as in Theorem 3.33, the Lyapunov transformation is constructed by applying the Gram–Schmidt procedure to each block $X_i(t)$ individually, yielding

$$X(t) = [Q_1(t), \dots, Q_k(t)] \operatorname{diag}(R_1(t), \dots, R_k(t)) =: L(t)Y(t).$$

Lemma 3.35 ([1, Remark 3.3.4])**.** *Let $X = [X_1, X_2]$. It holds*

$$\frac{G(X)}{G(X_1)G(X_2)} = \prod_{i=1}^{l} \sin^2 \alpha_i(\operatorname{im} X_1, \operatorname{im} X_2),$$

where $\alpha_1 \le \dots \le \alpha_l$ are the principle angles.

Lemma 3.36. *If equation (3.1) is 1-dim. strongly fast invertible and $X = [X_1, X_2]$ is a fundamental matrix such that $X_1 \in \mathbb{R}^{d\times 1}$ has a higher characteristic exponent than the columns of $X_2 \in \mathbb{R}^{d\times(d-1)}$, then*

$$\inf_t \sin^2 \alpha_1(\operatorname{im} X_1(t), \operatorname{im} X_2(t)) > 0. \tag{3.6}$$

Proof. Let $X = [X_1, X_2]$ be as in the claim. We first apply the Gram–Schmidt procedure to bring X into upper triangular form. Indeed, the Gram–Schmidt procedure is an orthogonal Lyapunov transformation (Proposition 3.2). Thus, it leaves the fraction in equation (3.5) and hence also $\sin^2 \alpha_1(\operatorname{im} X_1, \operatorname{im} X_2)$ invariant. So, it is sufficient to check equation (3.6) for

$$Y = [Y_1, Y_2] = \begin{pmatrix} y_{11} & Y_{12} \\ 0 & Y_{22} \end{pmatrix}$$

in which y_{11} has a higher characteristic exponent than $\|Y_{12}\|$ and $\|Y_{22}\|$.

Given $s \ge 0$, we always find $t_s \ge s$ such that

$$\max_{\alpha \neq 0} \frac{\left| \frac{y_{11}(s)}{y_{11}(t_s)} Y_{12}(t_s)\alpha \right|}{\|Y_{22}(s)\alpha\|} \le 1$$

and $\|Y(t_s)\| \le 2|y_{11}(t_s)|$. In order to make use of strong fast invertibility, we compute the Cauchy matrix:

$$Y(t,s) = \begin{pmatrix} y_{11}(t,s) & Y_{12}(t,s) \\ 0 & Y_{22}(t,s) \end{pmatrix} = \begin{pmatrix} \frac{y_{11}(t)}{y_{11}(s)} & Y_{12}(t)Y_{22}(s)^{-1} - \frac{y_{11}(t)}{y_{11}(s)} Y_{12}(s)Y_{22}(s)^{-1} \\ 0 & Y_{22}(t)Y_{22}(s)^{-1} \end{pmatrix}.$$

Now, for $s \ge 0$ and $\alpha \ne 0$ it holds

$$\begin{aligned} \frac{|Y_{12}(s)\alpha|}{\|Y_{22}(s)\alpha\|} &\le 1 + \left| \frac{\frac{y_{11}(s)}{y_{11}(t_s)} Y_{12}(t_s)\alpha}{\|Y_{22}(s)\alpha\|} - \frac{Y_{12}(s)\alpha}{\|Y_{22}(s)\alpha\|} \right| = 1 + \frac{\left| \frac{y_{11}(s)}{y_{11}(t_s)} Y_{12}(t_s,s)Y_{22}(s)\alpha \right|}{\|Y_{22}(s)\alpha\|} \\ &\le 1 + \frac{\|Y_{12}(t_s,s)\|\,|y_{11}(s)|}{|y_{11}(t_s)|} \le 1 + 2\frac{\|Y(t_s,s)\|\,\|Y(s)\|}{\|Y(t_s)\|} \le \underbrace{1 + 2c_{FI,L}\,\|Y(0)\|\,\|Y(0)^{-1}\|}_{=:c}. \end{aligned}$$

Since

$$\left(\frac{\langle Y_1(t), Y_2(t)\alpha\rangle}{\|Y_1(t)\|\ \|Y_2(t)\alpha\|}\right)^2 = \left(\frac{\langle e_1, Y_2(t)\alpha\rangle}{\|Y_2(t)\alpha\|}\right)^2 = \frac{|Y_{12}(t)\alpha|^2}{|Y_{12}(t)\alpha|^2 + \|Y_{22}(t)\alpha\|^2} = \frac{1}{1+\left(\frac{|Y_{12}(t)\alpha|}{\|Y_{22}(t)\alpha\|}\right)^{-2}}$$
$$\leq \frac{1}{1+c^{-2}},$$

we have

$$\sin^2\alpha_1(\operatorname{im} Y_1, \operatorname{im} Y_2) = 1 - \cos^2\alpha_1(\operatorname{im} Y_1, \operatorname{im} Y_2) = 1 - \max_{\alpha\neq 0}\left(\frac{\langle Y_1(t), Y_2(t)\alpha\rangle}{\|Y_1(t)\|\ \|Y_2(t)\alpha\|}\right)^2$$
$$\geq 1 - \frac{1}{1+c^{-2}} > 0. \qquad \square$$

Proof of Proposition 3.32. Let $X(t) = [X_1, \dots, X_p]$ be an ordered normal fundamental matrix so that the columns of X_i have characteristic exponent λ_i. We first show that equation (3.5) holds with $n_1 = L$ and $n_2 = d - L$ if the system is strongly fast invertible at dim. $L = d_1 + \cdots + d_l$ and then apply Theorem 3.33 successively to arrive at the desired form.

Fix $L = d_1 + \cdots + d_l$ and set

$$U := \operatorname{im}[X_1, \dots, X_l] \quad \text{and} \quad V := \operatorname{im}[X_{l+1}, \dots, X_p].$$

Lemma 3.35 and Corollary 2.5 imply

$$\frac{G(X)}{G([X_1,\dots,X_l])G([X_{l+1},\dots,X_p])} = \prod_{i=1}^{\min(L,d-L)} \sin^2\alpha_i(U,V) = \sin^2\alpha_1(\hat{U},\hat{V}),$$

where

$$\hat{U} = \operatorname{span}\{(\wedge^L X)(e_1 \wedge \cdots \wedge e_L)\}$$

and

$$\hat{V} = \operatorname{span}\{(\wedge^L X)(e_{i_1} \wedge \cdots \wedge e_{i_L}) \mid \{i_1,\dots,i_L\} \neq \{1,\dots,L\}\}.$$

Now, let $Y = [Y_1, Y_2]$ be a matrix representation of $\wedge^L X$ with respect to the standard basis on $\wedge^L \mathbb{R}^d$. Order the induced basis starting with $e_1 \wedge \cdots \wedge e_L$ so that the first column Y_1 of Y represents $(\wedge^L X)(e_1 \wedge \cdots \wedge e_L)$. Then,

$$\sin^2\alpha_1(\hat{U},\hat{V}) = \sin^2\alpha_1(\operatorname{im} Y_1, \operatorname{im} Y_2).$$

Due to regularity, Y_1 has characteristic exponent $\Lambda_1 + \cdots + \Lambda_L$ and the columns of Y_2 have lower characteristic exponents. Hence, Lemma 3.36 and the above arguments imply

$$\inf_t \frac{G(X)}{G([X_1,\dots,X_l])G([X_{l+1},\dots,X_p])} > 0.$$

To arrive at the block diagonal form claimed in the proposition, we apply the transformation from Remark 3.34 successively. First, we use strong fast invertibility at dim. d_1 to make the first d_1 columns of X orthogonal to the rest. Assuming orthogonality of the first d_1 columns, strong fast invertibility at dim. $d_1 + d_2$ implies

$$0 < \inf_t \frac{G(X)}{G([X_{d_1}, X_{d_2}])G([X_{d_3},\dots,X_{d_p}])} = \inf_t \frac{G(X)}{G(X_{d_1})G(X_{d_2})G([X_{d_3},\dots,X_{d_p}])}.$$

Thus, we may apply another Lyapunov transformation to decouple the next subsystem and so on. $\square$

Using Proposition 3.32, we now may prove our characterization result for strong fast invertibility.

Theorem 3.37. *Assume equation* (3.1) *has Lyapunov exponents* $\lambda_1 > \cdots > \lambda_p$ *with multiplicities* $d_1 + \cdots + d_p = d$. *If the system is regular and strongly fast invertible at dim.* $d_1 + \cdots + d_l$ *for* $l = 1, \ldots, p$, *then there exists a Lyapunov transformation reducing the system to block diagonal form*

$$\dot{y} = \operatorname{diag}(B_1(t), \ldots, B_p(t))y,$$

where $B_i(t) \in \mathbb{R}^{d_i \times d_i}$ *is upper triangular, such that the following hold:*

(i) all non-trivial solutions of $\dot{y}_i = B_i(t)y_i$ *have characteristic exponent* λ_i,

(ii) there is a constant $b > 0$ *such that*

$$\|Y_i(t,s)^{-1}\|^{-1} \geq b\|Y_{i+1}(t,s)\| \tag{3.7}$$

for all $t \geq s$, *where* $Y_i(t,s)$ *denotes the Cauchy matrix of* $\dot{y}_i = B_i(t)y_i$.

Conversely, any block diagonal system $\dot{y} = B(t)y$ *satisfying* (i) *and* (ii) *is strongly fast invertible at dim.* $d_1 + \cdots + d_l$ *for* $l = 1, \ldots, p$.

