On the Lattice of Clones on Three Elements

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Conference on Universal Algebra and Lattice Theory

Outline

Lattice of Clones.

Definitions

$E_k = \{0, 1, 2, \dots, k-1\}$.

$P_k^n = \{f \mid f : E_k^n \to E_k\}$, $P_k = \bigcup\limits_{n \geq 1} P_k^n$.

P_k is the set of all functions in k-valued logic.

$F \subseteq P_k$ is a clone if F is closed under superposition and contains all projections.

J_k is the set of all projections.

The clones form an algebraic lattice whose least element is J_k and whose greatest element is P_k.

Background

- All clones in two-valued logic were described (E. Post, 1921, 1941)
- There exists a continuum of clones in k-valued logic for $k \geq 3$ (Ju. I. Janov, A. A. Muchnik, 1959).
- All maximal (also known as precomplete) clones in three-valued logic were found (S. V. Jablonskij, 1955)
- All minimal clones in three-valued logic were found (B. Csákány, 1983)

Background

- All 158 submaximal clones in three-valued logic were described (D. Lau, H. Machida, J. Demetrovics, L. Hannak, S. S. Marchenkov, J. Bagyinszki).
- All maximal clones except the clone of all linear functions contain a continuum of subclones (J. Demetrovics, L. Hannak, S. S. Marchenkov, 1983).
- The cardinality of the set of all clones in a given submaximal clone in three-valued logic was found (A. Bulatov, A. Krokhin, K. Safin, E. Sukhanov).

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Self-Dual Functions

Definitions

An n-ary function in three-valued logic is called self-dual if

$$f(x_1+1, x_2+1, \ldots, x_n+1) = f(x_1, x_2, \ldots, x_n) + 1$$

for all $x_1, x_2, \ldots, x_n \in \{0, 1, 2\}$.
$+$ is addition modulo 3.

- Self-dual functions are functions that preserve the relation $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$.
- The set of all self-dual functions is a maximal clone in three-valued logic

Main Result

- Many clones of self-dual functions were found (S. S. Marchenkov, J. Demetrovich, L. Hannak, 1980).
- There exists a continuum of clones of self-dual functions (S. S. Marchenkov, 1983)

Main Result

All clones of self-dual functions in three-valued logic are described

- This is the first maximal clone besides the clone of all linear functions that has such description.

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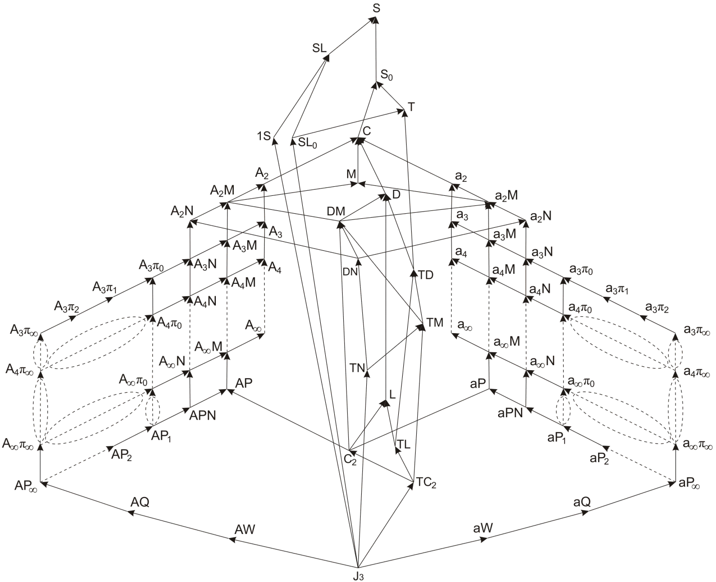
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AP_2
C_2
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AQ
aQ
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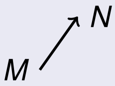
J_3

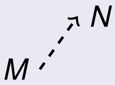
Meaning of Lines

solid line



M is a maximal clone in N.

