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NONOSCILLATORY SOLUTIONS OF THE FOUR-DIMENSIONAL DIFFERENCE SYSTEM

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Abstract. We study asymptotic properties of nonoscillatory solutions for a four-dimensional system

$$\begin{aligned} \Delta x_n &= C_n\, y_n^{\frac{1}{\gamma}} \\ \Delta y_n &= B_n\, z_n^{\frac{1}{\beta}} \\ \Delta z_n &= A_n\, w_n^{\frac{1}{\alpha}} \\ \Delta w_n &= D_n\, x_{n+\tau}^{\delta}. \end{aligned}$$

In particular, we give sufficient conditions that any bounded nonoscillatory solution tends to zero and any unbounded nonoscillatory solution tends to infinity in all its components.
This paper is in final form and no version of it will be submitted for publication elsewhere.

1. Introduction

In this paper, we study asymptotic behavior of solutions of a four-dimensional system

$$\text{(S)} \qquad \begin{aligned} \Delta x_n &= C_n\, y_n^{\frac{1}{\gamma}} \\ \Delta y_n &= B_n\, z_n^{\frac{1}{\beta}} \\ \Delta z_n &= A_n\, w_n^{\frac{1}{\alpha}} \\ \Delta w_n &= D_n\, x_{n+\tau}^{\delta} \end{aligned}$$

where $n \in \mathbb{N}$, $\alpha, \beta, \gamma, \delta$ are the ratios of odd positive integers, τ is nonnegative integer, and $\{A_n\}, \{B_n\}, \{C_n\}, \{D_n\}$ are positive real sequences defined for $n \in \mathbb{N}$ such that

$$\text{(1.1)} \qquad \sum_{n=1}^{\infty} A_n = \infty, \quad \sum_{n=1}^{\infty} C_n = \infty.$$

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The system (S) is a prototype of even-order systems and represents a large class of difference equations. By using the notation

$$A_n = a_n^{-\frac{1}{\alpha}} \quad B_n = b_n^{-\frac{1}{\beta}} \quad C_n = c_n^{-\frac{1}{\gamma}} \quad D_n = d_n,$$

the system (S) can be written as a fourth-order nonlinear difference equation of the form

$$\Delta\left(a_n\left(\Delta\left(b_n\left(\Delta\left(c_n\left(\Delta x_n\right)^{\gamma}\right)\right)^{\beta}\right)\right)^{\alpha}\right) - d_n x_{n+\tau}^{\delta} = 0. \tag{E}$$

Vice versa, if x is a solution of (E) and

$$x_n^{[1]} = c_n\left(\Delta x_n\right)^{\gamma}, \quad x_n^{[2]} = b_n\left(\Delta x_n^{[1]}\right)^{\beta}, \quad x_n^{[3]} = a_n\left(\Delta x_n^{[2]}\right)^{\alpha}$$

are the so called quasi-differences of x, then the vector $(x, x^{[1]}, x^{[2]}, x^{[3]})$ is a solution of (S).

When $d_n < 0$, asymptotic and oscillatory properties of (E) have been widely investigated in the literature, see e.g.[2]–[11] and references therein. When $d_n > 0$, according of our knowledge, equation (E) or system (S) has not been investigated.

If $\alpha = \beta = \gamma = 1$ and $\tau = 2$, then (S) reduces to the difference equation

$$\Delta\left(a_n \Delta\left(b_n \Delta\left(c_n \Delta x_n\right)\right)\right) - d_n x_{n+2}^{\delta} = 0 \tag{1.2}$$

which special case is the fourth order formally self-adjoint difference equation

$$\Delta^2\left(b_n \Delta^2 x_n\right) - d_n x_{n+2} = 0. \tag{1.3}$$

Observe that equation (1.2) is usually considered under the assumption

$$\sum_{n=1}^{\infty} \frac{1}{a_n} = \sum_{n=1}^{\infty} \frac{1}{b_n} = \sum_{n=1}^{\infty} \frac{1}{c_n} = \infty,$$

i.e. the difference operator is in the so called canonical form.

