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A note on asymptotics and nonoscillation of linear q-difference equations

Pavel Řehák*

Institute of Mathematics
Academy of Sciences of the Czech Republic
Žižkova 22, CZ-61662 Brno, Czech Republic
rehak@math.cas.cz

Abstract

We study the linear second order q-difference equation $y(q^2t) + a(t)y(qt) + b(t)y(t) = 0$ on the q-uniform lattice $\{q^k : k \in \mathbb{N}_0\}$ with $q > 1$, where $b(t) \neq 0$. We establish various conditions guaranteeing the existence of solutions satisfying certain estimates resp. (non)oscillation of all solutions resp. q-regular boundedness of solutions resp. q-regular variation of solutions. Such results may provide quite precise information about their asymptotic behavior. Some of our results generalize existing Kneser type criteria and asymptotic formulas, which were stated for the equation $D_q^2 y(qt) + p(t)y(qt) = 0$, D_q being the Jackson derivative. In the proofs however we use an original approach.

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1 Introduction

Consider the linear second order q-difference equation

$$y(q^2t) + a(t)y(qt) + b(t)y(t) = 0 \tag{1}$$

on $q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$ with $q > 1$, where $b(t) \neq 0$. We establish conditions guaranteeing the existence of a solution to (1), which satisfies certain effective estimate. Putting additional conditions, we then derive more precise estimates and we show that solutions are (non)oscillatory, resp. q-regularly bounded, resp. q-regularly varying. As a corollary we get sharp Kneser type criteria. Other our results generalize

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some known asymptotic formulas which were stated for certain associated self-adjoint equations. In the proofs however we use an original approach (including rather simple methods), which shows some advantages of our "three-term" setting.

The paper is organized as follows. In the next section we present basic facts about q-calculus, provide some information on equation (1), and briefly recall the theory of q-regular variation. In Section 3 we formulate the main results and give comments on them, including a comparison with existing results. The last section contains the proofs.

2 Basic concepts and preliminaries

We start with brief recalling some basic facts about q-calculus. For material on this topic see [2, 10, 12]. See also [7] for the calculus on time scales which somehow contains q-calculus. Since we work on the lattice $q^{\mathbb{N}_0}$ (which is a time scale), we may follow essentially a "time scale dialect" of q-calculus. The q-derivative of a function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is defined by $D_q f(t) = [f(qt) - f(t)]/[(q-1)t]$. We use the notation $[a]_q = (q^a - 1)/(q-1)$ for $a \in \mathbb{R}$. In view of the definition of $[a]_q$, it is natural to introduce the notation $[\infty]_q = \infty$, $[-\infty]_q = 1/(1-q)$. For $p : q^{\mathbb{N}_0} \to \mathbb{R}$ satisfying $1 + (q-1)tp(t) \neq 0$ for all $t \in q^{\mathbb{N}_0}$ we denote $e_p(t,s) = \prod_{u \in [s,t) \cap q^{\mathbb{N}_0}} [(q-1)up(u) + 1]$ for $s < t$, $e_p(t,s) = 1/e_p(s,t)$ for $s > t$, and $e_p(t,t) = 1$, where $s, t \in q^{\mathbb{N}_0}$. A function $e(\cdot, a)$ is called a q-exponential function, and is the solution of the IVP $D_q y = p(t)y$, $y(a) = 1$, $t \in q^{\mathbb{N}_0}$. Intervals having the subscript q denote the intervals in $q^{\mathbb{N}_0}$, e.g., $[a, \infty)_q = \{a, aq, aq^2, \dots\}$ with $a \in q^{\mathbb{N}_0}$.

We will continue with stating some fundamental properties of (1), which will be useful in our proofs. Along with (1) consider the Riccati type equation

$$w(qt) + a(t) + \frac{b(t)}{w(t)} = 0. \tag{2}$$

It is easy to see that if $y(t) \neq 0$ is a solution of (1) on $[T, \infty)_q$, then w defined by $w(t) = y(qt)/y(t)$ is a solution of (2) on $[T, \infty)_q$. Conversely, if w is a solution of (2) on $[T, \infty)_q$, then y defined by $y(t) = C \prod_{s \in [T,t)_q} w(s)$, $C \in \mathbb{R} \setminus \{0\}$, is a nonzero solution (1) on $[T, \infty)_q$. Clearly, an eventually positive solution of (2) corresponds to a solution of (1) which is eventually of one sign. Of course, there can be developed another forms of Riccati type substitutions for (1). One of them is discussed later, see Remark 1 (v).

