TERM CONDITIONS AND NOTIONS OF ABELIANNESS

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In these lecture notes we present the basic definitions which give rise to notions of abelianness in general algebras. These notions are in large part motivated by the success in group theory of categorizing groups according to whether they are abelian, nilpotent, solvable, and so on. In the general case, it turns out that two different notions of abelianness have proven to be useful. The first is *normal abelianness*, which coincides in groups with the usual notion of abelianness and which serves to separate vector spaces and essentially unary algebras (which are abelian in this sense) from, for example, boolean algebras, lattices and semilattices (which are entirely lacking in abelianness). The second notion is *strong abelianness*, which separates essentially unary algebras (which are strongly abelian) from vector spaces and abelian groups (which are entirely lacking in strong abelianness).

Our presentation starts from Hobby-McKenzie [2] Chapter 3, but also draws from the point of view of [1] and especially [3]. We begin by defining α, β-matrices and defining the centralizer relation in terms of them. The basic properties of this relation are presented, including the curious fact that the shape of a congruence lattice can force abelianness to occur. Following Kearnes and Kiss [3], we then define the rectangulation and strong centralization relations as the route to strong abelianness, and present their basic properties.

It turns out (though we don't show it) that the centralizer and strong centralizer relations, and the notions of abelianness which arise from them, are not terribly well-behaved in general. However, the reader will learn in later lectures that these notions have surprising and quite useful properties when restricted to finite algebras, and we give a hint of this in the last section.

1. Matrices

We start with an important construction.

Definition 1.1. If $\mathbf{A}$ is an algebra and $\alpha, \beta \in \operatorname{Con} \mathbf{A}$, then

$$\begin{aligned} A^{2\times 2} &= \left\{ \begin{bmatrix} x & y \\ z & y \end{bmatrix} \ : \ x, y, z, u \in A \right\} \\ \mathbf{A}^{2\times 2} &= \text{the subpower of } \mathbf{A} \text{ with universe } A^{2\times 2} \\ Trv(\alpha,\beta) &= \left\{ \begin{bmatrix} a & a \\ b & b \end{bmatrix} \ : \ a \stackrel{\alpha}{\equiv} b \right\} \cup \left\{ \begin{bmatrix} c & d \\ c & d \end{bmatrix} \ : \ c \stackrel{\beta}{\equiv} d \right\} \\ Mat(\alpha,\beta) &= \operatorname{Sg}^{\mathbf{A}^{2\times 2}}(Trv(\alpha,\beta)). \end{aligned}$$

The elements of $Mat(\alpha,\beta)$ are called **α,β-matrices** of $\mathbf{A}$. $Trv(\alpha,\beta)$ is the set of **trivial** α,β-matrices.

An explicit characterization of the α,β-matrices is given in the following lemma.

Lemma 1.2. *For any algebra $\mathbf{A}$ and congruences α, β,*

$$\begin{aligned} Mat(\alpha,\beta) &= \left\{ \begin{bmatrix} t(\mathbf{a},\mathbf{c}) & t(\mathbf{a},\mathbf{d}) \\ t(\mathbf{b},\mathbf{c}) & t(\mathbf{b},\mathbf{d}) \end{bmatrix} \ : \ t \in \operatorname{Clo}_{m+n} \mathbf{A},\ a_i \stackrel{\alpha}{\equiv} b_i,\ c_j \stackrel{\beta}{\equiv} d_j \right\} \\ &= \left\{ \begin{bmatrix} p(\mathbf{a},\mathbf{c}) & p(\mathbf{a},\mathbf{d}) \\ p(\mathbf{b},\mathbf{c}) & p(\mathbf{b},\mathbf{d}) \end{bmatrix} \ : \ p \in \operatorname{Pol}_{m+n} \mathbf{A},\ a_i \stackrel{\alpha}{\equiv} b_i,\ c_j \stackrel{\beta}{\equiv} d_j \right\}. \end{aligned}$$

