Hints for Exercise Set #1

(1) Let $\mathbf{A} = \langle A; \mathcal{C} \rangle$ be a finite algebra with $\mathbf{A}^\perp = \langle A; \mathcal{R} \rangle$.
   (a) Show how to derive composition of binary relations from the operations $\cap, \times, proj, \Pi, =$ of $\mathcal{R}$.
   (b) Show that the elements of $\mathrm{Con}\,(\mathbf{A})$ are equationally definable elements of $\mathcal{R}$, and that the lattice operations of $\mathrm{Con}\,(\mathbf{A})$ are operations of $\mathcal{R}$.
   (c) If $e^2 = e \in C_1$, and $U = e(A)$, show that restriction to U is a homomorphism of $\mathrm{Con}\,(\mathbf{A})$ into $\mathrm{Con}\,(\mathbf{A}|_U)$.
   (d) Give an example to show that the homomorphism from part (c) need not be surjective.
   (e) Show that if $\mathrm{Sg}^{\mathbf{A}}(e(A)) = A$, then the homomorphism from part (c) is surjective.

Hints.

(a) Suppose that $\theta, \psi \in \mathcal{R}_2$ are compatible binary relations. Let $proj^n_{i_1,\ldots,i_k}$ denote the instance of $proj$ that projects an n-ary relation onto the coordinates $i_1, \ldots, i_k$. Show that $proj^2_1(=) = A$, and then that

$$\theta \circ \psi = proj^4_{1,4}((\theta \times \psi) \cap (A \times (=) \times A)).$$

(b) $\theta \in \mathcal{R}$ is a congruence iff it satisfies $\theta \cap (=) = (=)$, $\theta^\cup = \theta$, and $\theta \circ \theta = \theta$. Since converse is an instance of Π, and $\circ$ is an operation of $\mathcal{R}$ by (a), this is an equational definition. Note that meet in $\mathrm{Con}\,(\mathbf{A})$ is given by $\cap$, and that join is given by $|A|$-fold composition.

(c) Restriction is a homomorphism of $\mathcal{R}$ onto $\mathcal{R}|_U$. Congruences are the compatible binary relations satisfying the equations in part (b). They restrict to relations satisfying the same equations, hence they restrict to congruences. Meet and join are operations of the relational clone, so restriction preserves these. Thus, restriction is a lattice homomorphism from $\mathrm{Con}\,(\mathbf{A})$ to $\mathrm{Con}\,(\mathbf{A}|_U)$.

(d) You can take $\mathbf{A} = \langle \{0, 1, 2, 3\}, * \rangle$ where $*$ is any binary operation of the form

$*$	0	1	2	3
0	0	0	0	1
1	1	1	1	2
2	2	2	2	–
3	–	–	–	x

with $x \in \{0, 1, 2\}$. Then take $e(x) = x * x$ and $U = e(A) = \{0, 1, 2\}$. Use the facts that $(1, 2) \in \mathrm{Cg}^{\mathbf{A}}(0, 1)$ and $(1, 2) \notin \mathrm{Cg}^{\mathbf{A}|_U}(0, 1)$ to conclude that no congruence of $\mathbf{A}$ restricts to $\mathrm{Cg}^{\mathbf{A}|_U}(0, 1)$.

(e) Since restriction is a surjective homomorphism $\mathcal{R} \to \mathcal{R}|_U$, if $\theta \in \mathrm{Con}\,(\mathbf{A}|_U)$ then there is a relation $T \in \mathcal{R}_2$ such that $T|_U = \theta$. Replacing T with the $|A|$-fold composition of $T \cap T^{\cup}$ if necessary, we may assume that T is symmetric and transitive. Also, since $T|_U = \theta$ contains the equality relation on U, it follows that T contains the compatible binary relation generated by $(=)|_U$. If $\mathrm{Sg}^{\mathbf{A}}(U) = A$, then the compatible binary relation generated by $(=)|_U$ is just $(=)|_A$. Hence if $\mathrm{Sg}^{\mathbf{A}}(U) = A$, then T is a congruence of $\mathbf{A}$ whose restriction to U is θ.