By a solution of the system (S) we mean a vector sequence (x, y, z, w) which satisfies the system (S) for $n \in \mathbb{N}$. A solution of the system (S) is said to be *oscillatory* if all of its component x, y, z, w are oscillatory. Otherwise, a solution is said to be *nonoscillatory.* The component x is said to be oscillatory if for any $n_0 \geq 1$ there exists $n > n_0$ such that $x_{n+1}x_n \leq 0$. The oscillation of the components y, z, w is defined by the same way. A solution of the system (S) is said to be *bounded* if all of its component x, y, z, w are bounded. Otherwise, a solution is said to be *unbounded.*

We say that the system (S) has *weak property B* if every nonoscillatory solution of (S) satisfies

$$x_n z_n > 0 \quad \text{and } y_n w_n > 0 \quad \text{for large } n, \tag{1.4}$$

and *property B* if any nonoscillatory solution of (S) satisfies either

$$\lim_{n\to\infty} |x_n| = \lim_{n\to\infty} |y_n| = \lim_{n\to\infty} |z_n| = \lim_{n\to\infty} |w_n| = \infty, \tag{1.5}$$

or

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = \lim_{n\to\infty} w_n = 0. \tag{1.6}$$

Property B is defined in accordance with those for higher order differential equations or for the system of differential equations, see [1] and references therein. Property B is an analogue of the so called property A which reads for the even order systems that all their solutions are oscillatory. For this reason, sometimes instead of the weak property B, the system (S), those nonoscillatory solutions satisfy (1.4), is said to be *almost oscillatory.* Solutions satisfying (1.4) and $x_n y_n > 0$ are called *strongly monotone solutions*, while solutions satisfying (1.4) and $x_n y_n < 0$ are called *Kneser solutions.* Hence, weak property B means that any nonoscillatory solution is either Kneser or strongly monotone solutions and property B means that these solutions are either unbounded or vanishing at infinity in all their components.

The aim of this paper is to investigate asymptotic behavior of nonoscillatory solutions of system (S). We give sufficient conditions that (S) has weak property B and property B. Both cases when the series $\sum B_n$ is divergent or convergent are considered and the role of the integer-valued argument τ is pointed out, as well. Our results can be applied to the linear system.

2. Preliminaries

First, we point out some basic properties of the system (S) given by (1.1).

Lemma 1. *Let (x, y, z, w) be a solution of system (S). The solution (x, y, z, w) is nonoscillatory if and only if any of its components x, y, z, w is either positive or negative for large n.*

Proof. It is sufficient to prove that if (x, y, z, w) is a nonoscillatory solution of (S), then all components are either positive or negative for large n. First assume that $x_n > 0$ for $n \geq n_0$ ($n_0 \in \mathbb{N}$). From the fourth equation of the system (S) we have that w_n is strictly increasing for $n \geq N_0$ and so, it is of one sign for large n. Proceeding by the same argument we get that z and y are monotone and of one sign for large n. The remaining cases when any of the components y, z, w is eventually positive or negative can be treated by the same way. □

Remark 1. *If the system (S) has a solution (x, y, z, w), then it has also a solution $(-x, -y, -z, -w)$. In view of Lemma 2, when studying nonoscillatory solutions, we can focus for solutions those first component is eventually positive.*

The following lemma describes the possible types of nonoscillatory solutions.

Lemma 2. *Any nonoscillatory solution (x, y, z, w) of system (S) with eventually positive x is one of the following types:*

type(a)	$x_n > 0$	$y_n > 0$	$z_n > 0$	$w_n > 0$	*for large n,*
type(b)	$x_n > 0$	$y_n > 0$	$z_n > 0$	$w_n < 0$	*for large n,*
type(c)	$x_n > 0$	$y_n < 0$	$z_n > 0$	$w_n < 0$	*for large n,*
type(d)	$x_n > 0$	$y_n > 0$	$z_n < 0$	$w_n < 0$	*for large n,*
type(e)	$x_n > 0$	$y_n < 0$	$z_n > 0$	$w_n > 0$	*for large n.*

