THE GLOBAL INFLUENCE OF MINIMAL ALGEBRAS

EMIL W. KISS

By now, we understand the structure of minimal algebras. How can we build on this knowledge to analyze a general algebra? The properties of minimal sets that we learned earlier help us to establish a relationship between global properties of the algebra, and the structure of its minimal sets. Let us first summarize what these properties are. Suppose that $\langle \alpha, \beta \rangle$ is a tame congruence quotient of a finite algebra.

(1) *Uniformity.* Any two $\langle \alpha, \beta \rangle$-minimal sets carry isomorphic induced algebras, and the isomorphism is established by a unary polynomial.
(2) *Neighborhood.* Every $\langle \alpha, \beta \rangle$-minimal set is the range of an idempotent unary polynomial.
(3) *Separation.* Any pair of elements in β that is not in α can be mapped to any $\langle \alpha, \beta \rangle$-trace by a unary polynomial in such a way that the resulting pair is still not in α.
(4) *Connectedness.* Any two elements that are β-related can be connected by a chain of $\langle \alpha, \beta \rangle$-traces and α-related pairs.

We try to show how these properties work together.

1. Abelian quotients

Suppose that $\mathbf{A}$ is a finite simple algebra of type $\mathbf{2}$, and f is a binary polynomial of $\mathbf{A}$. Let us try to build the "multiplication table" of this polynomial.

We know that the induced algebras on the traces are polynomially equivalent to a vectorspace. All binary polynomials of a vectorspace have the form

$$\lambda x + \mu y + c \,,$$

where λ, μ are scalars, and c is some fixed vector. Thus if N is a trace, and it happens that $f(N, N) \subseteq N$, then $f|_N$ is a polynomial of the induced algebra on N, and so it has the form above. This means that we have understood a fragment of the multiplication table of f.

What can we do if $f(N, N) \subseteq N$ does not hold? We know that the trace N is the range of an idempotent unary polynomial e. Therefore $ef(N, N) \subseteq N$ surely holds, and so we have information on the way ef works on N. Thus we shall investigate the operation $g = ef$ first, and understand f itself later. In other words, we investigate binary polynomials g whose range is contained in N.

We want information about $g(a, b)$, where a and b are arbitrary elements of A. By Connectedness, $a \in M$ and $b \in K$ for some traces M and K. We show that on the set $M \times K$ the polynomial g also behaves in an "affine" way: for $u \in M$ and $v \in K$ we have

$$g(u,v) = \lambda'(u) + \mu'(v) + c\,,$$

where λ' and μ' are unary polynomials mapping A to N, and c is some element of N.

We prepare a "local coordinate system". By uniformity, there exist unary polynomials p and q such that $p : N \to M$ and $q : N \to K$ are bijections. Thus the binary polynomial $g(p(x), q(y))$ maps $N \times N$ to N, and is therefore an induced polynomial on N. Hence

$$g(p(x), q(y)) = \lambda x + \mu y + c\,.$$

The polynomial $p : N \to M$ has an inverse p', which is also a polynomial. Thus, setting $\lambda'(u) = \lambda p'(u)$, and defining μ' similarly, we get the desired formula.

Next we try to patch these local representations together. Note that both mappings λ' and μ', as well as the constant c depend on both M and K in the formula above. We plan to find a single unary polynomial λ, and a single unary polynomial μ such that

$$g(u,v) = \lambda(u) + \mu(v) + c$$

holds for every u and v in A. The tool to prove this is Connectedness.

Consider the unary polynomial $h(u) = g(u,b) - g(u,b')$, where $b, b' \in K$. By the formula just proved, this mapping is constant when $u \in M$. This is true for every M (while K is kept fixed), but the constant we get depends on M. However, on overlapping traces this constant must be the same (substitute an element u in the overlap). So by Connectedness, h is constant on the entire algebra.

The set of all pairs (b, b') for which $h(u) = g(u,b) - g(u,b')$ does not depend on u obviously gives a transitive relation. Hence, again by Connectedness, we get that $h(u) = g(u,b) - g(u,b')$ is constant, no matter how we choose the elements $b, b' \in A$. Therefore no matter how we fix elements $a, b \in A$, the binary polynomial $g(u,v) - g(u,b) - g(a,v)$ does not depend on u and on v on A! If we denote this constant value by c, then

$$g(u,v) = g(u,b) + g(a,v) + c\,.$$

Hence $\lambda(u) = g(u,b)$ and $\mu(v) = g(a,v)$ works in the formula that we had to prove.

If we consider $\mu(v) + c$ as a single unary polynomial, then the result just proved can be expressed by saying that every binary polynomial mapping into a trace is the sum of two unary ones. The same argument works for polynomials of arbitrary arity. Let us record what we have shown so far.

