Gumm terms imply cyclic terms for finite algebras

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Terms

Def. A ternary operation is *Malt'sev* if $x \approx q(x,y,y)$ and $q(x,x,y) \approx y$.

Def. The ternary operations $s_0, s_1, \dots, s_{2m}, q$ are *Gumm* terms if

$$\begin{aligned} x &\approx s_0(x,y,z) && \\ s_i(x,y,x) &\approx x && \text{for all } i \\ s_i(x,x,y) &\approx s_{i+1}(x,x,y) && \text{for } i \text{ odd} \\ s_i(x,y,y) &\approx s_{i+1}(x,y,y) && \text{for } i \text{ even} \\ s_{2m}(x,y,y) &\approx q(x,y,y) && \\ q(x,x,y) &\approx y. && \end{aligned}$$

Def. An operation t of arity $n \geq 2$ is *cyclic* if $t(x,x,\dots,x) \approx x$ and

$$t(x_1, x_2, \dots, x_n) \approx t(x_2, x_3, \dots, x_1).$$

Def. An operation t of arity $n \geq 2$ is *weak near-unanimity* if $t(x,x,\dots,x) \approx x$ and

$$t(y,x,\dots,x) \approx t(x,y,x,\dots,x) \approx \cdots \approx t(x,\dots,x,y).$$

Semantic Theorems

Thm (McKenzie, —). *A finite algebra* $\mathbf{A}$ *lies in a variety omitting types* $\mathbf{1}$ *and* $\mathbf{2}$ *(congruence meet semi-distributive) iff* $\mathbf{A}$ *has weak near-unanimity terms of arity* n *for almost all* n*.*

Thm (McKenzie, —). *A finite algebra* $\mathbf{A}$ *lies in a variety omitting type* $\mathbf{1}$ *iff* $\mathbf{A}$ *has a weak near-unanimity term of some arity.*

Thm (Barto, Kozik, Niven). *If a finite algebra* $\mathbf{A}$ *has Jónsson terms (lies in a congruence distributive variety), then* $\mathbf{A}$ *has cyclic terms of arity* p *for all primes* $p > |A|$*.*

Thm. *If a finite algebra* $\mathbf{A}$ *has Gumm terms (lies in a congruence modular variety), then* $\mathbf{A}$ *has cyclic terms of arity* p *for all primes* $p > |A|$*.*

Example. *The infinite cylic group* $(\mathbb{Z}; +)$ *has no cylic term.*

Example. *Any finite algebra* $(A; t)$ *with the ternary discriminator operation has no cylic term of length less than or equal to* $|A|$*.*

Example. *Any semilattice has cylic terms* $t = x_1 \wedge x_2 \wedge \cdots \wedge x_n$ *for all arities.*

Bounded Width and Cyclic Terms

$$\mathbf{P} \leq \mathbf{A} \times \mathbf{A} \times \mathbf{A},$$
$$\mathbf{Q} \leq \quad \mathbf{A} \times \mathbf{A} \times \mathbf{A},$$
$$\mathbf{R} \leq \mathbf{A} \times \quad \mathbf{A} \times \mathbf{A},$$
$$\mathbf{S} \leq \mathbf{A} \times \mathbf{A} \times \quad \mathbf{A},$$

consistent relations

$$P|_{\{2,3\}} = Q|_{\{2,3\}}$$
$$Q|_{\{3,4\}} = R|_{\{3,4\}}$$
$$R|_{\{4,1\}} = S|_{\{4,1\}}$$
$$S|_{\{1,2\}} = P|_{\{1,2\}}$$

Is there $abcd \in A^4$ such that $abc \in P$, $bcd \in Q$ $acd \in R$ and $abd \in S$?