Proof. Exercise. □

Problems

1. Prove that
$$Mat(\alpha,\beta) \subseteq \left\{ \begin{bmatrix} x & y \\ z & u \end{bmatrix} \in A^{2\times 2} \ : \ x \stackrel{\beta}{\equiv} y \stackrel{\alpha}{\equiv} u \stackrel{\beta}{\equiv} z \stackrel{\alpha}{\equiv} x \right\}.$$
That is, the columns of an α,β-matrix are always α-related and the rows are always β-related.
2. Prove that if $\mathbf{A}$ is essentially unary then $Mat(\alpha,\beta) = Trv(\alpha,\beta)$.
3. Prove that if $\mathbf{A} = \langle A; +, -, 0, (f_r)_{r\in R}\rangle$ is a module over a ring, then
$$Mat(1_A, 1_A) = \left\{ \begin{bmatrix} x & y \\ z & u \end{bmatrix} \ : \ x - y - z + u = 0 \right\}.$$
More generally,
$$Mat(\alpha,\beta) = \left\{ \begin{bmatrix} x & y \\ z & u \end{bmatrix} \ : \ x \stackrel{\beta}{\equiv} y \stackrel{\alpha}{\equiv} u \stackrel{\beta}{\equiv} z \stackrel{\alpha}{\equiv} x \text{ and } x - y - z + u = 0 \right\}.$$
4. Prove that if $\mathbf{A}$ is a boolean algebra, then $Mat(1_A, 1_A) = A^{2\times 2}$. Generalize.

5. Prove that if $\mathbf{A}$ is a lattice then
$$\left\{ \begin{bmatrix} a & a \\ a & b \end{bmatrix} : (a,b) \in \alpha \cap \beta \right\} \subseteq Mat(\alpha, \beta).$$
Generalize.

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2. The Centralizer Relation

Definition 2.1. Let $\mathbf{A}$ be an algebra and $\alpha, \beta, \delta \in \mathrm{Con}\,\mathbf{A}$.

1. We say that α **centralizes** β **modulo** δ, and write $C(\alpha, \beta; \delta)$, if for all $\begin{bmatrix} x & y \\ z & u \end{bmatrix}$ in $Mat(\alpha, \beta)$, $x \overset{\delta}{\equiv} y$ iff $z \overset{\delta}{\equiv} u$.
2. If $\delta \leq \alpha$, then α is **abelian over** δ if $C(\alpha, \alpha; \delta)$.
3. α is **abelian** if $C(\alpha, \alpha; 0_A)$.
4. $\mathbf{A}$ is **abelian** if $C(1_A, 1_A; 0_A)$.

Lemma 2.2. *Suppose* $\mathbf{A}$ *is an algebra and* $\alpha, \beta, \alpha', \beta', \delta, \delta_i \in \mathrm{Con}\,\mathbf{A}$ *(*$i \in I$*).*

1. $C(\alpha, \beta; \alpha)$ *and* $C(\alpha, \beta; \beta)$.
2. *If* $C(\alpha, \beta; \delta_i)$ *for all* $i \in I$*, then* $C(\alpha, \beta; \bigwedge_{i \in I} \delta_i)$.
3. *If* $C(\alpha, \beta; \delta)$ *and* $\alpha' \leq \alpha$ *and* $\beta' \leq \beta$*, then* $C(\alpha', \beta'; \delta)$.

Proof. Exercise. □

The next lemma gives a refinement of the definition of the centralizer relation which is useful in practice (and is the "classical" definition in the literature).

Lemma 2.3. *Suppose* $\mathbf{A}$ *is an algebra and* $\alpha, \beta, \delta \in \mathrm{Con}\,\mathbf{A}$. *Then the following are equivalent:*

1. $C(\alpha, \beta; \delta)$.
2. *For all* $t \in \mathrm{Clo}_{1+n}\,\mathbf{A}$ *and all* $a \overset{\alpha}{\equiv} b$ *and* $c_j \overset{\beta}{\equiv} d_j$*, if* $t(a, \mathbf{c}) \overset{\delta}{\equiv} t(a, \mathbf{d})$ *then* $t(b, \mathbf{c}) \overset{\delta}{\equiv} t(b, \mathbf{d})$.

Jargon: if $\delta = 0_A$, then condition (2) in the above lemma is often referred to as the α, β**-term condition**.