Claim 1.1. *Let* $\mathbf{A}$ *be a simple algebra of type* $\mathbf{2}$*. Then every polynomial* f *whose range lies in a trace* N *of* $\mathbf{A}$ *can be written as*

$$f(x_1, \ldots, x_n) = f_1(x_1) + \cdots + f_n(x_n)\,,$$

where every f_i *is a unary polynomial of* $\mathbf{A}$ *whose range lies in* N*.*

So far, we only considered polynomials mapping into a trace. The following argument shows how Separation helps to prove something useful for arbitrary polynomials. We show that the algebra $\mathbf{A}$ must be abelian. Indeed, suppose that

$$f(\mathbf{a}, \mathbf{c}) = f(\mathbf{a}, \mathbf{d})$$

holds for some polynomial f, but

$$f(\mathbf{b}, \mathbf{c}) \neq f(\mathbf{b}, \mathbf{d}).$$

Now we use Separation. Since $f(\mathbf{b}, \mathbf{c}) \neq f(\mathbf{b}, \mathbf{d})$, there is a unary polynomial p mapping A into a trace such that it separates these two elements, that is,

$$pf(\mathbf{b}, \mathbf{c}) \neq pf(\mathbf{b}, \mathbf{d}),$$

while of course we still have

$$pf(\mathbf{a}, \mathbf{c}) = pf(\mathbf{a}, \mathbf{d}).$$

That is, we have reduced the failure of the abelian property to a polynomial pf mapping into a trace. But we know the structure of such polynomials: they are sums of unaries. So if

$$pf(\mathbf{x}, \mathbf{y}) = \sum g_i(x_i) + \sum h_j(y_j) = g(\mathbf{x}) + h(\mathbf{y}),$$

then

$$g(\mathbf{a}) + h(\mathbf{c}) = g(\mathbf{a}) + h(\mathbf{d}).$$

These are elements of a vectorspace, so by adding $g(\mathbf{b}) - g(\mathbf{a})$ we get

$$g(\mathbf{b}) + h(\mathbf{c}) = g(\mathbf{b}) + h(\mathbf{d}),$$

that is,

$$pf(\mathbf{b}, \mathbf{c}) = pf(\mathbf{b}, \mathbf{d}),$$

which is a contradiction. Thus $\mathbf{A}$ is abelian indeed.

It is easy to do the same argument for a tame congruence quotient rather than a simple algebra. Thus we get the following (see [2], Theorem 5.5):

Theorem 1.2. *Every tame quotient of type* **2** *is abelian.*

2. Quasi-affine algebras

General abelian algebras do not behave nicely at all. Therefore what we have proved so far has room for improvement. Let us return to our binary operation f on a simple algebra $\mathbf{A}$ of type 2. We have already understood the structure of ef. What can we say about f itself?

Any Separation-like property suggests an embedding into a product. Let N be a trace, let $p_1, \ldots, p_k$ be a list of all unary polynomials of $\mathbf{A}$ whose range lies in N, and consider the mapping $A \to N^k$ defined by

$$\varphi : a \mapsto (p_1(a), \ldots, p_k(a)).$$

By Separation, this map is injective, we denote its range by B. So we have embedded A into a direct power of a vectorspace. However, the mappings p_i are not homomorphisms, they are polynomials! Therefore this embedding is not a homomorphism either, we do not get a subalgebra. Nevertheless, we shall be able to rectify this situation to get a useful embedding theorem for **A**.

Let us try to transform the operation f over to the set B along φ. We see that

$$f(x,y) \mapsto (p_1 f(x,y), \ldots, p_k f(x,y))\,.$$

Now $p_1 f(x,y)$ maps into N, hence it is the sum of unaries: $p_1 f(x,y) = q(x) + r(y)$, where q and r also map A into N. Hence, q and r are among the functions $p_1, \ldots, p_k$. Say $q = p_i$ and $r = p_j$, thus

$$p_1 f(x,y) = p_i(x) + p_j(y)\,.$$

On the other hand,

$$\varphi : x \mapsto (p_1(x), \ldots, p_k(x))$$

and

$$\varphi : y \mapsto (p_1(y), \ldots, p_k(y))\,.$$

Hence the rule to compute the first component of the operation corresponding to f on B is this: take the i-th component of the first argument, the j-th component of the second argument, and add them. A similar rule works in all other coordinates, that is,

$$p_i f(x,y) = p_{\alpha(i)}(x) + p_{\beta(i)}(y) \qquad (\alpha(i), \beta(i) \in \{1, 2, \ldots, k\})\,.$$

Now the operation corresponding to f on the set B is

$$U(\mathbf{x}) + V(\mathbf{y})\,,$$

where

$$U(x_1, \ldots, x_k) = (x_{\alpha(1)}, \ldots, x_{\alpha(k)})$$

and

$$V(y_1, \ldots, y_k) = (y_{\beta(1)}, \ldots, x_{\beta(k)})\,.$$

Now the important point is that U and V are obviously **linear maps** on the vectorspace N^k. Hence if we consider N^k as a module over the ring of $k \times k$ matrices, then $U(\mathbf{x}) + V(\mathbf{y})$ is a module polynomial. Again, the argument works if the arity of f is arbitrary. Therefore we proved the following statement.

