Conference on Algorithmic Complexity and Universal Algebra

University of Shahid Beheshti

University of Szeged
Hungary, July 16–20, 2007

RELATIONS AMONG IMPLICATION ALGEBRAS AND SOME ORDERED ALGEBRAS

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Abstract

As we know several algebras with one binary and one nullary operations introduced as an algebraic-logic structure. One of them is implication algebra which is applied in other branches of sciences like computational intelligence, lattice theory, Meta theory, fuzzy theory, etc. In this paper, first we introduce some results of implication algebras and after that we verify the relation among the implication algebras with BCK-algebras, gBCK- algebras, Hilbert algebras and lattice implication algebras.

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1. Introduction

As we know several algebra with one binary and nullary operation introduced as an algebraic-logic which obtained from classical or non-classical logic. one of them is implication algebra introduced by J. C. Abott in 1967 which he discuss algebraic systems with a single binary operation modeled on the Boolean operation a* b = a` U b where a` is the Boolean complement of a. Such algebra is satisfy three basic equalities motivated from laws of implication in the absolute propositional calculus and are called implication algebras.

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As we know Resconi et al. (1990) built the Meta theory of uncertainly based upon modal logic, with the instrument of the modal logic, the fuzzy set theory and the fuzzy measure can be represented by a common model. The semantic of the fuzzy sets is given by the conceptual definition of possible worlds in modal logic. an isomorphism was established between the structure of the probability calculus in the physical domain and the fuzzy set theory in the conceptual domain of the worlds, and obtain the semantic structure to describe the evidence theory. Korlelainen (1999) in the topological approach to L-fuzzy set theory introduced the fuzzy modifier operator connected with accessibility relation.

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Lukasiewicz and other created in a pure conceptual domain, the many valued logic that Resconi et al. can describe in a symbolic way, by lattice structures. Zadeh with the definition of the continuous membership function created fuzzy sets for vagueness the operations of which are different from the usual set theory as intersection, union and complement.

L-fuzzy sets are built when the membership function assumes symbolic values L. Meta theory of uncertainty based upon modal logic is built to unify measures of vagueness uncertainty and inconsistence in natural language. So Meta-theory of uncertainty uses modal logic as logic instrument.

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Resconi(2004) suggest in his paper to rewrite the main part of the implication algebra by modal logic in such a way as to write a common language between the traditional fuzzy logic and the implication algebra.
Also Xu et al. (2003) with the lattice implication algebras as a meta language formalized the L-fuzzy set theory.
Xu has established the concept of lattice implication algebra by combinaning lattice and implication operator, which is a new logical-algebraic system and discuss its algebraic properties, also have studied the properties of lattice implication algebras.

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Further more has discussed the relation among the lattice implication algebra with MV-algebra, BL-algebra, R0-algebra and multiple valued logic.

Implication algebras have a strong logical- algebraic structure and as we know nonclassical logic has become a considerable formal tool for computer sciences and artificial intelligence information and uncertain information many-valued logic a great extension and development of classical logic has always been a crucial direction in non-classical logic, so the study of implication algebras and properties of it is very interested for many researchers especially mathematician which are working on applied sciences.

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On the other hand the concept of BCK- algebras which introduced by K. Iseki (1966) are based by use the sets theory and propositional calculus. Now this accept is applied in fuzzy logic, semantic language, etc. In this paper we will elaborate implication algebras with BCK- algebras, gBCK- algebras and Hilbert algebras chronologically and we will study their properties in brief. More over we will study relation between bounded implication algebras with lattice distributive implication algebra.

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Definition 2.1.

An implication algebra is a set X with a binary operation "*" which satisfies the following axioms:

(I1) $(x * y) * x = x$,

(I2) $(x * y) * y = (y * x) * x$,

(I3) $x * (y * z) = y * (x * z)$.

for all $x, y, z \in X$. [Abbott, 1967]

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Definition 2.2.

A BCK-algebra is a set X with a binary operation "*" and constant "1" which satisfies the following axioms:

(BCK1) $(y * z) * ((z * x)) * (y * x)) = 1$,

(BCK2) $y * ((y * x) * x) = 1$,

(BCK3) $x * x = 1$,

(BCK4) $x * y = y * x = 1$ imply, $x = y$,

(BCK5) $x * 1 = 1$.

for all $x, y, z \in X$. [Imai and Iseki, 1966]

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Lemma 2.3.

