A Course in Tame Congruence Theory
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LECTURE II:
The Classification Problem

0. Two-element algebras
1. Irreducible algebras in Post varieties
2. Term minimal algebras

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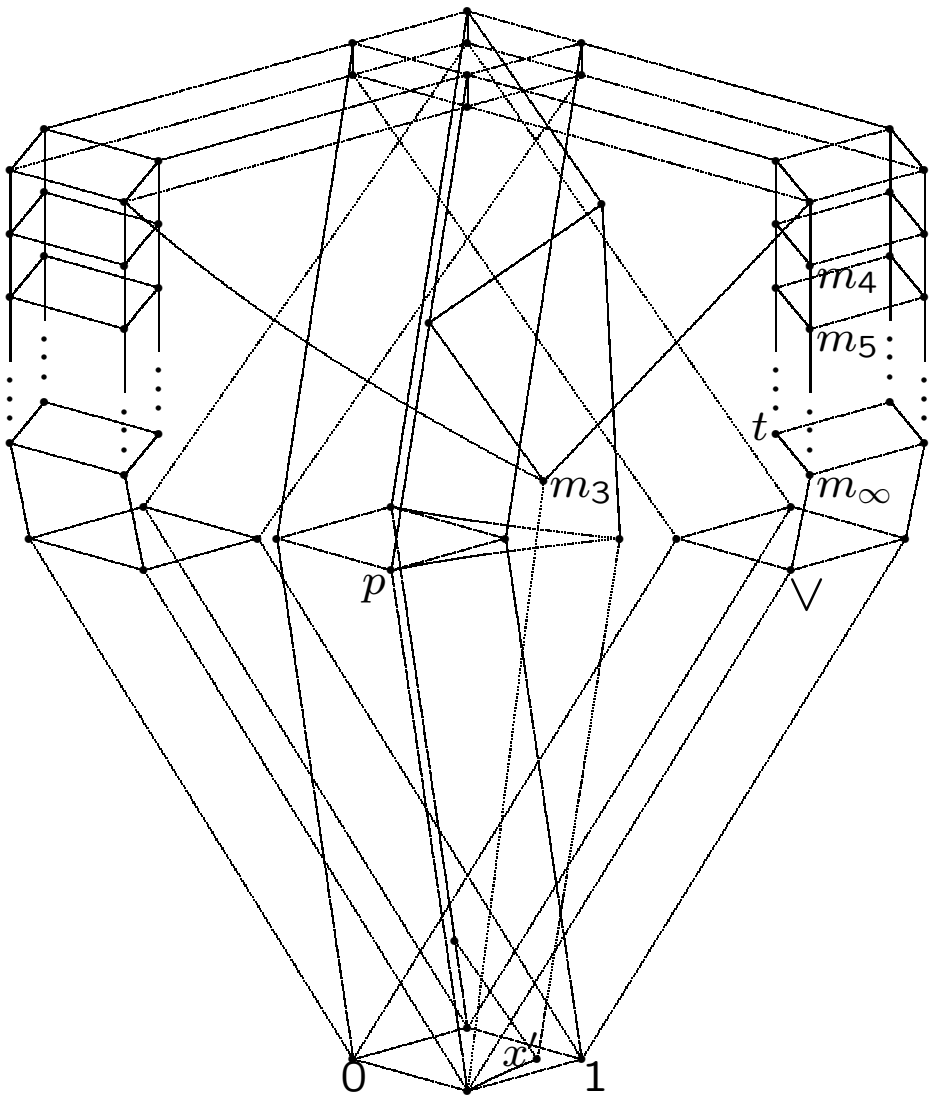
0. Two-element Algebras

E. Post (1920, 1941)



Post's Lattice

Notation:

$x \vee y, x \wedge y, x', 0, 1$: the Boolean operations
join, meet, complement, 0, 1

$m_n(x_1, \ldots, x_n) = \bigvee_{1 \leq i < j \leq n} (x_i \wedge x_j)$:
n-ary near unanimity

$m_\infty(x, y, z) = x \vee (y \wedge z)$

$t(x, y, z) = x \vee (y \wedge z')$

$p(x, y, z) = x + y + z$: ternary addition mod 2

Polynomial clones on the two-element set

1. unary: $[0, 1]$ and $[', 0, 1]$,
2. affine: $[+, 0, 1]$,
3. Boolean: $[\vee, \wedge, ', 0, 1]$,
4. lattice: $[\vee, \wedge, 0, 1]$,
5. semilattice: $[\vee, 0, 1]$ and $[\wedge, 0, 1]$.