It is also important to see relations between equations in the form (1) and in the self-adjoint form

$$D_q\big(r(t) D_q y(t)\big) + p(t)y(qt) = 0, \tag{3}$$

where $r(t) \neq 0$. It is not difficult to see that (3) can always be written in the form (1), where

$$a(t) = q(q-1)^2 t^2 \frac{p(t)}{r(qt)} - 1 - \frac{qr(t)}{r(qt)}, \quad b(t) = \frac{qr(t)}{r(qt)}. \tag{4}$$

Conversely, any "three-term" q-difference equation (1) can be written in a self-adjoint form provided we choose

$$r(t) = C \prod_{s\in[1,t)_q} \frac{q}{b(s)}, \quad p(t) = \frac{C\,(a(t)+1+b(t))}{q(q-1)^2t^2} \prod_{s\in[1,qt)_q} \frac{q}{b(s)}, \tag{5}$$

where C is an arbitrary nonzero real constant. These relations can be further rewritten by using $\prod_{s\in[1,t)_q} q = t$. An equation in the form (3) can be understood as a q-counterpart of the Sturm-Liouville differential equation

$$(r(t)y')' + p(t)y = 0, \tag{6}$$

which has been extensively studied, see e.g. [19]. Besides, it can be seen as a special case of the linear dynamic equation $(r(t)y^\Delta(t))^\Delta + p(t)y(\sigma(t)) = 0$ on time scales, where y^Δ denotes the delta derivative of y and σ is the forward jump operator, see e.g. [7].

Now let us deal with an "intuitive" definition of a generalized zero of a solution to (3) or (1), i.e., the situation when a solution has a zero or changes its sign (within a given interval $[t, qt]$). A nonoscillatory solution (on $[1,\infty)_q$) is then a solution having eventually no generalized zeros, i.e., is eventually of one sign; otherwise this solution is said to be oscillatory. It is not difficult to find an equation (1) or (3), having two nontrivial solutions, one oscillating and another one nonoscillating. From the Sturmian theory for (6) it follows that zeros of two linearly independent solutions of (6) separate each other. Thus this property seems to be violated for q-discrete counterparts of (6). However, the definition of a generalized zero can be modified in the following sense: An interval $(t, qt]$ is said to contain the *generalized zero* of a solution y of (3) if $y(t) \neq 0$ and $r(t)y(t)y(qt) \leq 0$. With this new definition it was shown that a Sturmian theory (in particular, a separation type result) for (3) works, see e.g. [15], where such a statement was proved in a more general setting – on time scales. The separation result says that generalized zeros of two linearly independent solutions to (3) separate each other (with the addendum that they cannot have a common zero but may have a common generalized zero). Thanks to this property we have the following equivalence: One solution of (3) is oscillatory if and only if every solution of (3) is oscillatory (where oscillation of a solution means that it has infinitely many generalized zeros). Hence we can comfortably introduce the concepts of *oscillation* and *nonoscillation of equation* (3). Of course, all these concepts can be appropriately adopted for equations in the form (1). Note that in the cases where r is positive, these concepts coincide with the "intuitive" ones.

Various aspects of linear q-difference equations were studied e.g. in [1, 2, 3, 4, 6, 8, 9, 11, 13, 17, 18]. For related topics see [10, 12] and the references therein.

We conclude this section with recalling the theory of q-regular variation, see e.g. [17]. A function $f : q^{\mathbb{N}_0} \to (0,\infty)$ is said to be *q-regularly varying of index ϑ*, $\vartheta \in \mathbb{R}$, if $\lim_{t\to\infty} f(qt)/f(t) = q^\vartheta$, we write $f \in \mathcal{RV}_q(\vartheta)$. If $\vartheta = 0$, then f is said to be *q-slowly varying*; we write $f \in \mathcal{SV}_q$. Here are some selected properties of $\mathcal{RV}_q$ functions: It holds $f \in \mathcal{RV}_q(\vartheta)$ if and only if $f(t) = t^\vartheta \delta(t) e_\psi(t,1)$, where $\delta : q^{\mathbb{N}_0} \to (0,\infty)$ tends to a positive constant (w.l.o.g., δ can be replaced by a positive constant) and $\psi : q^{\mathbb{N}_0} \to \mathbb{R}$

satisfies $\lim_{t\to\infty} t\psi(t) = 0$. Further, $f \in \mathcal{RV}_q(\vartheta)$ if and only if $\lim_{t\to\infty} tD_q f(t)/f(t) = [\vartheta]_q$. If $f_i \in \mathcal{RV}_q(\vartheta_i)$, $i = 1, 2$, then $\lim_{t\to\infty} f_1(t)/t^{\vartheta_1-\varepsilon} = \infty$, $\lim_{t\to\infty} f_1(t)/t^{\vartheta_1+\varepsilon} = 0$ for every $\varepsilon > 0$, $\lim_{t\to\infty} \ln f_1(t)/\ln t = \vartheta_1$, $f_1^\gamma \in \mathcal{RV}_q(\gamma\vartheta_1)$, $f_1 f_2 \in \mathcal{RV}_q(\vartheta_1 + \vartheta_2)$, and $1/f_1 \in \mathcal{RV}_q(-\vartheta_1)$. For other properties see, e.g., [17].

Note that in contrast to the classical theory of regular variation (i.e., for functions of a real variable or of an integer variable, see e.g. [5]), the theory of q-regularly varying functions differs in several basic aspects, is simpler, and provides new types of powerful tools, because the range $q^{\mathbb{N}_0}$ is somehow natural setting for regularly varying behavior, see [17].