In any BCK-algebra (X, *, 1), the following identities hold:

(i) $1 * x = x$,

(ii) $z * (y * x) = y * (z * x)$.

for all $x, y, z \in X$.

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Definition 2.4.

A gBCK-algebra is a set G with a binary operation "*" and constant "1" which satisfies the following axioms:

(G1) $1 * x = x$,

(G2) $x * x = 1$,

(G3) $z * (y * x) = y * (z * x)$,

(G4) $z * (y * x) = (z * y) * (z * x)$,

for all $x, y, z \in G$. [Hong, Jun, Ozturk, 2003]

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Note 1. Any BCK -algebra (X,*, 1) is not necessary a gBCK-algebra.

Example 2.5.

Let X = {a, b, 1} and operation “*” on X is defined as follows:

$*$	a	b	1
a	1	a	1
b	1	1	1
1	a	b	1

Then (X,*, 1) is a BCK- algebra but it is not a gBCK-algebra, since a * (a * b) ≠ (a * a) * (a * b).

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Note 2. Any gBCK -algebra (G,*, 1) is not necessary a BCK-algebra.

Example 2.6.

Let X = {a, b, 1} and operation "*" on X is defined as follows:

$*$	a	b	1
a	1	1	1
b	1	1	1
1	a	b	1

It is routine to check that (X,*, 1) is a gBCK- algebra but it is not a BCK-algebra, since a * b = b * a = 1 but a ≠ b.

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Lemma 2.7.

In any gBCK-algebra (G,*, 1), the following identities hold:

(i) $(y * z) * ((z * x) * (y * x)) = 1$,

(ii) $(z * x) * ((y * z) * (y * z)) = 1$,

(iii) $x * (x * y) = x * y$,

(iv) $x * (y * x) = 1$,

(v) $(x * (y * z)) * ((x * y) * (x * z)) = 1$,

(vi) $x * 1 = 1$,

(x) $y * ((y * x) * x) = 1$.

for all $x, y, z \in G$.

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Definition 2.8.

A gBCK- algebra (G,*, 1) is said to be commutative if

$$(x * y) * y = (y * x) * x,$$

for all $x, y \in G$.

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Lemma 2.9.

In any commutative gBCK-algebra $(G,*,1)$, the following identities is hold:

(i) $(x * y) * x = x$,

(ii) $x * x = y * y$,

(iii) If $x * y = y * x = 1$ then, $x = y$.

for all $x, y, z \in G$.

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Definition 2.10.

A Hilbert algebra is an algebra (H, *, 1) of type (2,0), where H is a nonempty set, “*” is a binary operation which satisfies the following axioms:

(H1) $x * (y * x) = 1$,
(H2) $(x * (y * z)) * ((x * y) * (x * z)) = 1$,
(H3) $x * y = 1$ and $y * x = 1$ imply, $x = y$.

for all $x, y, z \in H$. [Henkin, Skolern, 1950-1959]

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Lemma 2.11.

If $(H, *, 1)$ is a Hilbert algebra, then:

(i) $x * x = 1$,

(ii) $1 * x = x$,

(iii) $x * 1 = 1$,

(iv) $x * (y * z) = y * (x * z)$,

(v) $x * (y * z) = (x * y) * (x * z)$.

for all $x, y, z \in H$.

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Definition 2.12.

By a lattice implicative algebra we mean a bounded lattice $(L, \vee, \wedge, 0, 1)$ with order-reversing involution " ` " and a binary operation " * " which satisfies the following axioms:

(LI1) $x * (y * z) = y * (x * z)$,

(LI2) $x * x = 1$,

(LI3) $x * y = y` * x`$,

(LI4) $x * y = 1 = y * x \Rightarrow x = y$,

(LI5) $(x * y) * y = (y * x) * x$,

(L1) $(x \vee y)*z = (x * z) \wedge (y * z)$,

(L2) $(x \wedge y)*z = (x * z) \vee (y * z)$.

for all $x, y, z \in L$. [Chen,Oliveira, 1995]

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Question

Is there any relations among these various algebras?