Theorem 2.1. *Let* **A** *be a simple algebra of type* **2**. *Then* **A** *is a subalgebra of a reduct of a module over an associative ring.*

Such algebras are called quasi-affine. Obviously, quasi-affine algebras are abelian, but the converse does not hold in general.

It is possible to get a "tighter", more economical quasi-affine representation of these algebras, which is more useful in applications. See [2], Theorem 13.5 and [7], Theorem 3.1. See also [17] and [5] about quasi affine embeddings of abelian algebras.

3. Affine subsets

Can we improve this result further? The best result would be to show that the algebra **A** itself is polynomially equivalent to a module, and not just to a subreduct, that is, it is *affine* rather than quasi-affine. This is not true in general: if we take any simple algebra of type **2**, and keep only those polynomials, whose range is contained in a trace, then we still have a simple algebra of type **2**, and it does not anymore have the operation $x - y + z$.

Nevertheless, in some ways the only obstacle is the one occurring in the previous example: that the ranges of the polynomials are too small. If we ensure that this does not happen, then we do get an affine algebra.

A classical way to ensure this is to assume that the algebra belongs to a congruence modular variety (a famous result of C. Herrmann, see [1], Corollary 5.9). One can also get this conclusion by assuming that the variety omits type **1** (see [2], Theorem 7.12). These results have been generalized by Á. Szendrei and K. Kearnes (see [11]):

Theorem 3.1. *Let $\mathcal{V}$ be a variety satisfying an idempotent Maltsev-condition that fails in sets (see* [2]*, Chapter 9). Then every abelian algebra in the variety is quasi-affine. If this Maltsev-condition fails in semilattices, then every abelian algebra is polynomially equivalent to a module.*

Instead of such global assumptions, consider the following situation. Suppose again that f is a polynomial of our type **2** simple algebra **A**, and N is a trace. Let $T = f(N, \ldots, N)$, this is a subset of A. Using the methods above, it can be shown that the induced algebra on T is affine, and its structure is very tight (see [8]):

Theorem 3.2. *Let β be a minimal type* **2** *congruence on a finite algebra* **A***, and N a $\langle 0, \beta\rangle$-trace. Then for every polynomial f, the set $T = f(N, \ldots, N)$ is a neighborhood, and the induced algebra on T is polynomially equivalent to a simple module over a full matrix ring (over a finite field).*

Such subsets are therefore called multidimensional traces, or *multitraces*. They have proved useful in the investigation of residually small varieties, and of minimal varieties. See the papers [8] and [9] for details.

4. Strongly abelian quotients

Let us see what happens when the finite simple algebra **A** has type different from **2**. The case of type **1** is similar to the case of **2**, only easier.

Theorem 4.1. *If* **A** *is a finite simple algebra of type* **1***, then every polynomial mapping into a trace depends on at most one variable. The algebra* **A** *is strongly abelian.*

This result allows us to decide if a simple algebra (or a tame quotient in general) has type **2** or type **1**, without looking at traces. We have just proved necessary conditions: abelianness and strong abelianness respectively. It turns out that these are sufficient, and in fact this is very easy to prove. We only have to observe that both the abelian and strong abelian property is preserved by forming induced algebras. However, traces in types **3**, **4**, **5** are not abelian, and a type **2** trace is not strongly abelian, since it has a Maltsev term. Therefore we proved essentially the following result (see [2], Theorem 5.6).

Theorem 4.2. *A tame congruence quotient has type* **1** *if and only if it is strongly abelian. It has type* **2** *iff it is abelian, but not strongly abelian.*

The benefit of considering tame quotients rather than prime quotients can be seen from the following example. Suppose that the algebra $\mathbf{A}$, instead of being simple, has congruence lattice isomorphic to $\mathbf{M}_3$. What can we say about $\mathbf{A}$?

Notice that $\mathbf{M}_3$ is simple, and its coatoms intersect to zero. Hence it is a tight lattice, which forces $\mathbf{A}$ to be a tame algebra. We have to prove that its type cannot be **3**, **4** or **5**. In these three nonabelian types the minimal sets have only two elements, and so the congruence lattice of the induced algebra on any trace is the two element lattice. But traces are neighborhoods, and we learned earlier that restriction to a neighborhood yields a lattice homomorphism. Hence we get a lattice homomorphism of $\mathbf{M}_3$ onto the two element lattice, which cannot happen, since $\mathbf{M}_3$ is simple.

