Exercises

1. Verify that every nonunary polynomial clone contains a binary operation that depends on both of its variables.
2. Use the result proved in the preceding exercise to show that the polynomial clones on the two-element set $\{0, 1\}$ are exactly the seven clones $[0, 1]$, $[', 0, 1]$, $[+, 0, 1]$, $[\vee, \wedge, ', 0, 1]$, $[\vee, \wedge, 0, 1]$, $[\vee, 0, 1]$, and $[\wedge, 0, 1]$.
3. Let $\mathbf{S} = \langle \{0, 1\}, t, 1\rangle$ where $t(x, y, z) = x \vee (y \wedge z')$, and let $\mathcal{V} = V(\mathbf{S})$.
   (a) Verify that $\mathbf{S}$ is equivalent to the two-element implication algebra $\langle \{0, 1\}, \rightarrow\rangle$, and has $\vee$ as a term operation.
   (b) Show that every finite algebra $\mathbf{A} \in \mathcal{V}$ can be embedded as a filter into the $\{t, 1\}$-reduct of a finite Boolean algebra.
   (c) Prove that if a finite algebra $\mathbf{A} \in \mathcal{V}$ has no least element with respect to the $\vee$-order, then
   - no proper neighborhood of $\mathbf{A}|_A$ contains all minimal elements of $\mathbf{A}$,
   - $\mathbf{A}|_A$ is irreducible.
4. Let $\mathbf{S} = \langle \{0, 1\}, m_3\rangle$ where $m_3(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$. Let $\mathcal{V} = V(\mathbf{S})$, and let $\mathbf{A}$ be a finite member of $\mathcal{V}$. For $a, b \in A$ let
$$I(a, b) = \{x \in A : m_3(a, x, b) = x\}.$$
   Prove that
   (a) $I(a, b)$ is a neighborhood of $\mathbf{A}$ for all distinct $a, b \in A$;
   (b) $I(a, b)$ is a distributive lattice with respect to the operations
$$x \sqcup y = m_3(x, y, b) \quad \text{and} \quad x \sqcap y = m_3(x, y, a);$$
   (c) $\langle I(a, b); m_3\rangle$ is polynomially equivalent to the lattice $\langle I(a, b); \sqcup, \sqcap\rangle$.
   Using this information find an irreducible cover for each finite algebra $\mathbf{A} \in \mathcal{V}$.
5. Determine all finite, simple, idempotent, affine algebras.
6. Determine all affine G^0-algebras.
7. Let $\mathbf{A}$ be an affine algebra, and let ${}_RA$ be a (unitary) module polynomially equivalent to $\mathbf{A}$. Prove that $\mathbf{A}$ is a G-algebra for some group G if and only if there exists a module homomorphism ϕ from ${}_RA$ into the radical of R such that a polynomial $\sum_{i=1}^{n} r_i x_i + a$ of ${}_RA$ is a term operation of $\mathbf{A}$ if and only if $\sum_{i=1}^{n} r_i + \phi(a) = 1$.
8. Show that an orderable, strictly simple, idempotent algebra has two elements. (Recall: an algebra is called *orderable* if it has a connected, compatible partial order.)
9. Show that strictly simple G-algebras are not orderable.
10. Show that the following conditions on a strictly simple G^0-algebra $\mathbf{A}$ are equivalent:

(a) **A** is orderable,
(b) the semilattice order is a compatible order of **A**,
(c) the binary operation

$$x \circ y = \begin{cases} x & \text{if } y = 0, \\ 0 & \text{otherwise} \end{cases}$$

is not a term operation of **A**.

11. Construct a 3-element simple algebra with two non-isomorphic term minimal sets.
12. Let **A** be a finite algebra with no nontrivial proper subalgebras. Prove that either all term minimal sets of **A** are isomorphic, or **A** has all the constants.

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