Proof. Let us first assume that the system is regular and strongly fast invertible at the respective dimensions. After applying Proposition 3.32, all that remains is to show that equation (3.7) is satisfied for the reduced system. To this end, fix $L = d_1 + \cdots + d_l$ and set $Y = Z_1 + Z_2$ with

$$Z_1 := \operatorname{diag}(Y_1, \ldots, Y_L, 0, \ldots, 0)$$

and

$$Z_2 := \operatorname{diag}(0, \ldots, 0, Y_{L+1}, \ldots, Y_p).$$

Due to Lemma 2.3, we have

$$\| \wedge^L Y\| = \max_k \|(\wedge^k Z_1) \wedge (\wedge^{L-k} Z_2)\|$$

and

$$\| \wedge^L Y\| \geq \| \wedge^{L-1} Z_1\| \, \|Z_2\| = \frac{\| \wedge^L Z_1\|}{\sigma_L(Z_1)} \|Z_2\|.$$

The singular value $\sigma_L(Z_1)$ can be estimated via

$$\sigma_L(Z_1) = \sigma_L(\operatorname{diag}(Y_1, \ldots, Y_l)) = \| \operatorname{diag}(Y_1^{-1}, \ldots, Y_l^{-1})\|^{-1} \leq \|Y_l^{-1}\|^{-1}.$$

Due to regularity, we have

$$\chi[\| \wedge^L Z_1\|] = \lambda_{1,L} > \lambda_{2,L} \geq \chi[\|(\wedge^k Z_1) \wedge (\wedge^{L-k} Z_2)\|]$$

for $k < L$, which implies

$$\| \wedge^L Y(t)\| = \| \wedge^L Z_1(t)\|$$

for t large enough. Moreover, it holds

$$\| \wedge^L Z_1(t,s)\| = \| \wedge^L \operatorname{diag}(Y_1(t,s), \ldots, Y_l(t,s))\| = \frac{\| \wedge^L \operatorname{diag}(Y_1(t), \ldots, Y_l(t))\|}{\| \wedge^L \operatorname{diag}(Y_1(s), \ldots, Y_l(s))\|} = \frac{\| \wedge^L Z_1(t)\|}{\| \wedge^L Z_1(s)\|}$$

since $\wedge^L \mathbb{R}^L$ is one-dimensional. Combining the previous estimates, we get

$$\begin{aligned}
\|Y_l(t,s)^{-1}\|^{-1} &\geq \frac{\|\wedge^L Z_1(t,s)\|}{\|\wedge^L Y(t,s)\|}\|Z_2(t,s)\| \geq \frac{\|\wedge^L Z_1(t,s)\|}{\|\wedge^L Y(t,s)\|}\|Y_{l+1}(t,s)\| \\
&= \frac{\|\wedge^L Z_1(t)\|}{\|\wedge^L Y(t,s)\|\,\|\wedge^L Z_1(s)\|}\|Y_{l+1}(t,s)\| \geq \frac{\|\wedge^L Y(t)\|}{\|\wedge^L Y(t,s)\|\,\|\wedge^L Y(s)\|}\|Y_{l+1}(t,s)\| \\
&\geq \frac{1}{c_{FI,L}\|\wedge^L Y(0)\|\,\|\wedge^L Y(0)^{-1}\|}\|Y_{l+1}(t,s)\|
\end{aligned}$$

for large $t > 0$, which proves equation (3.7).

Next, we show that any block diagonal system as in Theorem 3.37 is strongly fast invertible at the respective dimensions. As before, fix $L = d_1 + \cdots + d_l$ and assume the decomposition $Y = Z_1 + Z_2$. Applying equation (3.7) repeatedly yields

$$\|Y_i^{-1}\|^{-1} \geq b\|Y_{i+1}\| \geq b\|Y_{i+1}^{-1}\|^{-1} \geq \cdots \geq b^k\|Y_{i+k}\|$$

and hence

$$\begin{aligned}
\sigma_L(Z_1) = \sigma_L(\operatorname{diag}(Y_1,\ldots,Y_l)) &= \|\operatorname{diag}(Y_1^{-1},\ldots,Y_l^{-1})\|^{-1} = \min_{i=1,\ldots,l}\|Y_i^{-1}\|^{-1} \\
&\geq b^{p-1}\max_{i=l+1,\ldots,p}\|Y_i\| = b^{p-1}\|Z_2\|
\end{aligned}$$

since $b \in (0,1]$ (set $s = t$ in equation (3.7)). Now, we estimate

$$\begin{aligned}
\|\wedge^L Y\| &= \max_k \|(\wedge^k Z_1)\wedge(\wedge^{L-k}Z_2)\| \leq \max_k \binom{d}{L}^{\frac{1}{2}}\|\wedge^k Z_1\|\,\|\wedge^{L-k}Z_2\| \\
&\leq \max_k \binom{d}{L}^{\frac{1}{2}}\|\wedge^k Z_1\|\,\|Z_2\|^{L-k} \leq \max_k \binom{d}{L}^{\frac{1}{2}}\|\wedge^k Z_1\| b^{-(p-1)(L-k)}\sigma_L(Z_1)^{L-k} \\
&\leq \underbrace{\max_k\left(\binom{d}{L}^{\frac{1}{2}} b^{-(p-1)(L-k)}\right)}_{=:c}\|\wedge^L Z_1\|.
\end{aligned}$$

Hence, we get

$$\sup_{t\geq s}\frac{\|\wedge^L Y(t,s)\|\,\|\wedge^L Y(s)\|}{\|\wedge^L Y(t)\|} \leq c^2\underbrace{\sup_{t\geq s}\frac{\|\wedge^L Z_1(t,s)\|\,\|\wedge^L Z_1(s)\|}{\|\wedge^L Z_1(t)\|}}_{=1} < \infty. \qquad \square$$

Remark 3.38. When testing for strong fast invertibility (resp. for stability of Lyapunov exponents), it is enough to check the conditions of Theorem 3.37 (resp. Theorem 3.24) after transforming the original system to any block diagonal system of the form

$$\dot{y} = \operatorname{diag}(B_1(t),\ldots,B_p(t))y$$

such that the non-trivial solutions of block $B_i(t) \in \mathbb{R}^{d_i\times d_i}$ have characteristic exponent λ_i.

Proof. Any Lyapunov transformation between systems of this form acts as Lyapunov transformations on the individual blocks, i.e., $L(t) = \operatorname{diag}(L_1(t),\ldots,L_p(t))$. This implies that equation (3.7) (resp. stability of the Lyapunov exponents of the subsystems and equation (3.4)) holds for one such system if and only if it holds for all of them. $\square$

An immediate consequence of Theorem 3.37 is that stability of Lyapunov exponents implies strong fast invertibility.

Corollary 3.39. *Assume equation (3.1) has Lyapunov exponents $\lambda_1 > \cdots > \lambda_p$ with multiplicities $d_1 + \cdots + d_p = d$. If the Lyapunov exponents are stable, then the system is strongly fast invertible at dim. $d_1 + \cdots + d_l$ for $l = 1, \ldots, p$.*

Remark 3.40. Example 3.6 shows that the converse of Corollary 3.39 is false in general (even with the help of regularity). There are regular systems that are strongly fast invertible at every dimension that do not have stable Lyapunov exponents.

In case the Lyapunov spectrum is simple, strong fast invertibility implies the existence of a fundamental system of solutions with uniformly separated growth.

Theorem 3.41. *If equation (3.1) is regular, has simple Lyapunov spectrum and is strongly fast invertible at every dimension, then there exists a fundamental matrix $X(t) = [x_1(t), \ldots, x_d(t)]$ and a constant $b > 0$ such that*

$$\frac{\|x_i(t)\|}{\|x_i(s)\|} \geq b \frac{\|x_{i+1}(t)\|}{\|x_{i+1}(s)\|} \tag{3.8}$$

for all $t \geq s$.

Proof. According to Theorem 3.37 there is a Lyapunov transformation $L(t)$ bringing the system to diagonal form with fundamental matrix $Y = \operatorname{diag}(y_{11}, \ldots, y_{dd})$ and a constant $b' > 0$ such that

$$\frac{|y_{ii}(t)|}{|y_{ii}(s)|} \geq b' \frac{|y_{(i+1)(i+1)}(t)|}{|y_{(i+1)(i+1)}(s)|}$$

for all $t \geq s$. The fundamental matrix $X = LY$ of the original system satisfies

$$\|x_i(t)\| = \|L(t)e_i\| \, |y_{ii}(t)|.$$

Since $L(t)$ is a Lyapunov transformation, the term $\|L(t)e_i\|$ is uniformly bounded from below and above by positive constants. equation (3.8) follows. □

Remark 3.42. There is a regular system with simple Lyapunov spectrum satisfying equation (3.8) that is not strongly fast invertible at every dimension.

Proof. Let $0 < \epsilon_n \leq 1/e$ be a sequence of numbers converging to zero. We set

$$c_n := \left(\frac{9}{4}\right)^n \prod_{k=1}^{n} \left(\frac{1}{\epsilon_k}\right)^{\frac{1}{\epsilon_k}}.$$