With only slightly more effort, we get the following corollary (see [2], Theorem 5.7):

Corollary 4.3. *Let* $\mathbf{A}$ *be a finite algebra and* $\alpha < \beta$ *congruences of* $\mathbf{A}$. *Suppose that the interval of the congruence lattice of* $\mathbf{A}$ *between* α *and* β *is a tight lattice of more than two elements. Then* β/α *is abelian.*

5. Matrix powers

There is an analogue of the quasi affine embedding theorem for the type **1** case, as well as of the theorem describing multitraces. To formulate these results, we have to generalize the concept of a module over a full matrix ring.

Let $\mathbf{A}$ be any algebra, we shall make the set A^k into an algebra. If we consider the direct product $\mathbf{A}^k$, then each of its n-ary terms is given by a term of the original algebra, acting componentwise:

$$\hat{t}\left(\begin{pmatrix} a_{11} \\ \dots \\ a_{k1} \end{pmatrix}, \dots, \begin{pmatrix} a_{1n} \\ \dots \\ a_{kn} \end{pmatrix}\right) = \begin{pmatrix} t(a_{11}, \dots, a_{1n}) \\ \dots \\ t(a_{k1}, \dots, a_{kn}) \end{pmatrix}.$$

Notice that we have the same term t in every component of the right hand side, and in the i-the component of the result, the elements substituted to t are the i-th components of the arguments of $\hat{t}$.

This is the situation in a direct power. But we can add much more operations to A^k in a natural way. We can have a different term in every component, and each of these terms may depend on all kn components that the n arguments altogether have. This way, we have constructed a function for each k-tuple of kn-ary terms of $\mathbf{A}$. If we add all these functions as basic operations, then we obtain the k-th matrix power of $\mathbf{A}$, which is denoted by $\mathbf{A}^{[k]}$ (see [2], Definition 13.2).

It can be proved that this matrix power $\mathbf{A}^{[k]}$ is similar to $\mathbf{A}$ in every respect: its subalgebras are exactly the matrix powers of the subalgebras of $\mathbf{A}$, its congruence lattice is isomorphic to that of $\mathbf{A}$, it is strongly abelian if and only if $\mathbf{A}$ is strongly abelian, etc. It is also easy to check that if $\mathbf{M}$ is an $\mathbf{R}$-module, then $\mathbf{M}^{[k]}$ will be $\mathbf{M}^k$ considered as an $\mathbf{R}^{k\times k}$-module.

We are in a position now to formulate the type **1** analogues of two previous results.

Theorem 5.1 ([2], Theorem 13.3). *Let* $\mathbf{A}$ *be a simple algebra of type* **1**. *Then* $\mathbf{A}$ *is a subalgebra of a reduct of the matrix power of a unary algebra (which is actually a set with only constant operations).*

Theorem 5.2 (see [8]). *Let* β *be a minimal type* **1** *congruence on a finite algebra* $\mathbf{A}$, *and* N *a* $\langle 0,\beta\rangle$*-trace. Then for every polynomial* f, *the set* $T = f(N,\dots,N)$ *is a neighborhood, and the induced algebra on* T *is polynomially equivalent to a matrix power of the induced algebra on* N.

6. Orderable quotients

Now let $\mathbf{A}$ be a finite simple algebra of type **4** or **5**. Instead of abelianness, they can be recognized by *orderability*.

Consider a trace $N = \{0, 1\}$ in $\mathbf{A}$. Let us try to build a compatible partial ordering $<$ for which $0 < 1$ holds. Certainly, if p is a unary polynomial, then $p(0) \le p(1)$ must hold. Hence all traces are ordered now. Let us simply take the transitive closure of this relation, that is, define $a \le b$ if and only if there is a chain

$$a = a_0, a_1, \dots, a_n = b$$

such that (a_i, a_{i+1}) is a polynomial image of $(0, 1)$. This is a compatible, transitive relation (and is connected, too, by Connectedness). To show that it is a partial ordering, we have to prove that it is antisymmetric.

Suppose that $a < b$ and $b < a$. By Separation, there exists a unary polynomial p such that $\{p(a), p(b)\} = \{0, 1\}$. Therefore we get that $0 < 1$ and $1 < 0$ holds too. In particular, there exists a chain

$$1 = a_0, a_1, \dots, a_n = 0$$

as above. But $\{0, 1\}$ is a Neighborhood, so there exists an idempotent unary polynomial e whose range is $\{0, 1\}$. Apply e to every element of this chain. The fact that e is idempotent ensures that it is the identity map on $\{0, 1\}$, hence this new chain still starts at 1, ends up at 0, and runs entirely within the two element set $\{0, 1\}$. Hence there exists and i for which $e(a_i) = 1$ and $e(a_{i+1}) = 0$. But (a_i, a_{i+1}) is a polynomial image of $(0, 1)$, so we have constructed a polynomial q that switches 0 and 1. This is impossible in types **4** and **5**.