1. Irreducible Algebras in Post Varieties

G. Boercker, K. Kearnes (2000)

Let $\mathbf{S}$ be a two-element algebra and $\mathcal{V} = V(\mathbf{S})$.
$\mathcal{V}_{\text{fin}}$ = class of finite members of $\mathcal{V}$.

Goal:
For each $\mathbf{A} \in \mathcal{V}_{\text{fin}}$ find a decomposition of $\mathbf{A}$ — more precisely, $\mathbf{A}|_A$ — into irreducible local algebras (irreducibility with constants).

Summary:
- $\mathbf{S}$ unary or affine or (bounded) semilattice: all $\mathbf{A} \in \mathcal{V}_{\text{fin}}$ are irreducible;
- $\mathbf{S}$ (bounded) distributive lattice: all $\mathbf{A} \in \mathcal{V}_{\text{fin}}$ decompose as lattices do;
- $\mathbf{S}$ quasiprimal: all $\mathbf{A} \in \mathcal{V}_{\text{fin}}$ are polynomially equivalent to Boolean algebras, and hence decompose as Boolean algebras do;

Summary (continued)**:**

- $\mathbf{S} = \langle \{0,1\}; t \rangle$ or $\mathbf{S} = \langle \{0,1\}; t, 1 \rangle$:
  if $\mathbf{A} \in \mathcal{V}_{\mathsf{fin}}$ has a least element (w.r.t. the $\vee$-order), $\mathbf{A}$ is polynomially equivalent to a Boolean algebra, so it decomposes as a Boolean algebra does;
  otherwise $\mathbf{A}$ is irreducible;
- $\mathbf{S} = \langle \{0,1\}; m_\infty \rangle$ or $\mathbf{S} = \langle \{0,1\}; m_\infty, 1 \rangle$:
  if $\mathbf{A} \in \mathcal{V}_{\mathsf{fin}}$ has a least element (w.r.t. the $\vee$-order), $\mathbf{A}$ is polynomially equivalent to a distributive lattice, so it decomposes as a lattice does;
  otherwise $\mathbf{A}$ is irreducible;
- $\mathbf{S} = \langle \{0,1\}; m_3 \rangle$ (median algebra):
  see Exercise 4.

Remaining cases: near unanimity algebras

$$\mathbf{S} = \langle \{0,1\}; m_n, t \rangle, \ \mathbf{S} = \langle \{0,1\}; m_n, t, 1 \rangle$$

and

$$\mathbf{S} = \langle \{0,1\}; m_n, m_\infty \rangle, \ \mathbf{S} = \langle \{0,1\}; m_n, m_\infty, 1 \rangle$$

($n \geq 3$).

More details on the first case:

Let $\mathbf{S}_n = \langle \{0,1\}; m_n, t \rangle$ and $\mathcal{V}_n = V(\mathbf{S}_n)$.

Proposition. *For every finite* $\mathbf{A} \in \mathcal{V}_n$ *there exists a Boolean algebra* $\mathbf{B}$ *such that*
(1) $\mathbf{A}$ *embeds in the* $\{m_n, t\}$*-reduct of* $\mathbf{B}$*, and*
(2) $\mathrm{Con}(\mathbf{A}) \cong \mathrm{Con}(\mathbf{B})$*.*
$\mathbf{B}$ *is uniquely determined (up to isomorphism), and*
(3) A *is a filter in* $\mathbf{B}$ *that contains all coatoms of* $\mathbf{B}$*.*

$\mathbf{B}$ will be called *the Boolean envelope of* $\mathbf{A}$.