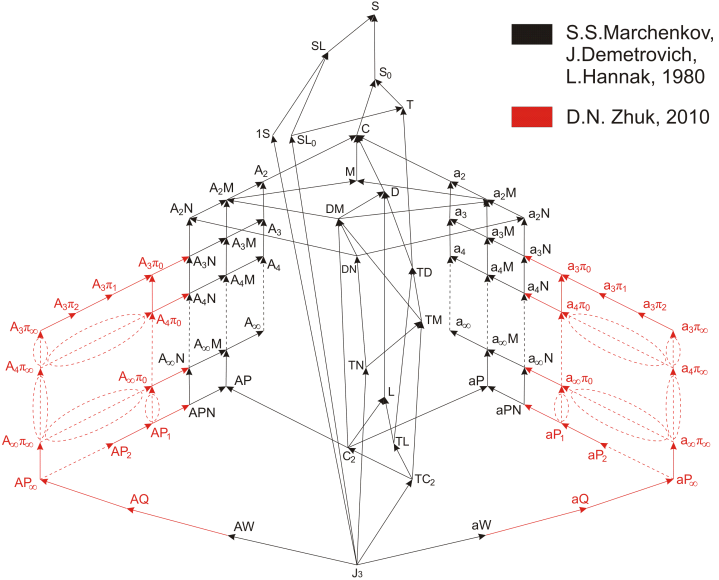
dotted line



$M \subset N$, sublattice between M and N is a countable chain.

dotted line inside a dotted ellipse

$M \subset N$, sublattice between M and N is not a chain.


S.S.Marchenkov, J.Demetrovich, L.Hannak, 1980
D.N. Zhuk, 2010
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T
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SL_0
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$A_3\pi_1$
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$a_4\pi_0$
$a_3\pi_2$
$A_3\pi_\infty$
$A_4\pi_0$
A_∞
TM
a_∞
$a_3\pi_\infty$
$A_4\pi_\infty$
$A_\infty M$
$a_\infty M$
$A_\infty N$
TN
$a_\infty N$
$a_4\pi_\infty$
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AP_2
C_2
aP_2
AP_∞
TC_2
aP_∞
AQ
aQ
AW
aW
J3

Predicates

Definition

A mapping $E_k^n \to \{0, 1\}$ is called an n-ary predicate.
$R_k^n = \{\rho \mid \rho : E_k^n \to \{0, 1\}\}$, $R_k = \bigcup_{n \geq 1} R_k^n$.

Definition

Inv(M) is the set of all predicates that are preserved by functions from M.

Definition

Pol(S) is the set of all functions that preserve predicates from S.

Duality

Let $\sigma : E_3 \longrightarrow E_3$, $\sigma(0) = 1$, $\sigma(1) = 0$, $\sigma(2) = 2$.

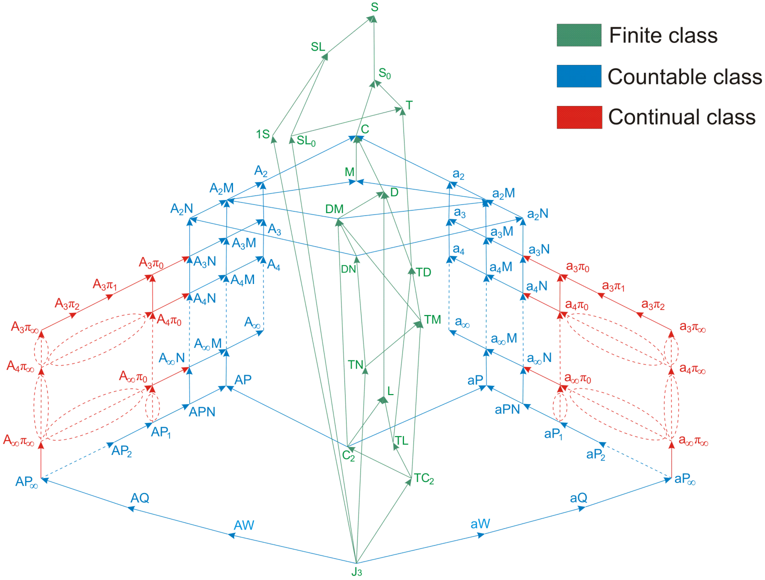
Definition

By f^* we denote the function that is dual (with respect to σ) to f :

$$f^*(x_1, \ldots, x_n) := \sigma^{-1}(f(\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_n))).$$

Suppose $M \subseteq P_3$, then put $M^* := \{f^* \mid f \in M\}$.

- Clones on the left side of the figure are dual (with respect to σ) to the clones on the right side of the figure


Finite class
Countable class
Continual class
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$a_\infty M$
$A_4\pi_\infty$
$A_\infty N$
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Set of Predicates Π

Suppose

- $m \in \{1, 2, 3, \dots, \}$, $n \in \{0, 1, 2, 3, \dots, \}$.
- $A_1, A_2, \dots, A_m \subseteq \{1, 2, \dots, n\}$,
- $A_1 \cup A_2 \cup \dots \cup A_m = \{1, 2, \dots, n\}$.