(2) For each $m > 1$ construct an m-element non-unary algebra $\mathbf{A}$ such that A is the only (nontrivial) neighborhood of $\mathbf{A}_A$.

Hints. Any unary polynomial of a nilpotent semigroup is constant or the identity, so the only nontrivial neighborhood is the whole algebra. Non-unary examples of this type can be constructed for any $m > 2$. For $m = 2$ choose any non-unary algebra of size 2.

(3) Show that a p-Sylow subgroup of a finite group $\mathbf{G}$ is a neighborhood of $\mathbf{G}_G$.

Hints. Let P be a p-Sylow subgroup of $\mathbf{G}$. Assume that $|P| = p^\ell$ and $|G| = p^\ell m$ where $p \not| m$. Define $e_0(x) = x^k$ where (using the Chinese Remainder Theorem) k is chosen so that $k \equiv 1 \pmod{p^\ell}$ and $k \equiv 0 \pmod{m}$. For any $g \in G$ we get that $e_0(g)$ has exponent p^ℓ, and therefore $e_0(G)$ is contained in the union of the p-Sylow subgroups. Moreover, $e_0(x)$ is the identity function on any p-Sylow subgroup, so e_0 is an idempotent which defines as a neighborhood the union of the p-Sylow subgroups of $\mathbf{G}$. Let $U = e_0(G)$ denote this union. Our first task will be to show that for every $u \in U - P$ there is a polynomial $e_u(x)$ with constants in P such that $e_u(G) \subseteq U$, $e_u(u) \in P$, and $e_u(x) = x$ on P. To show this, choose some $u \in U - P$ and let H be the subgroup of $\mathbf{G}$ generated by P and u. Of course, P is a p-Sylow subgroup of H and $u \in U - P$ is contained in some other p-Sylow subgroup of H. Every element $h \in H$ is expressible as $h = p_0 u p_1 u \cdots u p_s$, where each p_i is from P and u occurs exactly s times for some positive integer s.

Claim 0.1. *There is some $h \in H$ whose order is prime to p which can be expressed as $p_0 u p_1 u \cdots u p_s$ with $p \not| s$.*

We prove this by contradiction: Assume that every element of H whose order is prime to p can be expressed in the form $p_0 u p_1 u \cdots u p_s$ only if s is divisible by p. Let K be the subgroup of H consisting of all elements $h \in H$ which can be expressed as $h = p_0 u p_1 u \cdots u p_s$ with s divisible by p. K contains every element of H whose order is prime to p, so K contains every q-Sylow subgroup of H where $q \neq p$. K also contains P. Since the cardinality of a finite group is determined by the cardinalities of the Sylow subgroups it contains, $|K| = |H|$, and therefore $K = H$. This forces u to be in K, from which we conclude that

$$u = p_0 u p_1 u \cdots u p_s$$

for some s divisible by p. Choose an element $h \in H$ whose order is prime to p. Then

$$h = q_0 u q_1 u \cdots u q_{t-1} u q_t = q_0 u q_1 u \cdots u q_{t-1} (p_0 u p_1 u \cdots u p_s) q_t$$

shows two ways of expressing h in our special form, and u occurs $(p-1)$ more times in the second expression than it does in the first expression. The number of times u occurs in one of these expressions cannot be divisible by p. This contradiction proves the claim.

Now, using the claim, choose an $h \in H$ whose order is prime to p and is such that

$$h = p_0 u p_1 u \cdots u p_s$$

with s prime to p. Define $\pi(x) = e_0(p_0 x p_1 x \cdots x p_s)$. Then $\pi(x)$ is a polynomial of $\mathbf{G}$ with constants in P, $\pi(u) = e_0(h) = 1 \in P$ and $\pi(G) \subseteq U$.

Claim 0.2. *$\pi(x)$ is a permutation of P.*

As pointed out by E. Aichinger during P. P. Pálfy's talk, a unary polynomial $\pi(x)$ of a p-group P is a permutation iff the number of occurrences of x is prime to p. This number is sk where s is defined just before the claim and k was chosen so that $e_0(x) = x^k$. Both s and k are prime to p, so sk is prime to p.