Theorem 6.1. *If the type of a finite simple algebra is* **4** *or* **5**, *then it admits a compatible, connected partial order. Conversely, if a finite simple algebra admits a compatible, connected partial order, then its type is* **1**, **4** *or* **5**.

The converse is again easy to prove, it depends on the fact that traces of type **2** and **3** are not orderable. This theorem can also be generalized to prime quotients instead of simple algebras, see [2], Theorem 5.26.

Is it possible to prove embedding theorems for simple algebras of nonabelian type? What happens with multitraces?

If we use the technique described in the previous section, then for type **3** we might hope to get an embedding into a subreduct of a matrix power of the two element Boolean algebra. However, this matrix power is a primal algebra, and of course every algebra is a reduct of a primal algebra. So this embedding would be meaningless. Similarly in type **4** we could only construct an embedding into an order-primal algebra. So these types do not give us an interesting embedding theorem.

The case of type **5** is more interesting, and is open. We do not even have a description of the polynomials mapping into a trace. A related paper is [10].

Multitraces behave well in type **3**, it is proved in [8] that they are neighborhoods with a primal induced algebra. No such statement holds in types **4** and **5**.

7. Centrality and nilpotence

Let us consider three, seemingly unrelated questions.

- In the congruence modular case, homomorphic images of abelian algebras are also abelian. What can we say in the general case?
- We have seen that if the congruence lattice of a finite algebra is $\mathbf{M}_3$, then the algebra is abelian. Now suppose that $\mathbf{M}_3$ is only a 0, 1-sublattice in the congruence lattice of a finite algebra. The properties of centrality ensure that there is plenty of abelianness around. However, Ross Willard has shown in [19] that such algebras need not be abelian. What do they look like?
- A much investigated question in the theory of general algebras is the number of n-ary terms in a given algebra, as a function of n. How does the growth rate of this function relate to the algebraic structure of $\mathbf{A}$?

The keyword connecting these is *nilpotence*. We shall not discuss the third question, the reader can find information and references concerning this problem in [2], Chapter 12, and in [6]. The other two problems have been solved by Keith Kearnes in his famous paper [3]. He discovered nontrivial properties of centrality for finite algebras, which allowed him to approach these problems.

In group theory, there are several definitions of nilpotence. One is the following. A normal subgroup N of a finite group G is nilpotent if there is a chain of normal subgroups

$$\{e\} = N_0 \leq N_1 \leq \cdots \leq N_k = G$$

such that N centralizes each factor, that is, $[N, N_i] \leq N_{i-1}$ holds for every $i \geq 1$. It follows that N centralizes every prime quotient in the congruence lattice of G.

This definition works fine in congruence modular varieties. However, if we go beyond modularity, then we lose the commutativity of the commutator. The statement that a congruence β centralizes a quotient $\delta < \theta$ has four different possible meanings.

(*i*) $C(\beta, \theta; \delta)$.
(*ii*) $[\beta, \theta] \leq \delta$.
(*iii*) $C(\theta, \beta; \delta)$.
(*iv*) $[\theta, \beta] \leq \delta$.

Besides commutativity, there is another important property of the commutator in the modular case that is lost when we go beyond modularity, namely that if two congruences centralize each other, then they do so in every factor algebra. (This is related to distributivity over join.) These two properties ensure that the above four statements are equivalent in any modular variety.

Indeed, $[\beta, \theta]$ is the intersection of all δ for which $C(\beta, \theta; \delta)$ holds. Therefore $(i) \implies (ii)$ in any algebra. But (ii) only means that β centralizes θ modulo a congruence *below* δ (namely modulo $[\beta, \theta]$), and to show (i) we need to move this centrality to the factor modulo δ, which is possible in the modular case, but not in general. Properties (iii) and (iv) have the same relationship. Of course (ii) and (iv) are equivalent by the commutativity of the commutator.

Thus we have (at least) two versions of nilpotence: left and right. We say that β is a left nilpotent congruence of $\mathbf{A}$ if there exists a chain of congruences

$$1_A = \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_k = 1_A$$

such that β centralizes each of these quotients on the left, that is, that $[\beta, \alpha_i] \leq \alpha_{i-1}$ holds for every $i \geq 1$. Right nilpotence is defined by the formula $[\alpha_i, \beta] \leq \alpha_{i-1}$. It is easy to see that we get the same concept of left nilpotence in any algebra if we replace condition $[\beta, \alpha_i] \leq \alpha_{i-1}$ by $C(\beta, \alpha_i; \alpha_{i-1})$, and a similar statement holds for right nilpotence. Only the monotonicity of the commutator is needed to show this. We can also assume that $\alpha_i \prec \alpha_{i+1}$ holds for every quotient in the chain above.