The numbers

$$\begin{aligned}
t_{n,0} &:= c_n, \\
t_{n,1} &:= c_n + \frac{2}{3(1-\epsilon_n)}, \\
t_{n,2} &:= c_n + \frac{2}{3(1-\epsilon_n)} + \frac{\log\left(\frac{1}{\epsilon_n}\right)}{1-\epsilon_n} - \frac{2}{3}, \\
t_{n,3} &:= c_n + \frac{2}{3(1-\epsilon_n)} + \frac{\log\left(\frac{1}{\epsilon_n}\right)}{1-\epsilon_n},
\end{aligned}$$

$$t_{n,4} := c_n + \frac{2}{3(1-\epsilon_n)} + \frac{\log\left(\frac{1}{\epsilon_n}\right)}{\epsilon_n(1-\epsilon_n)},$$
$$t_{n,5} := c_n + \frac{2}{3(1-\epsilon_n)} + \frac{\log\left(\frac{1}{\epsilon_n}\right)}{\epsilon_n(1-\epsilon_n)} + 1,$$
$$t_{n,6} := c_n + \frac{5}{3(1-\epsilon_n)} + \frac{\log\left(\frac{1}{\epsilon_n}\right)}{\epsilon_n(1-\epsilon_n)},$$

satisfy

$$t_{n,0} \leq \cdots \leq t_{n,6} \leq t_{n+1,0}.$$

On the partition introduced by these numbers, we define functions

$$f(t) := \begin{cases} 1, & t \in [0, t_{1,0}) \\ c_{n-1} p_{n,f}(t - t_{n,0}), & t \in [t_{n,0}, t_{n,1}) \\ c_{n-1} \frac{3}{2} e^{(1-\epsilon_n)(t - t_{n,1})}, & t \in [t_{n,1}, t_{n,4}) \\ c_{n-1} \frac{3}{2} \left(\frac{1}{\epsilon_n}\right)^{\frac{1}{\epsilon_n}} q_{n,f}(t - t_{n,4}), & t \in [t_{n,4}, t_{n,6}) \\ c_n, & t \in [t_{n,6}, t_{n+1,0}) \end{cases}$$

and

$$g(t) := \begin{cases} 1, & t \in [0, t_{1,2}) \\ c_{n-1} p_{n,g}(t - t_{n,2}), & t \in [t_{n,2}, t_{n,3}) \\ c_{n-1} \frac{3}{2} e^{t - t_{n,3}}, & t \in [t_{n,3}, t_{n,4}) \\ c_{n-1} \frac{3}{2} \left(\frac{1}{\epsilon_n}\right)^{\frac{1}{\epsilon_n}} q_{n,g}(t - t_{n,4}), & t \in [t_{n,4}, t_{n,5}) \\ c_n, & t \in [t_{n,5}, t_{n+1,2}) \end{cases}$$

using the polynomials

$$p_{n,f}(t) := \frac{9}{8}(1-\epsilon_n)^2 t^2 + 1,$$
$$p_{n,g}(t) := \frac{9}{8} t^2 + 1,$$
$$q_{n,f}(t) := -\frac{1}{2}(1-\epsilon_n)^2 t^2 + (1-\epsilon_n) t + 1,$$
$$q_{n,g}(t) := -\frac{1}{2} t^2 + t + 1,$$

for smoothing. The functions have the following properties:

- $f, g \in C^1(\mathbb{R}_{\geq 0}, \mathbb{R})$,
- f, g are monotonically increasing,
- $|f' - f|/|g|$ bounded,
- $|g'|/|g|$ bounded,
- $1 \leq f(t), g(t) \leq t$ for $t \geq 1$.

In particular,

$$\dot{x} = A(t)x := \begin{pmatrix} 1 & \frac{f'-f}{g} \\ 0 & \frac{g'}{g} \end{pmatrix} x$$

is a continuous, bounded system with fundamental matrix

$$X(t) = [x_1(t), x_2(t)] = \begin{pmatrix} e^t & f \\ 0 & g \end{pmatrix}.$$

The solution $x_2(t)$ has characteristic exponent zero and the system is regular.

To show property equation (3.8), we first remark that

$$\frac{|p'_{n,f}|}{|p_{n,f}|}, \frac{|q'_{n,f}|}{|q_{n,f}|} \leq 1$$

on the respective intervals used in the definition of f. Since

$$(\log f(t))' = \frac{f'(t)}{f(t)} \leq 1,$$

the mean value theorem implies

$$\frac{f(t)}{f(s)} \leq e^{t-s}$$

for $t \geq s$. In combination with $g(t) \leq \frac{3}{2} f(t)$, we get equation (3.8):

$$\frac{\|x_2(t)\|^2}{\|x_2(s)\|^2} = \frac{f^2(t) + g^2(t)}{f^2(s) + g^2(s)} \leq \frac{13}{4} \frac{f^2(t)}{f^2(s)} \leq \frac{13}{4} \frac{\|x_1(t)\|^2}{\|x_1(s)\|^2}$$

for $t \geq s$.

To show that the system is not 1-dim. strongly fast invertible, we prove that there is no Lyapunov transformation bringing the system into diagonal form. For our particular fundamental matrix, it holds

$$\left(\frac{\langle x_1, x_2 \rangle}{\|x_1\| \, \|x_2\|} \right)^2 = \frac{1}{1 + \left(\frac{f}{g} \right)^{-2}}.$$

Since $f(t_{n,3})/g(t_{n,3}) = 1/\epsilon_n \to \infty$, the above expression converges to 1 along the subsequence $(t_{n,3})_n$. Now, let $Y = [y_1, y_2]$ be an arbitrary fundamental matrix. By expanding y_1 and y_2 with respect to our particular solutions and by using

$$\lim_{t \to \infty} \frac{\|x_2(t)\|}{\|x_1(t)\|} = 0,$$

one computes

$$\lim_{n \to \infty} \left(\frac{\langle y_1(t_{n,3}), y_2(t_{n,3}) \rangle}{\|y_1(t_{n,3})\| \, \|y_2(t_{n,3})\|} \right)^2 = 1.$$

Thus,

$$\inf_t \frac{G(Y)}{G(y_1)G(y_2)} = 1 - \sup_t \left(\frac{\langle y_1, y_2 \rangle}{\|y_1\| \, \|y_2\|} \right)^2 = 0,$$

and Theorem 3.33 implies that there is no Lyapunov transformation bringing the system into diagonal form. In particular, the system cannot be 1-dim. strongly fast invertible according to Theorem 3.37. $\square$

3.5.3 Perturbation theory

We now establish several perturbation results for strongly fast invertible systems. Since we plan to use them to derive convergence theorems for Benettin's algorithm and since numerical integration can be represented as a piecewise continuous dynamical system, the perturbation results are formulated for systems

$$\dot{x} = A(t)x \tag{3.9}$$

such that A is bounded and piecewise continuous. Moreover, we assume all perturbations in this subsection to be of the same class, i.e., bounded and piecewise continuous.

While strong fast invertibility is weaker than the stability of Lyapunov exponents, it is enough to ensure upper semicontinuity of $\lambda_{1,L}$.

Theorem 3.43. *Assume equation* (3.9) *is L-dim. strongly fast invertible. For any perturbation $Q(t)$, the largest Lyapunov exponent of the induced system satisfies*

$$\tilde{\lambda}_{1,L} \leq \lambda_{1,L} + Lc_{FI,L} \limsup_{t\to\infty} \frac{1}{t} \int_0^t \|Q(\tau)\|\, d\tau \leq \lambda_{1,L} + Lc_{FI,L}\|Q\|_{L^\infty}$$

Remark 3.44. Remember that

$$\lambda_{1,L} = \Lambda_1 + \cdots + \Lambda_L$$

for regular systems.

Proof. Let $\tilde{X}(t)$ be a fundamental matrix of the perturbed system. $\wedge^L \tilde{X}$ solves the induced equation

$$\frac{d}{dt}(\wedge^L \tilde{X}) = \widehat{A+Q}^L(\wedge^L \tilde{X}) = \hat{A}^L(\wedge^L X) + \hat{Q}^L(\wedge^L X)$$

and can be seen as a solution of a perturbed system to

$$\frac{d}{dt}(\wedge^L X) = \widehat{A}^L(\wedge^L X).$$

The variation of constants formula implies

$$\wedge^L \tilde{X}(t,s) = \wedge^L X(t,s) + \int_s^t (\wedge^L X(t,\tau))\hat{Q}^L(\tau)(\wedge^L \tilde{X}(\tau,s))\, d\tau.$$

Using $\|\hat{Q}^L(t)\| \leq L\|Q(t)\|$, we first estimate

$$\begin{aligned}
\frac{\| \wedge^L \tilde{X}(t,s)\|}{\| \wedge^L X(t,s)\|} &\leq 1 + L\int_s^t \frac{\| \wedge^L X(t,\tau)\| \, \| \wedge^L X(\tau,s)\|}{\| \wedge^L X(t,s)\|} \|Q(\tau)\| \frac{\| \wedge^L \tilde{X}(\tau,s)\|}{\| \wedge^L X(\tau,s)\|}\, d\tau \\
&\leq 1 + Lc_{FI,L} \int_s^t \|Q(\tau)\| \frac{\| \wedge^L \tilde{X}(\tau,s)\|}{\| \wedge^L X(\tau,s)\|}\, d\tau
\end{aligned}$$

and then apply Grönwall's inequality to get

$$\frac{\| \wedge^L \tilde{X}(t,s)\|}{\| \wedge^L X(t,s)\|} \leq e^{Lc_{FI,L}\int_s^t \|Q(\tau)\|\, d\tau}.$$

The claim now follows from Proposition 3.9. $\square$

Lemma 3.45. *Perturbations on finite intervals do not change the Lyapunov spectrum of a system or its induced systems.*

Proof. Since characteristic exponents are asymptotic quantities and since one may choose fundamental matrices $X(t)$ and $\tilde{X}(t)$ that coincide for large t, the claim follows. □

Theorem 3.46. *Assume equation (3.9) is L-dim. strongly fast invertible. If $Q(t)$ is a perturbation such that*

$$\|Q(t)\| \to 0 \text{ as } t \to \infty,$$

then

$$\tilde{\lambda}_{1,L} \leq \lambda_{1,L}.$$

Proof. The theorem is a direct consequence of Theorem 3.43. □

Equality in Theorem 3.46 can be achieved for L^1-perturbations.

Theorem 3.47. *Assume equation (3.9) is L-dim. strongly fast invertible. If $Q(t)$ is a perturbation such that*

$$\int_0^\infty \|Q(t)\|\, dt < \infty,$$

then

$$\tilde{\lambda}_{1,L} = \lambda_{1,L}.$$

The theorem follows directly from Theorem 3.46 using the following lemma.

Lemma 3.48. *Assume equation (3.9) is L-dim. strongly fast invertible. If $Q(t)$ is a perturbation such that*

$$\int_0^\infty \|Q(t)\|\, dt < \infty,$$

then the perturbed system is L-dim. strongly fast invertible. Moreover, $c_{FI,L}$ is continuous with respect to L^1-perturbations.

