SIXTY FOUR PROBLEMS IN UNIVERSAL ALGEBRA

1. Algebras

Problem 1.1. Is every finite lattice the congruence lattice of a finite algebra?

Problem 1.2. Find a lattice that is the congruence lattice of an algebra with countably many operations, but is not the congruence lattice of any algebra with finitely many operations.

Problem 1.3. Is every algebraic distributive lattice the congruence lattice of a lattice (or, of an algebra from a congruence distributive variety)?

Problem 1.4. Which finite algebras are dualizable?

2. Varieties

Problem 2.1. Which varieties have the property that all infinite members are elementarily equivalent?

Problem 2.2. Is there a residually finite variety that is not congruence distributive, but every finitely generated subvariety is congruence distributive?

Problem 2.3. Which lattices are subvariety lattices of varieties?

Problem 2.4. Which varieties are stable (i.e. each member has a stable theory)?

Problem 2.5. What kind of properties are preserved by Morita equivalence?

3. Residual Smallness

For a variety $\mathcal{V}$, let

$$\mathrm{resb}(\mathcal{V}) = \min\{\kappa \mid \text{every subdirectly irreducible algebra in } \mathcal{V} \text{ has } < \kappa \text{ elements.}\}$$

This is a cardinal number, or ∞. If $\mathrm{resb}(\mathcal{V}) = \infty$, then $\mathcal{V}$ is *residually large*. Otherwise it is *residually small*.

Problem 3.1. Characterize the finitely generated residually small varieties.

Problem 3.2. Is the class of finite groupoids $\mathbf{A}$ for which $\mathrm{resb}(\mathcal{V}(\mathbf{A})) = \infty$ recursively enumerable?

Problem 3.3. Is it possible to have $\mathrm{resb}(\mathcal{V}(\mathbf{A})) = \omega$ if $\mathbf{A}$ is finite and of finite signature?

Problem 3.4. For which finite $\mathbf{A}$ is the class of subdirectly irreducible members of $\mathcal{V}(\mathbf{A})$ first-order axiomatizable?

Problem 3.5. If $\mathbf{A}$ is finite and has finite signature, and $\mathrm{resb}(\mathcal{V}(\mathbf{A})) < \omega$, does it follow that $\mathbf{A}$ is finitely based?

Problem 3.6. If $\mathbf{A}$ is finite and has finite signature, and the class of subdirectly irreducible members of $\mathcal{V}(\mathbf{A})$ is first-order axiomatizable, does it follow that $\mathbf{A}$ is finitely based?

Problem 3.7. Suppose that $\mathbf{A}$ is a finite algebra, and $\mathcal{V}(\mathbf{A})$ omits type $\mathbf{5}$.

- If $\mathrm{resb}(\mathcal{V}(\mathbf{A})) \geq \omega$, does it follow that $\mathrm{resb}(\mathcal{V}(\mathbf{A})) = \infty$? (This is open even if $\mathbf{A}$ is nilpotent.)
- Is $\mathrm{resb}(\mathcal{V}(\mathbf{A}))$ algorithmically computable?

Problem 3.8. Is it true that a residually small congruence modular variety satisfies all congruence identities satisfied by the variety of abelian groups?

Problem 3.9. Is there a residually finite congruence modular variety of finite signature that has an infinite set of pairwise nonisomorphic subdirectly irreducible algebras?

Problem 3.10. Is there a residually finite variety of finite signature that omits type $\mathbf{1}$ and has an infinite set of pairwise nonisomorphic subdirectly irreducible algebras?

Problem 3.11. Describe the residually small minimal varieties $\mathcal{V}(\mathbf{A})$ where $\mathbf{A}$ is a strictly simple algebra of type $\mathbf{5}$.

4. Finite Bases

Problem 4.1. Is there an inherently nonfinitely based abelian algebra?

Problem 4.2. Describe necessary and sufficient conditions for a nilpotent algebra in a congruence modular variety to be finitely based.

Problem 4.3. Is every E-minimal algebra of finite type finitely based?

