Free spectra of semigroups (Part II)

jointwork with Csaba Szabó

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Structure

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Theorem (J. Berman (1995))

For each $k \geq 2$ there exist positive constants $d_1, \ldots, d_5$ and c_1, c_2, c_4, such that if $\mathbf{A}$ k-element simple algebra and $\mathcal{V} = \mathrm{Var}(\mathbf{A})$, then for every sufficiently large n,

1. *if* $\mathrm{typ}(\mathbf{A}) = \mathbf{1}$, *then* $d_1 n \leq |\mathbf{F}_{\mathcal{V}}(n)| \leq c_1 n^{\log_2 k}$;
2. *if* $\mathrm{typ}(\mathbf{A}) = \mathbf{2}$, *then* $d_2 k^n \leq |\mathbf{F}_{\mathcal{V}}(n)| \leq c_2 k^{(k-1)n}$;
3. *if* $\mathrm{typ}(\mathbf{A}) = \mathbf{3}$, *then* $k^{d_3 k^n} \leq |\mathbf{F}_{\mathcal{V}}(n)| \leq k^{k^n}$;
4. *if* $\mathrm{typ}(\mathbf{A}) = \mathbf{4}$, *then* $k^{d_4 k^n/\sqrt{n}} \leq |\mathbf{F}_{\mathcal{V}}(n)| \leq k^{c_4 k^n/\sqrt{n}}$;
5. *if* $\mathrm{typ}(\mathbf{A}) = \mathbf{5}$, *then* $d_5 k^n \leq |\mathbf{F}_{\mathcal{V}}(n)| \leq k^{\sigma(n)}$, *where*

$$\sigma(n) = \frac{nk}{n - k(k-1)^3} \binom{n}{(k-1)^3} (k-1)^{n-(k-1)^3}.$$

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A semigroup is called combinatorial if it does not contain a nontrivial group as a subgroup.

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A finite Rees matrix semigroup $\mathcal{M}^0(\mathbf{G}; I, \Lambda; M)$ is combinatorial iff the group $\mathbf{G}$ is trivial ($\mathbf{G} = \{1\}$).

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Definition

A variety is called exact if it is generated by a finite 0-simple semigroup.

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$\mathcal{SL}$	**Y**	$[1]$	
$\mathcal{LNB}$	**L**	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	
$\mathcal{RNB}$	**R**	$\begin{bmatrix} 1 & 1 \end{bmatrix}$	
$\mathcal{NB}$	**N**	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	
$\mathcal{B}$	$\mathbf{B_2}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	
$\mathcal{LNB}_2$	$\mathbf{L_2}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$	
$\mathcal{RNB}_2$	$\mathbf{R_2}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	
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$$|\mathbf{F}_{\mathcal{V}}(n)| = |\mathrm{Clo}_n\mathbf{A}| = \sum_{k=0}^{n} \binom{n}{k} p_k(\mathbf{A})$$

Proposition (S. Seif, Cs. Szabó (2003))

Let $\mathbf{S} \in \{\mathbf{Y}, \mathbf{L}, \mathbf{R}, \mathbf{N}\}$*. The sandwich matrix* M *of* $\mathbf{S}$ *is an* $r \times l$ *all 1 matrix. Let* $p = p(x_1, \ldots, x_n) = x_{i_1} \ldots x_{i_k}$ *and* $q = q(x_1, \ldots, x_n) = x_{j_1} \ldots x_{j_m}$ *be two terms. Then* $p \equiv q$ *over* $\mathbf{S}$ *if and only if the following conditions are satisfied:*

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$$[i, \lambda][j, \gamma] = \begin{cases} [i, \gamma] & \text{if } \lambda = j; \\ 0 & \text{if } \lambda \neq j. \end{cases}$$

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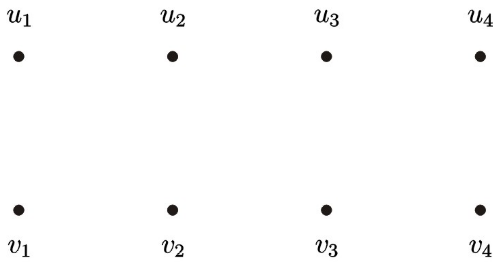

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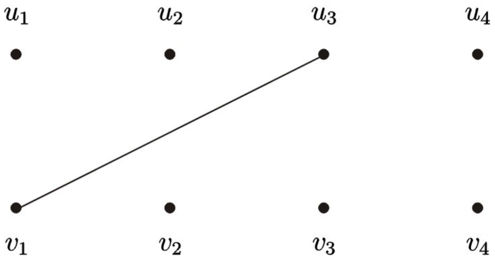

