The Equivalence Problem for Finite Structures

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Beginning

Syracuse, early 90's

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- synchronizing chemical experiments

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The Computer Scientist showed that this problem traces back to equivalences of terms over commutative rings.

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Finite Automatons, Formal Languages,
Montreal, Brno, Ekaterinburg

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- Syntactic Monoids

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It was Shown that this problem traces back to equivalences of terms over monoids.

Some Definitions and Examples

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Let $\mathbf{A}$ be an algebra and let t and s be two terms over $\mathbf{A}$.

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- $[[x,y],[x,z]]^2 \overset{?}{\equiv} 1$ over S_4
- Yes, $[x,y] \in A_4 \implies [[x,y],[x,z]] \in A_4'$
  $A_4' \simeq Z_2 \times Z_2$

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- What is the complexity of $\mathrm{TERM\text{-}EQ}$?
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- What is the complexity of the problem for certain class of structures?
- Goal: Prove dichotomy: $\mathrm{TERM\text{-}EQ}$ is either in P or coNP-complete

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$TERM\text{-}EQ$ is in P for metacyclic groups (semidirect product of cyclic groups).

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SAT can be formulated

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- for almost every at most 6 element monoid the problem is in P

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Multiplication:

$$\langle i, \lambda\rangle\langle j, \mu\rangle = \begin{cases} \langle i, \mu\rangle, & \text{if } \mathbf{M}(\lambda, j) = 1 \\ 0, & \text{if } \mathbf{M}(\lambda, j) = 0 \end{cases}$$

and

$$0 \cdot s = 0 = s \cdot 0 \qquad \forall s \in S_{\mathbf{M}}$$

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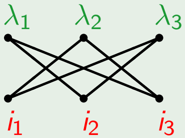

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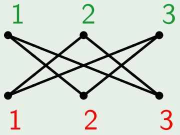

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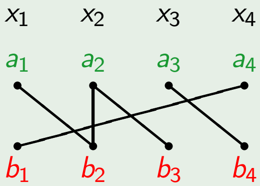

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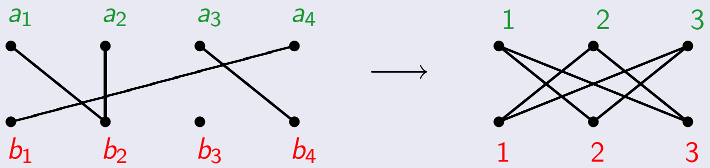

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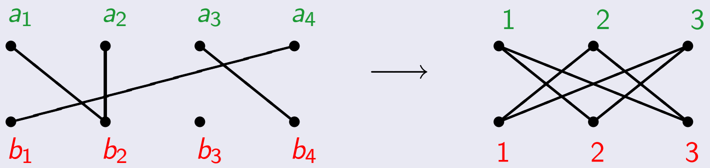

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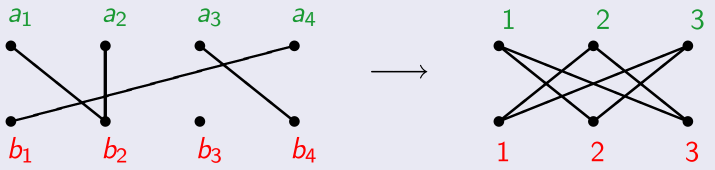

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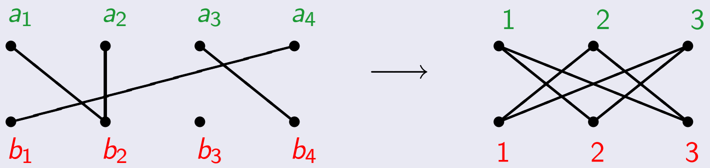

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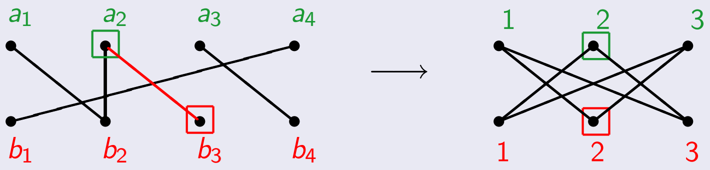

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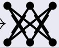
$G_t \to$  $\quad a_x \mapsto \lambda$, $b_x \mapsto i$ if $x = \langle i, \lambda \rangle$

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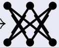
$G_t \to$  $\quad a_x \mapsto \lambda$, $b_x \mapsto i$ if $x = \langle i, \lambda \rangle$