Proof. Let (x, y, z, w) be a solution of (S) such that $x_n > 0$ for large n. First, assume that there exists solution such that $y_n < 0$, $z_n < 0$ for all large n. From the second equation of (S) we have $\Delta y_n < 0$ and this implies that exist $k > 0$ such that $y_n \leq -k$ for large n. By summation of the first equation of (S) we have

$$x_n - x_{n_0} = \sum_{i=n_0}^{n-1} C_i \cdot y_i^{\frac{1}{\gamma}} \leq -k^{\frac{1}{\gamma}} \sum_{i=n_0}^{n-1} C_i.$$

Passing $n \to \infty$, we get $\lim x_n = -\infty$, which is a contradiction with the fact $x_n > 0$. Now let us suppose that there exists solution such that $z_n < 0$, $w_n > 0$ for large n. Since w is eventually positive increasing, there exist $k > 0$ such that $w_n \geq k$. By summation of the third equation of (S) we have

$$z_n - z_{n_0} = \sum_{i=n_0}^{n-1} A_i \cdot w_i^{\frac{1}{\alpha}} \geq k^{\frac{1}{\alpha}} \sum_{i=n_0}^{n-1} A_i,$$

so, passing $n \to \infty$, we get a contradiction with the fact $z_n < 0$. □

Remark 2. *Strongly monotone solutions are of type (a) and Kneser solutions are of type (c).*

Lemma 3. *(i) Any solution of type (b) or (c) satisfies* $\lim_{n\to\infty} w_n = 0$.
(ii) Any solution of type (c) or (e) satisfies $\lim_{n\to\infty} y_n = 0$.
(iii) If solution of type (d) is bounded, then $\lim_{n\to\infty} y_n = 0$ *and* $\lim_{n\to\infty} w_n = 0$.

Proof. Claim (i). Suppose that (x, y, z, w) is a nonoscillatory solution of type (b) or (c). Then w is eventually negative increasing, thus there exists $\lim_{n\to\infty} w_n = h \leq 0$. Suppose $h < 0$. Then $w_n \leq h$ and from the third equation of the system (S) we obtain

$$z_n = z_{n_0} + \sum_{j=n_0}^{n-1} A_j w_j^{\frac{1}{\alpha}} \leq z_{n_0} + h^{\frac{1}{\alpha}} \cdot \sum_{j=n_0}^{n-1} A_j,$$

from where $z_n \to -\infty$ as $n \to \infty$. This contradicts the fact that $z_n > 0$, and therefore $\lim_{n\to\infty} w_n = 0$.

Claim (ii). If (x, y, z, w) is of type (c) or (e), then there exists $\lim_{n\to\infty} y_n = k$, $k \leq 0$. If $k < 0$, then from the first equation we get $x_n \to -\infty$ as $n \to \infty$, which is a contradiction with the positiveness of x.

Claim (iii). Obviously, y and w are bounded. If $y_n \geq h > 0$, then by summation of the first equation we get $x \to \infty$, which is a contradiction. Similarly, if $w \leq k < 0$, then from the third equation we get a contradiction with the boundedness of z. □

Lemma 4. *(i) Any solution of type (a) or (b) satisfies* $\lim_{n\to\infty} x_n = \infty$.
(ii) Any solution of type (a) or (e) satisfies $\lim_{n\to\infty} z_n = \infty$.

Proof. Let (x, y, z, w) be a solution of type (a) or (b). Then there exists $n_0 \in \mathbb{N}$ such that $x_n > 0$, $y_n > 0$ and $z_n > 0$ for $n \geq n_0$. Thus there exists $k > 0$ such that $y_n \geq k$ for $n \geq n_0$ and from the first equation

$$x_n - x_{n_0} = \sum_{i=n_0}^{n-1} C_i \cdot y_i^{\frac{1}{\gamma}} \geq k^{\frac{1}{\gamma}} \sum_{i=n_0}^{n-1} C_i.$$

Passing $n \to \infty$ we get $x_n \to \infty$ for $n \to \infty$.