Another way to prove this claim is to use Exercise 4.37 (6) of Hobby-McKenzie which shows that p-groups are E-minimal. This means that for any unary polynomial $f(x)$ of P, either f is a permutation of P or else some iterate $f^k(x) = f \circ f \circ \cdots \circ f(x)$ is a constant function. One can show that $\pi(x)$ is a permutation by showing that if a and b are distinct elements which are congruent modulo the center of P, then $\pi^i(a)$ and $\pi^i(b)$ are also distinct elements of P which are congruent modulo the center of P. It suffices to prove this for $i = 1$, so we will prove (the contrapositive of) this statement. Assume that a and b are congruent modulo the center of P. Then $b = az$ for some z in the center. If we have $\pi(a) = \pi(b)$, then

$$p_0 a p_1 a \cdots a p_s = \pi(a) = \pi(b) = p_0 b p_1 b \cdots b p_s = p_0 a p_1 a \cdots a p_s z^s,$$

so $z^s = 1$. Since s is prime to p and z has order dividing p^ℓ, this implies that $z = 1$ and so $a = b$. This proves that π is 1-1 on each coset of the center of P, and therefore $\pi(x)$ is 1-1 on P. Claim 0.2 is established (again).

Since $\pi(u) \in P$, $\pi(G) \subseteq U$ and π is a permutation of P, some iterate $e_u(x) = \pi^j(x)$ maps u into P, satisfies $e_u(G) \subseteq U$ and also $e_u(x) = x$ on P. These were all properties of e_u that we wanted to establish.

Claim 0.3. *P is a neighborhood.*

Construct a sequence of polynomials $E_0(x), E_1(x), \ldots$ with constants in P which have the properties that

- $P \subseteq E_i(G) \subseteq U$ for all i;
- $E_i(x) = x$ on P for all i; and

- if $E_i(G) \neq P$, then $|E_{i+1}(G)| < |E_i(G)|$.

It follows that for i sufficiently large we will have an idempotent polynomial $E_i(x)$ with constants in P which defines P as a neighborhood.

To construct the E_i, begin by taking $E_0(x) = e_0(x)$. Then, assuming that E_j has already been defined, define $E_{j+1}(x)$ to be

(1) $e_u \circ E_j(x)$ for some $u \in E_j(G) - P$, if there is such a u; or
(2) $E_j(x)$ if $E_j(G) = P$.

Since the kernel of e_u restricts nontrivially to the image of E_j in item (1), it follows that $|E_{j+1}(G)| < |E_j(G)|$ in this case. This completes the proof for Claim 0.3.

(4) Show that the Jacobson radical of a finite ring $\mathbf{R}$ is a neighborhood of $\mathbf{R}_R$.

Hints. Let $\mathbf{R}$ be a finite ring with radical J. Fix an integer $k > 0$ such that $\mathbf{R}$ satisfies the equation $x^k = x^{2k}$. For each subset $S \subseteq R \times R$ define a unary polynomial

$$P_S(x) = \left(1 + \sum_{(a,b)\in S} axb\right)^k.$$

Claim 0.4. *Any polynomial of the form* $e(x) = x\left(\prod_{S\subseteq R\times R} P_S(x)\right)$, *where the product is over all subsets* $S \subseteq R \times R$ *in some fixed order, is an idempotent polynomial of* $\mathbf{R}$ *whose image is* J.

We need to show that $e(R) \subseteq J$ and that $e(x) = x$ on J. To show the latter, assume that $s \in J$. Then for any $S \subseteq R \times R$ the element $\left(\sum_{(a,b)\in S} asb\right) \in J$, so $(1 + \sum asb) \in 1 + J$. It is known that $1 + J$ consists of units. (For any $j \in J$ we have $(1 + j)^{-1} = 1 - j + j^2 - j^3 + \cdots$, which is a finite sum since any $j \in J$ is nilpotent.) The element $P_S(s) = (1 + \sum asb)^k$ is an idempotent unit, since $\mathbf{R} \models x^k = x^{2k}$, so $P_S(s) = 1$ for any $S \subseteq R \times R$ and any $s \in J$. Thus $e(s) = s\,(\prod 1) = s$ when $s \in J$.

