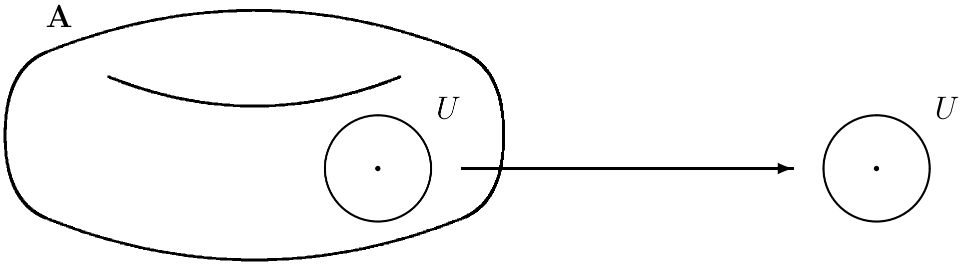
LECTURE I:
TAME CONGRUENCE THEORY IS A LOCALIZATION THEORY

KEITH A. KEARNES

1. Introduction

The purpose of this lecture is to describe the intuition behind tame congruence theory. The most important piece of information to take from this lecture is that tame congruence theory is a localization theory. This means that the theory is based on a method for selecting small subsets of an algebra, restricting structure to that subset, calculating locally, and piecing together local data to solve globally stated problems.



There are three main ingredients to a localization theory.

(1) Localization: Identify subsets which support good local approximations. Describe how to restrict structure to these "neighborhoods".
(2) Classification: Describe how to calculate locally.
(3) Globalization: Describe how to combine local results.

In this lecture we describe ingredients (1) and (3). Ingredient (2) will be discussed in later lectures.

2. Localization

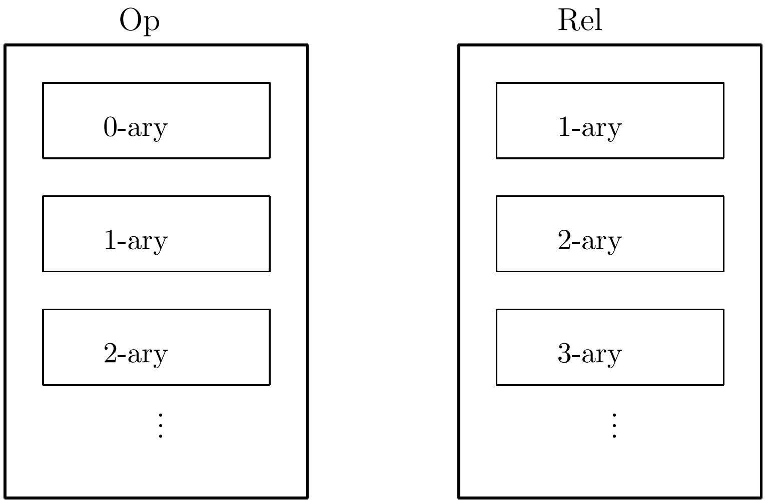
Let A be a finite set. Let Op be the graded set of all finitary operations on A and let Rel be the graded set of all finitary relations on A. The relation

$$\text{compatibility} \subseteq \mathrm{Op} \times \mathrm{Rel}$$

defined by

$$(f, R) \in \text{compatibility} \Longleftrightarrow f(R, R, \ldots, R) \subseteq R$$

determines a Galois connection between Op and Rel.



Lemma 2.1. *A graded subset $\mathcal{C}$ of* Op *is closed in this Galois connection if it is a collection of operations closed under*

(i) *composition* (*comp*)*; and*
(ii) *the projections* (p_i^n).

A subset $\mathcal{R}$ of Rel *is closed in this Galois connection if it is a collection of relations closed under*

(i) *intersection* ($\cap$)*;*
(ii) *products* ($\times$)*;*
(iii) *projection onto a subset of coordinates* (*proj*)*;*
(iv) *permutation of coordinates* (Π)*; and*
(v) *the equality relation* ($=$).