The second statement follows from the third equation using a similar argument. ☐

Summarizing, by Lemma 4 the unbounded solutions are of type (a),(b) and (e). Solutions of type (d) can be bounded or unbounded.

3. Weak property B

Our investigation is motivated by the following simple criterion in order to have (S) property B.

Proposition 1. *If*

$$\sum_{n=1}^{\infty} B_n = \infty, \tag{3.1}$$

then any nonoscillatory solution of (S) with eventually positive x is of type (a), (b) or (c), and moreover, solutions of type (c) satisfies

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = \lim_{n\to\infty} w_n = 0,$$

and solutions of type (a)

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = \infty.$$

In addition, if

$$\sum_{n=1}^{\infty} D_n = \infty, \tag{3.2}$$

then (S) has property B.

Proof. In view of Lemma 2 we show that there exist no type (d) and (e) solutions. Let (x, y, z, w) be a solution of type (d) or (e). Then y is bounded and z is either positive increasing or negative decreasing. By summation of the second equation we get that y is unbounded, which is contradiction.

Let (3.2) hold. If (x, y, z, w) is of type (b) solution, then x is bounded away from zero and from the fourth equation we get that $w \to \infty$, which is a contradiction. Hence any nonoscillatory solution is either strongly monotone or Kneser.

Asymptotic properties of type (c) solutions follow from Lemma 3 and reasoning as in the proof of this lemma with the components z (and x if (3.2) holds). Similarly, properties of type (a) solutions follow follow from Lemma 4 and reasoning as in the proof of this lemma with the components y, w. ☐

In view of Proposition 1, in the sequel, we assume

$$\sum_{n=1}^{\infty} D_n < \infty.$$

Theorem 1. *If either*

$$\sum_{n=1}^{\infty} D_n \left(\sum_{j=n_0}^{n-1+\tau} C_j \right)^{\delta} = \infty \tag{3.3}$$

or

$$(3.4)\qquad \sum_{n=1}^{\infty} D_n \left(\sum_{j=n_0}^{n-1+\tau} C_j\right)^{\delta} < \infty, \quad \sum_{n=1}^{\infty} A_n \left(\sum_{j=n}^{\infty} D_j \left(\sum_{k=1}^{j-1+\tau} C_k\right)^{\delta}\right)^{\frac{1}{\alpha}} = \infty,$$

then solutions of type (b) do not exist. In addition, if (3.1) *holds, then system (S) has weak property B.*

Proof. Assume that (x, y, z, w) is type (b) solution. Then z and w are bounded and there exists $k > 0$ and $n_0 \geq 1$ such that $y_n \geq k$ and $x_n \geq 0$ for $n \geq n_0$. Thus from the first equation we have

$$(3.5)\qquad x_n \geq k^{1/\gamma} \sum_{i=n_0}^{n-1} C_i \text{ for } n \geq n_0.$$

Moreover, by Lemma 3, $\lim_{n\to\infty} w_n = 0$.

If (3.3) holds, then by summation of the fourth equation of (S) we get

$$(3.6)\qquad w_n - w_{n_0} = \sum_{i=n_0}^{n-1} D_i x_{i+\tau}^{\delta} \geq k^{\delta/\gamma} \sum_{i=n_0}^{n-1} D_i \left(\sum_{j=n_0}^{i-1+\tau} C_j\right)^{\delta}$$

and passing $n \to \infty$ we get the contradiction with the boundedness of w.

If (3.4) holds, then by summation of the fourth equation of (S) from n to ∞ we get

$$-w_n = \sum_{i=n}^{\infty} D_i x_{i+\tau}^{\delta} \geq k^{\delta/\gamma} \sum_{i=n}^{\infty} D_i \left(\sum_{j=n_0}^{i-1+\tau} C_j\right)^{\delta},$$

which yields, by summation of the third equation of (S),

$$z_{n_0} - z_n = +\sum_{j=n_0}^{n-1} A_j \left(-w_j\right)^{\frac{1}{\alpha}} \geq k^{\delta/\alpha\gamma} \sum_{j=n_0}^{n-1} A_i \left(\sum_{j=i}^{\infty} D_j \left(\sum_{k=n_0}^{j-1+\tau} C_k\right)^{\delta}\right)^{\frac{1}{\alpha}}.$$

Passing $n \to \infty$ we get the contradiction with the boundedness of z.