Theorem. *Let* $\mathbf{A}$ *be a finite member of* $\mathcal{V}_n$*, and let* $\mathbf{B}$ *be its Boolean envelope.*

If $A = B$ *(i.e.,* $\mathbf{A}$ *has a least element w.r.t the* $\vee$*-order), then* $\mathbf{A}$ *is polynomially equivalent to a Boolean algebra, so* $\mathbf{A}|_A$ *decomposes as a Boolean algebra does.*

If $A \subset B$ *and* $\mathbf{B}$ *has at least* n *atoms, then* $\mathbf{A}|_A$ *has a cover* $\mathcal{U}$ *such that*

- *each* $U \in \mathcal{U}$ *is a proper subalgebra of* $\mathbf{A}$*,*
- *each* $\mathbf{A}|_U$ $(U \in \mathcal{U})$ *is equivalent to* $\mathbf{U}|_U$*.*

If $A \subset B$ *and* $\mathbf{B}$ *has fewer than* n *atoms, then* $\mathbf{A}|_A$ *is irreducible.*

2. Term Minimal Algebras

$(= \langle 0, 1 \rangle$-irreducible, without constants$)$

C. Bergman, R. McKenzie (1990)
Á. Szendrei (1987, 1992, 1994),
K. Kearnes, Á. Szendrei (1997)

Definition. An algebra $\mathbf{T}$ is *term minimal* if $\mathbf{T}$ is finite and every idempotent unary term operation of $\mathbf{T}$ is the identity or constant:

$$e^2 = e \in \mathrm{Clo}_1 \mathbf{T} \quad \Longrightarrow \quad e = \mathrm{id} \text{ or } e = \mathrm{const.}$$

Examples.

- two-element algebras,
- affine algebras over finite local rings
  (e.g. GF(q), $\mathbf{Z}_q$, q a prime power)
- idempotent (!) algebras
  ($\models f(x, \ldots, x) = x$ for all $f \in \mathrm{Clo}\,\mathbf{T}$).

The e functor for finite algebras with no nontrivial proper subalgebras

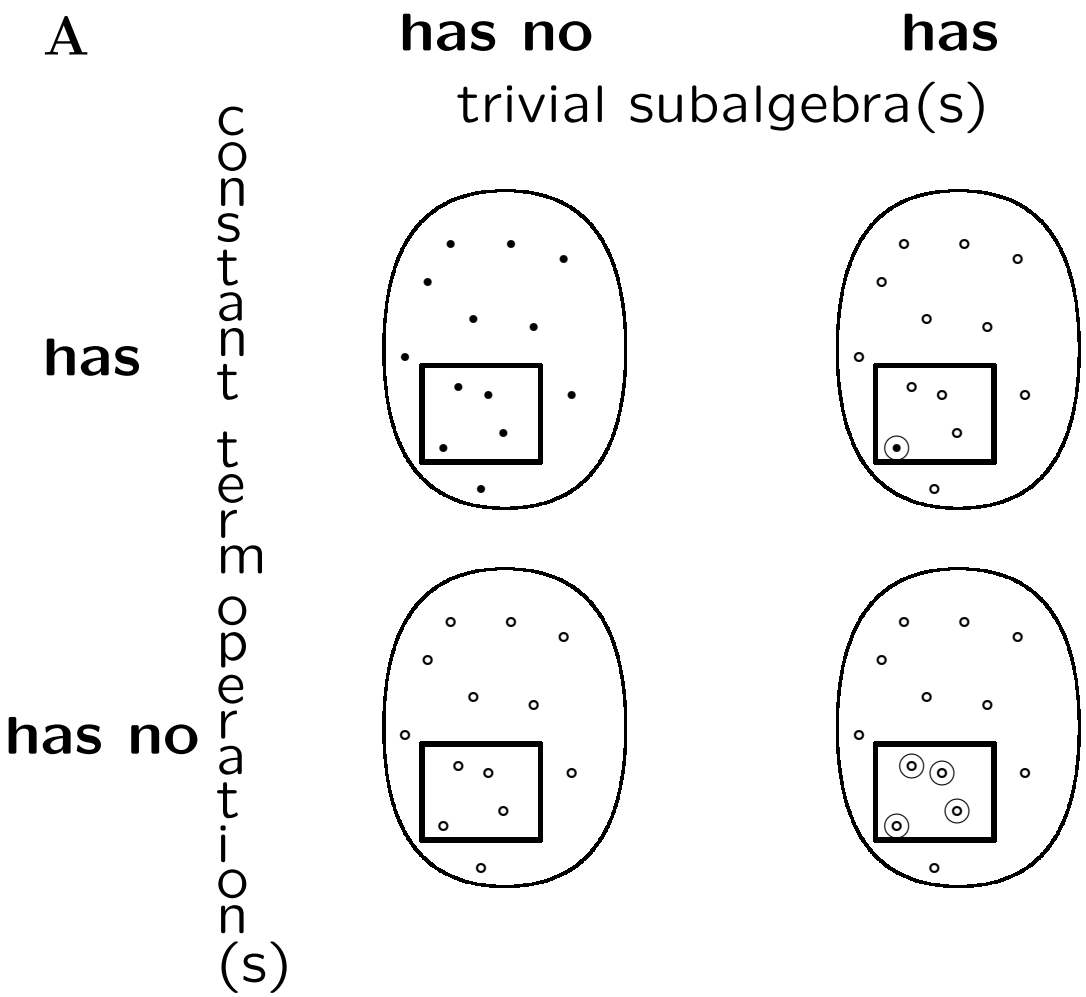
Let $e^2 = e \in \mathrm{Clo}_1 \mathbf{A}$.