We have defined q-regular variation at infinity. If we consider a function $f : q^{\mathbb{Z}} \to (0, \infty)$, $q^{\mathbb{Z}} := \{q^k : k \in \mathbb{Z}\}$, then $f(t)$ is said to be q-regularly varying at zero if $f(1/t)$ is q-regularly varying at infinity. But it is apparent that it is sufficient to develop just the theory of q-regular variation at infinity. Note that from the continuous theory or the discrete theory the concept of a normalized regular variation is known. Because of the above mentioned properties, there is no need to introduce a normality in the q-calculus case, since every q-regularly varying function is automatically normalized.

If we relax the condition in the definition of q-regular variation, we obtain the concept of q-regular boundedness: A function $f : q^{\mathbb{N}_0} \to (0, \infty)$ is said to be *q-regularly bounded* if $0 < \liminf_{t\to\infty} f(qt)/f(t) \le \limsup_{t\to\infty} f(qt)/f(t) < \infty$. The totality of q-regularly bounded functions is denoted by $\mathcal{RB}_q$. It is clear that $\bigcup_{\vartheta\in\mathbb{R}} \mathcal{RV}_q(\vartheta) \subset \mathcal{RB}_q$. We select the following properties: It holds $f \in \mathcal{RB}_q$ if and only if $f(t) = \delta(t)e_\psi(t, 1)$, where $C_1 \le \delta(t) \le C_2$ and $D_1 \le t\psi(t) \le D_2$ with some $0 < C_1 \le C_2 < \infty$ and $[-\infty]_q < D_1 \le D_2 < [\infty]_q$. Without loss of generality, in particular in the only if part, the function δ can be replaced by a positive constant. It holds $f \in \mathcal{RB}_q$ if and only if for $f : q^{\mathbb{N}_0} \to (0, \infty)$ there exist $\gamma_1, \gamma_2 \in \mathbb{R}$, $\gamma_1 < \gamma_2$, such that $f(t)/t^{\gamma_1}$ is eventually (almost) increasing and $f(t)/t^{\gamma_2}$ is eventually (almost) decreasing. If $f, g \in \mathcal{RB}_q$, then $f + g, fg, f/g \in \mathcal{RB}_q$. Similarly as above, we can introduce q-regular boundedness at zero.

3 Main results

We start with the most general statement where no sign conditions on the coefficients are assumed. The existence of a solution to (1) is guaranteed, which satisfies certain effective estimate in terms of the coefficient a.

Theorem 1. *If there exists $\zeta \in (0, \infty)$ such that*

$$\frac{\zeta^2|b(t)|}{a^2(t)} + \left|\frac{a(qt)}{a(t)}\right| \le \zeta \quad \textit{for large } t, \tag{7}$$

then (1) *possesses a solution $\tilde{y}$ such that*

$$\zeta|\tilde{y}(qt)/\tilde{y}(t)| \ge |a(t)| \quad \textit{eventually.}$$

Elaborating further the main idea of the proof of Theorem 1, we can show that if a sign condition on the coefficients is somehow strengthened, then sufficient condition

(7) can be relaxed. Moreover, we are able to get an information about (non)oscillation of (1). We offer also variants of this sufficient condition, and later we discuss their optimality.

Theorem 2. *(i) If $a(t)a(qt) > 0$ and there exists $\zeta \in (0, \infty)$ such that*

$$\zeta \geq \begin{cases} \frac{\zeta^2 b(t)}{a^2(t)} + \frac{a(qt)}{a(t)} & \text{when } b(t) > 0 \\ \frac{a(qt)}{a(t)} & \text{when } b(t) < 0 \end{cases} \quad \text{for large } t, \tag{8}$$

then (1) *possesses a solution* $\tilde{y}$ *such that*

$$\zeta \tilde{y}(qt)/(\tilde{y}(t)a(t)) \leq -1 \quad \text{eventually.}$$

(ii) If $a(t)a(qt) > 0$ and there exists $C \in (0, \infty)$ such that

$$|a(t)| \geq C \text{ and } b(t) \leq C^2/4 \text{ for large } t, \tag{9}$$

then (1) *possesses a solution* $\tilde{y}$ *such that*

$$2\tilde{y}(qt)/\tilde{y}(t) \geq C \text{ if } a(t) < 0 \quad \text{and} \quad 2\tilde{y}(qt)/\tilde{y}(t) \leq -C \text{ if } a(t) > 0 \quad \text{eventually.}$$

(iii) If, in addition to (8) *or* (9)*, $a(t) < 0$ and $b(t) > 0$ for large t, then all nontrivial solutions of* (1) *are eventually of one sign (i.e.,* (1) *is nonoscillatory).*

If, in addition to (8)*, $a(t) > 0$ and $b(t) > 0$ for large t, then* (1) *possesses a solution $\tilde{y}$ such that $\tilde{y}(t)\tilde{y}(qt) < 0$ for large t and for any solution y of* (1) *it holds $y(t)y(qt) \leq 0$ at infinitely many t's (i.e.,* (1) *is oscillatory).*

Remark 1. (i) Condition (8) is implied, for instance, by

$$4b(t) \leq a^2(t) \text{ and } |a(t)| \text{ is nonincreasing for large } t \tag{10}$$

for large t.