In view of Proposition 1, solutions of type (d) and (e) do not exist, thus (S) has weak property B. □

Assumption (3.1) can be relaxed and the following extension of Theorem 1 holds.

Theorem 2. *Assume* (3.3) *and*

$$(3.7)\qquad \sum_{n=1}^{\infty} B_n \left(\sum_{k=1}^{n-1} A_k \left(\sum_{j=k}^{\infty} D_j\right)^{1/\alpha}\right)^{1/\beta} = \infty.$$

Then the system (S) has weak property B.

Proof. By Theorem 1, the assumption (3.3) ensures that solutions of type (b) do not exist.

Obviously, (3.7) implies that

$$\sum_{n=1}^{\infty} B_n \left(\sum_{k=1}^{n-1} A_k \right)^{1/\beta} = \infty.$$

Assume that (x, y, z, w) is type (e) solution. Then x and y are bounded and there exist $k > 0$ and $n_0 \geq 1$ such that $w_n \geq k$ for $n \geq n_0$. Thus

$$z_n \geq k^{1/\alpha} \sum_{i=n_0}^{n-1} A_i \text{ for } n \geq n_0, \tag{3.8}$$

and using the second equation we have

$$y_n - y_{n_0} \geq k^{1/\alpha\beta} \sum_{i=n_0}^{n-1} B_i \left(\sum_{k=1}^{i-1} A_k \right)^{1/\beta} . \tag{3.9}$$

Passing $n \to \infty$ we get the contradiction that y is bounded. Hence, solutions of type (e) do not exist.

Let (x, y, z, w) be type (d) solution. Then y and w are bounded and there exist $n_0 \geq 1$ and positive constant k, l such that $x_n \geq k$ and $-z_n \geq l$ for $n \geq n_0$. If $y_n \geq h > 0$ for large n, then using the estimations (3.5) and (3.6) with constant h and passing $n \to \infty$, the condition (3.3) yields that w_n is unbounded, which is a contradiction. Thus $\lim_{n\to\infty} y_n = 0$. Similarly, if $-w_n \geq h > 0$ for large n, then using estimations (3.8) with constant h where z is replaced by $-z$, and by summation of the second equation of (S) from n to ∞

$$y_n = \sum_{i=n}^{\infty} B_i (-z_i^{1/\beta}) \geq h^{1/\alpha\beta} \sum_{i=n}^{\infty} B_i \left(\sum_{j=n_0}^{i-1} A_j \right)^{1/\beta} , \tag{3.10}$$

from where we get, as $n \to \infty$, a contradiction that y is bounded. Hence, also $\lim_{n\to\infty} w_n = 0$. From here and the fact that $x_n \geq k$ we get

$$-w_n \geq k^{\delta} \sum_{i=n}^{\infty} D_i, \quad z_{n_0} - z_n \geq k^{\delta/\alpha} \sum_{k=1}^{n-1} A_k \left(\sum_{j=k}^{\infty} D_j \right)^{1/\alpha} . \tag{3.11}$$

By summation of the second equation and substituing into z, we get

$$y_{n_0} - y_n = \sum_{j=n_0}^{n-1} B_j \left(-z_j\right)^{\frac{1}{\beta}} \geq k^{\frac{\delta}{\alpha\beta}} \sum_{i=n_0}^{n-1} B_i \left(\sum_{j=1}^{i-1} A_j \left(\sum_{k=j}^{\infty} D_k \right)^{1/\alpha} \right)^{1/\beta} . \tag{3.12}$$

Passing $n \to \infty$, assumption (3.7) yields a contradiction with the boundedness of y. Hence, also solutions of type (d) do not exist and any nonoscillatory solutions with the positive first component are of type (a) or (c). □

4. Bounded and unbounded solutions

In this section we study bounded and unbounded solutions. We start with the properties of bounded solutions.