To show that $e(R) \subseteq J$, choose an arbitrary $r \in R$. In any finite (or Artinian) ring J is the intersection of the maximal 2-sided ideals of $\mathbf{R}$. Thus it will suffice to show that $e(r) \in M$ where M is an arbitrarily chosen maximal ideal. For this we can work in the quotient ring $\overline{\mathbf{R}} = \mathbf{R}/M$, which is simple. In this ring we must show that for an arbitrary $\overline{r} \in \overline{\mathbf{R}}$ it is the case that

$$\overline{e}(\overline{r}) = \overline{r}\left(\prod_{S\subseteq R\times R} \overline{P_S}(\overline{r})\right) = \overline{0}.$$

If $\overline{r} = \overline{0}$, then the leading $\overline{r}$ in this expression gives the desired conclusion. If $\overline{r} \neq \overline{0}$, then the ideal generated by $\overline{r}$ is $\overline{\mathbf{R}}$, therefore

$$-\overline{1} = \sum_{(a,b)\in S} \overline{a}\overline{r}\overline{b}$$

for some $S \subseteq R \times R$. This shows that $\overline{P_S}(\overline{r}) = \overline{0}$ for this choice of S, so $\overline{e}(\overline{r}) = \overline{0}$. Hence $e(r) \in M$ for each $r \in R$ and each maximal ideal M, which proves that $e(R) \subseteq J$. The claim is proved.

(5) Let $\mathbf{A}$ be a finite algebra with congruences $\alpha \prec \beta$. Show that a neighborhood U of $\mathbf{A}_A$ is $\langle \alpha, \beta \rangle$-irreducible if and only if it is $\langle \alpha, \beta \rangle$-minimal.

Hints. Recall that a neighborhood U of $\mathbf{A}_A$ is $\langle \alpha, \beta \rangle$-irreducible if it is minimal under inclusion for $\alpha|_U \neq \beta|_U$, and that it is $\langle \alpha, \beta \rangle$-minimal if every unary polynomial of $\mathbf{A}|_U$ is a permutation or collapses $\beta|_U$ into $\alpha|_U$. For any $\langle \alpha, \beta \rangle$ the property of $\langle \alpha, \beta \rangle$-minimality is stronger than $\langle \alpha, \beta \rangle$-minimality. We must show that they are equivalent if $\alpha \prec \beta$.

There is no change in assumptions or conclusions if we replace $\mathbf{A}$ by $\mathbf{A}|_U$, so assume that $A = U$. Assume that A is $\langle \alpha, \beta \rangle$-irreducible but not $\langle \alpha, \beta \rangle$-minimal. Since it is not $\langle \alpha, \beta \rangle$-minimal there is a nonpermutation $f \in \mathrm{Pol}_1(\mathbf{A})$ such that $f(\beta) \nsubseteq \alpha$. Among all such f's, we choose one whose range $f(A) = V$ is minimal under inclusion. Since $f(\beta) \nsubseteq \alpha$, there exists $(a, b) \in \beta$ such that $(f(a), f(b)) \notin \alpha$. The congruence $\alpha \vee \mathrm{Cg}(f(a), f(b))$ properly contains α and is contained in β, so $\alpha \vee \mathrm{Cg}(f(a), f(b)) = \beta$. Thus $(a, b) \in (\alpha \vee \mathrm{Cg}(f(a), f(b))) - \alpha$. There is a Maltsev chain connecting a to b consisting of α-links and links of the form $\{pf(a), pf(b)\}$ where p is a unary polynomial. Applying f to this chain produces a Maltsev chain connecting $f(a)$ to $f(b)$ consisting of α-links and links of the form $\{fpf(a), fpf(b)\}$ where p is a unary polynomial. Since $(f(a), f(b)) \notin \alpha$, there is some p such that $(fpf(a), fpf(b)) \notin \alpha$. For this p the polynomial $fpf(x)$ has range contained in $f(A) = V$, yet $fpf(\beta) \nsubseteq \alpha$ since $(fpf(a), fpf(b)) \notin \alpha$. By the minimality of V we get $fpf(A) = V$. Thus, $fp(x)$ has range V and is a permutation of V. Let $e(x)$ be an idempotent iterate of fp. We have $e(A) = V$, so V is a neighborhood. We also have $(f(a), f(b)) \in \beta|_V - \alpha$. Since f is a nonpermutation, $|V| = |f(A)| < |A|$, so we have constructed a neighborhood V smaller than A that separates α and β. This contradicts the $\langle \alpha, \beta \rangle$-irreducibility of A.

