Hints to the Exercises of Lecture II

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1. Suppose that $f(x_1, x_2, \ldots, x_n)$ $(n \geq 2)$ depends on its first variable, and let $a_2, \ldots, a_n$ be constants such that $f(x, a_2, \ldots, a_n)$ depends on x. Putting y in the second argument, and then moving it to the right one position at a time show that either $f(x, b_2, \ldots, b_{i-1}, y, b_{i+1}, \ldots, b_n)$ depends on both variables for some i $(2 \leq i \leq n)$ and some constants $b_2, \ldots, b_n$, or f fails to depend on its variables $x_2, \ldots, x_n$.

2. Let C be a nonunary polynomial clone on $\{0, 1\}$. Show:

(1) C contains $+$ or $\vee$ or $\wedge$. (Use the claim in Exercise 1 and examine all binary operations on $\{0, 1\}$.)

(2) If C properly contains $[0, 1, +]$, then C contains multiplication modulo 2 ($= \wedge$), and hence $C = [0, 1, \vee, \wedge, ']$. (Use the fact that every operation on $\{0, 1\}$ is a polynomial function over the two-element field.)

From now on we can assume that $\wedge \in C$.

(3) If C contains an operation that is not monotone, then C contains $'$, and hence $C = [0, 1, \vee, \wedge, ']$.

(4) If all operations in C are monotone, then $C \subseteq [0, 1, \vee, \wedge]$. (Find appropriate disjunctive normal form.)

(5) Finally, prove that if all operations in C are monotone and C properly contains $[0, 1, \wedge]$, then $C = [0, 1, \vee, \wedge]$. (Use disjunctive normal form.)

3. (a) $x \to y = x' \vee y = t(y, 1, x)$; $t(x, y, z) = (y \to z) \to x$, $1 = x \to x$; $x \vee y = t(x, y, x)$.

(b) Notice that $\mathcal{V}$ is congruence distributive, and prove that $\mathbf{A}$ can be embedded into the $\{t, 1\}$-reduct of a finite Boolean algebra $\mathbf{B}$ in such a way that A is a filter in $\mathbf{B}$ and A contains all coatoms of $\mathbf{B}$.

(c) Prove and use the following fact: if a unary polynomial operation p of $\mathbf{A}$ is not a permutation, then for some atom b of $\mathbf{B}$ from part (b) $p(x) \geq b$ for all $x \in A$.

4. (a) The polynomial operation $e(x) = m_3(a, x, b)$ of $\mathbf{A}$ is idempotent.

(b,c) Several identities for m_3 are required to check the claim. Since $\mathcal{V}$ is generated by $\mathbf{S}$, it suffices to verify that the identites used for m_3 hold in $\mathbf{S}$.

An irreducible cover consists of neighborhoods of the form $I(a, b)$ such that the corresponding bounded lattice has a join irreducible largest element and a meet irreducible least element.

5. Let $\mathbf{A}$ be an affine algebra. By definition, $\mathbf{A}$ is polynomially equivalent to a (faithful, unitary) module ${}_RA$. Prove or take my word for it that this implies: $x - y + z$ is a term operation of $\mathbf{A}$. Using this fact show that every module polynomial $\sum_{i=1}^{k} r_i x_i$ with $\sum_{i=1}^{k} r_i = 1$ is a term operation of $\mathbf{A}$. (Consider the case $k = 2$ first.) Conclude that if $\mathbf{A}$ is idempotent, then the term operations of $\mathbf{A}$ are exactly the module polynomials $\sum_{i=1}^{k} r_i x_i$ with $\sum_{i=1}^{k} r_i = 1$. If $\mathbf{A}$ is a finite, simple, affine algebra, then so is ${}_RA$, hence A is an n-dimensional vector space over a finite field K for some n, and R is the $n \times n$ matrix ring over K with the usual action on the vector space A.

6. Show first that if an affine algebra $\mathbf{A}$ has only one proper subalgebra, namely $\{0\}$, and the constant operation with value 0 is a term operation of $\mathbf{A}$, then $\mathbf{A}$ is term equivalent to a module ${}_RA$. If, moreover, $\mathbf{A}$ is a G^0-algebra, then conclude that R is a field, and ${}_RA$ is a one-dimensional vector space.

7. Every unary term operation of $\mathbf{A}$ is of the form $f(x) = rx + a$ for some $r \in R$ and $a \in A$. Clearly, $f(0) = a$. So, if $\mathbf{A}$ is a G-algebra, then for every $a \in A$ there exists a unique $r_a \in R$ such that $r_a x + a \in \mathrm{Clo}_1\mathbf{A}$. Let $\phi\colon A \to R$, $a \mapsto 1 - r_a$. Then

$$\mathrm{Clo}_1\mathbf{A} = \big\{\big(1 - \phi(a)\big)x + a : a \in A\big\}.$$

By the hints to Exercise 5, $x - y + z$ and $rx + (1-r)y$ ($r \in R$) are term operations of $\mathbf{A}$. Substitute unary term operations into these operations to show that ϕ is a module homomorphism ${}_RA \to {}_RR$. Since $\mathrm{Clo}_1\mathbf{A}$ consists of permutations, $1 - r\phi(a) = 1 - \phi(ra)$ is invertible in R for all $a \in A$ and $r \in R$. This implies that $\phi(a)$ is in the radical of R for all $a \in A$. By identifying variables one can see that every term operation of $\mathbf{A}$ is of the form described in the exercise. To prove that all such module polynomials are indeed term operations of $\mathbf{A}$, combine unary and idempotent term operations of $\mathbf{A}$.