(ii) Similarly as Theorem 2 (ii), we can prove that if there are $C, \zeta \in (0, \infty)$ such that $a(t) \leq -C$ and $b(t) \leq C^2/\zeta^2$, then (1) possesses a solution $\tilde{y}$ such that $\zeta\tilde{y}(qt)/\tilde{y}(t) \geq C$. We present this statement in order to show that such a variant of Theorem 2 (ii) with the parameter ζ does not yield a generalization since its value can be optimally chosen (namely $\zeta = 2$) and then we get just Theorem 2 (ii). Indeed, assume $w(t) \geq C/\zeta$ and, as in the proof, we want to show that $w(qt) \geq C/\zeta$. We have $w(qt) = -a(t) - b(t)/w(t) \geq C - C/\zeta \geq C/\zeta$, where the last inequality is equivalent to $\zeta \geq 2$. Thus the parameter ζ needs to be in $[2, \infty)$. However any of its values greater than 2 means a more restrictive assumption on b (since then $C^2/\zeta^2 < C^2/4$) and, moreover, gives a worse estimate of $\tilde{y}(qt)/\tilde{y}(t)$ (since $C/2 > C/\zeta$).

(iii) It is interesting to see the nonoscillation result from Theorem 2 in terms of self-adjoint equation (3) under some special conditions. We claim: If $r(t) = t^\gamma$, $\gamma \in \mathbb{R}$, and

$$t^2 p(t) \leq \frac{\left(\sqrt{qr(t)} - \sqrt{r(qt)}\right)^2}{q(q-1)^2} \tag{11}$$

for large t, then (3) is nonoscillatory. To show it, we translate the problem from the "self-adjoint" setting to the "three-term" setting and show that the sufficient conditions for nonoscillation from Theorem 2 (iii) (more precisely, (9), $a(t) < 0$, $b(t) > 0$) are satisfied: Set $C = 2q^{(1-\gamma)/2}$. In view of the second identity in (4), we have $b(t) = q^{1-\gamma} = C^2/4$. Further, the first identity in (4) and (11) yield

$$\begin{aligned} a(t) &= \frac{1}{r(qt)}\Big(q(q-1)^2t^2p(t) - r(qt) - qr(t)\Big) \\ &\leq \frac{1}{r(qt)}\Big(\Big(\sqrt{qr(t)} - \sqrt{r(qt)}\Big)^2 - r(qt) - qr(t)\Big) \\ &= -\frac{2\sqrt{qr(t)r(qt)}}{r(qt)} = -\frac{2\sqrt{qr(t)}}{\sqrt{r(qt)}} = -2^{\frac{1-\gamma}{2}} = -C < 0 \end{aligned}$$

for large t.

Similarly we can show that $4b(t) \leq a^2(t)$ (i.e., the first condition in (10)) is equivalent to (11) – we emphasize that this holds for a general positive r and negative a. Thus, in view of (10), one can obtain another (general) version of Kneser type nonoscillation criterion, namely in the form of the conditions (11) and monotonicity of a (rewritten in terms of the associated self-adjoint equation).

Note that, with $r(t) = t^\gamma$, (11) reads as

$$t^{2-\gamma}p(t) \leq q^{\gamma-1}\left[\frac{1-\gamma}{2}\right]_q^2,$$

and, with $r(t) \equiv 1$, it reduces to

$$t^2p(t) \leq \frac{1}{q(\sqrt{q}+1)^2} \tag{12}$$

for large t, in which we recognize a q-version of the well known Kneser criterion (see e.g. [19] for the differential equations setting). Related results for q-difference equations can be found in [8, 16]. Observe how the constant on the right-hand side of (12) tends to $1/4$ as $q \to 1$, which is the critical constant known from the continuous theory.

(iv) Since the constant on the right hand side in (12) is known to be the best possible (see [8, 16]), we can conclude that also original conditions in the three-term setting are somehow sharp.

(v) In all our proofs, an important role is played by relations between equation (1) and the Riccati type equation (2). But, as already mentioned, there can be developed also another forms of Riccati type substitutions. For instance, with $a(t) < 0$ and $b(t) > 0$, a nonzero solution y is related to a positive solution z of the Riccati type equation

$$\frac{b(qt)}{a(t)a(qt)}z(qt) - 1 + \frac{1}{z(t)} = 0 \tag{13}$$

by the substitution $z(t) = (-a(t)/b(t))(y(qt)/y(t))$. Similarly as in the proof of Theorem 2, it is not difficult to construct inductively a solution z of (13), which satisfies $z(t) \geq 2$ for large t, provided

$$4b(t) \leq a(t)a(t/q) \quad \text{for large } t. \tag{14}$$

Thus we get the following variant of Theorem 2:

Theorem 2'. *If $a(t) < 0$, $b(t) > 0$, and* (14) *hold for large t, then* (1) *possesses a solution $\tilde{y}$ such that $\tilde{y}(qt)/\tilde{y}(t) \geq -2b(t)/a(t)$ eventually, and* (1) *is nonoscillatory.*

It is interesting to observe that (14) is implied by (10).

If we further strengthen previous conditions for nonoscillation of (1), then q-regular boundedness of positive solutions to (1) is guaranteed.