Proof. Clearly (1) implies (2). To prove that (2) implies (1), first note that (2) is equivalent to

(2)$'$. For all $t \in \mathrm{Clo}_{n+1}\,\mathbf{A}$ and all $a \overset{\alpha}{\equiv} b$ and $c_j \overset{\beta}{\equiv} d_j$ and $i \leq n$,

$$\begin{aligned} &t(c_0, \dots, c_{i-1}, \underline{a}, c_{i+1}, \dots, c_n) \overset{\delta}{\equiv} t(d_0, \dots, d_{i-1}, \underline{a}, d_{i+1}, \dots, d_n) \\ &\quad \Rightarrow \quad t(c_0, \dots, c_{i-1}, \underline{b}, c_{i+1}, \dots, c_n) \overset{\delta}{\equiv} t(d_0, \dots, d_{i-1}, \underline{b}, d_{i+1}, \dots, d_n). \end{aligned}$$

Now suppose $\begin{bmatrix} x & y \\ z & u \end{bmatrix} \in Mat(\alpha,\beta)$ and, say, $x \overset{\delta}{\equiv} y$. Pick (using Lemma 1.2) $t \in \mathrm{Clo}_{m+n}\,\mathbf{A}$ and $a_i \overset{\alpha}{\equiv} b_i$ and $c_j \overset{\beta}{\equiv} d_j$ such that

$$\begin{bmatrix} x & y \\ z & u \end{bmatrix} = \begin{bmatrix} t(\mathbf{a},\mathbf{c}) & t(\mathbf{a},\mathbf{d}) \\ t(\mathbf{b},\mathbf{c}) & t(\mathbf{b},\mathbf{d}) \end{bmatrix}.$$

We have

$$t(\underline{a_1}, a_2, \dots, a_m, \mathbf{c}) \overset{\delta}{\equiv} t(\underline{a_1}, a_2, \dots, a_m, \mathbf{d}),$$

and using $(2)'$ we get

$$t(\underline{b_1}, a_2, \dots, a_m, \mathbf{c}) \overset{\delta}{\equiv} t(\underline{b_1}, a_2, \dots, a_m, \mathbf{d}).$$

Let's rewrite the last line as

$$t(b_1, \underline{a_2}, \dots, a_m, \mathbf{c}) \overset{\delta}{\equiv} t(b_1, \underline{a_2}, \dots, a_m, \mathbf{d});$$

using $(2)'$ again we get

$$t(b_1, \underline{b_2}, \dots, a_m, \mathbf{c}) \overset{\delta}{\equiv} t(b_1, \underline{b_2}, \dots, a_m, \mathbf{d}).$$

Repeating this $m-2$ more times will yield

$$t(b_1, b_2, \dots, \underline{b_m}, \mathbf{c}) \overset{\delta}{\equiv} t(b_1, b_2, \dots, \underline{b_m}, \mathbf{d}),$$

i.e., $z \overset{\delta}{\equiv} u$ as required. A similar argument shows $z \overset{\delta}{\equiv} u \Rightarrow x \overset{\delta}{\equiv} y$. $\square$

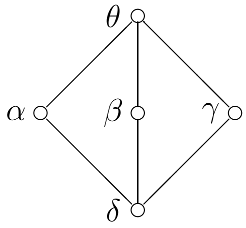
We now explore the idea that the shape of a congruence lattice can influence the existence of abelianness.

Lemma 2.4. *Suppose* $\mathbf{A}$ *is an algebra and* $\alpha_i, \beta, \delta \in \mathrm{Con}\,\mathbf{A}$ $(i \in I)$. *If* $C(\alpha_i, \beta; \delta)$ *for all* $i \in I$, *then* $C(\bigvee_{i \in I} \alpha_i, \beta; \delta)$.

