Automatic algebras, finitely based or not

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A problem

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The Finite Algebra Membership Problem for $\mathbf{A}$

INSTANCE: A finite algebra $\mathbf{B}$ of the same signature as $\mathbf{A}$.

QUESTION: Does $\mathbf{B}$ belong to $\mathcal{V}$?

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In 1952, Jan Kalicki pointed out that there are always algorithms to solve these problems. In 2000, Bergman and Słutzki showed that it is always possible to find algorithms which have no worse than doubly exponential time complexity.

A problem

For some finite algebras **A** this problem can be solved in polynomial time—for example, when **A** is finitely based.

For other choices of **A** the problem may require doubly exponential time, as has been recently announced by Marcin Kozik.

A problem

A Problem to Try

Is there an algorithm that determines for a finite algebra $\mathbf{A}$ whether the finite algebra membership problem for $\mathbf{A}$ is solvable in polynomial time?

Algebras arise

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Algebras associated with graphs, hypergraphs, tournaments, Turing machines, and other combinatorial structures have all emerged in the last several decades, leading to concepts and results useful from the more general perspective.

Finite Automata

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A **finite automaton** is a system $\langle \Sigma, Q, \delta, q_0, F \rangle$ where

- Σ is a nonempty finite set, referred to as the **alphabet**,
- Q is a nonempty finite set, referred to as the set of **states**,
- the sets Σ and Q are disjoint,
- δ is a function from a subset of $\Sigma \times Q$ to Q and it is called the **transition function**,
- q_0, referred to as the **initial state**, belongs to Q, and
- $F \subseteq Q$ is referred to as the collection of **final states**.

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Automatic algebras

The function δ is a partial operation on $\Sigma \cup Q$. By adding a new element 0 as a default value we can extend δ to a total binary operation on $\Sigma \cup Q \cup \{0\}$.

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Given an automaton $M = \langle \Sigma, Q, \delta, q_0, F \rangle$ associated **automatic** algebra $\mathbf{A}(M) = \langle \Sigma \cup Q \cup \{0\}, \cdot \rangle$ is the algebra that satisfies the following stipulations: $0 \notin \Sigma \cup Q$ and

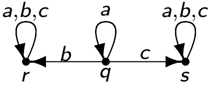
$$a \cdot r = \begin{cases} \delta(a, r) & \text{if } \delta \text{ is defined at } (a, r) \\ 0 & \text{otherwise.} \end{cases}$$

Displaying automata and their algebras

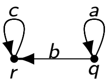
Finite automata (and their algebras) can be displayed via diagrams. These diagrams are certain directed graphs with labelled edges. The states of the automaton are the vertices of the graph. An edge directed from q to r with label a is in the diagram provided $r = a \cdot q = \delta(a, q)$.

Displaying automata and their algebras

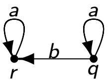
Here is a display three automata L, R, and M_0 associated with automatic algebras to be found in the literature. The diagram V would depict an autmaton except that one of the "states" would also be a "letter".



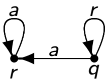
L



R



M_0



V

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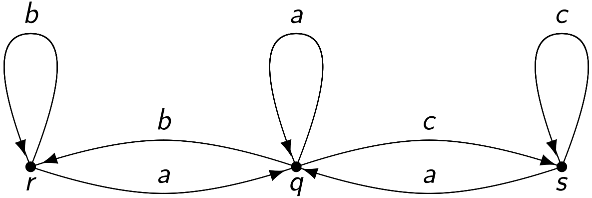
One reason to investigate automatic algebras is to develop methods for distinguishing which finite algebras are finitely based, which are inherently nonfinitely based, and which are nonfinitely based but fail to be inherently nonfinitely based. It may also be fruitful to develop an understanding of which automatic algebras are dualizable.

Automatic algebras are 2-step strongly solvable

This was observed by Keith Kearnes and Ross Willard in 1994. At the same time they proved that the automatic algebra associated with the automaton displayed below is inherently nonfinitely based.



Languages accepted by automata

An automaton M **accepts** a word w on the alphabet Σ provided there is a directed path from the initial state to some final state so that w is can be read off the edges as they are traversed in going from the initial state to the final state. Let $\mathcal{L}$ be a set of words on Σ. We say that M **accepts the language** $\mathcal{L}$ provided $\mathcal{L} = \{w \mid M \text{ accepts } w\}$.

Three finite basis theorems

Theorem

Let $\mathbf{A}$ *be an automatic algebra. If the language accepted by the automaton* M *is finite whenever* $\mathbf{A} = \mathbf{A}(M)$*, then* $\mathbf{A}$ *is finitely based.*

Three finite basis theorems

We say a letter b is a **bridge** letter for the language $\mathcal{L}$ provided

- b occurs no more than once in every word belonging to $\mathcal{L}$, and
- b occurs as the rightmost letter in arbitrarily long words in $\mathcal{L}$.

Theorem

Let M be an automaton that accepts an infinite language with a bridge letter. Let $\mathbf{P}$ be a nontrivial algebra in which $x \cdot y \approx x$ holds. The algebra $\mathbf{A}(M) \times \mathbf{P}$ is not finitely based.

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This theorem extends a 1996 result of V. L. Murskii. Note that $\mathbf{P}$ can be taken to be a two-element algebra.

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Which languages can replace a^*bc^* in the theorem above?

Bilinear algebras made from automata

Let M be a finite automata and $\mathfrak{K}$ be a field. Form a nonassociative bilinear algebra $\mathfrak{K}(M)$ by regarding the default element 0 as the zero vector and the set $\Sigma \cup Q$ as a basis for a vector space over $\mathfrak{K}$.

The elements of $\mathfrak{K}(M)$ then become the linear combinations of the basis vectors and the product on $\mathfrak{K}(M)$ is the natural extension of the automatic algebra product in $\mathbf{A}(M)$.

It is notable that Isaev's 1989 example of an inherently nonfinitely based finite algebra that generates a congruence modular variety is exactly $\mathfrak{K}(R)$.

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What is it about the automaton R that causes both $\mathbf{A}(R)$ and $\mathfrak{K}(R)$ to be inherently nonfinitely based?


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