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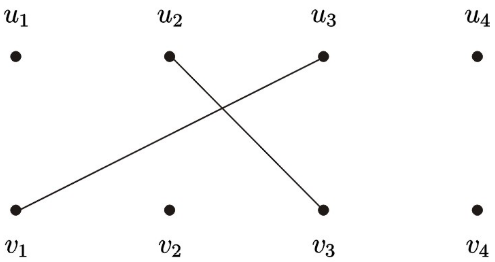

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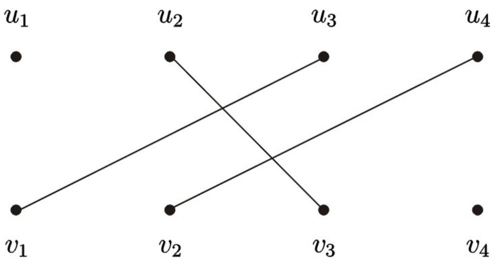

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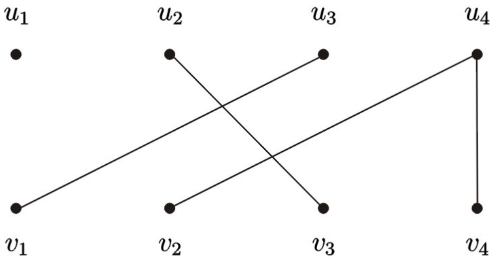

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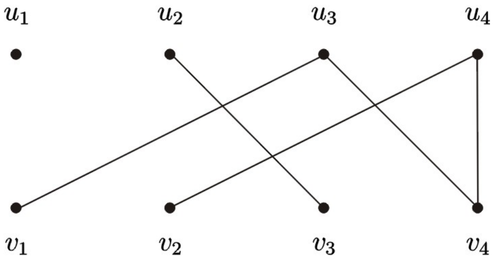

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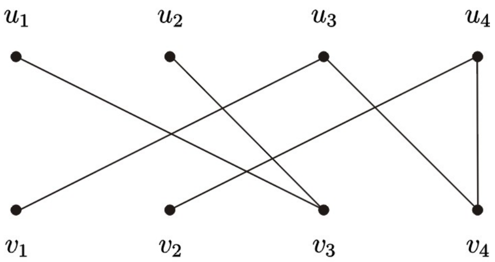

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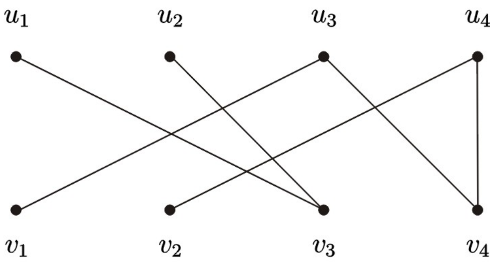

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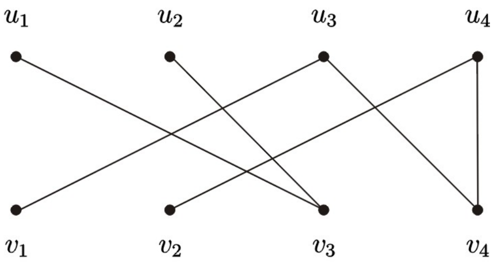

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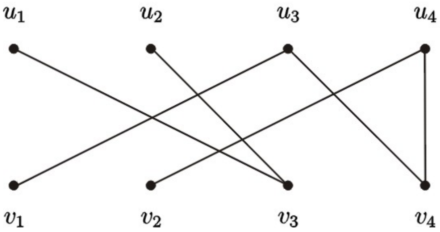

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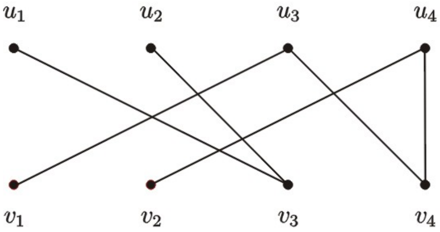

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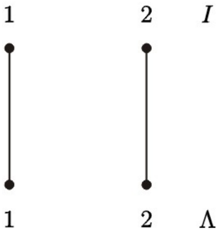