Theorem 3. *Let $\liminf_{t\to\infty} a(t) > -\infty$ and $\liminf_{t\to\infty} b(t) > 0$. Assume that $a(t) < 0$ for large t and* (1) *is nonoscillatory (which can be guaranteed e.g. by* (8) *or* (9) *or* (14)*). Then all eventually positive solutions of* (1) *(which indeed exist) are q-regularly bounded.*

Finally we strengthen conditions in the sense of the existence of certain limits of the coefficients of (1). This leads to q-regularly varying behavior (with known index) of positive solutions to (1).

Theorem 4. *Let the limits*

$$\lim_{t\to\infty} a(t) = A \in (-\infty, 0) \quad \text{and} \quad \lim_{t\to\infty} b(t) = B \in (0, \infty)$$

exist with $4B \leq A^2$. In the case $4B = A^2$ assume that (8) *or $a(t) \leq -A$ and $b(t) \leq A^2/4$ hold for large t. Then* (1) *possesses solutions y_1 and y_2 with $y_i \in \mathcal{RV}_q(\log_q \lambda_i)$, $i = 1, 2$, where*

$$\lambda_1 = \left(-A + \sqrt{A^2 - 4B}\right)/2 \quad \text{and} \quad \lambda_2 = \left(-A - \sqrt{A^2 - 4B}\right)/2.$$

Moreover, all nontrivial solutions of (1) *are eventually of one sign and for any eventually positive solution y of* (1) *it holds $y \in \mathcal{RV}_q(\log_q \lambda_1) \cup \mathcal{RV}_q(\log_q \lambda_2)$.*

Remark 2. (i) In terms of the coefficients of the corresponding self-adjoint equation (3), the condition $\lim_{t\to\infty} b(t) = B \in (0, \infty)$ means that $r \in \mathcal{RV}_q(\log_q(q/B))$. The existence of the limit $\lim_{t\to\infty} a(t) = A$ then says that p is asymptotically equivalent to certain constant multiple of $r(t)/t^2$. Thus, possibly up to sign, p is q-regularly varying too, with the index $\log_q(q/B) - 2$.

(ii) Consider the equation

$$D_q^2 y(t) + p(t)y(qt) = 0. \tag{15}$$

This equation is related to (1) by $p(t) = (a(t) + q + 1)/(q(q-1)^2 t^2)$ and $b(t) \equiv q$. In [17] we proved that, under the assumption $t^2 p(t) \leq 1/(q(\sqrt{q}+1)^2)$ we have: If the limit

$$\lim_{t\to\infty} t^2 p(t) = P \in \left(-\infty, \frac{1}{q(\sqrt{q}+1)^2}\right) \tag{16}$$

exists, then (15) has a fundamental set of solutions $y_i \in \mathcal{RV}_q(\vartheta_i)$, $i = 1, 2$, with $\vartheta_i = \log_q[(q-1)\mu_i+1]$, μ_i, $i = 1, 2$, being the (real) roots of $(\mu-\mu^2)/[\mu(q-1)+1] = qP$. It is easy to see that (16) expressed in terms of a takes the form $\lim_{t\to\infty} a(t) = A \in \left(-\infty, -2\sqrt{q}\right)$. Thus the result in [17] is a special case of Theorem 4, and recall that it can be viewed as a q-version of the sufficient condition for $y'' + p(t)y = 0$ to have regularly varying solutions, see, e.g., [14]. In both settings this condition can be easily shown to be also necessary. Note that the condition in the differential equations case is in a certain integral form (indeed, it reads as $\lim_{t\to\infty} t \int_t^\infty p(s)\,\mathrm{d}s \in (0, 1/4)$), in contrast to the q-case; for an explanation of this discrepancy see [17].

Remark 3. In connection with our results it is interesting to observe one important feature concerning the "three term" q-difference equation with constant coefficients

$$y(q^2t) + Ay(qt) + By(t) = 0, \tag{17}$$

where $A, B \in \mathbb{R}$. Let us consider, for definiteness, the case where $A < 0$, $B > 0$, and $A^2 - 4B \geq 0$. Let $\lambda_1 \geq \lambda_2 > 0$ be the (real) roots of $\lambda^2 + A\lambda + B = 0$. Then $y_1(t) = \lambda_1^{\log_q t} = t^{\log_q \lambda_1}$ and $y_2(t) = \lambda_2^{\log_q t} = t^{\log_q \lambda_2}$ are solutions of (17), and with $A^2 - 4B > 0$ they form the fundamental system of (17). We see, that in the contrast, e.g., to the case of classical three term recurrence relations of the form $y_{k+2} + Ay_{k+1} + By_k = 0$, power functions play a key role in searching solutions of (17). Note that the Euler type q-difference equation $D_q^2 y(t) + (\gamma/t^2)y(qt) = 0$, γ being a parameter, has in some cases real solutions in the form of power functions; equations of Euler type are important in oscillation theory. Especially, (17) with the critical value of $\gamma = 1/(q(\sqrt{q}+1)^2)$ has the (nonoscillatory) solution $y(t) = \sqrt{t}$. Further recall that q-regularly varying functions behave like a product of a power function and the factor which varies "more slowly" than the power function and Theorem 4 says that equations with coefficients "close" to constants have just q-regularly varying solutions. Hence, in view of these facts, we can see usefulness – not known in the theory of classical difference equations – of three term forms, when studying asymptotic behavior of solutions in the framework of q-regular variation and some oscillatory properties of linear q-difference equations. However, it is worthy of note, that in some other aspects, the self-adjoint form may has its advantages.

