Remarks on certain parallel linear behavior of mixed volumes and mixed discriminants

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Let $\mathcal{M}^n$ denote the set of real symmetric positive semidefinite $n \times n$ matrices. The mixed discriminant $D : (\mathcal{M}^n)^n \longrightarrow \mathbb{R}$ is the unique symmetric function for which

$$\det(\lambda_1 A_1 + \ldots \lambda_m A_m) = \sum_{i_1,\ldots,i_n=1}^{m} \lambda_{i_1} \ldots \lambda_{i_n} D(A_{i_1}, \ldots, A_{i_n}) \tag{1}$$

for $m \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_m \geqslant 0$, and $A_1, \ldots, A_m \in \mathcal{M}^n$.

Panov proved that the following two statements are equivalent for positive semidefinite symmetric matrices $A_1, \ldots, A_m \in \mathcal{M}^n$. For a matrix $A \in \mathcal{M}^n$, let $P(A)$ denote the linear subspace of $\mathbb{R}^n$ spanned by the eigenvectors of A having positive associated eigenvalues.

i) $D(A_1, \ldots, A_n) > 0$,

ii) If $K_i = \dim P(A_i) \cap B^n$, where B^n is the unit ball in $\mathbb{R}^n$, then

$$V(K_1, \ldots, K_n) > 0.$$

Here $V(K_1, \ldots, K_n)$ denotes the mixed volume of the convex bodies $K_1, \ldots, K_n$.

Florentin, Milman and Schneider provided a characterization of mixed discriminants among all functions $(\mathcal{M}^n)^n \longrightarrow \mathbb{R}$ by means of the non-negativity, additivity in each variable and the fact that the function is zero if any two arguments are proportional matrices of rank one.

In the setting of convex bodies, Milman and Schneider characterized the mixed volume of centrally symmetric convex bodies in $\mathbb{R}^n$ was as the only function, up to a multiplicative constant, of n centrally symmetric convex bodies which is Minkowski additive and increasing (with respect to set inclusion) in each variable and which vanishes if two of its arguments are parallel segments.

The latter shows a strong analogy of defining, structural and behavioral aspects of mixed discriminants and mixed volumes. There are some differences though, which seem not to allow to change the convex bodies framework to the symmetric positive semidefinite matrices one.

In this talk we will thoroughly introduce mixed discriminants, mixed volumes and the known connections among those. Then, we will consider some refinement of inequalities, such as the Aleksandrov-Fenchel inequality, which are known in the geometrical setting, aiming for an analogous version within the context of mixed discriminants, in a broad sense. We will address quantities associated to positive semidefinite matrices, resembling those naturally appearing in the geometric setting.