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Lemma 3.1.

Let $(X,*,1)$ be an implication algebra and $x, y, z \in X$. Then,

(i) $x \leq y$ imply $y * z \leq x * z$,

(ii) $x \leq y$ imply $z * x \leq z * y$,

(iii) $x * y \leq (y * z) * (x * z)$ and $y * z \leq (x * y) * (x * z)$,

(iv) $x \leq y$ and $y \leq z$ imply $x \leq z$.

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Lemma 3.2.

In any implication algebra (X,*), the following identities hold:

(i) $x * (x * y) = x * y$,

(ii) $x * x = y * y$,

(iii) There exists a unique element 1 in X, for all $x \in X$,

(iv) $x * x = 1$, $1 * x = x$ and $x * 1 = 1$,

(v) If $x * y = 1$ and $y * x = 1$ then, $x = y$.

for all $x, y \in X$.

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Definition 3.3.

Let (X,*, 1) be a BCK-algebra. Then X is called a implicative BCK-algebra, if

$$x = (x * y) * x,$$

for all $x, y \in X$.

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Lemma 3.4.

Let (X,*, 1) be an implication algebra . Then

$$x * (y * x) = 1,$$

for all x, y $\in$ X.

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Theorem 3.5.

Any implication algebra is an implicative BCK-algebra.

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Theorem 3.6.

Any implicative BCK-algebra is an implication algebra .

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Corollary 3.7.

Implication algebras are equivalent to the implicative BCK-algebras.

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Theorem 3.8.

Implication algebras are equivalent to the commutative gBCK algebras.

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Theorem 3.9.

(i) Any implication algebra is a Hilbert algebra.

(ii) Any commutative Hilbert algebra is an implication algebra.

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Example 3.10.

Let X = {a, b, c, 1} and operation "*" on X is defined as follows:

$*$	a	b	c	1
a	1	b	c	1
b	1	b	c	1
c	1	1	c	1
1	1	1	1	1

Then, (X,* , 1) is a Hilbert algebra which is not commutative . Moreover, X is not an implication algebra, since, $(a * b) * a \neq a$. Hence, the commutative condition is necessary in the Theorem 3.9(ii).

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Corollary 3.11.

The following statements are equivalent:

(X,*, 1) is an implication algebra,

(X,*, 1) is a commutative gBCK−algebra,

(X,*, 1) is an implicative BCK−algebra,

(X,*, 1) is a commutative Hilbert algebra.

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Corollary 3.11.

$$\begin{tikzcd}
\text{Implication algebra} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \text{Implicative BCK-algebra} \arrow[d, leftrightarrow] \\
\text{Commutative gBCK-algebra} \arrow[r, leftrightarrow] & \text{Commutative Hilbert algebra}
\end{tikzcd}$$

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Definition 3.12.

Let (X,*, 1) be an implication algebra and

$$B(a, b) = \{x : b \leq a * x\}$$

for a, b $\in$ X. If for all a, b $\in$ X, B (a, b) has a least element, written $a \otimes b$, then implication algebra X is called to be with condition (P).

It is clear that, 1, a, b $\in$ B (a, b).

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Theorem 3.13.

Let $(X,*,1)$ be a bounded implication algebra with unit m. Then,

(BI1): $x \vee y = (x * y) * y$,

(BI2): $x \wedge y = ((x * m) \vee (y * m)) * m$,

for all $x, y \in X$.

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Lemma 3.14.

Any bounded implication algebra $(X,*, 1)$ is with condition (P) and for all $a, b \in X$, we have,

$$a \otimes b = a \wedge b.$$

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Theorem 3.15.

Any bounded implication algebra is a distributive lattice implicative algebra.

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Note 3. The converse of theorem 3.15 is not correct in general.

Example 3.16.

Let X = {0,a, b, c, 1} and operation "*" on X is defined as follows:

$*$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	a	b	c	1	1
1	0	a	b	c	1

x	x'
0	1
a	c
b	b
c	a
1	0

Then, (X, ∨, ∧, 0, 1) is distributive lattice implicative algebra but it is not a bounded implication algebra, since (b * a) * b = c * b = c ≠ b.

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