4 Proofs

Proof of Theorem 1. Let $T \in q^{\mathbb{N}_0}$ be such that (7) holds for $t \in [T, \infty)_q$. Let us construct the function w by defining $|w(T)| \in [|a(T)|/\zeta, \infty)$ and $w(qt) = -a(t) - b(t)/w(t)$ for $t \in [T, \infty)_q$. Then w is well defined and satisfies (2) with $|w(t)| \geq |a(t)|/\zeta$ for $t \in [T, \infty)_q$. Indeed, let $|w(t)| \geq |a(t)|/\zeta$. Then

$$|w(qt)| = \left| -a(t) - \frac{b(t)}{w(t)} \right| \geq |a(t)| - \frac{|b(t)|}{|w(t)|} \geq |a(t)| - \frac{\zeta|b(t)|}{|a(t)|} \geq \frac{|a(qt)|}{\zeta},$$

for $t \in [T, \infty)_q$, in view of (7). Define $\tilde{y}$ by $\tilde{y}(t) = \prod_{s\in[T,t)_q} w(s)$. Then $\tilde{y}$ is a solution of (1), which is nonzero and satisfies $|\tilde{y}(qt)/\tilde{y}(t)| = |w(t)| \geq |a(t)|/\zeta$ for $t \in [T, \infty)_q$. $\square$

Proof of Theorem 2. (i) Assume that (8) and $a(t) < 0$ hold for $t \in [T, \infty)_q$. The case $a(t) > 0$ can be treated similarly — in such a case we look for a solution w of (2) satisfying $w(t) \leq -a(t)/\zeta$ eventually. Let us construct the function w by defining $w(T) \in [-a(T)/\zeta, \infty)$ and $w(qt) = -a(t) - b(t)/w(t)$ for $t \in [T, \infty)_q$. Similarly as in the proof of Theorem 1 we can show that w is well defined, solves (2) and satisfies $w(t) \geq -a(t)/\zeta$ for $t \in [T, \infty)_q$. Indeed, let $w(t) \geq -a(t)/\zeta$. If $b(t) > 0$, then the inequality in (8) is equivalent to $-a(t) \geq -\zeta b(t)/a(t) - a(qt)/\zeta$, and so $w(qt) = -a(t) - b(t)/w(t) \geq -a(t) + \zeta b(t)/a(t) \geq -a(qt)/\zeta$ for $t \in [T, \infty)_q$. If $b(t) < 0$, then the inequality in (8) is equivalent to $-a(t) \geq -a(qt)/\zeta$, and so $w(qt) = -a(t) - b(t)/w(t) \geq -a(t) \geq -a(qt)/\zeta$ for $t \in [T, \infty)_q$. Thus the $\tilde{y}$ defined by $\tilde{y}(t) = \prod_{s \in [T,t)_q} w(s)$ is nonzero, solves (1), and satisfies $\zeta \tilde{y}(qt)/\tilde{y}(t) \geq -a(t)$ for $t \in [T, \infty)_q$.

(ii) Assume that (9) and $a(t) < 0$ hold for $t \in [T, \infty)_q$. The case $a(t) > 0$ can be treated similarly. Let us construct the function w by defining $w(T) \in [C/2, \infty)$ and $w(qt) = -a(t) - b(t)/w(t)$ for $t \in [T, \infty)_q$. Similarly as above we can show that w is well defined, solves (2) and satisfies $w(t) \geq C/2$ for $t \in [T, \infty)_q$. Indeed, let $w(t) \geq C/2$. Then

$$w(qt) \geq C - \frac{C^2/4}{C/2} = C/2$$

for $t \in [T, \infty)_q$, in view of (9). Similarly as above, such a w generates a solution $\tilde{y}$ of (1) satisfying $\tilde{y}(qt)/\tilde{y}(t) \geq C/2$ for $t \in [T, \infty)_q$.

(iii) Now assume that $b(t) > 0$ for large t, say $t \in [T, \infty)_q$. Define the coefficients r and p of (3) by (5), where $C = 1$ and the interval $[1, t)_q$ is replaced by $[T, t)_q$. Then $r(t) > 0$, and $\tilde{y}$ solves (3). Since $\tilde{y}(t)\tilde{y}(qt)$ is eventually positive resp. negative provided $a(t) < 0$ resp. $a(t) > 0$, the Sturm type separation theorem yields that $y(t)y(qt) > 0$ holds eventually resp. $y(t)y(qt) \leq 0$ holds at infinitely many t's for any nontrivial solution y of (3) and so of (1). $\square$