Problem 4.4. Which finite modules have a finite basis for their quasi-identities?

Problem 4.5. Let $\mathbf{A}$ be a finite nilpotent algebra in a congruence modular variety. If $\mathbf{A}$ has a finite basis for its quasi-identities, is $\mathbf{A}$ abelian?

5. Clone Theory

Problem 5.1. Describe all maximal TC clones on finite sets.

Problem 5.2. Find a Rosenberg-type characterization of quasiprimal algebras; that is, describe all clones that are maximal for the property of not containing the ternary discriminator.

Problem 5.3. Determine all minimal clones.

Problem 5.4. Which permutation groups $\langle X; G\rangle$ have the property that the only clone on X with unary part equal to G is the essentially unary clone?

Problem 5.5. Is the lattice of clones on a finite set of cardinality > 2 simple?

Problem 5.6. Do there exist uncountably many clones containing a fixed Maltsev operation on a finite set?

6. Maltsev Conditions

Problem 6.1. Is it true that $\mathcal{V}$ is congruence join-semidistributive if and only if it is congruence meet-semidistributive and satisfies a congruence identity?

Problem 6.2. Let $\mathbf{G}$ be a p-group. If $\mathbf{G}$ generates a variety that satisfies all congruence identities true in the variety of abelian groups, must the nilpotence class of $\mathbf{G}$ be $< p$?

Problem 6.3. Is congruence modularity a join-prime Maltsev condition?

Problem 6.4. Is the property of having a semilattice operation a join-prime Maltsev condition?

Problem 6.5. Is it true that a variety is congruence n-permutable for some n if and only if it satisfies an idempotent linear Maltsev condition that fails in the variety of distributive lattices?

7. Free spectra

For a variety $\mathcal{V}$ let $f_{\mathcal{V}}(n)$ be the cardinality of the free algebra on n generators (this is the free spectrum function of $\mathcal{V}$). Let $g_{\mathcal{V}}(n)$ be the number of n-generated algebras in the variety up to isomorphism.

Problem 7.1. True or false: If $f_{\mathcal{V}}(n)$ is the free spectrum of a variety of monoids, and $\log(f_{\mathcal{V}}(n))$ is sub-exponential, then $\log(f_{\mathcal{V}}(n))$ is asymptotically equivalent to either $n\log(n)$ or a polynomial.

Problem 7.2. Determine which locally finite varieties $\mathcal{V}$ omitting type $\mathbf{1}$ have the property that $\log(f_{\mathcal{V}}(n))$ is sub-exponential.

Problem 7.3. Characterize locally finite varieties for which $f(n) \leq c^n$ for some constant c. (Solved by K. Kearnes in the case, when the variety omits type **1**.)

Problem 7.4. Characterize locally finite varieties for which $g(n) \leq n^c$ for some constant c. (Solved by P. Idziak and R. McKenzie in the case, when the variety omits type **1**.)

Problem 7.5. Investigate varieties where the function

$$t_{\mathcal{V}}(n) = \text{the number of quantifier-free } n\text{-types}$$

grows slowly. Are the results here essentially the same as the Berman-Idziak-McKenzie results on slow growing g-spectrum?

Problem 7.6. Assume that $\mathcal{V}$ is a finitely generated variety of finite signature. If $f_{\mathcal{V}}(n) \leq c^n$ for some constant c, is the generating function $F(z) = \sum f_{\mathcal{V}}(n)z^n$ of $f_{\mathcal{V}}$ a rational function?

8. Decidability

Problem 8.1. For which finite $\mathbf{A}$ is the theory of $\mathsf{H}\,\mathsf{S}\,\mathsf{P}_{fin}(\mathbf{A})$ decidable?

Problem 8.2. Is it decidable whether a finite algebra has a near unanimity function?

Problem 8.3. Is it decidable whether a finite ring is of finite representation type?

Problem 8.4. Is it decidable whether a finite algebra generates a finitely based quasivariety?

Problem 8.5. Is it decidable whether a finite algebra generates a finitely based abelian variety?