(6) Show that if U and V are irreducible neighborhoods that cover one another, then they are isomorphic.

Hints. Choose $e^2 = e, f^2 = f \in \mathcal{C}_1(\mathbf{A})$ such that $e(A) = U$ and $f(A) = V$. Recall that U covers V if there are terms λ, ρ_i such that $f(x) = \lambda(e\rho_1(x), \ldots, e\rho_m(x))$. Similarly, V covers U if there are terms Λ, R_i such that $e(x) = \Lambda(fR_1(x), \ldots, fR_n(x))$.

Choose a pair $\langle S,T\rangle$ of n-ary relations with $S \subsetneq T$ such that U is $\langle S,T\rangle$-irreducible. By replacing T by a compatible relation $T' \subseteq T$ that is minimal under inclusion for $T'|_U \not\subseteq S$, and then replacing S by $T' \cap S$, we may assume that U is $\langle S,T\rangle$-irreducible where $T|_U$ is join-irreducible and $S|_U \prec T|_U$ in the $\cap$-order on the n-ary relations of $\mathcal{R}(\mathbf{A}|_U)$. Choose $\overline{u} \in T|_U - S$. Since $\Lambda(fR_1(\overline{u}), \ldots, fR_n(\overline{u})) = e(\overline{u}) = \overline{u} \notin S$, it must be that some $fR_i(\overline{u})$ is not in S. Write $\overline{v}$ for some such tuple. Since $f(A) = V$ it must be that $\overline{v} \in T|_V - S$. Now we redo the same argument in the other direction using the other decomposition equation. S does not contain

$$\overline{v} = f(\overline{v}) = \lambda(e\rho_1(\overline{v}), \ldots, e\rho_m(\overline{v})),$$

so some $e\rho_j(\overline{v})$ is in $T|_U - S$. Since $T|_U$ is join-irreducible and $S|_U \prec T|_U$ in the $\cap$-order of $\mathcal{R}_n(\mathbf{A}|_U)$, $T|_U$ is generated by any element in $T|_U - S$. In particular, $T|_U$ is the relation on $\mathbf{A}|_U$ generated by the tuple $e\rho_j(\overline{v})$. Since $\overline{u} \in T|_U$ we get that there is a $et \in \mathcal{C}_1(\mathbf{A}|_U)$ such that $ete\rho_j(\overline{v}) = \overline{u}$.

Let $g(x) = fR_i(x)$ and let $h(x) = ete\rho_j(x)$. Since $g = fg$ and $h = eh$ we have $g(U) \subseteq f(A) = V$ and $h(V) \subseteq e(A) = U$. We have shown that $g(\overline{u}) = \overline{v}$ and $h(\overline{v}) = \overline{u}$, so $hg \in \mathcal{C}_1(\mathbf{A}|_U)$ and $hg(\overline{u}) = \overline{u}$. Let $U' \subseteq U$ be the image of an idempotent iterate $(hg)^k(x)$ of hk. Since $hk(\overline{u}) = \overline{u}$ we must have $\overline{u} \in T|_{U'} - S$, so U' is a subneighborhood of U that separates S and T. By the $\langle S,T\rangle$-irreducibility of U we get that $U' = U$, and this means that $(hg)^k(x) = x$ on U. Hence $g(hg)^{k-1}h(x)$ is an idempotent with range $V' \subseteq g(A) \subseteq V$. Moreover $g : U \to V'$ and $(hg)^{k-1}h : V' \to U$ are isomorphisms. Therefore we have that V contains a subneighborhood V' that is isomorphic to U. By the symmetry in the hypotheses, U contains a subneighborhood isomorphic to V. By finiteness U is isomorphic to V.