Definition 2.2. The closed subsets of Op called *clones*. The closed subsets of Rel are called *relational clones*. An *algebra* is a pair $\mathbf{A} = \langle A; \mathcal{C} \rangle$ where $\mathcal{C}$ is a clone on A. We write $\mathbf{A}^\perp$ for $\langle A; \mathcal{R} \rangle$ where $\mathcal{R} = \mathcal{C}^\perp$. Let $\mathbf{A}^{\perp\perp} = \mathbf{A}$.

Note that either of the structures $\mathbf{A}$ or $\mathbf{A}^\perp$ determines the other. If one wanted to study local approximations of an algebra considered in its operational form, $\mathbf{A}$, one might consider subsets $B \subseteq A$ such that restriction to B is a homomorphism from the clone of $\mathbf{A}$ to the clone of all operations on the set B. Such subsets are called "subalgebras". This type of localization theory leads to the study of an algebra by its system of subalgebras. TCT may be viewed as a theory that studies local approximations of an algebra considered in its relational form, $\mathbf{A}^\perp$.

Theorem 2.3. *If $\mathbf{A}^\perp = \langle A; \mathcal{R}\rangle$ and $U \subseteq A$, then restriction to U is a homomorphism of*

$$\mathcal{R} = \langle R; \cap, \times, proj, \Pi, =\rangle$$

into the relational clone of all relations on U iff $U = e(A)$ for some $e \in C_1$ for which

$$\mathbf{A} \models e(e(x)) = e(x).$$

Sketch of Proof. For one direction, show that if $U = e(A)$, then restriction to U preserves $\cap, \times, proj, \Pi$, and $=$.

For the other direction, show that if restriction to U preserves projection of A-ary relations onto the coordinates in U, then there is an $e \in C_1$ such that $e^2 = e$ and $e(A) = U$. Specifically, let

$$T = \{(t(a_i))_{i<|A|} \mid t \in C_1(\mathbf{A})\}$$

be the A-ary relation consisting of graphs of unary terms, let $proj_U$ be projection onto the coordinates in U, and let ρ_U be restriction to U. Note that $\rho_U(proj_U(T))$ contains the graph of the identity function on U. If $\rho_U(proj_U(T)) = proj_U(\rho_U(T))$, then $proj_U(\rho_U(T))$ also contains the graph of the identity function on U. Thus T must contain the graph of a unary term whose range is in U and which is the identity function on U; i.e., an idempotent unary term with range U.

The previous result identifies which subsets are appropriate for localization:

Definition 2.4. Let $\mathbf{A}$ be an algebra. A set $U \subseteq A$ is a *neighborhood* if $U = e(A)$ for some idempotent $e \in C_1$.

Definition 2.5. Let $U, V \subseteq A$ be neighborhoods of $\mathbf{A}$. U is *isomorphic* to V (written $U \simeq V$) if $\langle U; \mathcal{R}|_U\rangle \cong \langle V; \mathcal{R}|_V\rangle$.

We can identify isomorphic localizations of an algebra $\mathbf{A}$ without computing $\mathbf{A}^\perp$ and trying to verify that $\mathbf{A}^\perp|_U \cong \mathbf{A}^\perp|_V$.

Lemma 2.6. *U is isomorphic to V if and only if there exist $s, t \in C_1$ such that $s : U \to V$ and $t : V \to U$ are inverse bijections.*

Sketch of Proof. If such $s, t \in C_1$ exist, then they are inverse relational morphisms $s : \langle U; \mathcal{R}|_U\rangle \to \langle V; \mathcal{R}|_V\rangle$ and $t : \langle V; \mathcal{R}|_V\rangle \to \langle U; \mathcal{R}|_U\rangle$.