8. Let $a < b$ be a covering pair in the connected compatible partial order of an idempotent algebra $\mathbf{A}$. Verify that $\{a, b\}$ is a subalgebra of $\mathbf{A}$.

9. A more general statement is true: If $\mathbf{A} = \langle A; \Gamma\rangle$ is a nontrivial unary algebra such that Γ is a permutation group on A and $\mathbf{A}$ has no proper subalgebras (i.e., Γ acts transitively on A), then $\mathbf{A}$ has no nontrivial compatible partial order. Suppose the contrary, consider a covering pair $a < b$ in a nontrivial compatible partial order of $\mathbf{A}$, and using the assumption on the subalgebras write b as $b = \gamma(a)$ for some $\gamma \in \Gamma$. Apply γ repeatedly, and derive a contradiction.

10. If $\mathbf{A} = \langle G^0; 0, \{g\cdot : g \in G\}, \dots\rangle$ is a strictly simple G^0-algebra, then the meet operation $\wedge$ of the height 1 semilattice on G^0 with least element 0 is a term operation of $\mathbf{A}$ (see the lecture). It suffices to prove (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (b), because the converse implications are trivial.
(a) $\Rightarrow$ (b): The $\wedge$-order and its converse are the only possible connected compatible partial orders of $\mathbf{A}$. To show this notice that $\mathrm{Clo}_1\mathbf{A}$ restricts to $G = A \setminus \{0\}$ as a transitive permutation group, hence the statement in the hint to Exercise 9 yields that the elements of G are pairwise incomparable in any connected compatible partial order of $\mathbf{A}$.
(c) $\Rightarrow$ (b): Prove the contrapositive: Suppose that the $\wedge$-order is not compatible. Then for some $p \in \mathrm{Pol}_1\mathbf{A}$ and $a \in G$ we have $0 < a$ but $p(0) \not< p(a)$. Thus $p(0) \in G$, and $p'(x) = (p(0))^{-1}p(ax) \wedge 1$ is a polynomial operation of $\mathbf{A}$ such that $p'(1) = 0$ and $p'(0) = 1$. Since $\mathbf{A}$ is generated by 1, $\mathbf{A}$ has a binary term operation t such that $p'(x) = t(1, x)$. Now recall that the only unary term operations of $\mathbf{A}$ are 0 and the left multiplications by elements of G. It follows that the term operation $x \circ y = \bigwedge_{g\in G} t(x, g^{-1}y)$ satisfies $0 \circ 1 = 0$, $1 \circ 0 = 1$, and $1 \circ g = 0$ for all $g \in G$. To show that $\circ$ is the operation described in (c), look at the unary term operations $0 \circ y$, $x \circ 0$, and $x \circ gx$ ($g \in G$).

11. Define the operations of $\mathbf{A} = \langle\{0, 1, 2\}; e, f, \circ\rangle$ so that $e = e^2$, $f = f^2$, the ranges of e, f are non-isomorphic two-element subalgebras in $\mathbf{A}$, and $\mathbf{A}$ is simple.

12. If $\mathbf{A}$ has no nontrivial proper subalgebras and $\mathbf{A}$ does not have all the constants, then one of the following conditions holds for $\mathbf{A}$ (see the lecture):

(i) $\mathbf{A}$ has no constants and no proper subalgebras;

(ii) $\mathbf{A}$ has a unique trivial subalgebra $\{0\}$, and the constant with value 0 is a term operation;

(iii) $\mathbf{A}$ has more than one trivial subalgebra, and the union U of trivial subalgebras is the range of a term operation $e^2 = e \in \mathrm{Clo}_1\mathbf{A}$.

In case (iii) the only term minimal set is U.

In case (ii) let $e(A)$, $f(A)$ be term minimal sets in $\mathbf{A}$ ($e^2 = e, f^2 = f \in \mathrm{Clo}_1\mathbf{A}$). We may assume that $|e(A)| \geq |f(A)|$. Let $a \in e(A)$, $b \in f(A)$, $a \neq 0$, $b \neq 0$. Since each of a, b generates $\mathbf{A}$, there exist $g, h \in \mathrm{Clo}_1\mathbf{A}$ such that $g(a) = b$ and $h(b) = a$. Then for $p = fge$ and $q = ehf$ we have $p(A) \subseteq f(A)$, $q(A) \subseteq e(A)$, and $qp(a) = a$. Since $\{0\}$ is a subalgebra of $\mathbf{A}$, we also have $qp(0) = 0$. Thus qp maps $e(A)$ into $e(A)$ and no power of it is constant. Let $(qp)^k$ be an idempotent power of qp. Since $e(A)$ is a term minimal set, it follows that $(qp)^k$ is the identity on $e(A)$. Hence p and $(qp)^{k-1}q$ are inverse (term) isomorphisms between $e(A)$ and $f(A)$.

Case (i) is similar to case (ii) only simpler, because $\mathbf{A}$ has no constant unary term operations.