Theorem 3. *If* (3.7) *holds, then every bounded nonoscillatory solution of (S) is Kneser solution and satisfies* $\lim_{n\to\infty} x_n = 0$.

Proof. Assume that (x, y, z, w) is a bounded nonoscillatory solution of type (d). Then there exist $n_0 \geq 1$ and positive constant k, l such that $x_n \geq k$ and $-z_n \geq l$ for $n \geq n_0$ and by Lemma 3 we have

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} w_n = 0. \tag{4.1}$$

Now, by the same argument as in the proof of Theorem 2 we get (3.11) and (3.12), which leads to a contradition with the boundedness of y. Therefore, solution of type (d) does not exist.

Let (x, y, z, w) be a solution of type (c). Then all of its components have the finite limit and (4.1) holds. If $x_n \geq k > 0$ for large n, then w_n satisfies (3.11) and using the same argument as in the proof of Theorem 2 we get a contradiction. Therefore, $\lim_{n\to\infty} x_n = 0$. □

Next result describes properties of strongly monotone solutions.

Theorem 4. *Assume*

$$\sum_{n=1}^{\infty} B_n \left(\sum_{k=1}^{n-1} A_k\right)^{1/\beta} = \infty \tag{4.2}$$

and

$$\sum_{n=1}^{\infty} D_n \left(\sum_{i=1}^{n-1+\tau} C_i \left(\sum_{j=1}^{i-1} B_j \left(\sum_{k=1}^{j-1} A_k\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\gamma}}\right)^{\delta} = \infty. \tag{4.3}$$

Then strongly monotone solutions of (S) satisfies (1.5).

Proof. Let (x, y, z, w) be a type (a) solution of (S). We prove that it satisfies (1.5). In view of Lemma 4 it is sufficient to prove that $\lim_{n\to\infty} y_n = \infty$ and $\lim_{n\to\infty} w_n = \infty$.

Since x, y, z are positive as for type (b) solutions, the estimation (3.5) holds. Moreover, w_n is increasing and positive and so there exists $k > 0$ such that $w_n \geq k$ for large n and

$$z_n - z_{n_0} = \sum_{i=n_0}^{n-1} A_i w_i^{\frac{1}{\alpha}} \geq k^{\frac{1}{\alpha}} \sum_{i=n_0}^{n-1} A_i.$$

Thus from the second equation of the system (S) we obtain

$$y_n \geq \sum_{i=n_0}^{n-1} B_i z_i^{\frac{1}{\beta}} \geq k^{\frac{1}{\alpha\beta}} \sum_{i=n_0}^{n-1} B_i \left(\sum_{j=n_0}^{i-1} A_j\right)^{\frac{1}{\beta}}$$

and therefore the sequence $y_n \to \infty$ for $n \to \infty$. Finally, from the fourth and first equation of the system (S) and using previous inequality we get

$$w_n \geq \sum_{i=n_0}^{n-1} D_i x_{i+\tau}^{\delta} \geq \sum_{i=n_0}^{n-1} D_i \left(\sum_{j=n_0}^{i-1+\tau} C_j \left(k^{\frac{1}{\alpha\beta}} \sum_{k=n_0}^{j-1} B_k \left(\sum_{t=n_0}^{k-1} A_t \right)^{\frac{1}{\beta}} \right)^{\frac{1}{\gamma}} \right)^{\delta}$$

$$w_n \geq k^{\frac{\delta}{\alpha\beta\gamma}} \sum_{i=n_0}^{n-1} D_i \left(\sum_{j=n_0}^{i-1+\tau} C_j \left(\sum_{k=n_0}^{j-1} B_k \left(\sum_{t=n_0}^{k-1} A_t \right)^{\frac{1}{\beta}} \right)^{\frac{1}{\gamma}} \right)^{\delta}$$

and so as $n \to \infty$ we get $\lim w_n = \infty$. □

5. Property B

Applying Theorems 2, 3, 4, we get the following conditions ensuring that system (S) has property B.