Conversely, assume that $U = e(A)$ and $V = f(A)$ for idempotents $e, f \in C_1$. If one has inverse morphisms $\sigma : \langle U; \mathcal{R}|_U\rangle \to \langle V; \mathcal{R}|_V\rangle$ and $\tau : \langle V; \mathcal{R}|_V\rangle \to \langle U; \mathcal{R}|_U\rangle$, then the functions $s = \sigma \circ e$ and $t = \tau \circ f$ are morphisms from $\mathbf{A}^\perp$ to itself which restrict to U and V to give σ and τ respectively. Since clones are the closed objects of the Galois connection between operations and relations, morphisms from $\mathbf{A}^\perp$ to itself are realized by unary terms, so $s, t \in C_1$ are terms which restrict to U and V to give the desired isomorphism.

It is clear how to restrict the relational structure of $\mathbf{A}^\perp = \langle A; \mathcal{R}\rangle$ to a neighborhood U: simply restrict each relation $S \in R_n$ to U in the usual way ($S|_U = S \cap U^n$). Thus $\mathbf{A}^\perp|_U = \langle U; \mathcal{R}|_U\rangle$. Since U is a neighborhood the restriction map is a relational clone homomorphism, so $\mathcal{R}|_U$ is a relational clone on U. As such it corresponds to an algebra $(\mathbf{A}^\perp|_U)^\perp$ on U.

$$\begin{array}{ccc} \mathbf{A} & \longrightarrow & \mathbf{A}^\perp = \langle A; \mathcal{R}\rangle \\ & & \downarrow \\ \mathbf{A}|_U = (\mathbf{A}^\perp|_U)^\perp & \longleftarrow & \mathbf{A}^\perp|_U = \langle U; \mathcal{R}|_U\rangle \end{array}$$

This leads us to our definition of the induced algebra on U.

Definition 2.7. If $U = e(A)$ is a neighborhood of $\mathbf{A}$, then *the algebra that* $\mathbf{A}$ *induces on* U, written $\mathbf{A}|_U$ or $e(\mathbf{A})$ is $(\mathbf{A}^\perp|_U)^\perp$.

Lemma 2.8. $\mathbf{A}|_U = \langle U; e(\mathcal{C})\rangle$ *where* $e(\mathcal{C}) = \{et \mid t \in \mathcal{C}\} = \bigcup_n \{t \in C_n \mid t(U^n) \subseteq U\}$.

3. Globalization

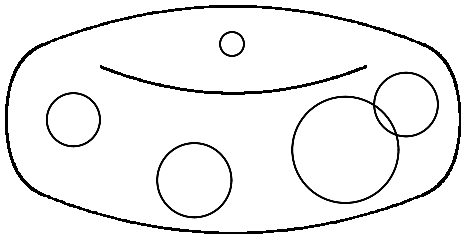
The companion to a localization theory is a globalization theory. It is natural to expect to attack a problem with a localization theory by translating the problem into a family of local problems, solving them locally, and then combining the local results into a global result. There is a construction called the matrix product, described below, which allows the reconstruction of an algebra up to definitional equivalence from sufficiently many localizations. This construction constitutes an adequate globalization theory.

Fully reconstructing an algebra from a family of localizations may be impractical in most cases, but at the same time a full reconstruction is rarely necessary. Instead, it is usually more convenient to completely translate a problem into a family of local problems, and then solve the local problems. For this method, the role of the globalization theory is merely to identify how many localizations are "enough". "Enough" means roughly: enough so that restricting attention to these local algebras represents no loss of information. A collection of neighborhoods will be called a cover if restriction to these neighborhoods represents no loss of information.

Definition 3.1. A *cover* of $\mathbf{A}$ is a set $\mathcal{U}$ of neighborhoods for which

$$\bigwedge_{U \in \mathcal{U}} S|_U = T|_U \Longrightarrow S = T$$

for all $S, T \in \mathcal{R}$.



Remark 3.2. So, $\mathcal{U}$ is a cover if the sequence of relational clone homomorphisms $\rho_U : \mathcal{R} \to \mathcal{R}|_U$ is jointly 1-1.