Proof. According to Lemma 3.29 strong fast invertibility can be tested on $[T, \infty)$ for any fixed $T > 0$. Thus, by setting $Q(t)$ to zero on a finite interval, we may assume that

$$\int_0^\infty \|Q(t)\|\, dt < \frac{\log(2)}{Lc_{FI,L}} \tag{3.10}$$

without affecting whether the perturbed system is strongly fast invertible or not. The variation of constants formula implies

$$\wedge^L \tilde{X}(t,s) - \wedge^L X(t,s) = \int_s^t (\wedge^L X(t,\tau))\hat{Q}^L(\tau)(\wedge^L \tilde{X}(\tau,s))\, d\tau.$$

Thus, it holds

$$\begin{aligned} 1 &+ \frac{\| \wedge^L \tilde{X}(t,s) - \wedge^L X(t,s)\|}{\| \wedge^L X(t,s)\|} \\ &\leq 1 + L\int_s^t \frac{\| \wedge^L X(t,\tau)\| \, \| \wedge^L X(\tau,s)\|}{\| \wedge^L X(t,s)\|} \|Q(\tau)\| \left(1 + \frac{\| \wedge^L \tilde{X}(\tau,s) - \wedge^L X(\tau,s)\|}{\|(\wedge^L X(\tau,s)\|}\right) d\tau. \end{aligned}$$

Through Grönwall's inequality we get

$$1 + \frac{\| \wedge^L \tilde{X}(t,s) - \wedge^L X(t,s)\|}{\| \wedge^L X(t,s)\|} \leq e^{Lc_{FI,L}\int_s^t \|Q(\tau)\|\, d\tau}$$

and due to equation (3.10)

$$\begin{aligned}\| \wedge^L \tilde{X}(t,s)\| &\geq \| \wedge^L X(t,s)\| \left(1 - \frac{\| \wedge^L \tilde{X}(t,s) - \wedge^L X(t,s)\|}{\| \wedge^L X(t,s)\|}\right) \\ &\geq \| \wedge^L X(t,s)\| \left(2 - e^{Lc_{FI,L}\int_s^t \|Q(\tau)\|\, d\tau}\right).\end{aligned}$$

Combining the last estimate with the one from the proof of Theorem 3.43 yields

$$\frac{\| \wedge^L \tilde{X}(t,\tau)\|\, \| \wedge^L \tilde{X}(\tau,s)\|}{\| \wedge^L \tilde{X}(t,s)\|} \leq \frac{\| \wedge^L X(t,\tau)\|\, \| \wedge^L X(\tau,s)\|}{\| \wedge^L X(t,s)\|} \cdot \frac{e^{Lc_{FI,L}\int_s^t \|Q(u)\|\, du}}{2 - e^{Lc_{FI,L}\int_s^t \|Q(u)\|\, du}},$$

which proves that the perturbed system is L-dim. strongly fast invertible.

Moreover, it follows that

$$\tilde{c}_{FI,L} \leq c_{FI,L} \frac{e^{Lc_{FI,L}\|Q\|_{L^1}}}{2 - e^{Lc_{FI,L}\|Q\|_{L^1}}},$$

which implies upper semicontinuity of $c_{FI,L}$ with respect to L^1-perturbations. Lower semicontinuity follows by switching roles, i.e., by viewing the original system as a perturbation of the perturbed system via $-Q(t)$. □

Corollary 3.49. *Assume equation (3.9) is regular and L-dim. strongly fast invertible. For every $\hat{v} \notin V'_{2,L}$, we find $\epsilon > 0$ such that*

$$\lim_{t\to\infty} \frac{1}{t} \log \|(\wedge^L \tilde{X}(t,0))\hat{v}\| = \Lambda_1 + \cdots + \Lambda_L$$

for any perturbation $Q(t)$ with

$$\int_0^\infty \|Q(t)\|\, dt < \epsilon.$$

Proof. Since characteristic exponents are invariant under multiplication of the solution by a nonzero constant, we may assume that $\|\hat{v}\| = 1$. It holds

$$\begin{aligned}\|(\wedge^L \tilde{X}(t,0))\hat{v}\| &\geq \|(\wedge^L X(t,0))\hat{v}\| - \|(\wedge^L \tilde{X}(t,0) - \wedge^L X(t,0))\hat{v}\| \\ &\geq \|(\wedge^L X(t,0))\hat{v}\| \left(1 - \frac{\| \wedge^L \tilde{X}(t,0) - \wedge^L X(t,0)\|}{\| \wedge^L X(t,0)\|} \frac{\| \wedge^L X(t,0)\|}{\|(\wedge^L X(t,0))\hat{v}\|}\right).\end{aligned}$$

Since the system is regular, Corollary 3.21 implies that

$$\lim_{t\to\infty} \frac{\| \wedge^L X(t,0)\|}{\|(\wedge^L X(t,0))\hat{v}\|} < \infty.$$

Now, according to the proof of Lemma 3.48, we may choose $\epsilon > 0$ small enough to ensure

$$\|(\wedge^L \tilde{X}(t,0))\hat{v}\| \geq c\|(\wedge^L X(t,0))\hat{v}\|$$

for some constant $c > 0$ independent of t. Hence, it holds

$$\liminf_{t\to\infty} \frac{1}{t} \log \|(\wedge^L \tilde{X}(t,0))\hat{v}\| \geq \liminf_{t\to\infty} \frac{1}{t} \log \|(\wedge^L X(t,0))\hat{v}\| = \lambda_{1,L}$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \log \|(\wedge^L \tilde{X}(t,0))\hat{v}\| \leq \tilde{\lambda}_{1,L} = \lambda_{1,L}$$

due to Theorem 3.47. The claim follows since regularity implies $\lambda_{1,L} = \Lambda_1 + \cdots + \Lambda_L$. □

4 Computation of Lyapunov exponents

In this section we derive convergence results for the computation of Lyapunov exponents via *Benettin's algorithm* [4,5].

Assume $h_{\max} < \infty$ and let

$$\Phi : \mathbb{R}_{\geq 0} \times [0, h_{\max}] \to GL(d, \mathbb{R})$$

be a (linear) one-step method that is consistent of order $p > 0$.

Definition 4.1. We call Φ *consistent* if there is a constant $c_\Phi > 0$ such that

$$\|\Phi(t,h) - X(t+h,t)\| \leq c_\Phi h^{p+1}$$

for all $t \geq 0$ and $0 \leq h \leq h_{\max}$.

Given stepsizes $0 < h_n \leq h_{\max}$, we shorten our notation by defining

$$\Phi_n := \Phi(t_{n-1}, h_n)$$

and

$$\Phi^n := \Phi_n \ldots \Phi_1.$$

A similar notation will be adopted for X.

4.1 Benettin's algorithm

The idea behind Benettin's algorithm can be explained via exterior products: Since $V_{2,L}$ is a proper subspace of the space of solutions of the induced system on $\wedge^L \mathbb{R}^d$, the solution to Lebesgue-almost every initial condition $v_1 \wedge \cdots \wedge v_L$ has characteristic exponent $\lambda_{1,L}$. Using $X(t)[v_1, \ldots, v_L] = Q(t)R(t)$, it holds

$$\chi[r_{11}(t) \ldots r_{LL}(t)] = \chi[(\wedge^L X(t))(v_1 \wedge \cdots \wedge v_L)] = \lambda_{1,L}$$

and hence, for regular systems, the Lyapunov exponents can be computed as

$$\Lambda_i = \lim_{t \to \infty} \frac{1}{t} \log r_{ii}(t)$$

for Lebesgue-almost every tuple of initial vectors, which is the core idea behind Benettin's algorithm.

The propagated vectors[8] (columns of V_n) are reorthonormalized periodically to prevent numerical singularities. If not, they could collapse onto the fastest expanding direction rendering them numerically indistinguishable, which makes it impossible to compute their associated volumes. Analytically, however, intermediate orthonormalizations do not play a role since the associated volumes stay the same. Indeed, we get the same output analytically if we perform the QR-decomposition only once at the end:

$$\Phi^n V_0 = \Phi_n \ldots \Phi_2 \Phi_1 V_0 = \Phi_n \ldots \Phi_2 V_1 R_1 = \cdots = V_n (R_n \ldots R_1).$$

[8]The propagated vectors from Benettin's algorithm are more than a mere byproduct for Lyapunov exponents. For instance, they have been exploited by Ginelli et al. [12] and by Wolfe–Samelson [24] in their algorithms to compute covariant Lyapunov vectors (see [18,19] for a theoretical analysis).

Algorithm 1: Benettin's algorithm [4,5]

Input: number of integration steps N, stepsizes $(h_n)_{n=1}^N$
Output: computed Lyapunov exponents $\mu_1, \ldots, \mu_d$

$V_0 = \text{rand(d)}$ `// set random initial vectors`
for n = 1:N **do**
  $W = \Phi_n V_{n-1}$ `// evolve`
  $[Q_n, R_n] = \text{qr}(W)$ `// orthonormalize`
  $V_n = Q_n$ `// set new vectors`
for i = 1:d **do**
  $\mu_i(N) = \frac{1}{t_N} \sum_{n=1}^N \log(R_n)_{ii}$ `// average`

One may adjust the frequency of orthonormalizations depending on how fast V_n becomes singular. Furthermore, in practice it makes sense to compute the Lyapunov exponents as a running average during the propagation loop to save memory and to monitor convergence properties.

While the algorithm works perfectly fine analytically, there are some numerical challenges. In practice several types of errors influence the output of Benettin's algorithm. For example, errors are introduced by numerical integration, by limiting the integration time, by finite-precision computing or through errors of the underlying equations[9].

4.2 Convergence results

For all results in this subsection, we assume:

- $0 < h_n \leq h_{\max}$,
- $t_n = h_1 + \cdots + h_n \to \infty$,
- Φ is consistent of order $p > 0$.