Proof of Theorem 3. Consider any nontrivial solution y of (1). Then $y(t)y(qt) > 0$ for large t, say $t \in [T, \infty)_q$, by Theorem 2. Set $w(t) = y(qt)/y(t)$. Then w is a positive solution of (2) on $[T, \infty)_q$. We will show that $\liminf_{t\to\infty} w(t) > 0$ and $\limsup_{t\to\infty} w(t) < \infty$. Assume by a contradiction that $\limsup_{t\to\infty} w(t) = \infty$. Then

$$\infty = \limsup_{t\to\infty} w(t) \leq \limsup_{t\to\infty} \left(w(qt) + \frac{b(t)}{w(t)} \right) = \limsup_{t\to\infty} (-a(t)) = -\liminf_{t\to\infty} a(t) < \infty,$$

a contradiction. Now assume by a contradiction that $\liminf_{t\to\infty} w(t) = 0$. Since $\liminf_{t\to\infty} b(t) > 0$, there exists $K > 0$ such that $b(t) \geq K$, $t \in [T, \infty)_q$. Hence,

$$\infty = \limsup_{t\to\infty} \frac{K}{w(t)} \leq \limsup_{t\to\infty} \frac{b(t)}{w(t)} \leq \limsup_{t\to\infty} \left(w(qt) + \frac{b(t)}{w(t)} \right) = \limsup_{t\to\infty} (-a(t)) < \infty,$$

a contradiction. Therefore $0 < \liminf_{t\to\infty} y(qt)/y(t) \leq \limsup_{t\to\infty} y(qt)/y(t) < \infty$. Since y was arbitrary, the statement follows. $\square$

Proof of Theorem 4. In the proof we distinguish the two cases (I) $A^2 > 4B$ and (II) $A^2 = 4B$.

(I) Let $T \in q^{\mathbb{N}_0}$ and $A_1, B_2 \in \mathbb{R}$ be such that $0 < A_1 \leq -a(t)$ and $b(t) \leq B_2$ for $t \in [T,\infty)_q$, $A_1^2 > 4B_2$, and $N := (A_1 + \sqrt{A_1^2 - 4B_2})/2 > \lambda_2$. Then $N = A_1 - B_2/N$. Construct a solution w_1 of (2) by defining $w_1(T) = N$ and $w_1(qt) = -a(t) - b(t)/w_1(t)$, $t \in [T,\infty)_q$. We note that if $w_1(t) \geq N$ for any $t \in [T,\infty)_q$, then $w_1(qt) \geq -a(t) - b(t)/N \geq A_1 - B_2/N = N$. Hence the function w_1 is well defined, and it is readily verified that w_1 satisfies (2). Denote $M_* = \liminf_{t\to\infty} w_1(t)$ and $M^* = \limsup_{t\to\infty} w_1(t)$. By taking lim sup as $t \to \infty$ in $w_1(qt) + a(t) = -b(t)/w_1(t)$ we get $M^* < \infty$, and so $M_*, M^* \in [N,\infty)$. The lim inf and lim sup as $t \to \infty$ in $w_1(qt) + a(t) = -b(t)/w_1(t)$ yield $M_* + A = -B/M_*$ and $M^* + A = -B/M^*$, respectively. Hence, $f(M_*) = -A = f(M^*)$, where $f(x) = x + B/x$. It is easy to see that f is convex on $(0,\infty)$ and $f(\lambda_1) = -A = f(\lambda_2)$. Since the values of M_*, M^* are strictly greater than λ_2, it must hold $M_* = M^* = \lambda_1$. Hence, $\lim_{t\to\infty} w_1(t) = \lambda_1$.

The existence of a positive solution $w_2(t)$ of (2), which tends to λ_2 as $t \to \infty$ will be shown by means of the Banach fixed point theorem. Let $T \in q^{\mathbb{N}_0}$ and $A_1, A_2, B_1, B_2 \in \mathbb{R}$ be such that $0 < A_1 \leq -a(t) \leq A_2$ and $0 < B_1 \leq b(t) \leq B_2$ for $t \in [T,\infty)_q$, $A_1^2 > 4B_2$, and $N_2 := (A_1 - \sqrt{A_1^2 - 4B_2})/2 < \lambda_1$. Denote $N_1 := (A_2 - \sqrt{A_2^2 - 4B_1})/2$. Without loss of generality, T can be the same as in the previous part of the proof. Observe that, with $x, y > 0$, $x \mapsto (x - \sqrt{x^2 - 4y})/2$ is decreasing while $y \mapsto (x - \sqrt{x^2 - 4y})/2$ is increasing. We have $N_2 < A_1/2 < A_1$, $N_1 = B_1/(A_2 - N_1)$, $N_2 = B_2/(A_1 - N_2)$, and $N_1 \leq \lambda_2 \leq N_2$. Denote $\Omega \in \{w \in \mathcal{X} : N_1 \leq w(t) \leq N_2 \text{ for } t \in [T,\infty)_q\}$. Let $\mathcal{T} : \Omega \to \mathcal{X}$ be the operator defined by $(\mathcal{T}w)(t) = b(t)/(-w(qt) - a(t))$. By means of the contraction mapping theorem we will prove that $\mathcal{T}$ has a fixed point in Ω. First we show that $\mathcal{T}\Omega \subseteq \Omega$. Let $w \in \Omega$. Then $(\mathcal{T}w)(t) \leq B_2/(A_1 - N_2) = N_2$ and $(\mathcal{T}w)(t) \geq B_1/(A_2 - N_1) = N_1$ for $t \in [T,\infty)_q$. Now we prove that $\mathcal{T}$ is a contraction mapping on Ω. Let $w, z \in \Omega$. Then