Problem 8.6. Is it decidable whether a finite algebra generates a variety with definable principal congruences?

9. Tame congruence theory and extensions

Problem 9.1. Which algebras are homomorphic images of finite (strongly) Abelian algebras?

Problem 9.2. Let $\mathbf{A}$ be a finite simple algebra of type **5** and let N be a trace. Describe the polynomials of $\mathbf{A}$ whose range lies in N.

Problem 9.3. Develop (and apply) tame congruence theory for terms instead of polynomials, and for tolerances instead of congruences.

Problem 9.4. Let P be the ordered set of types, and let I, J be order ideals in P. Describe what it means for a variety to omit the types in I and omit the tails for the types in J.

Problem 9.5. Investigate when one minimal set is contained in another.

10. Commutator theory

Problem 10.1. Is it true that every subvariety of a finitely generated congruence modular variety is finitely generated?

Problem 10.2. Use the linear commutator to explore the structure of varieties satisfying a nontrivial idempotent Maltsev condition (or a congruence identity). In what ways do these behave differently than modular varieties?

Problem 10.3. Investigate the structure of weakly Abelian varieties. Use this understanding to investigate their residual character.

Problem 10.4. Find a Klukovits-type characterization of weakly Abelian varieties.

Problem 10.5. Is it true that every idempotent Abelian algebra is quasi-affine?

Problem 10.6. Let $\mathcal{V}$ be a locally finite variety that omits type $\mathbf{1}$. Is it true that if $[\alpha, \beta] = [\beta, \alpha]$ for all congruences α, β of algebras in $\mathcal{V}$, then $\mathcal{V}$ has a difference term?

Problem 10.7. Are there natural conditions on a variety $\mathcal{V}$ under which the implications

$$[\alpha, \beta] = [\alpha, \gamma] \Longrightarrow [\alpha, \beta] = [\alpha, \beta \vee \gamma] \quad \text{and} \quad [\beta, \alpha] = [\gamma, \alpha] \Longrightarrow [\beta, \alpha] = [\beta \vee \gamma, \alpha]$$

hold throughout the variety $\mathcal{V}$? (Consider, e.g., the condition '$\mathcal{V}$ has a difference term'.)

Problem 10.8. Which locally finite, congruence modular varieties have only finitely many critical algebras?

11. Algorithms

Problem 11.1. Does there exist a polynomial time algorithm to determine whether a given finite algebra has a Maltsev term?

Problem 11.2. Apply versions of tame congruence theory to determine when the term-equivalence problem (or polynomial equivalence problem) for a finite algebra is NP-complete.

12. Outside connections

Find nontrivial applications of universal algebra in other areas of mathematics: not just in logic and computer science, but in classical algebra, too!

Problem 12.1. Describe the minimal sets in finite groups and semigroups.

13. More technical problems (Not part of the main list!)

Problem 13.1. Is every minimal, idempotent, congruence distributive variety congruence n-permutable for some n or else generated by a 2-element algebra? (Cf. Problem **??**)

Problem 13.2. How far is the congruence generated by the square of a type **2** body from its centralizer?

Problem 13.3. Is the congruence generated by the body of a type **2** $\langle 0, \alpha\rangle$-minimal set nilpotent?

Problem 13.4. What are those varieties where type **2** bodies are

- traces for some tame quotient?
- vector spaces?

Problem 13.5. If α, β are congruences of a finite algebra $\mathbf{A}$, let $\alpha \sim \beta$ denote the following property: for every minimal set $U = B \cup T$ in $\mathbf{A}$ where B is the body and T is the tail of U,

$$\alpha|_U \subseteq B^2 \cup T^2 \quad \text{if and only if} \quad \beta|_U \subseteq B^2 \cup T^2.$$

What does $\sim$ mean?

Problem 13.6. Let $\mathcal{V}$ be a congruence modular variety such that every nontrivial algebra in $\mathcal{V}$ has a nontrivial center. Is it true that $\mathcal{V}$ is nilpotent?

Problem 13.7. Does every variety contain an idempotent simple algebra or a one-generated simple algebra?