Theorem 3.3. *The following are equivalent.*

(i) $\mathcal{U}$ *is cover of* $\mathbf{A}$.

(ii) $\mathbf{A}$ *satisfies an equation of the form*

$$\lambda(e_1\rho_1(x), \dots, e_m\rho_m(x)) = x$$

where $e_i(A) \in \mathcal{U}$ *for all* i.

(iii) $\mathbf{A}^\perp$ *is a retract of a product of relational structures from the set*

$$\left\{\mathbf{A}^\perp|_U \mid U \in \mathcal{U}\right\}.$$

Sketch of Proof. For (i) implies (ii): Let $T = \{(t(a_i))_{i<|A|} \mid t \in C_1(\mathbf{A})\}$ be the A-ary relation consisting of graphs of unary terms. Let

$$S = \{(t(a_i))_{i<|A|} \mid t(x) = \lambda(e_1\rho_1(x), \dots, e_m\rho_m(x)) \in C_1(\mathbf{A}), e_i^2 = e_i, e_i(A) \in \mathcal{U}\}$$

be the relation consisting of graphs of *certain* unary terms. (Note: For any $U_i = e_i(A)$ the relation $T|_{U_i}$ consists of the graphs of unary terms $e_i\rho_i$, where ρ_i is an arbitrary unary term. Thus, S is the compatible relation of $\mathbf{A}$ generated by all sets $T|_{U_i}$. As such we have $T|_{U_i} \subseteq S \subseteq T$ for all i.)

Note that $S|_U = T|_U$ for all $U \in \mathcal{U}$. If (i) holds than this implies that $S = T$, so S contains the graph of the identity function. This implies that (ii) holds.

If (ii) holds, then $\Lambda = \lambda(x_1, \dots, x_m)$ and $ER = (e_1\rho_1(x), \dots, e_m\rho_m(x))$ are morphisms between $\mathbf{A}^\perp$ and $\mathbf{A}^\perp|_{U_1} \times \cdots \times \mathbf{A}^\perp|_{U_m}$ satisfying $\Lambda \circ ER = \mathrm{id}_A$. Thus, (iii) holds.

Now assume that (iii) holds. Choose compatible relations $S \subseteq T$ such that $S|_U = T|_U$ for all $U \in \mathcal{U}$. Thus $S = T$ in $\mathbf{A}^\perp|_{U_1} \times \cdots \times \mathbf{A}^\perp|_{U_m}$, and hence in any retract. From (iii) we get that $S = T$, establishing that $\mathcal{U}$ is a cover.

Now we introduce the matrix product.

Definition 3.4. Let $\mathcal{U} = \{U_1, \dots, U_m\}$ be a set of neighborhoods of $\mathbf{A}$. The *matrix product* of $\mathbf{A}|_{U_1}, \dots, \mathbf{A}|_{U_m}$ is

$$\mathbf{A}|_{U_1} \boxtimes \cdots \boxtimes \mathbf{A}|_{U_m} = \left(\left(\mathbf{A}^\perp|_{U_1}\right) \times \cdots \times \left(\mathbf{A}^\perp|_{U_m}\right)\right)^\perp.$$