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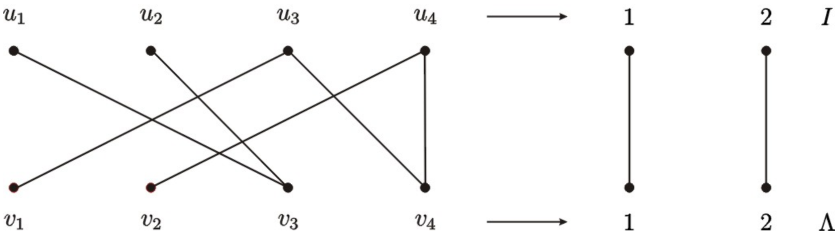

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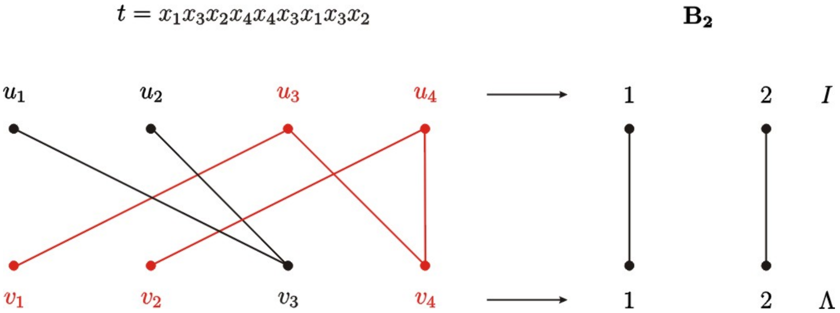

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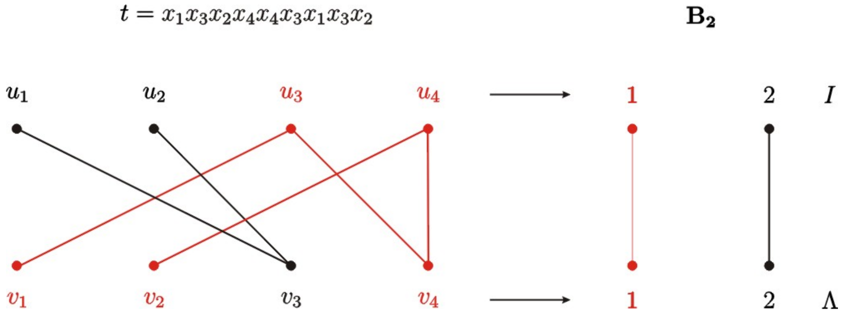

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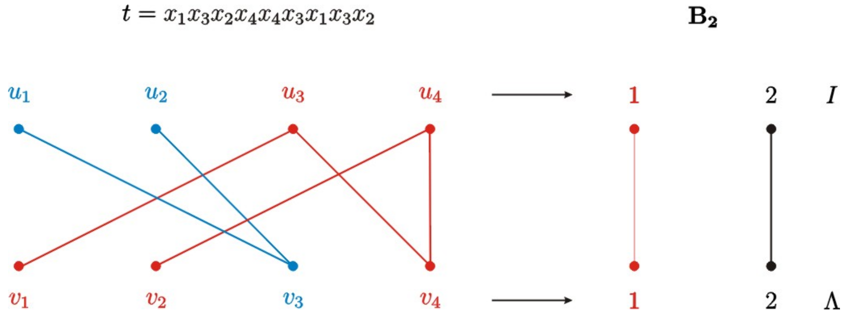

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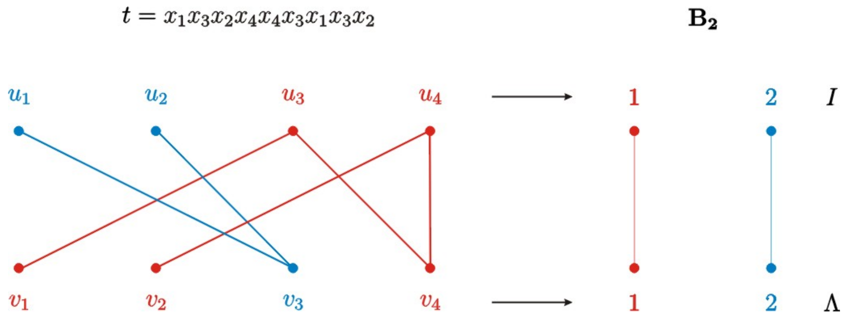