Corollary 1. *Let assumptions of Theorem 2 hold. If either $\sum B_n$ is divergent or $\sum B_n$ is convergent and*

$$\sum_{n=1}^{\infty} C_n \left(\sum_{i=n}^{\infty} B_i \right)^{1/\gamma} = \infty,$$

then system (S) has property B.

Proof. Condition (3.7) implies that (4.2) holds and condition (3.3) implies that (4.3) holds. It remains to prove $lim_{n\to\infty} z_n = 0$ for type (c) solutions. If $z_n \geq l > 0$ then by summation of the second equation we get

$$-y_n \geq l^{1/\beta} \sum_{i=n}^{\infty} B_i.$$

Then from the first equation we get

$$x_{n_0} - x_n = \sum_{i=n_0}^{n-1} C_i \left(-y_i\right)^{1/\gamma} \geq l^{1/\beta\gamma} \sum_{i=n_0}^{n-1} C_i \left(\sum_{j=i}^{\infty} B_j \right)^{1/\gamma}$$

and passing $n \to \infty$ we get the contradiction with the boundedness of x. Now the conclusion follows from Theorems 2–4. □

Remark 3. *Assume* (3.7) *and* (3.3) *with $\tau = 0$. Then, because* (3.3) *holds for $\tau > 0$, system (S) has property B for any $\tau \geq 0$.*

Consider a four-dimensional symmetric system

$$\text{(T)}\qquad \begin{aligned} \Delta x_n &= A_n \cdot y_n^{\frac{1}{\alpha}} \\ \Delta y_n &= B_n \cdot z_n^{\frac{1}{\beta}} \\ \Delta z_n &= A_n \cdot w_n^{\frac{1}{\alpha}} \\ \Delta w_n &= B_n \cdot x_{n+\tau}^{\frac{1}{\beta}}. \end{aligned}$$

Applying Corollary 1 to this system, we get the following result.

Corollary 2. *If either* (3.1) *or*

$$\sum_{n=1}^{\infty} B_n < \infty, \quad \sum_{n=1}^{\infty} B_n \left(\sum_{k=1}^{n-1} A_k \left(\sum_{j=k}^{\infty} B_j \right)^{1/\alpha} \right)^{1/\beta} = \infty,$$

then system (T) has property B for any $\tau \geq 0$.

Proof. Condition (3.3) reduces to (4.2). Obviously, in view of (3.7), this condition is satisfied, and so the conclusion follows from Corollary 1. □

Remark 4. *Our results can be applied to equation (E). A solution x of (E) is called nonoscillatory if x is eventually positive or eventually negative. By Lemma 2 a solution (x, y, z, w) of system (S) is nonoscillatory if and only if x is nonoscillatory solution of equation (E). Property B reads for (E) as the property that any nonoscillatory solution of (E) satisfies either* $\lim x_n^{[i]} = 0$ *or* $\lim \left| x_n^{[i]} \right| = \infty$ *for all* $i = 0, 1, 2, 3$*, where* $x_n^{[0]} = x_n$.

6. Concluding remarks

Here we discuss the role of the integer-valued argument τ in (S) to the behavior of nonoscillatory solutions.

(1) The argument τ appears in conditions (3.3) and (4.3). The condition (3.3) ensures the nonexistence of type (b) solutions (Theorem 1) and that solutions of type (d) satisfy $\lim y_n = 0$ (proof of Theorem 2). It is a question if Theorem 2 and Corollary 1 remain to hold if (3.3) is replaced by (3.4).

(2) Assume that the series $\sum B$ is divergent. By Theorem 1, if (3.4) holds, then (S) has a weak property B. It would be interesting to study the existence of solutions of type (b).

(3) Any solution of (S) with the positive initial conditions is strongly monotone. Hence, system (S) has always these solutions. It is an open problem whether Kneser solutions exist and if the argument τ can change the (non)existence of Kneser solutions.

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