$$\mathbf{A} \quad \mapsto \quad e(\mathbf{A}) = \langle e(A); \{et|_{e(A)} : t \in \mathrm{Clo}\,\mathbf{A}\}\rangle$$

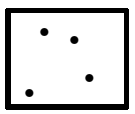
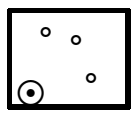
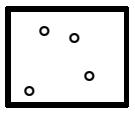
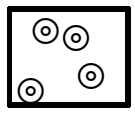
Then $e(\mathbf{A})$ is term minimal iff e(A) is minimal among the ranges of nonconstant idempotent unary term operations of $\mathbf{A}$. In this case $e(A)$ is called a *term minimal set of* $\mathbf{A}$.

Proposition. *Let* $\mathbf{A}$ *be finite with no nontriv. proper subalgebras. Let* $e^2 = e \in \mathrm{Clo}_1 \mathbf{A}$.
(0) $e(\mathbf{A})$ *has no nontriv. proper subalgebras.*
(1) $\mathbf{A}$, $e(\mathbf{A})$ *have the same triv. subalgebras.*
(2) $e\colon \mathrm{Con}\,\mathbf{A} \to \mathrm{Con}\,e(\mathbf{A})$ *is a surjective hom.*
(3) *Either all term minimal sets of* $\mathbf{A}$ *are isomorphic, or* $\mathbf{A}$ *has all the constants.*
(3′) *If* $\mathbf{A}$ *is simple, then all term minimal sets of* $\mathbf{A}$ *are isomorphic.*

Finite algebras $\mathbf{A}$ with no nontrivial proper subalgebras



Term minimal algebras $\mathbf{T}$ with no nontrivial proper subalgebras

$\mathbf{T} = e(\mathbf{A})$	**has no** trivial subalgebra(s)	**has**
has constant term operation(s)	$\mathbf{T}$ is E-minimal 	$\mathrm{Clo}_1\,\mathbf{T} =$ $\{0\} \cup$ perm.gr. 
has no	$\mathrm{Clo}_1\,\mathbf{T} =$ transitive perm.gr. 	$\mathrm{Clo}_1\,\mathbf{T} = \{\mathrm{id}\}$ ($\mathbf{T}$ idempotent) 

E-minimal algebras

An algebra $\mathbf{A}$ is *E-minimal* if $\mathbf{A}$ is finite and every idempotent unary polynomial operation of $\mathbf{A}$ is the identity or constant.

P. P. Pálfy (1984)
D. Hobby, R. McKenzie (1988)
E. W. Kiss (1997)

From the basics of TCT (cf. the lecture tomorrow morning):

Theorem. *If $\mathbf{T}$ is a simple E-minimal algebra with more than 2 elements, then*
(1) $\mathbf{T}$ *is essentially unary*
(poly. equiv. to $\langle T; G\rangle$ for a perm. group),
or
(2) $\mathbf{T}$ *is affine*
(poly. equiv. to a one-dim. vector space).

G-algebras

For a group G, an algebra $\mathbf{G} = \langle G, F \rangle$ is a *G-algebra* if $\mathrm{Clo}_1 \mathbf{G} = L_G$, the group of left multiplications by elements of G.