The basis for Kearnes's results is the discovery that if $\delta \prec \theta$ (or more generally, if this is a tame quotient), then there are nontrivial relationships between the properties $(i) - (iv)$ defined above. The (nontrivial) proofs use the techniques discussed earlier, they are found in [3].

Theorem 7.1. *Let* $\mathbf{A}$ *be a finite algebra,* β *a congruence of* $\mathbf{A}$*, and* $\delta < \theta$ *a tame quotient. Let* $(i) - (iv)$ *be the properties above. Then*

(1) *if the type of* $\langle \delta, \theta \rangle$ *is not* $\mathbf{1}$*, then* $(ii) \implies (iii) \iff (iv)$*;*
(2) *if the type of* $\langle \delta, \theta \rangle$ *is not* $\mathbf{1}$ *or* $\mathbf{2}$*, then in addition we have* $(ii) \iff (iii)$.

If β *is left nilpotent, then*

(1) *if the type of* $\langle \delta, \theta \rangle$ *is not* $\mathbf{1}$*, then* $(i) - (iv)$ *are all equivalent;*
(2) *if the type of* $\langle \delta, \theta \rangle$ *is* $\mathbf{1}$*, then* $(iv) \implies (i) \iff (ii)$.

Corollary 7.2. *Every right nilpotent congruence on a finite algebra is left nilpotent.*

Now we are in the position to handle algebras having $\mathbf{M}_3$ as a 0, 1-sublattice in their congruence lattice. In fact, we can weaken this condition.

Theorem 7.3. *Suppose that* $\mathbf{A}$ *is a finite algebra and*

$$\bigvee \left\{ \beta \in \operatorname{Con} \mathbf{A} \;\middle|\; \bigvee \{\gamma \mid \gamma \wedge \beta = 0_A\} = 1_A \right\} = 1_A\,.$$

Then all homomorphic images of $\mathbf{A}$ *are left nilpotent.*

We see that it is type **1** that does not behave well (as is the case in general). A more careful analysis is needed to understand their behavior. This has been done in the same work of Kearnes. He introduced two new concepts: *coherence* and *regularity.* These are properties of tame quotients of type **1**.

Let $\delta \prec \theta$ and β be congruences. A technical definition of the β-coherence of $\langle \delta, \theta \rangle$ is the requirement that for any two $\langle \delta, \theta \rangle$-traces N and M contained in the same θ-class, any two β-twin idempotent unary polynomials of $\mathbf{A}$ mapping M onto N agree on M modulo δ. This is the condition needed to ensure that if β centralizes all traces on the left, then it centralizes the entire quotient. Both $[\theta, \beta] \le \delta$ and $[\beta, \theta] \le \delta$ are sufficient to ensure coherence. The significance of this concept is shown by the following result.

Theorem 7.4. *If* $\mathbf{A}$ *is left nilpotent, and every prime quotient is coherent for every congruence, then every homomorphic image of* $\mathbf{A}$ *is also left nilpotent.*

A very important example of left nilpotent and coherent algebras is the class of E-minimal algebras of type **1** and **2** (see [3], Corollary 4.11, [2], Chapter 13, and [13], Theorem 4.4). Another condition to ensure coherence is the one in Theorem 7.3.

Regularity is stronger than coherence. A technical definition of the β-regularity of $\langle \delta, \theta \rangle$ is the requirement that for any $\langle \delta, \theta \rangle$-trace N, any two β-twin unary polynomials of $\mathbf{A}$ mapping N onto N that agree at one point modulo δ are identical on N modulo δ. This is the condition needed to ensure that $(iv) \implies (iii)$ and $(ii) \implies (iii)$. If β is left nilpotent, then regularity implies that all conditions $(i) - (iv)$ are equivalent.

Theorem 7.5. *If* $\mathbf{A}$ *is left nilpotent, and every prime quotient is* 1_A*-regular, then every homomorphic image of* $\mathbf{A}$ *is (left and) right nilpotent.*

A group-theoretic argument shows that abelianness is sufficient to ensure regularity, in fact, if $[\beta, \beta] \wedge \theta \le \delta$, then $\langle \delta, \theta \rangle$ is β-regular. This enables us to say something nontrivial about homomorphic images of abelian algebras.

Theorem 7.6. *If* $\mathbf{A}$ *is a finite abelian algebra, then all homomorphic images of* $\mathbf{A}$ *are (left and) right nilpotent.*

There is no known characterization of the homomorphic images of finite abelian algebras. The condition given by the previous theorem is not sufficient.

Nilpotence plays an important role in other areas too, like the already mentioned free spectra, or the investigation of residually small varieties. Still another such topic is discussed in the following section.