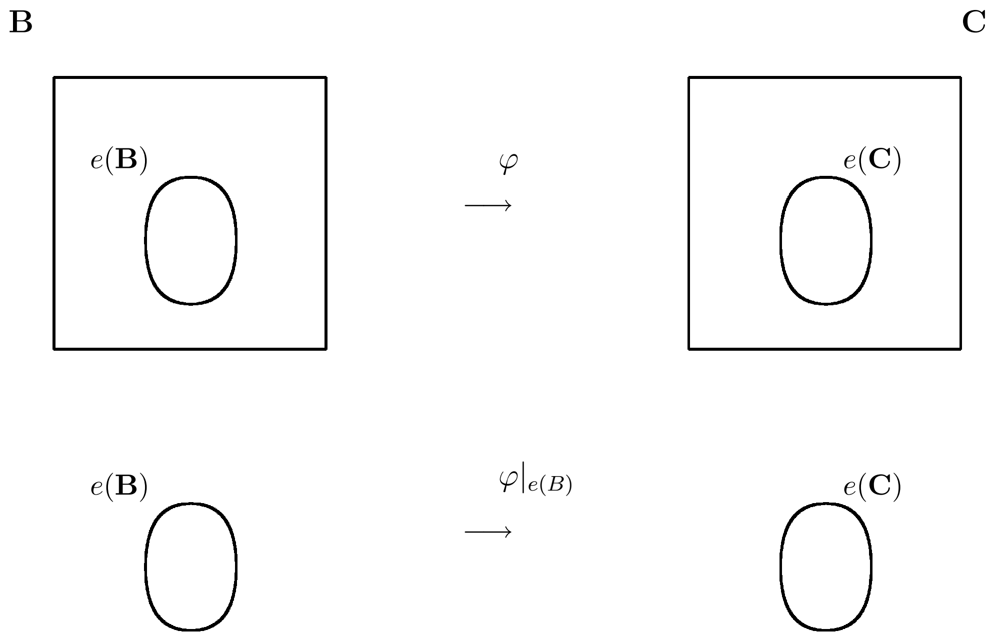
Lemma 3.5. $\mathbf{A}|_{U_1} \boxtimes \cdots \boxtimes \mathbf{A}|_{U_m}$ *is the algebra on* $U_1 \times \cdots \times U_m$ *with operations*

$$F\underbrace{\left(\begin{bmatrix} x_{11} \\ \vdots \\ x_{m1} \end{bmatrix}, \dots, \begin{bmatrix} x_{1n} \\ \vdots \\ x_{mn} \end{bmatrix}\right)}_{M} = \begin{bmatrix} e_1 f_1(M) \\ \vdots \\ e_m f_m(M) \end{bmatrix}$$

where $e_i^2 = e_i \in C_1(\mathbf{A}), f_i \in C_{mn}(\mathbf{A}),$ *and* $e_i(A) = U_i$.

4. The Localization Functor

Let $\mathcal{V}$ be a variety of algebras, and let $e(x)$ be a unary term for which $\mathcal{V} \models e(e(x)) = e(x)$. The class of algebras of the form $e(\mathbf{A})$ for $\mathbf{A} \in \mathcal{V}$ is a variety, which we denote by $e(\mathcal{V})$. As indicated in the following diagram, the assignments $\mathbf{B} \mapsto e(\mathbf{B})$, $\varphi \mapsto \varphi|_{e(B)}$ is a functor from $\mathcal{V}$ to $e(\mathcal{V})$.



The fact that these assignments determine a functor follows from the equation $\varphi(e(b)) = e(\varphi(b))$. Thus, if $\mathbf{A}$ is an algebra, $e^2 = e \in C_1(\mathbf{A})$, and $U = e(A)$, then localization to U is part of functor from $\mathcal{V}(\mathbf{A})$ to $\mathcal{V}(\mathbf{A}|_U)$. It is not hard to generalize this to the observation that if $\mathcal{U} = \{U_1, \dots, U_m\}$ is a set of neighborhoods of $\mathbf{A}$, then there is a functor from $\mathcal{V}(\mathbf{A})$ to $\mathcal{V}(\mathbf{A}|_{U_1} \boxtimes \cdots \boxtimes \mathbf{A}|_{U_m})$ which maps $\mathbf{A}$ to the matrix product $\mathbf{A}|_{U_1} \boxtimes \cdots \boxtimes \mathbf{A}|_{U_m}$.