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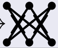
$G_t \to$  $\quad a_x \mapsto \lambda$, $b_x \mapsto i$ if $x = \langle i, \lambda \rangle$

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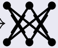
- $t(\vec{a}) \neq 0 \iff G_t \to$  is a homomorphism;

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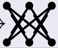
$G_t \to$  $\quad a_x \mapsto \lambda$, $b_x \mapsto i$ if $x = \langle i, \lambda \rangle$

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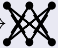
- $t(\vec{a}) \neq 0 \iff G_t \to$  is a homomorphism;
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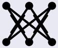
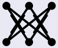
- $G_t \to$  is a homomorphism $\iff$ $G_s \to$  is a homomorphism;

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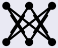
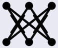
- $G_t \to$  is a homomorphism $\iff$ $G_s \to$  is a homomorphism;
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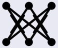
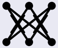
- $G_t \to$  is a homomorphism $\iff$ $G_s \to$  is a homomorphism;
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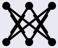
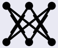
- $G_t \to$  is a homomorphism $\iff$ $G_s \to$  is a homomorphism;
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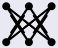
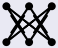
- $G_t \to$  is a homomorphism $\iff$ $G_s \to$  is a homomorphism;
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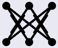
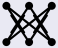
- $G_t \to$  is a homomorphism $\iff$ $G_s \to$  is a homomorphism;
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Multiplication:

$$\langle i, g, \lambda\rangle\langle j, h, \mu\rangle = \begin{cases} \langle i, g\mathbf{M}(\lambda, j)h, \mu\rangle, & \text{if } \mathbf{M}(\lambda, j) \in G \\ 0, & \text{if } \mathbf{M}(\lambda, j) = 0 \end{cases}$$

and

$$0 \cdot s = 0 = s \cdot 0 \qquad \forall s \in S_{\mathbf{M}}$$

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$Z_2 = \langle a \rangle$

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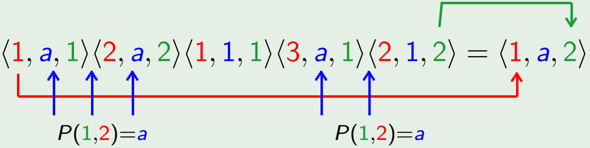

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Multiplying a row or column of the matrix by a group-element doesn't change the semigroup

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Translating to Graphs

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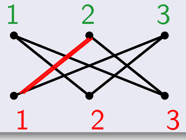
$$P = \begin{pmatrix} 0 & a & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$



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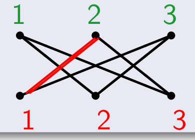

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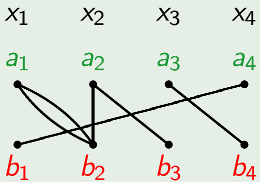

The Equivalaence Problem over S_P

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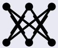
- 2HOM() $\overset{\text{poly}}{\Longleftrightarrow}$ TERM-EQ(S_P)

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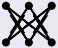
- 2HOM() $\overset{\text{poly}}{\iff}$ TERM-EQ(S_P)
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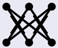
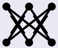
- 2HOM() $\overset{\text{poly}}{\Longleftrightarrow}$ TERM-EQ(S_P)
- 2HOM() coNP-complete.

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Different approaches of the Equivalence Problem over Rings

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Hilbert Theorem 90's

There exist an element of norm 1 in $\mathbb{F}_p^\alpha$ over $\mathbb{F}_p$.

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w is a word that proves coNP-completeness for $M_{n_1}(\mathbb{F}_1) \oplus \cdots \oplus M_{n_k}(\mathbb{F}_k)$ then
w^n proves the coNP-completeness for $\mathcal{R}$

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Theorem

For any CSP(B) $\exists$ a combinatorial 0-simple semigroup S such that:
$\mathit{POL\text{-}EQ}(S) \equiv \mathit{CSP}(B)$

- unfortunately $\mathrm{TERM\text{-}EQ}$ is in P for these structures (Seif, Szabó)
- 0-simple semigroups
  - S_P $\surd$ (Goldberg, VV)
  - Hope: For any CSP(B) $\exists$ a 0-simple semigroup S such that: $\mathrm{POL\text{-}EQ}(S) \equiv \mathrm{CSP}(B)$?
- NP-complete for bands $\surd$ (Klima)
- NP-complete for the Brandt monoid $\surd$ (Klima, Seif)
- Combinatorial semigroups ?

Vége = The End

What about your favorite structure?