Before deriving the desired convergence theorems for Benettin's algorithm, we introduce two auxiliary systems: a piecewise constant approximation of our original system and a piecewise constant system representing the numerical integration.

If $\|X_n - I\| < 1$, the logarithm of X_n exists and is equal to

$$\log X_n = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(X_n - I)^k}{k}.$$

We set

$$A_n := \frac{1}{h_n} \log X_n.$$

A_n is well-defined for small stepsizes since

$$\|X_n - I\| \leq \int_{t_{n-1}}^{t_n} \|A(s) X(s, t_{n-1})\| \, ds \leq M e^{h_{\max} M} h_n.$$

[9]If the linear system is derived as the linearization along a solution of a nonlinear system, integration errors from computing the solution of the nonlinear system can result in errors in the derived linear equations of order $O(1)$. In that case our perturbation results for linear theory are ineffective and nonlinear theory is necessary to study the computational error.

Similarly, we set

$$B_n := \frac{1}{h_n} \log \Phi_n$$

whenever $\|\Phi_n - I\| < 1$. Consistency of the numerical integrator and the previous estimate imply that B_n is well-defined for small stepsizes as well.

We have the following relation between A_n and B_n:

Lemma 4.2. *There is a constant*[10] $c > 0$ *independent of the stepsizes such that*

$$\|A_n - B_n\| \leq c h_n^p$$

for small h_n.

Proof. Assume h_n is small enough such that

$$\max(\|\Phi_n - I\|, \|X_n - \Phi_n\|) \leq \frac{1}{4}. \tag{4.1}$$

The logarithms of X_n and Φ_n exist. Moreover, it holds

$$\|\log(X_n) - \log(\Phi_n)\| \leq \sum_{k=1}^{\infty} \frac{\|(X_n - I)^k - (\Phi_n - I)^k\|}{k}.$$

Using

$$\|(M + E)^k - M^k)\| \leq \|E\| 2^k \max(\|M\|, \|E\|)^{k-1}$$

with $M := \Phi_n - I$ and $E := X_n - \Phi_n$, we estimate

$$\|A_n - B_n\| = \frac{1}{h_n} \|\log(X_n) - \log(\Phi_n)\| \leq \frac{1}{h_n} \left(\sum_{k=1}^{\infty} \frac{4}{2^k k} \right) \|X_n - \Phi_n\|.$$

The claim follows from consistency of the numerical integrator. □

We may regard A_n as a perturbation of the original system on $[t_{n-1}, t_n)$. The size of the perturbation depends on the continuity property of A.

Lemma 4.3. *If* A *is globally Lipschitz continuous, then there is a constant* $c > 0$ *independent of the stepsizes such that*

$$\sup_{t \in [t_{n-1}, t_n)} \|A(t) - A_n\| \leq c h_n$$

for small h_n.

Proof. If h_n is small enough such that equation (4.1) holds, then

$$\|X_n - I\| \leq \frac{1}{2}$$

and

$$\|\log(X_n) - (X_n - I)\| \leq \left(\sum_{k=2}^{\infty} \frac{\|X_n - I\|^{k-2}}{k} \right) \|X_n - I\|^2 \leq \underbrace{\left(\sum_{k=2}^{\infty} \frac{4}{2^k k} \right) M^2 e^{2 h_{\max} M}}_{=:C} h_n^2.$$

[10] A more detailed analysis of the constant for systems with stable Lyapunov exponents can be found in [10].

Thus, it holds

$$\begin{aligned}\|A(t) - A_n\| = \left\|A(t) - \frac{1}{h_n}\log X_n\right\| &\leq \left\|A(t) - \frac{1}{h_n}(X_n - I)\right\| + Ch_n \\ &\leq \frac{1}{h_n}\int_{t_{n-1}}^{t_n} \|A(t) - A(s)X(s,t_{n-1})\|\,ds + Ch_n.\end{aligned}$$

for $t \in [t_{n-1}, t_n)$. Now, the claim follows from

$$A(t) - A(s)X(s,t_{n-1}) = A(t) - A(s) + A(s)(X(t_{n-1},t_{n-1}) - X(s,t_{n-1}))$$

via Lipschitz continuity of A. □

In the following, we only consider stepsizes such that $h_n \to 0$. While we cannot guarantee that A_n and B_n are well-defined for small n, there is $N \geq 0$ such that A_n and B_n are well-defined for $n \geq N$. We define

$$A_{pc}(t) := \begin{cases} 0, & t \in [0, t_{N-1}) \\ A_n, & t \in [t_{n-1}, t_n) \text{ for } n \geq N \end{cases}$$

and

$$B_{pc}(t) := \begin{cases} 0, & t \in [0, t_{N-1}) \\ B_n. & t \in [t_{n-1}, t_n) \text{ for } n \geq N \end{cases}$$

By possibly increasing N, we ensure that A_{pc} and B_{pc} are bounded and can be estimated using the previous lemmata.[11]

Let us denote the Cauchy matrix corresponding to A_{pc} by $X_{pc}(t,s)$. If $n \geq m \geq N-1$, then

$$X_{pc}(t_n, t_m) = e^{h_n A_n} \dots e^{h_{m+1}A_{m+1}} = X(t_n, t_m).$$

Hence, A_{pc} can be seen as a piecewise constant approximation of our original system (for large t).

Lemma 4.4. *The Lyapunov spectra of the original system and the piecewise constant approximation coincide. The same holds for their induced systems. Moreover, if the original system is regular, then the piecewise constant approximation is regular.*

Proof. Since

$$X_{pc}(t, t_{N-1}) = e^{(t-t_{n-1})A_n} X(t, t_{n-1})^{-1} X(t, t_{N-1})$$

for $t \in [t_{n-1}, t_n)$ and $n \geq N$, the growth rates of solutions with the same initial condition differ by at most a constant:

$$\left(\frac{\|X_{pc}(t,t_{N-1})v\|}{\|X(t,t_{N-1})v\|}\right)^{\pm 1} \leq e^{h_{\max}(\|A_{pc}\|_\infty + M)}$$

for any $v \neq 0$. Hence, the Lyapunov spectra of the original system and the piecewise constant approximation coincide. The remaining statements follow similarly. □

[11] This is the case if equation (4.1) is satisfied for $n \geq N$.

We write $\tilde{X}_{pc}(t,s)$ for the Cauchy matrix corresponding to B_{pc} and note that

$$\tilde{X}_{pc}(t_n,t_m) = \Phi_n \dots \Phi_{m+1}$$

for $n \geq m \geq N-1$. In particular, fixing the fundamental matrix $\tilde{X}_{pc}(t)$ satisfying $\tilde{X}_{pc}(t_{N-1}) = \Phi^{N-1}$, we have

$$\tilde{X}_{pc}(t_n) = \tilde{X}_{pc}(t_n,t_{N-1})\tilde{X}_{pc}(t_{N-1}) = \Phi_n \dots \Phi_N \Phi^{N-1} = \Phi^n$$

for $n \geq N-1$. Thus, B_{pc} can be understood as representing the numerical integration. It follows that

$$\begin{aligned}\mu_1(n) + \dots + \mu_L(n) &= \sum_{i=1}^{L} \frac{1}{t_n} \sum_{k=1}^{n} \log(R_k)_{ii} = \frac{1}{t_n} \log\left(\prod_{i=1}^{L} (R_n \dots R_1)_{ii}\right) \\ &= \frac{1}{t_n} \log \|(\wedge^L \Phi_n)(v_1 \wedge \dots \wedge v_L)\| = \frac{1}{t_n} \log \|(\wedge^L \tilde{X}_{pc}(t_n))(v_1 \wedge \dots \wedge v_L)\|\end{aligned}$$

for $n \geq N-1$. Since the stepsizes and B_{pc} are bounded, we have

$$\begin{aligned}\limsup_{n\to\infty} \mu_1(n) + \dots + \mu_L(n) &= \limsup_{n\to\infty} \frac{1}{t_n} \log \|(\wedge^L \tilde{X}_{pc}(t_n))(v_1 \wedge \dots \wedge v_L)\| \\ &= \limsup_{t\to\infty} \frac{1}{t} \log \|(\wedge^L \tilde{X}_{pc}(t))(v_1 \wedge \dots \wedge v_L)\|.\end{aligned}$$

Thus, Benettin's algorithm computes $\tilde{\lambda}_{1,L}$ of B_{pc} for Lebesgue-almost every tuple of initial vectors:

$$\limsup_{n\to\infty} \mu_1(n) + \dots + \mu_L(n) = \tilde{\lambda}_{1,L}. \tag{4.2}$$

In particular, if B_{pc} is regular, then

$$\lim_{n\to\infty} \mu_1(n) + \dots + \mu_L(n) = \tilde{\Lambda}_1 + \dots + \tilde{\Lambda}_L. \tag{4.3}$$

Theorem 4.5. *Assume equation (3.1) is globally Lipschitz continuous and has stable Lyapunov exponents. If $h_n \to 0$, then*

$$\limsup_{n\to\infty} \mu_1(n) + \dots + \mu_i(n) \leq \Lambda_1 + \dots + \Lambda_i$$

for all i and Lebesgue-almost every tuple of initial vectors.[12]

Proof. Since there are constant $c_1, c_2 > 0$ independent of the stepsizes such that

$$\sup_{t\in[t_{n-1},t_n)} \|A(t) - B_{pc}(t)\| \leq \sup_{t\in[t_{n-1},t_n)} \|A(t) - A_n\| + \|A_n - B_n\| \leq c_1 h_n + c_2 h_n^p$$

for $n \geq N$, B_{pc} is a perturbation of the original system such that $\|Q(t)\| \to 0$. In particular, Theorem 3.26 implies that B_{pc} has the same Lyapunov spectrum as the original system. Now, equation (4.2) and Proposition 3.20 imply the theorem. □

[12] A version of the estimate for finite time can be found in [11]. The authors relate the error to the departure from normality of the R-matrices obtained during Benettin's algorithm.