Theorem 4.1. *If* $\mathcal{U} = \{U_1, \dots, U_m\}$ *is a cover of* $\mathbf{A}$, *then there is a categorical equivalence from* $\mathcal{V}(\mathbf{A})$ *to* $\mathcal{V}(\mathbf{A}|_{U_1} \boxtimes \cdots \boxtimes \mathbf{A}|_{U_m})$ *which maps* $\mathbf{A}$ *to* $\mathbf{A}|_{U_1} \boxtimes \cdots \boxtimes \mathbf{A}|_{U_m}$.

5. Irreducible and Minimal Sets

From now on $\mathbf{A}$ is finite. If $\mathcal{U} = \{U_1, \ldots, U_m\}$ covers $\mathbf{A}$, then (we have seen that) $\mathbf{A}$ is "reconstructible" from $\mathbf{A}|_{U_1}, \ldots, \mathbf{A}|_{U_m}$. We may try to further decompose each $\mathbf{A}|_{U_i}$ by the same method. This leads to the concept of a refinement of a cover.

Definition 5.1. The set $\mathcal{V}$ of neighborhoods *covers* the neighborhood U if

$$\bigwedge_{V \in \mathcal{V}} S|_V = T|_V \Longrightarrow S|_U = T|_U$$

for all $S, T \in \mathcal{R}$.

Equivalently,

$$\mathbf{A} \models \lambda(e_1\rho_1(x), \ldots, e_n\rho_n(x)) = e(x)$$

with $e_i(A) \in \mathcal{V}$ and $e(A) = U$.

Definition 5.2. $\mathcal{V}$ *refines* $\mathcal{U}$ if each $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$ and $\mathcal{V}$ covers each $U \in \mathcal{U}$.

Theorem 5.3. *Any finite algebra has a unique irredundant nonrefinable cover up to isomorphism.*

Sketch of Proof. For each n, and each join irreducible relation T with lower cover S in the meet semilattice R_n, choose a neighborhood $U \subseteq A$. Show that the choice of U is determined up to isomorphism by $\langle S, T\rangle$. Let $\mathcal{U}$ be the set of maximal neighborhoods from this collection. Show that every cover can be refined to one like this one.

Covers of finite algebras cannot be refined indefinitely. We must eventually reach a point (identified in the previous theorem) where we have a cover $\mathcal{U}$ where each $U \in \mathcal{U}$ fails to be covered by its proper subneighborhoods. Equivalently, any cover of the algebra $\mathbf{A}|_U$ must include the universe of the algebra. This leads to the definition of an irreducible neighborhood.

Definition 5.4. A neighborhood $U \subseteq A$ is *irreducible* if every cover of $\mathbf{A}|_U$ contains the set U. U is $\langle S, T\rangle$*-irreducible* if it is minimal under inclusion among neighborhoods V for which $S|_V \neq T|_V$.

Theorem 5.5. *A neighborhood is irreducible iff it is $\langle S, T\rangle$-irreducible for some $S \subsetneq T$.*

Sketch of Proof. It suffices to prove it in the case where $U = A$. Fix an enumeration $(a_i)_{i<|A|}$ of A. Let $T = \{(t(a_i))_{i<|A|} \mid t \in C_1(\mathbf{A})\}$ and let

$$S = \{(t(a_i))_{i<|A|} \mid t(x) = \lambda(e_1\rho_1(x), \ldots, e_m\rho_m(x)) \in C_1(\mathbf{A}), e_i^2 = e_i, e_i(A) \neq A\}.$$

Show that $S|_V = T|_V$ for all $V \subsetneq A$, but $S \neq T$ if $U = A$ is irreducible.

It turns out that irreducible sets satisfy a stronger proper called minimality. This stronger property is useful for classifying the structure on an irreducible set.

Definition 5.6. A neighborhood $U \subseteq A$ is $\langle S, T\rangle$-*minimal* if every $t \in C_1(\mathbf{A}|_U)$ is a permutation of U or satisfies $t(T) \subseteq S$.