Proof. Let $\alpha = \bigvee_{i\in I} \alpha_i$. We shall prove $C(\alpha,\beta;\delta)$ by verifying condition (2) in the statement of Lemma 2.3. So let $t \in \mathrm{Clo}_{1+n}\,\mathbf{A}$ and $a \overset{\alpha}{\equiv} b$ and $c_j \overset{\beta}{\equiv} d_j$ and suppose that $t(a,\mathbf{c}) \overset{\delta}{\equiv} t(a,\mathbf{d})$. By definition of α, we can choose $i_1, \dots, i_p \in I$ and $a_0, \dots a_p \in A$ so that $a = a_0 \overset{\alpha_{i_1}}{\equiv} a_1 \overset{\alpha_{i_2}}{\equiv} a_2 \overset{\alpha_{i_3}}{\equiv} \cdots \overset{\alpha_{i_p}}{\equiv} a_p = b$. We have assumed that $t(a_0,\mathbf{c}) \overset{\delta}{\equiv} t(a_0,\mathbf{d})$.

Inductively assume that we have $t(a_{k-1},\mathbf{c}) \overset{\delta}{\equiv} t(a_{k-1},\mathbf{d})$. Using $C(\alpha_{i_k},\beta;\delta)$ we get $t(a_k,\mathbf{c}) \overset{\delta}{\equiv} t(a_k,\mathbf{d})$. By induction we get $t(b,\mathbf{c}) \overset{\delta}{\equiv} t(b,\mathbf{d})$ as desired. $\square$

Theorem 2.5. *Suppose in an algebra* $\mathbf{A}$ *that the congruences* $\alpha,\beta,\gamma,\delta,\theta$ *form a sublattice of* $\mathrm{Con}\,\mathbf{A}$ *isomorphic to* M_3 *as shown.*

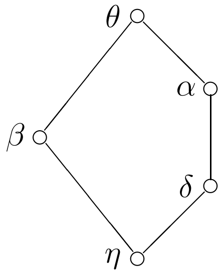


Then each of α, β, γ is abelian over δ.

Proof. $C(\alpha, \beta; \alpha)$ and $C(\alpha, \beta; \beta)$, and hence $C(\alpha, \beta; \alpha \wedge \beta)$ by Lemma 2.2; in other words, $C(\alpha, \beta; \delta)$. An identical argument shows $C(\gamma, \beta; \delta)$. Hence by Lemma 2.4, $C(\alpha \vee \gamma, \beta; \delta)$; i.e., $C(\theta, \beta; \delta)$. Then using Lemma 2.2(4) we get $C(\beta, \beta; \delta)$. This proves that β is abelian over δ which, by symmetry, suffices. $\square$

Problems

6. Assume **A** is either essentially unary or is a module over a ring. Prove that $C(\alpha, \beta; \delta)$ for all $\alpha, \beta, \delta \in \operatorname{Con} \mathbf{A}$. Conclude that **A** is abelian.
7. Assume **A** is either a boolean algebra or a lattice. Prove that $C(\alpha, \beta; \delta)$ iff $\alpha \wedge \beta \leq \delta$. Conclude that α is abelian over δ only if $\alpha = \delta$.
8. Suppose $\beta \wedge [\alpha \vee (\beta \wedge \delta)] \leq \delta$.
   (a) Prove that $C(\alpha, \beta; \delta)$.
   (b) If $\beta \vee \delta = \delta \circ \beta \circ \delta$, prove that $C(\alpha, \beta \vee \delta; \delta)$.
9. Suppose in an algebra that the congruences $\alpha, \beta, \delta, \eta, \theta$ form a sublattice of $\operatorname{Con} \mathbf{A}$ isomorphic to the pentagon as shown.



   Prove that $C(\alpha, \beta; \delta)$.

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3. Rectangulation and the Strong Centralizer Relation

Definition 3.1. Let **A** be an algebra and $\alpha, \beta, \delta \in \operatorname{Con} \mathbf{A}$.

1. We say that α **rectangulates** β **modulo** δ, and write $R(\alpha, \beta; \delta)$, if for all $\begin{bmatrix} x & y \\ z & u \end{bmatrix}$ in $Mat(\alpha, \beta)$, if $x \overset{\delta}{\equiv} u$ then $x \overset{\delta}{\equiv} y \overset{\delta}{\equiv} z \overset{\delta}{\equiv} u$.

2. We say that α **strongly centralizes** β **modulo** δ, and write $S(\alpha,\beta;\delta)$, if $C(\alpha,\beta;\delta)$ and $R(\alpha,\beta;\delta)$.
3. If $\delta \le \alpha$, then α is **strongly abelian over** δ if $S(\alpha,\alpha;\delta)$.
4. α is **strongly abelian** if $S(\alpha,\alpha;0_A)$.
5. $\mathbf{A}$ is **strongly abelian** if $S(1_A,1_A;0_A)$.