In case of $n = 0$ we have $A_1 = A_2 = \dots = A_m = \varnothing$.

Let $\pi_{A_1, A_2, \dots, A_m}$ be the predicate of arity $m + n$ such that

$$\pi_{A_1, A_2, \dots, A_m}(a_1, \dots, a_m, b_1, \dots, b_n) = 1$$

iff the following conditions hold:

1. $\forall i (a_i = 1 \vee (a_i = 0 \wedge (\forall j \in A_i : \ b_j \in \{0, 1\})))$;
2. at least one of the values $a_1, \dots, a_m, b_1, \dots, b_n$ is not equal to 0.

The set of all such predicates we denote by Π

Quasiorder

- We define a quasiorder $\lesssim$ on the set Π;

Definition

$F \subseteq \Pi$ is a down-set if the following condition holds

$$\forall \rho \in F \ \forall \sigma \in \Pi \ (\sigma \lesssim \rho \implies \sigma \in F).$$

Continual Class of Clones

For $F \subseteq \Pi$ we put

$$Clone(F) = Pol\left(F \cup \left\{ \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \end{pmatrix} \right\}\right),$$

$$Clone^*(F) = (Clone(F))^*,$$

The continual class of clones consists of all clones

$$Clone(F),\ Clone^*(F),$$

where $F \subseteq \Pi$ is a nonempty down-set.

Continual Class of Clones

Theorem

Suppose F_1, F_2 are nonempty down-sets, then

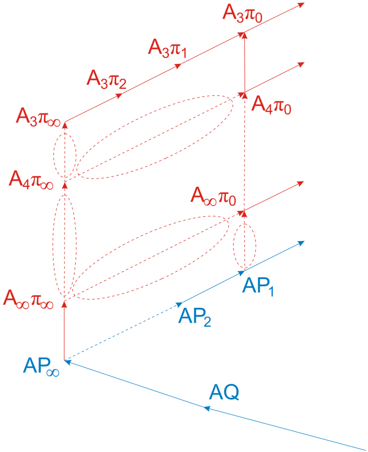
$$Clone(F_1) \subseteq Clone(F_2) \iff F_1 \supseteq F_2.$$

Corollary

Suppose F_1, F_2 are nonempty down-sets, then

$$F_1 \neq F_2 \implies Clone(F_1) \neq Clone(F_2).$$

- Each clone from continual class is uniquely defined by a nonempty down-set $F \subseteq \Pi$.