Lemma 5.7. *If U is $\langle S, T\rangle$-minimal, then U is $\langle S, T\rangle$-irreducible.*

Sketch of Proof. If U is not $\langle S, T\rangle$-irreducible, then there is a proper subneighborhood $V \subsetneq U$ such that $S|_V \neq T|_V$. There is an idempotent $e \in C_1(\mathbf{A})$ such that $e(A) = V$, and necessarily $e(T) = T|_V \not\subseteq S|_V = e(S)$. The term e is not a permutation and does not collapse T into S, so U is not $\langle S, T\rangle$-minimal.

$\langle S, T\rangle$-minimality is sometimes strictly stronger than $\langle S, T\rangle$-irreducibility, as one sees by considering finite groups. (The $\langle 0, 1\rangle$-minimal groups are the groups of prime exponent. The $\langle 0, 1\rangle$-irreducible groups are the groups of prime power exponent.) However, the concept of $\langle S, T\rangle$-*irreducibility for some* $\langle S, T\rangle$ is equivalent to the property of $\langle S, T\rangle$-*minimality for some pair* $\langle S, T\rangle$:

Theorem 5.8. *If U is $\langle S, T\rangle$-irreducible, then it is $\langle S', T'\rangle$-minimal for some $S' \subsetneq T' \subseteq T$, $S' \subseteq S$, $T' \not\subseteq S$.*

Sketch of Proof. Choose $T' \subseteq T$ minimal for $T'|_U \not\subseteq S|_U$, and let $S' = T' \cap S$.

The previous result leads to an internal characterization of irreducibility.

Corollary 5.9. *A neighborhood $U \subseteq A$ is irreducible iff whenever $f_i \in C_1(\mathbf{A}|_U)$ are nonpermutations and $g \in C_n(\mathbf{A}|_U)$, then $g(f_1(x), \ldots, f_n(x))$ is a nonpermutation.*

Sketch of Proof. U is irreducible iff it is $\langle S, T\rangle$-minimal for some S and T. The nonpermutations in $C_1(\mathbf{A})$ are precisely the terms that collapse T into S. This set is a subalgebra of the algebra of unary terms.

Conversely, if the nonpermutations are a subalgebra, then no decomposition equation $\lambda(e_1\rho_1(x), \ldots, e_m\rho_m(x)) = x$ is possible unless some e_i is the identity.

6. Final Comments

The theory in [1] concentrates on polynomial clones on finite sets with respect to pairs $\langle \alpha, \beta\rangle$ of congruences with $\alpha \subseteq \beta$. These notes concentrate on arbitrary clones on (usually finite) sets with respect to arbitrary pairs $\langle S, T\rangle$ of compatible relations with $S \subseteq T$. We will learn in later lectures that there is an adequate classification of $\langle \alpha, \beta\rangle$-minimal algebras for finite algebras of the form $\mathbf{A}_A$ when $\alpha \prec \beta$ are congruences, and it is this that makes TCT a powerful theory. It is not known if an adequate description of $\langle S, T\rangle$-minimal algebras is possible, even for the case that $\mathbf{A} = \mathbf{A}_A$ is finite and $S \prec T$.

(N.B.) In these notes we have deviated from standard notation in some ways. In [1], and most papers of applications of tame congruence theory, the induced algebra (denoted $\mathbf{A}|_U$) always refers to what would be expressed as $(\mathbf{A}_A)|_U$ in these notes. Moreover, what we call "isomorphism" of neighborhoods ($U \simeq V$) is usually called "polynomial isomorphism".

Exercises

(1) Let $\mathbf{A} = \langle A; \mathcal{C}\rangle$ be a finite algebra with $\mathbf{A}^\perp = \langle A; \mathcal{R}\rangle$.
   (a) Show how to derive composition of binary relations from the operations $\cap, \times, proj, \Pi, =$ of $\mathcal{R}$.
   (b) Show that the elements of $\mathrm{Con}\,(\mathbf{A})$ are equationally definable elements of $\mathcal{R}$, and that the lattice operations of $\mathrm{Con}\,(\mathbf{A})$ are operations of $\mathcal{R}$.
   (c) If $e^2 = e \in C_1$, and $U = e(A)$, show that restriction to U is a homomorphism of $\mathrm{Con}\,(\mathbf{A})$ into $\mathrm{Con}\,(\mathbf{A}|_U)$.
   (d) Give an example to show that the homomorphism from part (c) need not be surjective.
   (e) Show that if $\mathrm{Sg}^{\mathbf{A}}(U) = A$, then the homomorphism from part (c) is surjective.