$$\begin{aligned}
|(\mathcal{T}w)(t) - (\mathcal{T}z)(t)| &= b(t)\left|\frac{1}{-w(qt) - a(t)} - \frac{1}{-z(qt) - a(t)}\right| \\
&\leq \frac{b(t)}{(-w(qt) - a(t))(-z(qt) - a(t))}\|w - z\| \\
&\leq \frac{B_2}{(A_1 - N_2)^2}\|w - z\|
\end{aligned}$$

for $t \in [T,\infty)_q$. Thus $\|\mathcal{T}w - \mathcal{T}z\| \leq \|w - z\| B_2/(A_1 - N_2)^2$. Now we need to show that $B_2/(A_1 - N_2)^2 < 1$. Since $N_2 = B_2/(A_1 - N_2)$, the required inequality is equivalent to $N_2 < A_1 - N_2$, which trivially follows from $N_2 < A_1/2$. The Banach fixed point now assures the existence of $w_2 \in \Omega$ such that $w_2 = \mathcal{T}w_2$, i.e., w_2 solves (2) with $N_1 \leq w_2(t) \leq N_2$ for $t \in [T,\infty)_q$. Denote $N_* = \liminf_{t\to\infty} w(t)$ and $N^* = \limsup_{t\to\infty} w(t)$. We have $N_*, N^* \in [N_1, N_2]$. The lim inf and lim sup as $t \to \infty$ in $w(qt) + a(t) = -b(t)/w(t)$ yield $N_* + A = -B/N_*$ and $N^* + A = -B/N^*$, respectively. Hence, $f(N_*) = -A = f(N^*)$ with $f(x) = x + B/x$. Since $N^* < \lambda_1$, in view of the properties of f described in the previous part, we get $N_* = N^* = \lambda_2$. Thus $\lim_{t\to\infty} w_2(t) = \lambda_2$.

Define y_i, $i=1,2$, by $y_i(t)=\prod_{s\in[T,t)_q} w_i(s)$. Then y_1,y_2 are solutions of (1), and $\lim_{t\to\infty} y_i(qt)/y_i(t)=\lambda_i$, $i=1,2$. Hence, $y_i\in\mathcal{RV}_q(\log_q\lambda_i)$, $i=1,2$.

It remains to show that any eventually positive solution y of (1) is in $\mathcal{RV}_q(\lambda_1)\cup\mathcal{RV}_q(\lambda_2)$. Recall that, excluding the trivial solution, (1) possesses only eventually positive and eventually negative solutions because it is nonoscillatory by Theorem 2. Since $\{y_1,y_2\}$ forms a fundamental system of (1), there exist $c_1,c_2\in\mathbb{R}$ such that $y=c_1y_1+c_2y_2$. If $c_1=0$, then $y=c_2y_2$ and so $c_2>0$ and $y\in\mathcal{RV}_q(\log_q\lambda_2)$. Now assume $c_1\neq 0$. From the representations of y_1,y_2, with $L_1,L_2\in\mathcal{SV}_q$, we have $y_2(t)/y_1(t)=t^{\log_q(\lambda_2/\lambda_1)}L_2(t)/L_1(t)\to 0$ as $t\to\infty$, since $\lambda_1>\lambda_2$. Further, $y_2(qt)/y_1(t)=(y_2(qt)/y_2(t))\cdot(y_2(t)/y_1(t))\to\lambda_2\cdot 0=0$ as $t\to\infty$. Hence,

$$\frac{y(qt)}{y(t)}=\frac{c_1y_1(qt)+c_2y_2(qt)}{c_1y_1(t)+c_2y_2(t)}=\frac{c_1y_1(qt)/y_1(t)+c_2y_2(qt)/y_1(t)}{c_1+c_2y_2(t)/y_1(t)}\sim\frac{y_1(qt)}{y_1(t)}\sim\lambda_1 \tag{18}$$

as $t\to\infty$, which implies $y\in\mathcal{RV}_q(\log_q\lambda_1)$. Since y was arbitrary, we get that every eventually positive solution of (1) is in $\mathcal{RV}_q(\log_q\lambda_1)$ or $\mathcal{RV}_q(\log_q\lambda_2)$.

(II) We now prove the case with $A^2=4B$. Nonoscillation of (1) is guaranteed by Theorem 2 (iii). Take any eventually positive solution y of (1). Then w defined by $w(t)=y(qt)/y(t)$ is a solution of (2), which is positive for large t. Similarly as in the first part of (i), with $\liminf_{t\to\infty}w(t)=K_*$ and $\limsup_{t\to\infty}w(t)=K^*$, we find that $K_*,K^*\in(0,\infty)$ and $f(K_*)=-A=f(K^*)$, f being the same as above. Recall that f is convex on $(0,\infty)$. Moreover, f has the only global minimum in $(0,\infty)$ at $x=-A/2$ since $B=A^2/4$, and $f(-A/2)=-A$. Hence, $K_*=K^*=-A/2(=\lambda_1=\lambda_2)$, and so $y\in\mathcal{RV}_q(\log_q(-A/2))$. The statement now follows from the fact that we worked with an arbitrary eventually positive solution of (1). □

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