8. Subalgebras of abelian algebras

The condition of abelianness is closely related to all subalgebras being congruence classes, which is called the Hamiltonian property. The definition of $\mathbf{A}$ being abelian can be trivially reformulated to say that the diagonal of $\mathbf{A}^2$ is a congruence class. The generic examples for abelian varieties (modules over a ring, unary algebras and their matrix powers) are all Hamiltonian. One wonders if tame congruence theory gives tools to investigate this topic, since this theory is based on polynomials, and the polynomial structure of an algebra is not closely related to the structure of subalgebras. Yet it was precisely tame congruence theory that allowed the first important breakthrough in this topic. This is a result of Matthew Valeriote ([18]).

Theorem 8.1. *A finite simple abelian algebra can have only trivial subalgebras (that is, such algebras are Hamiltonian).*

It is interesting to note that the quasi affine embedding theorem does not seem to help when proving this statement. Valeriote's result has been generalized by Ralph McKenzie ([15]).

Theorem 8.2. *If* $\mathbf{A}$ *is a finite abelian algebra, then every maximal subalgebra of* $\mathbf{A}$ *is a congruence class.*

If we think of group theory, then the condition of maximal subgroups being normal is not abelianness, it characterizes nilpotence by Wielandt's theorem. However, we have to be careful if we want to generalize this result. If we add all constants to an algebra, then its nilpotence status will not change, but the subalgebras disappear. So although it might be true that in nilpotent algebras, all maximal subalgebras are congruence classes, we cannot expect a converse. However, we do get a converse if we go up to the square of the original algebra. This is a result of Keith Kearnes ([4]). He discovered a condition that is weaker than the forms of nilpotence discussed previously. It requires that 1_A centralize all traces for all prime quotients.

Theorem 8.3. *If a finite algebra* $\mathbf{A}$ *satisfies the weak nilpotence condition mentioned above, in particular if it is left nilpotent, then all maximal subalgebras are congruence classes. Conversely, if every subalgebra of* $\mathbf{A}^2$ *has the property that all maximal subalgebras are congruence classes, then* $\mathbf{A}$ *satisfies this nilpotence condition.*

For locally finite varieties, the abelian property and the Hamiltonian property are equivalent (see [14]):

Theorem 8.4. *A locally finite variety is Hamiltonian if and only if it is abelian.*

The proof of this result does not use tame congruence theory. It has been applied by Keith Kearnes and Ross Willard to prove the following interesting statement (see [12]):

Theorem 8.5. *Every locally finite abelian variety is generated by a finite algebra.*

Practice problems

1. What is the type of a finite simple lattice?

2. Let $\mathbf{G}$ be a finite, undirected, connected graph. Construct a simple algebra of type **1**, and one of type **2** on G whose minimal sets are exactly the edges of $\mathbf{G}$.

3. Let $\mathbf{A}$ be a finite simple algebra of type **2**. Prove that

(1) any two distinct elements can be mapped to two distinct elements of any trace N with an idempotent unary polynomial whose range is N;
(2) the intersection of any two traces can have at most one element.

4. Prove Theorem 4.1.

5. Prove that every finite algebra whose congruence lattice is $\mathbf{M}_7$ is strongly abelian.

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6. Let A be a finite set containing an element 0. Add all finitary functions f as basic operations that satisfy the following property: the value of f is 0 if and only if at least one argument is 0. Prove that the resulting algebra $\mathbf{A}$ is simple, and determine its type and its minimal sets. Which are the connected compatible partial orders?

7. Let $\mathbf{G}$ be an undirected graph. Define an algebra $\mathbf{A}(\mathbf{G})$ on $G \cup \{0\}$ that has a single binary multiplication for which the product xy is x if there is an edge between x and y, and all other products are 0. Let $\mathbf{C}_n$ be a cycle of length n. Prove that $\mathbf{A}(\mathbf{C}_n)$ is simple, and determine its type and its minimal sets.

8. Let $\mathbf{S}$ be the two element semilattice, and $\mathbf{A}$ a matrix power of $\mathbf{S}$.

(1) Prove that $\mathbf{A}$ has a semilattice operation $\wedge$ such that for every polynomial f and vectors $\mathbf{a} = (a_1, \ldots, a_k)$, $\mathbf{b} = (b_1, \ldots, b_k)$ we have
$$f(a_1 \wedge b_1, \ldots, a_k \wedge b_k) = f(\mathbf{a}) \wedge f(\mathbf{b})$$
(that is, $\wedge : \mathbf{A}^2 \to \mathbf{A}$ is a homomorphism).
(2) Prove that $\mathbf{A}$ has a compatible partial ordering $\leq$ which satisfies the following condition: for every polynomial f, vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ of $\mathbf{A}$, and $u \in \mathbf{A}$,
$$f(\mathbf{a}, \mathbf{c}), f(\mathbf{b}, \mathbf{d}) \geq u \implies f(\mathbf{a}, \mathbf{d}), f(\mathbf{b}, \mathbf{c}) \geq u\,.$$

9. Among the algebras $\mathbf{A}$ and $\mathbf{A}(\mathbf{C}_n)$ in Exercises 6 and 7, which ones can be embedded into a matrix power of the two element semilattice?