(2) For each $m > 1$ construct an m-element non-unary algebra $\mathbf{A}$ such that A is the only neighborhood of $\mathbf{A}_A$.

(3) Show that a p-Sylow subgroup of a finite group $\mathbf{G}$ is a neighborhood of $\mathbf{G}_G$.

(4) Show that the Jacobson radical of a finite ring $\mathbf{R}$ is a neighborhood of $\mathbf{R}_R$.

(5) Let $\mathbf{A} = \mathbf{L}_L$ where $\mathbf{L}$ is a finite lattice.
   (a) Show that any interval in $\mathbf{L}$ is a neighborhood of $\mathbf{A}$.
   (b) Show that $\mathbf{A}$ is covered by its intervals $I[a, b]$ where a is meet irreducible and b is join irreducible.
   (c) Show that if $\mathbf{L} = I[a, b]$ with a meet irreducible and b join irreducible, then $\mathbf{A}$ is $\langle S, T\rangle$-minimal for $T = L \times L$ and $S = T - \{(a, b), (b, a)\}$. (Your argument should include a proof that S is compatible.)
   (d) Use localization to show that every compatible reflexive binary relation on a finite relatively complemented lattice is transitive. (In particular, relatively complemented lattices have permuting congruences.)

(6) Let $\mathbf{A}$ be a finite algebra with congruences $\alpha \prec \beta$. Show that a neighborhood U of $\mathbf{A}_A$ is $\langle \alpha, \beta\rangle$-irreducible if and only if it is $\langle \alpha, \beta\rangle$-minimal.

(7) Show that if U and V are irreducible neighborhoods that cover one another, then they are isomorphic.

(8) Assume that $\mathcal{U} = \{U_1, \ldots, U_m\}$ is a cover of $\mathbf{A}$.

(a) Show that if $\mathbf{A}$ has a majority term, then so does each $\mathbf{A}|_{U_i}$.
(b) Show that if each $\mathbf{A}|_{U_i}$ has a majority term, then so does $\mathbf{A}$.

(9) Suppose that $\mathbf{A}$ is finite and $\mathbf{A}_A$ generates a congruence distributive variety.
  (a) Show that any neighborhood U of $\mathbf{A}_A$ that is minimal under inclusion has 2 elements, and $\mathbf{A}|_U$ has a majority polynomial. (Hard.)
  (b) Show that the following are equivalent.
    (i) The set of neighborhoods minimal under inclusion is a cover of $\mathbf{A}_A$.
    (ii) $\mathbf{A}$ has a majority polynomial, and all compatible reflexive binary relations of $\mathbf{A}$ are transitive.

(10) Assume that $\mathcal{V}$ is a locally finite variety, and that there is some k that bounds the size of the irreducible sets of $\mathbf{A}_A$ for every finite $\mathbf{A} \in \mathcal{V}$.
  (a) Show that $\mathcal{V}$ is n-permutable for some n.
  (b) Show that $\mathcal{V}$ is congruence distributive. (Hard.)
  (c) Must $\mathcal{V}$ have a near unanimity term? (Open.)

References

[1] D. Hobby and R. McKenzie, *The Structure of Finite Algebras*, Contemporary Mathematics v. 76, American Mathematical Society, 1988.

(Keith A. Kearnes) Department of Mathematics, University of Colorado, Boulder, CO 80309–0395, USA.

E-mail address: `kearnes@euclid.colorado.edu`