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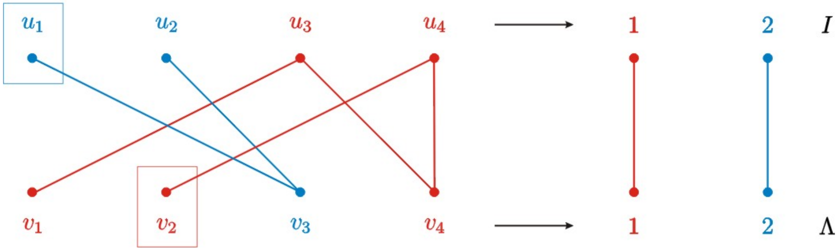

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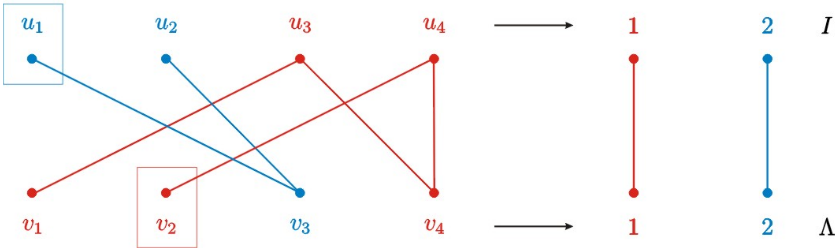

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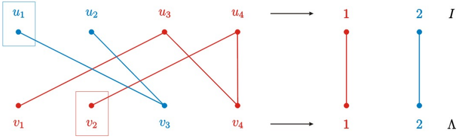

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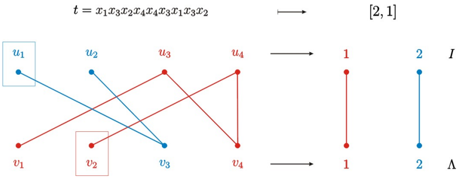

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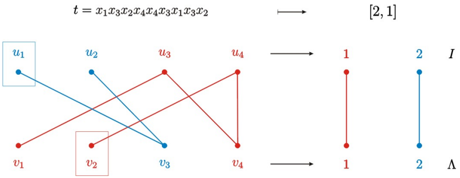

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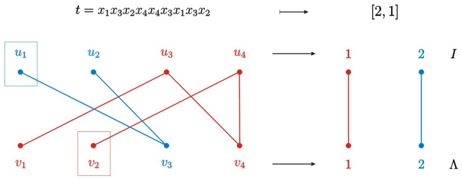

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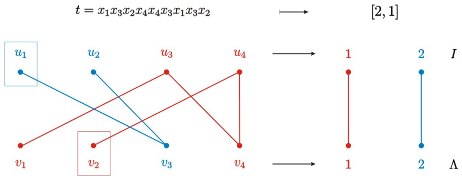

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$\mathcal{RNB}$	$\mathbf{R}$	$\begin{bmatrix} 1 & 1 \end{bmatrix}$	$n2^{n-1}$
$\mathcal{NB}$	$\mathbf{N}$	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$n(n+1)2^{n-2}$
$\mathcal{B}$	$\mathbf{B_2}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\sim_{\log} n^{2n}$
$\mathcal{LNB}_2$	$\mathbf{L_2}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$	$\sim_{\log} n^{2n}$
$\mathcal{RNB}_2$	$\mathbf{R_2}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	$\sim_{\log} n^{2n}$
$\mathcal{NB}_2$	$\mathbf{N_2}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$	$\sim_{\log} n^{2n}$
$\mathcal{A}$	$\mathbf{A_2}$	$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$	$\sim n^2 2^{n^2}$

Variety	generating semigroup	matrix (M)	free spectra of variety
$\mathcal{SL}$	$\mathbf{Y}$	$[1]$	$2^n - 1$
$\mathcal{LNB}$	$\mathbf{L}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$n2^{n-1}$
$\mathcal{RNB}$	$\mathbf{R}$	$\begin{bmatrix} 1 & 1 \end{bmatrix}$	$n2^{n-1}$
$\mathcal{NB}$	$\mathbf{N}$	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$n(n+1)2^{n-2}$
$\mathcal{B}$	$\mathbf{B_2}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\sim_{\log} n^{2n}$
$\mathcal{LNB}_2$	$\mathbf{L_2}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$	$\sim_{\log} n^{2n}$
$\mathcal{RNB}_2$	$\mathbf{R_2}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	$\sim_{\log} n^{2n}$
$\mathcal{NB}_2$	$\mathbf{N_2}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$	$\sim_{\log} n^{2n}$
$\mathcal{A}$	$\mathbf{A_2}$	$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$	$\sim n^2 2^{n^2}$