(7) Assume that $\mathcal{U} = \{U_1, \ldots, U_m\}$ is a cover of $\mathbf{A}$.
   (a) Show that if $\mathbf{A}$ has a majority term, then so does each $\mathbf{A}|_{U_i}$.
   (b) Show that if each $\mathbf{A}|_{U_i}$ has a majority term, then so does $\mathbf{A}$.

Hints.

(a) If $U_i = e_i(A)$ and $m(x,y,z) \in C_3(\mathbf{A})$ is a majority operation, then $e_i m(x,y,z) \in C_3(\mathbf{A}|_{U_i})$ is a majority operation of $\mathbf{A}|_{U_i}$

(b) Assume that $m_i(x,y,z) \in C_3(\mathbf{A}|_{U_i})$ and that $\lambda(e_1\rho_1(x), \ldots, e_n\rho_n(x)) = x$ on $\mathbf{A}$. Then

$$\lambda(m_1(e_1\rho_1(x), e_1\rho_1(y), e_1\rho_1(z)), \ldots, m_n(e_n\rho_n(x), e_n\rho_n(y), e_n\rho_n(z)))$$

is a majority term on $\mathbf{A}$. (Notes: This type of argument shows that $\mathbf{A}$ satisfies a given linear Maltsev condition if and only each $\mathbf{A}|_{U_i}$ satisfies it. Also, there is a less computational argument: if one first argues that localization and matrix products preserve the satisfaction of any linear Maltsev condition, then one gets that the satisfaction of any linear Maltsev condition is preserved and reflected by localization to sets in a cover. To prove that the satisfaction of a linear Maltsev condition is preserved by localization, use the argument from (a). To prove that it is preserved by matrix products use the fact that any matrix product contains the operations of the nonindexed product.)

(8) Suppose that $\mathbf{A}$ is finite and $\mathbf{A}_A$ generates a congruence distributive variety.
   (a) Show that any neighborhood U of $\mathbf{A}_A$ that is minimal under inclusion has 2 elements, and $\mathbf{A}|_U$ has a majority polynomial. (Hard.)
   (b) Show that the following are equivalent.
      (i) The set of neighborhoods minimal under inclusion is a cover of $\mathbf{A}_A$.
      (ii) $\mathbf{A}$ has a majority polynomial, and all compatible reflexive binary relations of $\mathbf{A}$ are transitive.

Hints.

(a) The fact that U is minimal under inclusion means exactly that $(\mathbf{A}_A)|_U$ is E-minimal. Using an argument like that in Exercise (7)(a) show that $\mathbf{A}|_U$ has Jónsson polynomials. Next show that an E-minimal algebra with Jónsson polynomials is simple. Conclude from Exercise (5) that $(\mathbf{A}_A)|_U$ is a minimal algebra. Now use Pálfy's Theorem classifying minimal algebras to show that $(\mathbf{A}_A)|_U$ has size 2 and a majority polynomial.

(b) Using part (a) and Exercise 7 one can show that (i) implies that $\mathbf{A}$ has a majority polynomial. Since any 2-element algebra whose clone contains the constants and a majority operation has the property that the compatible reflexive binary relations are transitive, and because this is an equational property of $\mathcal{R}(\mathbf{A}_A)$, $\mathbf{A}$ will also have this property. This proves that (i) implies (ii).

To prove that (ii) implies (i) it suffices to show that if $\mathbf{A}$ has a majority polynomial and all compatible reflexive binary relations are transitive, then the only irreducible neighborhoods of $\mathbf{A}_A$ have 2 elements. In fact, it suffices to prove this in the case when A itself is an irreducible neighborhood. Thus, we must show that if $\mathbf{A}_A$ is irreducible, has a majority polynomial, and all reflexive compatible binary relations are transitive, then $|A| = 2$.