$$\begin{array}{ccccc}
a_1 & b_1 & c_1 & - & \in P \\
- & b_1 & c_1 & d_1 & \in Q \\
a_2 & - & c_1 & d_1 & \in R \\
a_2 & b_2 & - & d_1 & \in S \\
a_2 & b_2 & c_2 & - & \in P \\
 & & & & \vdots \\
a_n & b_n & c_n & - & \in P \\
- & b_n & c_n & d_n & \in Q \\
a_1 & - & c_n & d_n & \in R \\
a_1 & b_1 & - & d_n & \in S \\
a_1 & b_1 & c_1 & - & \in P
\end{array}$$

$$\begin{array}{cccc}
a_1 & b_1 & c_1 & \in P \\
a_2 & b_2 & c_2 & \in P \\
 & \vdots & & \\
a_n & b_n & c_n & \in P \\
\hline
a & b & c & \in P
\end{array}$$

$$\begin{array}{cccc}
a_2 & b_2 & d_1 & \in S \\
a_3 & b_3 & d_2 & \in S \\
 & \vdots & & \\
a_1 & b_1 & d_n & \in S \\
\hline
a & b & d & \in S
\end{array}$$

where $a = t(a_1, \ldots, a_n)$, $b = t(b_1, \ldots, b_n)$, etc.

The Nicely Connected Graph

- Let $\mathbf{A}$ be a finite algebra of minimal size with Gumm terms and without cyclic term of arity p for some prime $p > |A|$.
- We can assume that $\mathbf{A}$ is idempotent (Gumm terms are the basic operations)
- There exists $\bar{a} \in A^p$ such that the subpower $\mathbf{B} \leq \mathbf{A}^p$ generated by the tuples

$$\begin{aligned} \bar{a} &= \langle a_1, a_2, a_3, \ldots, a_p \rangle \\ \sigma(\bar{a}) &= \langle a_2, a_3, a_4, \ldots, a_1 \rangle \\ &\vdots \\ \sigma^{p-1}(\bar{a}) &= \langle a_p, a_1, a_2, \ldots, a_{p-1} \rangle \end{aligned}$$

  contains no constant tuple $\langle c, c, \ldots, c \rangle$
- $\mathbf{B}$ is subdirect and closed under σ
- $\mathbf{A}$ is simple

- If $\mathbf{A}$ is abelian, then
  - $\mathbf{A}$ has a Malt'sev term (by Commutator Theory)
  - $\mathbf{B} \cong \mathbf{A}^k$ for some $k \leq p$ (by Fleischer's Lemma)
  - p does not divide $|B|$
  - σ on B has a one-element orbit, a contradiction
- Thus $\mathbf{A}$ is non-abelian and non-Malt'sev
- $\operatorname{Con} \mathbf{B}$ is distributive (by Commutator Theory)
- Define the graph $G = (V, E)$ where

$$V = \{ \langle b_1, \ldots, b_{p-1} \rangle : \bar{b} \in B \}$$
$$E = \{ (\langle b_1, \ldots, b_{p-1} \rangle, \langle b_2, \ldots, b_p \rangle) : \bar{b} \in B \}$$

- Every vertex is on a cycle of length p (via the σ automorphism)
- G is strongly connected and contains a cycle of length $kp + 1$ for some integer k (from the distributivity of $\operatorname{Con} \mathbf{B}$ using projection kernels)
- The greatest common divisor of the length of cycles in G is 1
- G contains no loop

The Loop Lemma

Lemma. *Let* $\mathbf{E} \leq \mathbf{V}^2$ *be a nontrivial strongly connected directed graph where*

(1) *the greatest common divisor of the length of loops is* 1, *and*

(2) $\mathbf{V}$ *has the property ...*

Then $\langle c, c\rangle \in E$ *for some* $c \in V$.

- Put $N_0 = \{v\}$ for some fixed $v \in V$, and

$$N_{i+1} = \{\, v \in V : u \in N_i \text{ and } \langle u, v\rangle \in E \,\}$$

- $\{v\} = N_0 \to N_1 \to \cdots \to N_i$ (exactly i-step reachable vertices)
- $N_k \subset N_{k+1} = V$ for some k.
- $N_k < V$ is a proper subuniverse (from idempotency)
- N_k is a Jónsson ideal, i.e. $p_i(u, x, v) \in N_k$ for all $u, v \in N_k$, $x \in V$, and i
- We define $\mathbf{C} \leq N_k$, and a congruence ϑ so that $\mathbf{C}/\vartheta \models q(x, y, y) \approx x$ (Malt'sev)
- If $\vartheta \neq 1$, then $\mathbf{A}$ is a homomorphic image of the Malt'sev algebra $\mathbf{C}/\vartheta$
- If $\vartheta = 1$, then G can be replaced with a smaller one
- By induction we always get a contradiction