Notes on minimal sets and their types

Paweł M. Idziak, June 2001

The material is based on Chapters 2 and 4 of [1].

From P.P.Pálfy's Theorem we know that each minimal algebra is polynomially equivalent to one of the following:

1. a finite set with a group action on it,

2. a finite vector space over a finite field,

3. a two element Boolean algebra,

4. a two element lattice,

5. a two element semilattice.

We are interested in minimal sets in finite algebras. Suppose $\alpha \prec \beta$ is a prime quotient in $\mathsf{Con}\,\mathbf{A}$. A set $U \subseteq A$ is called (α, β)-minimal if it is minimal, with respect to inclusion, in the set $\{f(A) : f \in \mathrm{Pol}_1\mathbf{A} \text{ and } f(\beta) \not\subseteq \alpha\}$. Basic properties of minimal sets are listed below:

1. all (α, β)-minimal sets are polynomially isomorphic,
2. each (α, β)-minimal set is a range of a unary idempotent polynomial,
3. if U is (α, β)-minimal and a unary polynomial f does not collapse $\beta|_U$ to α then $f(U)$ is (α, β)-minimal, and $f|_U$ establishes a polynomial isomorphism between U and $f(U)$,
4. if U is (α, β)-minimal then any pair $(a, b) \in \beta - \alpha$ can be mapped by a unary polynomial f into U in a way that $(fa, fb) \in \beta|_U - \alpha$, in particular $f(A) = U$,
5. every unary polynomial that does not collapse β to α is a polynomial isomorphism between two (α, β)-minimal sets.

By an (α, β)-trace in the (α, β)-minimal set U we mean any set of the form $a/\beta \cap U$ not contained in a/α. The body, or (α, β)-body, of U is the union of all (α, β)-traces. The rest of U is its tail.

If the prime quotient $\alpha_0 \prec \beta_0$ transposes up to $\alpha_1 \prec \beta_1$, i.e., $\alpha_1 \wedge \beta_0 = \alpha_0$ and $\alpha_1 \vee \beta_0 = \beta_1$ then those two prime quotients have the same minimal sets. However the respective traces can be different.

Moreover, if $\theta \subseteq \alpha \prec \beta$ then $(\alpha/\theta, \beta/\theta)$-minimal sets in $\mathbf{A}/\theta$ are precisely the quotients $U/\theta|_U$, where U ranges over (α, β)-minimal sets of $\mathbf{A}$.

If U is an (α, β)-minimal set in the algebra $\mathbf{A}$ then $\alpha|_U \prec \beta|_U$ in the congruence lattice of the induced algebra $\mathbf{A}|_U$ and this algebra $\mathbf{A}|_U$ is (α, β)-minimal. Thus to understand the structure of (α, β)-minimal sets it suffices to understand (α, β)-minimal algebras.

Moreover, if N is an (α, β)-trace of an (α, β)-minimal algebra $\mathbf{A}$ then $\mathbf{A}|_N/\alpha|_N$ is a minimal algebra, and the type of this minimal algebra is said to be the type of the trace N.

The following list useful properties of minimal algebras and their traces:

- all traces in an (α, β)-minimal algebra are polynomially isomorphic and therefore have the same type,
- if an (α, β)-minimal algebra contains at least two (α, β)-traces then they are of type **1** or **2**.

This allows to introduce the type of the prime quotient $\alpha \prec \beta$.

The structure of the (α, β)-minimal algebras (sets) is described depending of their type. We assume that $\mathbf{C}$ is a finite (α, β)-minimal algebra with the body B and the tail T.

TYPE 5

- for the unique (α, β)-trace N of $\mathbf{C}$ the algebra $\mathbf{C}|_N/\alpha|_N$ is polynomially isomorphic to the 2-element semilattice,

- there are elements $0, 1 \in N$ and a binary polynomial $p(x, y)$ such that

  - $N = 0/\alpha|_N \cup 1/\alpha|_N$,
  - $1/\alpha = \{1\}$,
  - N is closed under p and $(N; p)/\alpha|_N$ is a semilattice with neutral element $\{1\}$, actually $\mathbf{C}|_N/\alpha|_N$ is polynomially equivalent to $(N; p)/\alpha|_N$
  - $p(x, x) = x = p(x, 1) = p(1, x)$ for all $x \in C$,
  - $p(x, p(x, y)) = p(x, y)$ for all $x, y \in C$,
  - $p(x, z) \overset{\alpha}{\equiv} x \overset{\alpha}{\equiv} p(z, x)$ for all $u \in 0/\alpha|_N$ and $x \in C - \{1\}$.