Step 1: Use the Baker-Pixley theorem to show that if $\mathbf{A}$ is irreducible and has majority polynomial then $\mathbf{A}$ is $\langle S, T\rangle$-minimal for binary relations $S \subsetneq T$ with T a compatible binary relation generated by a single pair $(0, 1)$.

Step 2: Using the polynomials $x \wedge y = m(0, x, y)$ and $x \vee y = m(x, y, 1)$, show that A is the disjoint union of $\{0, 1\}$ and a set M where $(0, m), (m, 1) \in S$ for every $m \in M$.

Step 3: If S is transitive and $M \neq \emptyset$, then $(0, 1) \in S$, which contradicts $S \subsetneq T$. Conclude that $M = \emptyset$, and therefore $A = \{0, 1\}$.

(9) Assume that $\mathcal{V}$ is a locally finite variety, and that there is some k that bounds the size of the irreducible sets of $\mathbf{A}_A$ for every finite $\mathbf{A} \in \mathcal{V}$.
   (a) Show that $\mathcal{V}$ is n-permutable for some n.

(b) Show that $\mathcal{V}$ is congruence distributive. (Hard.)
(c) Must $\mathcal{V}$ have a near unanimity term? (Open.)

Hints.

(a) If $\mathcal{V}$ is not n-permutable for any n, then there are algebras in $\mathcal{V}$ with congruences θ and ψ where $S = \theta \circ_n \psi \subsetneq \theta \circ_{n+1} \psi = T$ for large values of n. Such S and T cannot be separated on sets of size k if $n > k$.

(b) If $\mathbf{A} \in \mathcal{V}$ is finite, then there is a natural embedding of $\mathrm{Con}\,(\mathbf{A})$ into the product $\prod_{\mathcal{U}} \mathrm{Con}\,(\mathbf{A}|_U)$ where $\mathcal{U}$ is the set of $\langle \alpha, \beta \rangle$-minimal sets of $\mathbf{A}$. This embedding shows that it is a general fact about that the lattice variety generated by all $\mathrm{Con}\,(\mathbf{A}), \mathbf{A} \in \mathcal{V}_{finite}$, is the same as the lattice variety generated by all $\mathrm{Con}\,(\mathbf{A}|_U)$ where $\mathbf{A} \in \mathcal{V}_{finite}$ and U is a minimal set of $\mathbf{A}$.

By assumption, each minimal set U has size at most k. Thus there are at most finitely many lattices of the form $\mathrm{Con}\,(\mathbf{A}|_U)$, so the congruence variety of $\mathcal{V}$ is a finitely generated lattice variety. One can find in papers of Ralph Freese different ways to prove that a finitely generated lattice variety is a congruence variety if and only if it is distributive. This finishes the argument.

There is a self-contained argument for this exercise. It involves proving that if any minimal set in any $\mathbf{A} \in \mathcal{V}_{finite}$ has a tail, then there are arbitrarily large minimal sets in finite algebras in $\mathcal{V}$, and then showing that if any minimal set has type $\mathbf{1}$, $\mathbf{2}$ or $\mathbf{5}$, then there are arbitrarily large minimal sets in algebras in $\mathcal{V}$. Finally one uses the theorem that if no minimal sets have tails, and the variety omits types $\mathbf{1}$, $\mathbf{2}$, or $\mathbf{5}$, then it is congruence distributive. The part about excluding tails is the only hard part, and it is too long to include here.

(c) ???

(Remark: if $\mathcal{V}$ is generated by a 2-element algebra, then there is a bound k on the sizes of irreducible neighborhoods of $\mathbf{A}_A$, $\mathbf{A} \in \mathcal{V}$, if and only if $\mathcal{V}$ is n-permutable and has a near unanimity term operation. The bound is $k = 2^m - 1$ where $m + 1$ is the number of variables of the near unanimity operation.)