Theorem. $\mathbf{T}$ *is a term minimal algebra with no constants and no proper subalgebras iff* $\mathbf{T}$ *is a homomorphic image of a G-algebra for some finite group G.*
If such an algebra $\mathbf{T}$ *is simple, then*
(1) $\mathbf{T}$ *is essentially unary, or*
(2) $\mathbf{T}$ *is an affine G-algebra*
(G elementary abelian p-group), or
(3) $\mathbf{T}$ *is a quasiprimal G-algebra.*

G^0-algebras

For a group G, let G^0 be the extension of G with a zero element 0. An algebra $\mathbf{C} = \langle G^0, F \rangle$ is a G^0-*algebra* if $\mathrm{Clo}_1 \mathbf{C} = \{0\} \cup L_G$ (here L_G acts on G^0).

Theorem. $\mathbf{T}$ *is a term minimal algebra with a unique trivial subalgebra (named by a constant) iff* $\mathbf{T}$ *is a homomorphic image of a* G^0*-algebra for some finite group* G*.*
If such an algebra $\mathbf{T}$ *is simple, then it is a* G^0*-algebra; moreover, for* $|T| > 2$ *we have that*
(2) $\mathbf{T}$ *is affine*
(equiv. to a one-dim. vector space), or
($*$) *the meet operation of the height* 1 *semilattice on* G^0 *with least element* 0 *is a term operation of* $\mathbf{T}$*.*

Call an algebra *orderable* if it has a connected, compatible partial order.

In class ($*$) the clones of orderable algebras and the clones of non-orderable algebras form intervals. Their sizes are $2^{\aleph_0}$ if $|T| > 2$.

Idempotent algebras with no nontrivial proper subalgebras

For an element $0 \in T$ the *n-dimensional cross* $\chi_n(0)$ on T is

$$\chi_n(0) = \{\langle t_1, \ldots, t_n \rangle \in T^n : \text{some } t_i = 0\}.$$

Let $\mathscr{X}_n(0) = [\chi_n(0)]$ (the clone of all operation on T which preserve $\chi_n(0)$), and let $\mathscr{X}_\omega(0) = \bigcap_{n=2}^{\infty} \mathscr{X}_n(0)$.

Theorem. *If $\mathbf{T}$ is an idempotent algebra with no nontrivial proper subalgebras, then $\mathbf{T}$ is term minimal and simple. If $|T| > 2$, then*
(2) *$\mathbf{T}$ is affine, or*
(3) *$\mathbf{T}$ is quasiprimal, or*
(3)′ *$\mathbf{T}$ has an element 0 such that*

- *0 is fixed by all automorphisms of $\mathbf{T}$, and*
- *for some k $(2 \leq k \leq \omega)$ the term operations of $\mathbf{T}$ are exactly those idempotent operations from $\mathscr{X}_k(0)$ which preserve the automorphisms of $\mathbf{T}$.*

Application:
Locally finite minimal varieties

A finite algebra $\mathbf{A}$ is *strictly simple* if it is simple and has no nontrivial proper subalgebras.

Fact. Every locally finite minimal variety is generated by a strictly simple algebra.

Theorem. *Let $\mathbf{T}$ be a strictly simple term minimal algebra.*
The following conditions are equivalent:
(a) $\mathcal{V}(\mathbf{T})$ *is minimal,*
(b) *if $\mathbf{T}$ is "abelian" (unary or affine), then it has a trivial subalgebra,*
(c) $\mathbf{T}$ *embeds into every nontrivial algebra in* $\mathcal{V}(\mathbf{T})$.

By 'lifting' this result along the functor e from $\mathbf{T} = e(\mathbf{A})$ to $\mathbf{A}$ we get

Theorem. *Let $\mathbf{A}$ be a strictly simple algebra and let $e^2 = e \in \mathrm{Clo}_1 \mathbf{A}$ such that $e(A)$ is a term minimal set of $\mathbf{A}$.*
The following conditions are equivalent:
(a) $\mathcal{V}(\mathbf{A})$ *is minimal.*
(b) *if $\mathbf{A}$ is "abelian" ($e(\mathbf{A})$ is unary or affine), then $\mathbf{A}$ has a trivial subalgebra, and*

$$\mathbf{B} \models e(x) = e(y) \text{ for no nontriv. } \mathbf{B} \in \mathcal{V}(\mathbf{A}).$$

(c) $\mathbf{A}$ *embeds into every nontrivial algebra in $\mathcal{V}(\mathbf{A})$.*

Main step of proof: (b) $\Rightarrow$ (c)