Lemma 3.2. *Suppose* $\mathbf{A}$ *is an algebra and* $\alpha,\beta,\alpha',\beta',\delta,\delta_i \in \operatorname{Con}\mathbf{A}$ *(*$i \in I$*).*

1. $S(\alpha,\beta;\alpha)$ *and* $S(\alpha,\beta;\beta)$.
2. *If* $S(\alpha,\beta;\delta_i)$ *for all* $i \in I$*, then* $S(\alpha,\beta;\bigwedge_{i\in I}\delta_i)$.
3. *If* $S(\alpha,\beta;\delta)$ *and* $\alpha' \le \alpha$ *and* $\beta' \le \beta$*, then* $S(\alpha',\beta';\delta)$.

Proof. Exercise. □

Lemma 3.3. *Suppose* $\mathbf{A}$ *is an algebra and* $\delta \le \alpha \in \operatorname{Con}\mathbf{A}$*. Then the following are equivalent:*

1. $S(\alpha,\alpha;\delta)$.
2. *For all* $t \in \operatorname{Clo}_{1+n}\mathbf{A}$ *and all* $a \overset{\alpha}{\equiv} b$ *and* $c_j \overset{\alpha}{\equiv} d_j \overset{\alpha}{\equiv} e_j$*, if* $t(a,\mathbf{c}) \overset{\delta}{\equiv} t(b,\mathbf{d})$ *then* $t(a,\mathbf{e}) \overset{\delta}{\equiv} t(b,\mathbf{e})$.

Jargon: if $\delta = 0_A$, then condition (2) in the above lemma is often referred to as the **strong term condition** for α.

Proof. (1) ⇒ (2): Suppose $S(\alpha,\alpha;\delta)$ and let t,a,b,c_j,d_j,e_j be as in (2) and suppose $t(a,\mathbf{c}) \overset{\delta}{\equiv} t(b,\mathbf{d})$. Check that

$$\begin{bmatrix} t(a,\mathbf{c}) & t(b,\mathbf{c}) \\ t(a,\mathbf{d}) & t(b,\mathbf{d}) \end{bmatrix}, \begin{bmatrix} t(a,\mathbf{c}) & t(b,\mathbf{c}) \\ t(a,\mathbf{e}) & t(b,\mathbf{e}) \end{bmatrix} \in Mat(\alpha,\alpha).$$

Applying $R(\alpha,\alpha;\delta)$ to the first matrix gives $t(a,\mathbf{c}) \overset{\delta}{\equiv} t(b,\mathbf{c})$. Then applying $C(\alpha,\alpha;\delta)$ to the second matrix gives $t(a,\mathbf{e}) \overset{\delta}{\equiv} t(b,\mathbf{e})$, proving (2).

To prove that (2) implies (1), first note that (2) is equivalent to

(2)′. For all $t \in \operatorname{Clo}_{n+1}\mathbf{A}$ and all $a \overset{\alpha}{\equiv} b$ and $c_j \overset{\alpha}{\equiv} d_j \overset{\alpha}{\equiv} e_j$ and $i \le n$,

$$\begin{aligned} &t(c_0,\dots,c_{i-1},\underline{a},c_{i+1},\dots,c_n) \overset{\delta}{\equiv} t(d_0,\dots,d_{i-1},\underline{b},d_{i+1},\dots,d_n) \\ &\quad\Rightarrow\quad t(e_0,\dots,e_{i-1},\underline{a},e_{i+1},\dots,e_n) \overset{\delta}{\equiv} t(e_0,\dots,e_{i-1},\underline{b},e_{i+1},\dots,e_n). \end{aligned}$$

We shall first use (2)′ to prove

(3). For all $t \in \operatorname{Clo}_{m+n}\mathbf{A}$ and all $a_i \overset{\alpha}{\equiv} b_i$ and $c_j \overset{\alpha}{\equiv} d_j \overset{\alpha}{\equiv} e_j$, if $t(\mathbf{a},\mathbf{c}) \overset{\delta}{\equiv} t(\mathbf{b},\mathbf{d})$ then $t(\mathbf{a},\mathbf{e}) \overset{\delta}{\equiv} t(\mathbf{b},\mathbf{e})$.