Theorem 4.6. *Assume equation* (3.1) *is globally Lipschitz continuous, regular and has stable Lyapunov exponents. If* $h_n \to 0$*, then*

$$\lim_{n\to\infty} \mu_i(n) = \Lambda_i$$

for all i *and Lebesgue-almost every tuple of initial vectors.*

Proof. The claim follows similarly to the last theorem using Proposition 3.27 and equation (4.3) instead of equation (4.2). □

Remark 4.7. If the stability of Lyapunov exponents transfers from A to A_{pc}, then the assumption that A is globally Lipschitz continuous can be dropped in Theorems 4.5 and 4.6.

Next, we derive convergence results for strongly fast invertible systems. For this, we need the following lemma:

Lemma 4.8. *The original system is L-dim. strongly fast invertible if and only if its piecewise constant approximation is L-dim. strongly fast invertible.*

Proof. The claim follows from

$$e^{(t_n-t)A_n} X_{pc}(t,s) e^{(s-t_{m-1})A_m} = X_{pc}(t_n, t_{m-1}) = X(t_n, t_{m-1}) = X(t_n,t)X(t,s)X(s,t_{m-1})$$

for $s \in [t_{m-1}, t_m)$ and $t \in [t_{n-1}, t_n)$, $n \geq m \geq N$, and the fact that strong fast invertibility can be tested on $[t_{N-1}, \infty)$ (see Lemma 3.29). □

Theorem 4.9. *Assume equation* (3.1) *is L-dim. strongly fast invertible. If* $h_n \to 0$*, then*

$$\limsup_{n\to\infty} \mu_1(n) + \cdots + \mu_L(n) \leq \Lambda_1 + \cdots + \Lambda_L$$

for Lebesgue-almost every tuple of initial vectors.

Proof. Since A_{pc} is L-dim. strongly fast invertible and its induced systems have the same Lyapunov spectra as the induced systems of the original system, Theorem 3.46 and Proposition 3.20 imply

$$\tilde{\lambda}_{1,L} \leq \lambda_{1,L} \leq \Lambda_1 + \cdots + \Lambda_L.$$

The claim follows from equation (4.2). □

Theorem 4.10. *Assume equation* (3.1) *is regular and L-dim. strongly fast invertible. If*

$$\sum_{n=1}^{\infty} h_n^{p+1} < \infty,$$

then

$$\limsup_{n\to\infty} \mu_1(n) + \cdots + \mu_L(n) = \Lambda_1 + \cdots + \Lambda_L$$

for Lebesgue-almost every tuple of initial vectors.

Proof. Lemma 4.2 and the stepsize condition ensure that

$$\int_0^\infty \|A_{pc}(t) - B_{pc}(t)\|\, dt < \infty.$$

Now, the proof is as in Theorem 4.9 using Theorem 3.47 and regularity:

$$\tilde{\lambda}_{1,L} = \lambda_{1,L} = \Lambda_1 + \cdots + \Lambda_L.$$ □

Since we do not know if regularity transfers form the original to the numerical system, Theorem 4.10 only ensures convergence to the Lyapunov exponents as a limes superior. However, when fixing a tuple of initial vectors, we can ensure convergence as a limit if the stepsizes decay fast enough.

Theorem 4.11. *Assume equation (3.1) is regular and L-dim. strongly fast invertible. For Lebesgue-almost every tuple of initial vectors, we have the following: If $\sum_{n=1}^{\infty} h_n^{p+1}$ is small enough, then*

$$\lim_{n\to\infty} \mu_1(n) + \cdots + \mu_L(n) = \Lambda_1 + \cdots + \Lambda_L.$$

Proof. The theorem follows from Corollary 3.49. □

Remark 4.12. The condition "$\sum_{n=1}^{\infty} h_n^{p+1}$ small enough" in Theorem 4.11 depends on the chosen initial vectors. Indeed, following the associated proofs, we require smaller stepsizes the smaller the first principle angle between $\mathrm{span}(v_1, \dots, v_L)$ and V'_{l+1}.

Finally, when applied to the right dimensions, Theorems 4.10 and 4.11 ensure that we may approximate the Lyapunov exponents of strongly fast invertible systems using Benettin's algorithm.

Corollary 4.13. *Assume equation (3.1) is regular and strongly fast invertible at dim. $d_1 + \cdots + d_l$ for $l = 1, \dots, p$. If*

$$\sum_{n=1}^{\infty} h_n^{p+1} < \infty,$$

then

$$d_i \lambda_i = \limsup_{n\to\infty} \mu_1(n) + \cdots + \mu_{d_1+\cdots+d_i}(n) - \limsup_{n\to\infty} \mu_1(n) + \cdots + \mu_{d_1+\cdots+d_{i-1}}(n)$$

for all i and for Lebesgue-almost every tuple of initial vectors. In particular, if the Lyapunov spectrum is simple, then

$$\lambda_i = \limsup_{n\to\infty} \mu_1(n) + \cdots + \mu_i(n) - \limsup_{n\to\infty} \mu_1(n) + \cdots + \mu_{i-1}(n)$$

for all i and for Lebesgue-almost every tuple of initial vectors.

Corollary 4.14. *Assume equation (3.1) is regular and strongly fast invertible at dim. $d_1 + \cdots + d_l$ for $l = 1, \dots, p$. For Lebesgue-almost every tuple of initial vectors, we have the following: If $\sum_{n=1}^{\infty} h_n^{p+1}$ is small enough, then*

$$d_i \lambda_i = \lim_{n\to\infty} \mu_{d_1+\cdots+d_{i-1}+1}(n) + \cdots + \mu_{d_1+\cdots+d_i}(n)$$

for all i. In particular, if the Lyapunov spectrum is simple, then

$$\lambda_i = \lim_{n\to\infty} \mu_i(n)$$

for all i.

5 Numerical examples

To showcase Benettin's algorithm, we present three numerical examples: a linear system, the Lorenz-63 and the Lorenz-96 model. While the linear system satisfies the assumptions required in our convergence analysis, the other two systems are nonlinear and exhibit additional errors that are not accounted for in our analysis. Moreover, not all Lyapunov exponents of the Lorenz models are known analytically.

All systems will be integrated using a fifth-order Runge–Kutta method (RK5) with

1. adaptive stepsizes[13],
2. constant stepsizes, and
3. two types of varying stepsizes[14] ($h_n = \frac{h_0}{\sqrt[2]{n}}$ and $h_n = \frac{h_0}{\sqrt[3]{n}}$).

In case of adaptive stepsizes, we apply the Dormand–Prince method (DP54) which uses a fourth-order Runge–Kutta method (RK4) in addition to RK5 to estimate the integration error. Since the computational costs per integration step for each approach are close[15], we can roughly associate the total computational costs to the number of integration steps to compare the performance of Benettin's algorithm for the different stepsizes.

The corresponding MATLAB code can be found in [20].

5.1 Linear system

Let

$$B(t) = \begin{pmatrix} \lambda_1 & f(t) & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$$

be a block-triangular matrix with $\lambda_1 > \lambda_2$ and a bounded continuous function f. One may check that the system $\dot{y} = B(t)y$ is regular and has stable Lyapunov exponents λ_1 and λ_2 due to Theorem 3.24[16]. In particular, the system is strongly fast invertible at dimensions $d_1 = 2$ and $d_1 + d_2 = 3$. These properties remain valid under Lyapunov transformations.

To get a specific example, we set $\lambda_1 = 1$, $\lambda_2 = 0$ and $f(t) = \sin(t) + 1/(t^2 + 1)$ and transform $\dot{y} = B(t)y$ to

$$\dot{x} = A(t)x := (LBL^{-1} + \dot{L}L^{-1})(t)x$$

[13]We use adaptive stepsizes in accordance with [9, Section 3.1] for linear systems and with [8, pp. 13–14] for nonlinear systems.

[14]We chose varying stepsizes such that

$$\sum_{n=1}^{\infty} h_n^{p+1} < \infty.$$

[15]DP54 uses the same function evaluations as RK5 and hence only increases the computational costs per step by a small amount. The possibly largest increase in costs for the adaptive method comes from stepsize corrections that require to repeat the integration step. However, at least in our examples, the increase was relatively low (a factor of ∼1.3 for the linear system and ∼1.0 for the Lorenz systems).

[16]Condition (iii) follows from

$$Y_1(t,s) = e^{\lambda_1(t-s)} \begin{pmatrix} 1 & \int_s^t f(\tau)\, d\tau \\ 0 & 1 \end{pmatrix}$$