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10. Let $\langle \delta, \theta \rangle$ be a type **2** prime quotient of a finite algebra $\mathbf{A}$, and β a congruence of $\mathbf{A}$. Prove that $[\beta, \theta] \leq \delta$ and $[\theta, \beta] \leq \delta$ both imply $C(\theta, \beta; \delta)$.

11. Suppose that $\mathbf{A}$ is a finite algebra such that $\mathsf{H}\,\mathsf{S}(\mathbf{A}^2)$ is abelian. Prove that if $\mathbf{A}$ is not strongly abelian, then it has a type **2** prime quotient.

12. Let U be a minimal set for a prime quotient $\langle \delta, \theta \rangle$ of a finite algebra $\mathbf{A}$, and e an idempotent unary polynomial whose range is U. Let $\beta = (\delta : \theta)$. Prove that $(c, d) \in \beta$ if and only if every binary polynomial $p(x, y)$ of $\mathbf{A}$ satisfies that $ep(x, c)$ is a permutation of U if and only if $ep(x, d)$ is a permutation of U.

Hints, answers, solutions

1. It is **4**. Lattices are nonabelian and orderable, excluding types **1**, **2** and **3**. If $a \prec b$, then $e(x) = (a \vee x) \wedge b$ is an idempotent unary polynomial with range $\{a, b\}$. Hence this is a trace, and the induced algebra has lattice operations, excluding type **5**.

2. To get a type **1** simple algebra, add all unary operations that map G into an edge. These form a semigroup under composition, hence we do not get new traces. To get a type **2** simple algebra, expand the previous algebra by a binary operation $+$ defined as follows. Let $\{0, 1\}$ be an edge, and $+$ addition mod 2 on this subset. Pick any idempotent function e whose range is $\{0, 1\}$, and extend $+$ to G by letting $x + y = e(x) + e(y)$. The algebra stays abelian, because the relation

$$(a, b) \equiv (c, d) \iff ef(a) - ef(b) = ef(c) - ef(d) \text{ for every unary polynomial } f$$

is a congruence on the square of the algebra, and the diagonal is a congruence class.

3. To prove (1) let e be an idempotent unary polynomial whose range is N. Suppose that $a \neq b \in A$. If $e(a) \neq e(b)$, then we are done. Let f be a unary polinomial with range N such that $f(a) \neq f(b)$. If f restricted to N is constant, then replace f by $e + f$, where $+$ is the addition of the vectorspace structure on N. This new f still separates a and b, and is not constant on N. Hence f is a permutation of N, whose inverse is a polynomial. Composing f with this inverse we get the desired map.

To prove (2), suppose that M is another trace with $|M \cap N| \geq 2$. Let e be an idempotent unary polynomial with range N, pick $u \in M - N$ and let $v = e(u) \in N$. Apply (1) to get an idempotent polynomial f with range N that separates u and v. Now $e - f$ is constant zero on N, hence it is constant on M, too, because of the overlap. Hence $e = f$ on M, but $e(u) = v = e(v)$, which is a contradiction.

5. The lattice $\mathbf{M}_7$ is tight of more than two elements, hence our algebra is tame and abelian. Resctriction to a trace is an onto lattice homomorphism, since $\mathbf{M}_7$ is simple. But the congruence lattice of a vector space cannot be $\mathbf{M}_7$, so the type is **1**.

6. The minimal sets are $\{0, a\}$ ($a \in A - \{0\}$), the type is **5** (use that all nonconstant polynomials preserve 0). The only connected compatible ordering is the one where $0 \leq a$ for every $a \in A$, and the elements of $A - \{0\}$ are pairwise incomparable.

7. The minimal sets are those two element subsets that contain 0, the type is **5**. (Observe that a nonabelian simple algebra with an absorbing element has type **5**.)

8. (2) can be derived from (1).

9. The answer is no if $|A| \geq 3$ by Exercise 8 (2). The answer is yes for $\mathbf{A}(\mathbf{C}_n)$. Consider $\mathbf{S}^{[n]}$, let e_i be the element whose i-th coordinate is 1, the rest is 0, and define a matrix power operation f such that the i-th coordinate of $f(\mathbf{a}, \mathbf{b})$ is $a_i \wedge b_{i+1}$.

10. Consult Lemma 2.9 of [13] for a hint. The proof is in Lemma 3.2 of [3].

11. This is a result of McKenzie and Valeriote, see [16], Theorem 7.1.

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Eötvös University, Department of Algebra and Number Theory, Hungary
E-mail address, Emil W. Kiss: `ewkiss@cs.elte.hu`