We prove this by showing inductively that $t(b_1,b_2,\dots,b_{i-1},a_i,\dots,a_m,\mathbf{e}) \overset{\delta}{\equiv} t(\mathbf{a},\mathbf{e})$. For $i = 0$ the claim is clear. Suppose $i < m$ and $t(b_1,\dots,b_{i-1},a_i,\dots,a_m,\mathbf{e}) \overset{\delta}{\equiv} t(\mathbf{a},\mathbf{e})$

holds. From $t(a_1, \dots, \underline{a_i}, \dots, a_m, \mathbf{c}) \overset{\delta}{\equiv} t(b_1, \dots, \underline{b_i}, \dots, b_m, \mathbf{d})$ and using $(2)'$, we get $t(\mathbf{a}, \mathbf{e}) \overset{\delta}{\equiv} t(a_1, \dots, a_{i-1}, \underline{b_i}, a_{i+1} \dots, a_m, \mathbf{e})$ and thus by transitivity

$$t(b_1, \dots, b_{i-1}, \underline{a_i}, a_{i+1}, \dots, a_m, \mathbf{e}) \overset{\delta}{\equiv} t(a_1, \dots, a_{i-1}, \underline{b_i}, a_{i+1} \dots, a_m, \mathbf{e}).$$

Another application of $(2)'$ yields

$$t(b_1, \dots, b_{i-1}, \underline{a_i}, a_{i+1}, \dots, a_m, \mathbf{e}) \overset{\delta}{\equiv} t(b_1, \dots, b_{i-1}, \underline{b_i}, a_{i+1} \dots, a_m, \mathbf{e}),$$

which completes the inductive argument. With $i = m$ we have established (3). We leave the proof that (3) $\Rightarrow$ (1) for the reader.

□

Problems

8. If $\mathbf{A}$ is essentially unary, prove that $R(\alpha, \beta; \delta)$ for all $\alpha, \beta, \delta \in \operatorname{Con} \mathbf{A}$. Conclude that $\mathbf{A}$ is strongly abelian.
9. If $\mathbf{A}$ is a module over a ring, prove that $R(\alpha, \beta; \delta) \Rightarrow \alpha \wedge \beta \leq \delta$. Conclude that α is strongly abelian over δ only if $\alpha = \delta$.

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4. Commutators, solvability, and notions of nilpotency

By analogy with group theory, we can develop notions of derived congruences, solvability and nilpotency from either the centralizer or the strong centralizer relation.

Definition 4.1. Suppose $\mathbf{A}$ is an algebra and $\alpha, \beta, \delta \in \operatorname{Con} \mathbf{A}$.

1. By Lemma 2.2(2) there is a smallest $\widehat{\delta} \in \operatorname{Con} \mathbf{A}$ satisfying $C(\alpha, \beta; \widehat{\delta})$. This smallest $\widehat{\delta}$ is called the **commutator of** α **and** β and is written $[\alpha, \beta]$.
2. By Lemma 3.2(2) there is a smallest $\widehat{\delta} \in \operatorname{Con} \mathbf{A}$ satisfying $S(\alpha, \beta; \widehat{\delta})$. This smallest $\widehat{\delta}$ is called the **strong commutator of** α **and** β and is written $[\alpha, \beta]_{\mathbf{S}}$.
3. By Lemma 2.4, given $\beta, \delta \in \operatorname{Con} \mathbf{A}$ there is a largest $\widehat{\alpha} \in \operatorname{Con} \mathbf{A}$ satisfying $C(\widehat{\alpha}, \beta; \delta)$. This largest $\widehat{\alpha}$ is called the **annihilator of** β **over** δ and is written $(\delta : \beta)$.
4. α is **solvable (strongly solvable) over** δ if there exists an ascending chain of congruences $\delta = \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n = \alpha$ such that α_{i+1} is abelian (strongly abelian) over α_i for each $i < n$.
5. α is **solvable (strongly solvable)** if it is solvable (strongly solvable) over 0_A.
6. α is **left-nilpotent (strongly left-nilpotent)** if there exists an ascending chain of congruences $0_A = \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n = \alpha$ such that $C(\alpha, \alpha_{i+1}; \alpha_i)$ ($S(\alpha, \alpha_{i+1}; \alpha_i)$) for each $i < n$.