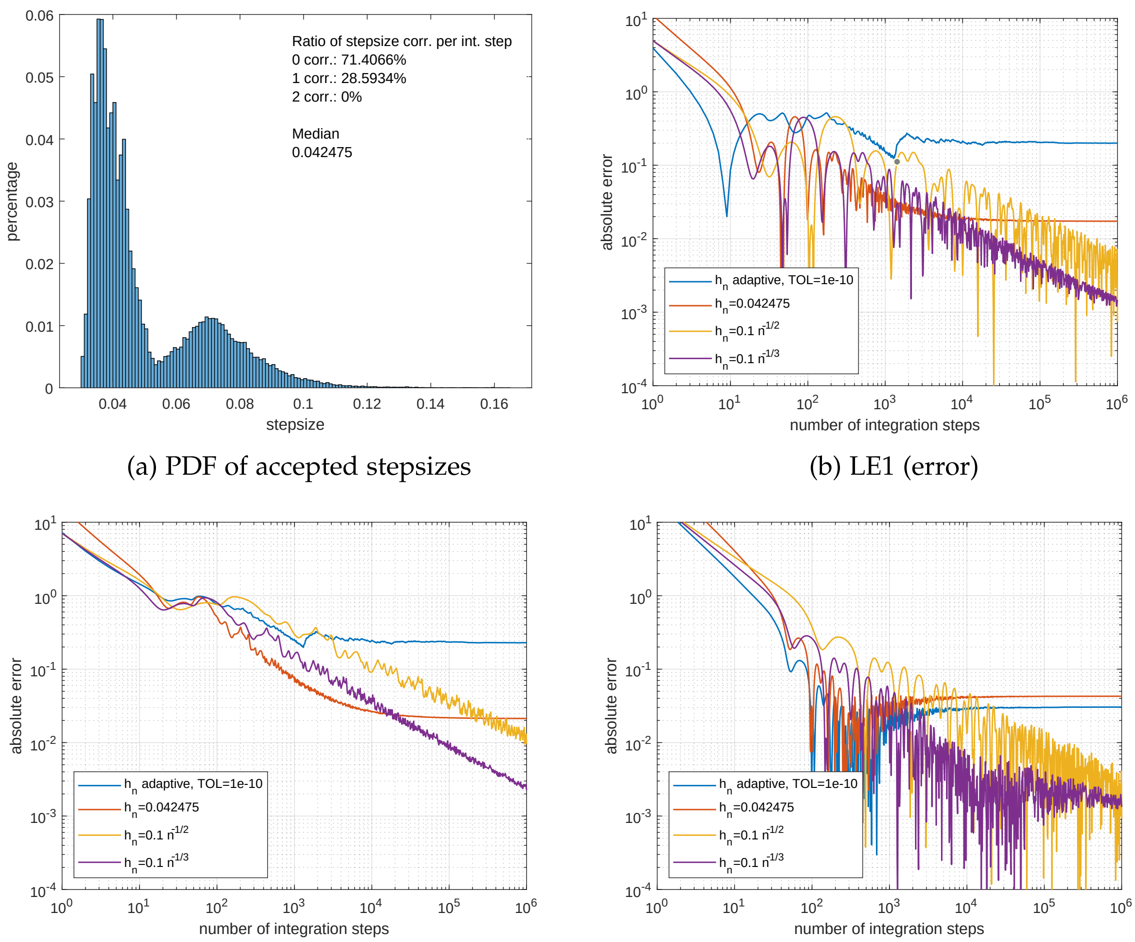
and $Y_2(t,s) = e^{\lambda_2(t-s)}$. Stability of the subsystem for λ_1 can be checked by showing $\overline{\omega}_1 = \lambda_1 = \Omega_1$ using that the constant function equal to λ_1 is a lower and upper function for the first subsystem.

using the composition $L(t)$ of two Lyapunov transformations

$$\begin{pmatrix} 1 & \sin(t) & \cos(t) \\ 0 & 1 & \sin(t) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\sqrt{2}t) & 0 & \sin(\sqrt{2}t) \\ 0 & 1 & 0 \\ -\sin(\sqrt{2}t) & 0 & \cos(\sqrt{2}t) \end{pmatrix}.$$

While our theory only ensures convergence of Benettin's algorithm for $2\lambda_1$ and λ_2, we apply the algorithm to compute each individual exponent without regarding the degeneracy. The Lyapunov exponents are approximated using the different stepsizes mentioned in the beginning of Section 5. We start with the adaptive approach and take the median of the accepted stepsizes (Figure 5.1a) as the constant stepsizes for the second approach. The approximated Lyapunov exponents of all approaches are compared to their exact values ($\Lambda_1 = 1$, $\Lambda_2 = 1$ and $\Lambda_3 = 0$) and plotted against the number of integration steps (Figures 5.1b, 5.1c and 5.1d).

For all three Lyapunov exponents, we see that adaptive and constant stepsizes lead to an accumulation of integration errors that persists in the limit. On the other hand, the errors obtained with varying stepsizes show a steady decay as we would expect from our theory.

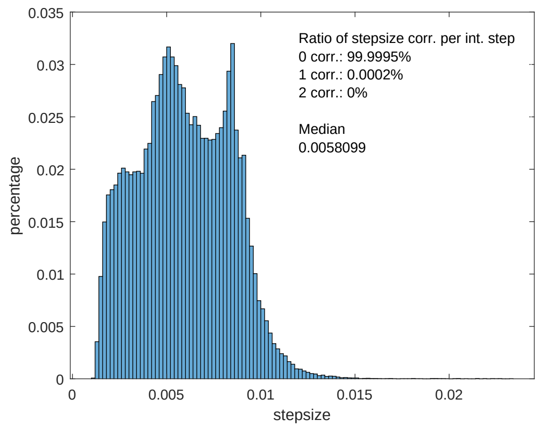


(a) PDF of accepted stepsizes (b) LE1 (error)

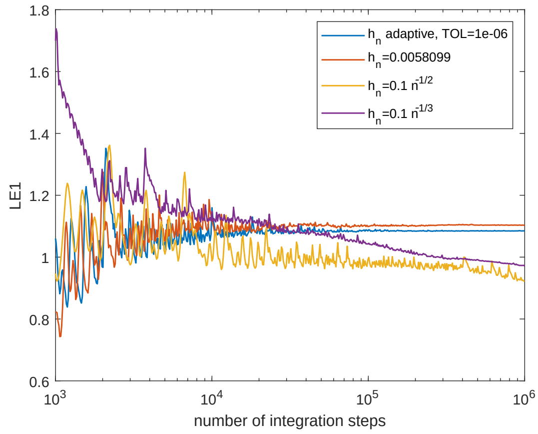
(c) LE2 (error) (d) LE3 (error)

Figure 5.1: The Lyapunov exponents of the system $\dot{x} = A(t)x$ from Subsection 5.1 are approximated using Benettin's algorithm with different stepsizes. The first plot shows the PDF of the accepted stepsizes of the adaptive approach, while the other plots show the errors between computed Lyapunov exponents and their exact values. To save memory, the PDF was formed over every 10th integration step.

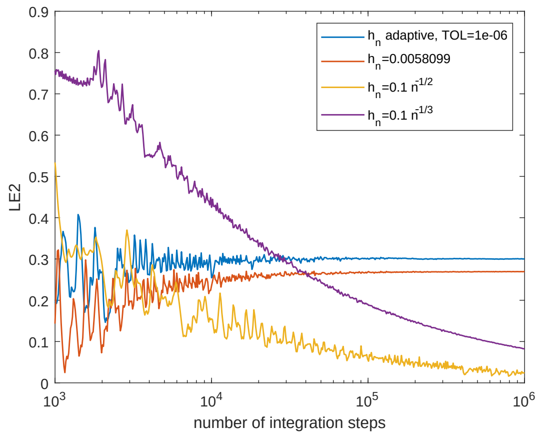
5.2 Lorenz-63 model



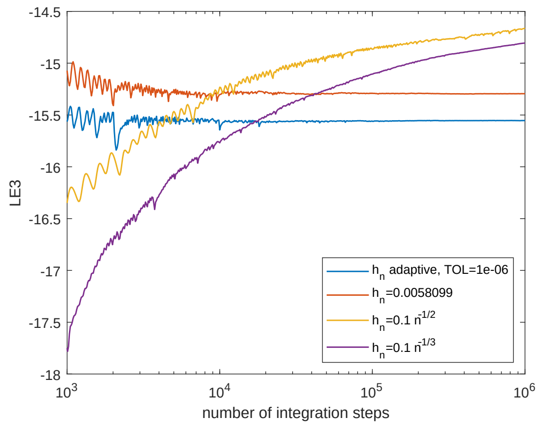
(a) PDF of accepted stepsizes



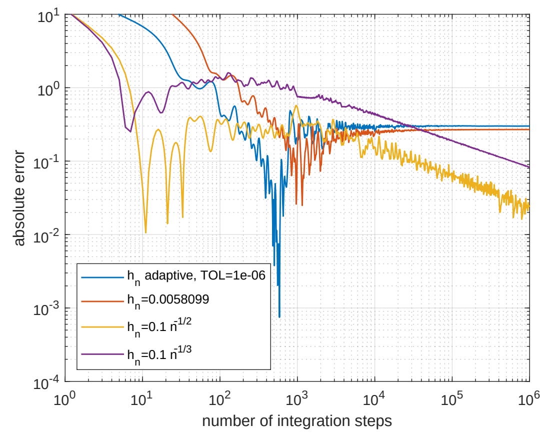
(b) LE1



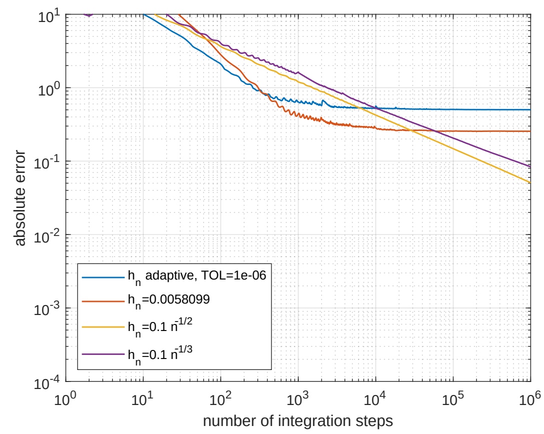
(c) LE2



(d) LE3



(e) LE2 (error)



(f) LE sum (error)

Figure 5.2: The Lyapunov exponents of the Lorenz-63 model from Subsection 5.2 are approximated using Benettin's algorithm with different stepsizes. The first plot shows the PDF of the accepted stepsizes of the adaptive approach, while the other plots show the computed Lyapunov exponents and their errors. To save memory, the PDF was formed over every 10th integration step.

Our second example is the Lorenz-63 model [15] (a reduced model for thermal convection) given by the equations

$$\left.\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= x_1(\rho - x_3) - x_2 \\ \dot{x}_3 &= x_1 x_2 - \beta x_3 \end{aligned}\right\} := f(x)$$

with $\sigma = 10$, $\beta = 8/3$ and $\rho = 28$ as the classical parameters. The system is famous for its unpredictable behavior despite being deterministic. It exhibits a so-called *strange attractor* (the *Lorenz attractor*), which is a minimal attracting flow-invariant set that is neither a steady state nor a periodic orbit and which has a positive largest Lyapunov exponent[17]. So far, the existence of the attractor and the existence of a physical ergodic invariant measure that admits a positive largest Lyapunov exponent is only known as a consequence of the computer-aided proof by Tucker [23].