TYPE 4 and 3

- for the unique (α, β)-trace N of $\mathbf{C}$ the algebra $\mathbf{C}|_N/\alpha|_N$ is polynomially isomorphic to the 2-element lattice,

- there are elements $0, 1 \in N$ and binary polynomials $p(x, y), q(x, y)$ (and additionally a unary polynomial $c(x)$ in type **3** case) such that

  - $N = \{0, 1\}$, in particular $\alpha|_N$ is trivial,
  - in type **4** case: N is closed under p, q and $(N; p, q)$ is a lattice, actually $\mathbf{C}|_N/\alpha|_N$ is polynomially equivalent to $(N; p, q)$
  - in type **3** case: N is closed under p, q, c and $(N; p, q, c)$ is a Boolean algebra, actually $\mathbf{C}|_N/\alpha|_N$ is polynomially equivalent to $(N; p, q, c)$
  - $p(x, x) = x = p(x, 1) = p(1, x)$ and $q(x, x) = x = q(x, 0) = q(0, x)$ for all $x \in C$,
  - $p(x, p(x, y)) = p(x, y)$ and $q(x, q(x, y)) = q(x, y))$ for all $x, y \in C$,
  - $p(x, 0) \overset{\alpha}{\equiv} x \overset{\alpha}{\equiv} p(0, x)$ and $q(x, 1) \overset{\alpha}{\equiv} x \overset{\alpha}{\equiv} q(1, x)$ for all $x \in T$.

TYPE 2

- for every (α, β)-trace N of $\mathbf{C}$ the algebra $\mathbf{C}|_N/\alpha|_N$ is polynomially isomorphic to a one dimensional vector space over a finite field,
- there is a ternary polynomial $d(x, y, z)$ of $\mathbf{C}$ satisfying
  - $d(x, x, x) = x$ for all $x \in C$,
  - $d(x, x, y) = y = d(y, x, x)$ for all $x \in B$ and $y \in C$,
  - there are no $t \in T$ and $b \in B$ such that $d(t, t, b) \overset{\beta}{\equiv} b$ or $d(b, t, t) \overset{\beta}{\equiv} b$,
  - B is closed under d,
  - for all $a, b \in B$ the unary polynomials $d(x, a, b), d(a, x, b)d(a, b, x)$ are permutations of C,

TYPE 1

- for every (α, β)-trace N of $\mathbf{C}$ the algebra $\mathbf{C}|_N/\alpha|_N$ is polynomially isomorphic to a simple G-set.

Exercises

1. Show that if a finite algebra has a global semilattice operation then it has no trace of type **1** or **2**.

2. Suppose that a finite algebra **A** generates a congruence distributive variety.

   (a) show that each of its trace is of type **3** or **4**,

   (b) show that minimal sets in **A** have no tails.

3. Suppose that a finite algebra **A** generates a congruence permutable variety.

   (a) show that each of its trace is of type **2** or **3**,

   (b) show that minimal sets in **A** have no tails.

4. Let $\alpha_1, \alpha_2, \alpha_3$ be all atoms in the congruence lattice of the group $\mathbf{Z}_2 \times \mathbf{Z}_2$. Determine $(0, \alpha_i)$- and $(\alpha_i, 1)$-minimal sets and their traces.

5. Describe minimal sets and traces in the ring $\mathbf{Z}_8$, or more generally in the rings $\mathbf{Z}_{p^k}$ for a prime number p.

6. Describe minimal sets and traces in the dihedral group $\mathbf{D}_4$.

7. Show that in any finite simple non Abelian group every two element subset is $(0, 1)$-minimal.

8. Let **V** be a one dimensional vector space over Galois field $\mathbf{GF}(p^k)$.

   (a) Observe that **V** is a minimal algebra with one trace.

   (b) **V** can be treated, in a natural way, as a k dimensional vector space over $\mathbf{GF}(p)$. Describe minimal sets and traces in this k dimensional vector space.

References

[1] D. Hobby and R. McKenzie, *The Structure of Finite Algebras*, Contemporary Mathematics vol. 76, Amer. Math. Soc., Providence, RI, 1988.