7. α is **right-nilpotent (strongly right-nilpotent)** if there exists an ascending chain of congruences $0_A = \alpha_0 \le \alpha_1 \le \cdots \le \alpha_n = \alpha$ such that $C(\alpha_{i+1}, \alpha; \alpha_i)$ ($S(\alpha_{i+1}, \alpha; \alpha_i)$) for each $i < n$.

Problems

10. Prove that in any algebra,
    (a) $[\alpha, \beta] \le [\alpha, \beta]_{\mathbf{S}} \le \alpha \wedge \beta$
    (b) $\alpha' \le \alpha$ and $\beta' \le \beta$ imply $[\alpha', \beta'] \le [\alpha, \beta]$ and $[\alpha', \beta']_{\mathbf{S}} \le [\alpha, \beta]_{\mathbf{S}}$.
11. Define $\mathbf{A} = \langle \mathbb{Z}_4, f(x,y) \rangle$ where $f(x,y) = x + 2y \mod 4$. Show that $\mathbf{A}$:
    (a) is abelian;
    (b) is not strongly abelian;
    (c) is strongly solvable;
    (d) has a nonabelian homomorphic image.
    Bonus: Is the nonabelian homomorphic image you found left-nilpotent? Right-nilpotent? Strongly left-nilpotent? Strongly right-nilpotent?
12. Given an algebra $\mathbf{A}$ and a congruence $\alpha \in \operatorname{Con} \mathbf{A}$, define $\alpha^{[0]} = \alpha$ and $\alpha^{[k+1]} = [\alpha^{[k]}, \alpha^{[k]}]$. Prove that α is solvable iff $\alpha^{[n]} = 0_A$ for some n.
13. Given an algebra $\mathbf{A}$ and a congruence $\alpha \in \operatorname{Con} \mathbf{A}$, define $\alpha^{(0]} = \alpha^{[0)} = \alpha$ and $\alpha^{(k+1]} = [\alpha, \alpha^{(k]}]$, $\alpha^{[k+1)} = [\alpha^{[k)}, \alpha]$. Prove that α is left- (right-) nilpotent iff $\alpha^{(n]} = 0_A$ ($\alpha^{[n)} = 0_A$) for some n.

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5. Postscript

The algebra in Problem 11 shows that the analogy with groups is not perfect, in that a homomorphic image of an abelian algebra need not be abelian. A striking illustration of this is offered by the fact that, in any similarity type, the absolutely free algebras are strongly abelian. As a consequence, *every* algebra is a homomorphic image of a strongly abelian algebra.

The centralizer relation and its corresponding commutator operation are very well-behaved in congruence modular varieties (though proving this is beyond the scope of these notes); for example, homomorphic images of abelian algebras in congruence modular varieties are always abelian. We will see in later lectures that, for combinatorial reasons, the centralizer and strong centralizer are moderately well-behaved in finite algebras. For example, we will see that homomorphic images of finite abelian (strongly abelian) algebras are at least solvable (strongly solvable). In the seminal paper [4] Kearnes improves this by showing that a homomorphic image of a finite abelian algebra is always right-nilpotent. As well, Kearnes shows that in finite algebras, right-nilpotent congruences are always left-nilpotent.

References

[1] R. Freese and R. McKenzie, *Commutator Theory for Congruence Modular Varieties*, London Math. Soc. Lecture Notes No. 125, 1987.
[2] D. Hobby and R. McKenzie, *The Structure of Finite Algebras*, Contemporary Mathematics, Vol. 76, Amer. Math. Soc., 1988.
[3] K. Kearnes and E. Kiss, Finite algebras of finite complexity, *Discrete Math.* **207** (1999), 89–135.
[4] K. Kearnes, An order-theoretic property of the commutator, *Internat. J. Algebra Comput.* **3** (1993), 491–533.

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