To compute the Lyapunov exponents, we require a background trajectory lying on the attractor. This is problematic, since the Lorenz attractor has Lebesgue measure zero. In particular, the initial state of our background trajectory will almost surely lie outside of the attractor. While the trajectory may be close to the attractor after a transient, a rigorous analysis would require continuity properties of the Lyapunov exponents that extend to a neighborhood of the attractor. However, in general, the Lyapunov exponents vary only measurably with the base point. Moreover, integration errors can lead to an error of order $O(1)$ in the background trajectory as well as in the derived linear system. This may drastically change the computed Lyapunov exponents.

Despite the open challenges, we try to approximate the Lyapunov exponents and compare our methods for the known cases, that is, the second Lyapunov exponent and the sum of all Lyapunov exponents. The second Lyapunov exponent corresponds to the tangent vector of the background trajectory, which is a solution of the linearized equations that is neither exponentially growing nor decaying, and thus must be zero. The sum of all three Lyapunov exponents must be equal to the trace of the Jacobian as it is constant. Indeed, it holds

$$\begin{aligned} \Lambda_1 + \Lambda_2 + \Lambda_3 &= \lim_{t\to\infty} \frac{1}{t} \log \left|\det\left(X(t)\right)\right| \\ &= \lim_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{tr}(Df(x(s)))ds = -(\sigma + 1 + \beta) \end{aligned}$$

for any solution $x(t)$ along which the linearized system $X(t)$ is regular. Exact values of the first and the third Lyapunov exponent are unknown.

The Lyapunov exponents are computed in a similar fashion to Subsection 5.1. This time, however, we require a background trajectory along which to propagate linear perturbations. In order to start close to the attractor, we first compute a transient starting from a randomly chosen state. After the transient, we couple the nonlinear system to its linearization. This way, we can compute the background trajectory and propagate linear perturbations simultaneously.

In this example, the adaptive approach requires almost no stepsize corrections (Figure 5.2a) and is thus well comparable to the other approaches in terms of computational costs per step. As before, we observe that Lyapunov exponents computed with adaptive and constant stepsizes seem to converge to different values depending on the given tolerance or stepsize (Figures 5.2b, 5.2c and 5.2d). A meaningful comparison between the different approaches is

[17]For nonlinear systems, the Lyapunov exponents are obtained from the linearized flow along nonlinear trajectories. If the nonlinear system has an ergodic invariant measure, the Lyapunov exponents coincide for almost every trajectory. Hence, we may associate Lyapunov exponents with the attractor (or measure) instead of the trajectory.

only possible for the second and the sum of all Lyapunov exponents (Figures 5.2e and 5.2f). We note that, similar to the linear example, the accuracy for adaptive and constant stepsizes are limited, while varying stepsizes seem to admit convergence properties that lead to the true Lyapunov exponents.

5.3 Lorenz-96 model

Our third example is the Lorenz-96 model [16], which was created to test numerical weather prediction. The model simulates the effects of external forcing, damping and advection on a scalar physical quantity, which lives on a periodic lattice representing a circle of latitude. Different from the Lorenz-63 model, the Lorenz-96 model treats each variable on the lattice the same. By changing the number of variables, one may study the effects of increased spacial resolution on the predictability (or chaoticity) of the system. There already are studies of spatiotemporal chaos in the Lorenz-96 model in which spectra of Lyapunov exponents have been computed [14].

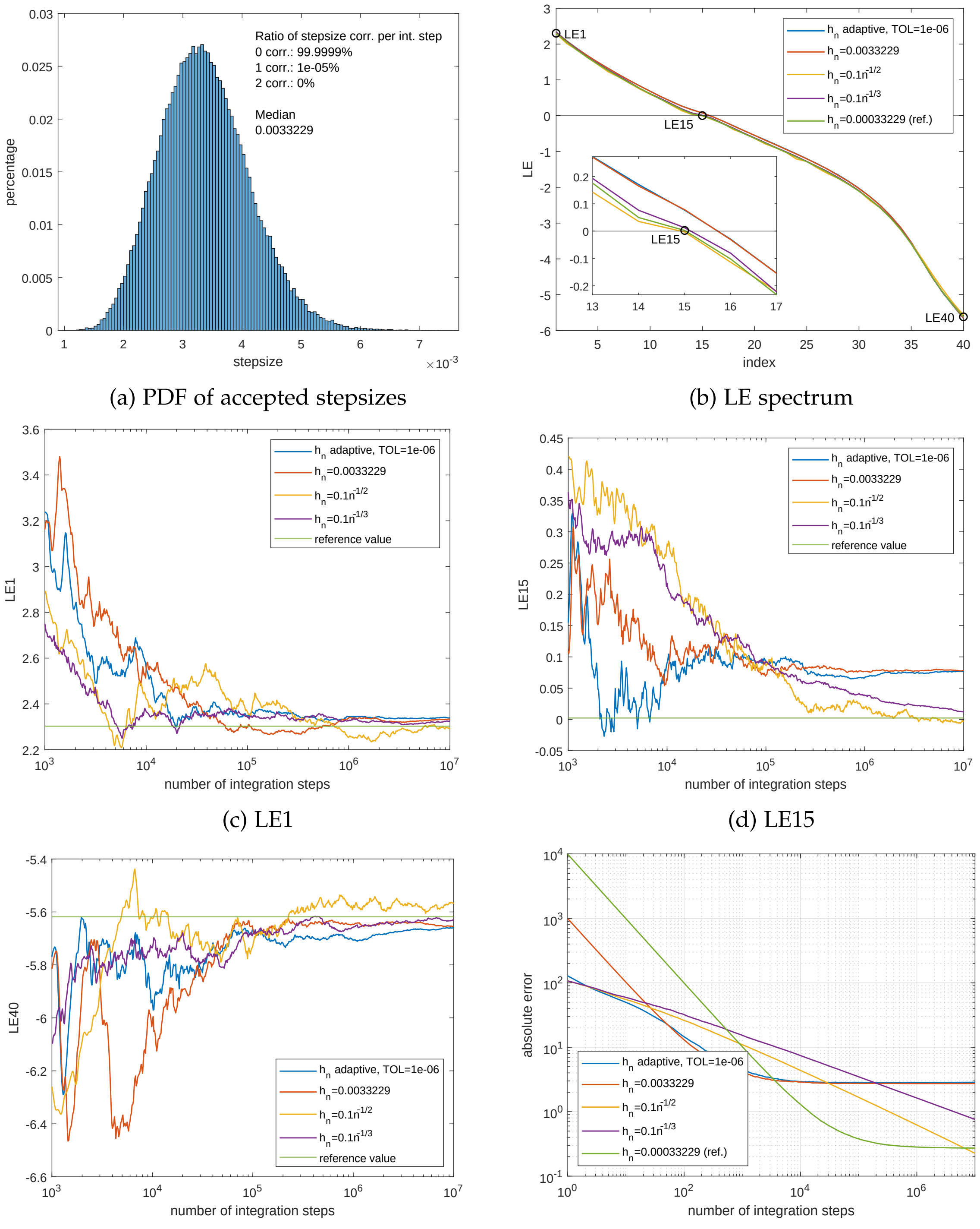
The Lorenz-96 model with $d \geq 4$ variables is given by the equations

$$\dot{x}_i = (x_{i+1} - x_{i-2})x_{i-1} - x_i + F$$

with periodic indices, i.e., $x_{i+kd} := x_i$ for $k \in \mathbb{Z}$. We choose $d = 40$ and a forcing parameter of $F = 10$ to allow for chaotic dynamics. Under the same conditions, [14] computed the Lyapunov spectrum and noticed that the shape of the spectrum does not change while increasing the number of variables from 40 to 50. They used constant stepsizes of $h_n = 1/64$ and integrated the trajectory for $T = 5 \times 10^5$ units of time or $N = 64T$ timesteps. The first $2/3$ of the trajectory were used as a transient to approach the attractor before computing the Lyapunov exponents.

Here, we explore the effects of different stepsizes on the Lyapunov spectrum in terms of the approaches mentioned in the beginning of Section 5. Similar to the Lorenz-63 model, we know that at least one Lyapunov exponent must vanish and that the sum of all Lyapunov exponents must be equal to the trace of the Jacobian, which is $-d$. However, it is not clear which index corresponds to the vanishing Lyapunov exponent. Instead of looking at each individual exponent, we compare the Lyapunov spectra of the different approaches to a reference spectrum computed with a high-resolution run. The latter is performed with ten times the number of integration steps than the other runs and with constant stepsizes that are ten times smaller than the constant stepsizes from the low-resolution run. While a higher resolution heuristically provides better estimates of the Lyapunov exponents, we still have to expect persistent errors.

Our computed Lyapunov spectra (Figure 5.3b) take on similar shapes to the spectrum in [14, Figure 5]. We see that the spectra computed with varying stepsizes are closer to the reference spectrum than the ones computes with adaptive or constant stepsizes. The differences between the spectra vary with the specific Lyapunov exponent. A more detailed picture showing the computed extremal Lyapunov exponents and the near-zero exponent of the high-resolution run can be seen in (Figures 5.3c, 5.3d and 5.3e). Concerning the sum of all Lyapunov exponents, we observe a similar behavior as in the other two examples (Figure 5.3f).



(a) PDF of accepted stepsizes

(b) LE spectrum

(c) LE1

(d) LE15

(e) LE40

(f) LE sum (error)

Figure 5.3: The Lyapunov exponents of the Lorenz-96 model from Subsection 5.3 are approximated using Benettin's algorithm with different stepsizes. The first plot shows the PDF of the accepted stepsizes of the adaptive approach, the second plot shows the computed spectra after 10^7 integration steps (10^8 steps for the reference), and the other plots compare the computed Lyapunov exponents to the reference (final value of the exponent after 10^8 steps). To save memory, the PDF was formed over every 100th integration step.

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