A3π0
A3π1
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AP1
AP2
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AP∞
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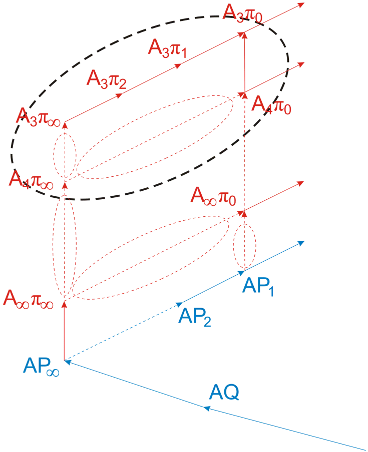
$A_3\pi_0$
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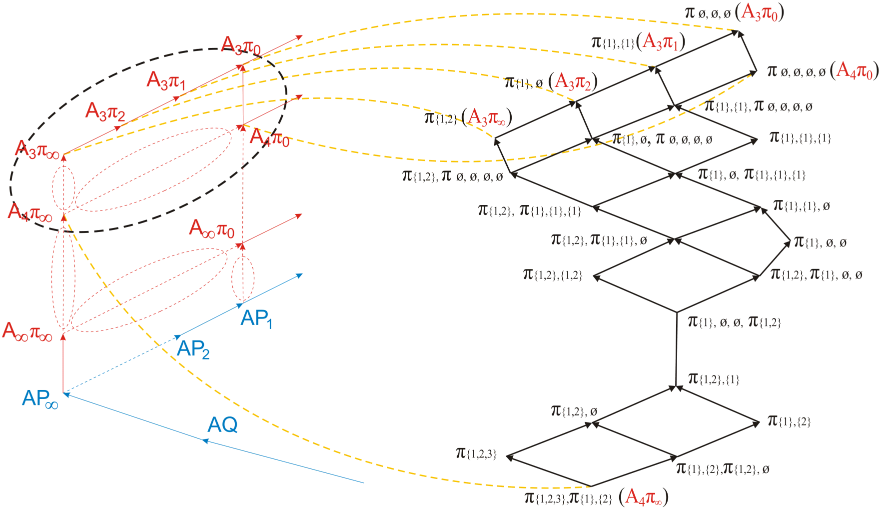
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$\pi_{ø,ø,ø}$ ($A_3\pi_0$)
$\pi_{\{1\},\{1\}}$ ($A_3\pi_1$)
$\pi_{ø,ø,ø,ø}$ ($A_4\pi_0$)
$\pi_{\{1\},ø}$ ($A_3\pi_2$)
$\pi_{\{1\},\{1\}}, \pi_{ø,ø,ø,ø}$
$\pi_{\{1,2\}}$ ($A_3\pi_\infty$)
$\pi_{\{1\},ø}, \pi_{ø,ø,ø,ø}$
$\pi_{\{1\},\{1\},\{1\}}$
$\pi_{\{1,2\}}, \pi_{ø,ø,ø,ø}$
$\pi_{\{1\},ø}, \pi_{\{1\},\{1\},\{1\}}$
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$\pi_{\{1\},ø,ø}, \pi_{\{1,2\}}$
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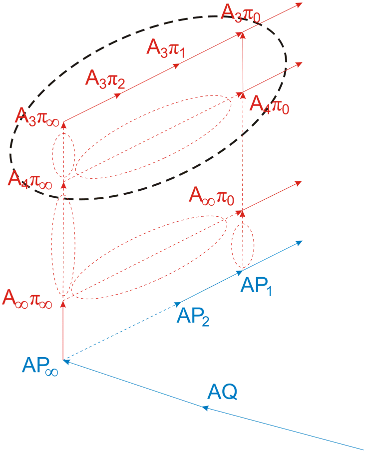
$\pi_{\{1,2\},ø}$
$\pi_{\{1\},\{2\}}$
$\pi_{\{1,2,3\}}$
$\pi_{\{1\},\{2\}}, \pi_{\{1,2\},ø}$
$\pi_{\{1,2,3\}}, \pi_{\{1\},\{2\}}$ ($A_4\pi_\infty$)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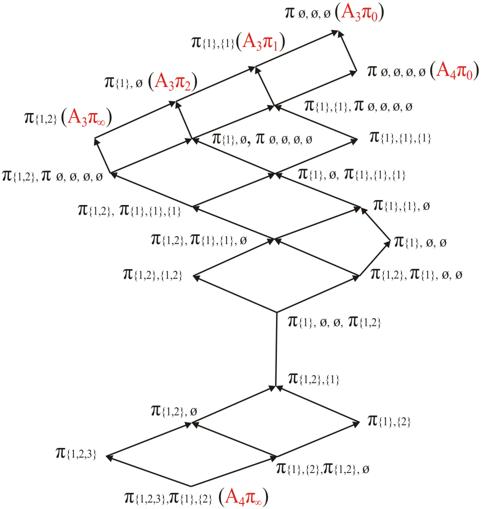
$\pi_{\varnothing,\varnothing,\varnothing}$ ($A_3\pi_0$)
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$\pi_{\{1\},\{2\}}, \pi_{\{1,2\},\varnothing}$
$\pi_{\{1,2,3\}}, \pi_{\{1\},\{2\}}$ ($A_4\pi_\infty$)

Pairwise Inclusion of the Clones Into Each Other

- For the finite and countable classes of clones pairwise inclusion is shown by the graph.
- For the continual class of clones we formulate theorems that describe pairwise inclusion.

Bases of Clones

Definition

B is a basis of M iff $[B] = M$ and $\forall B' \subset B \;:\; [B'] \neq M$.

- It is proved that every clone of self-dual functions has a basis (finite or infinite)
- All clones with a finite basis (or finitely generated clones) are found
- All clones from finite and countable classes have a finite basis (are finitely generated)
- A basis for each clone from the finite and countable classes is found
- For the continual class of clones we formulate a simple criterion for the existence of a finite basis

Relation Degree

Definition

Relation degree of a clone is the minimal arity of relations such that the clone can be defined as the set of all functions that preserve these relations.

- The relation degree of each clone of self-dual functions is found
- The values of relation degree prove that our description is minimal. That is, these clones can not be defined by relations of smaller arity.

Cardinality of principal filters and principal ideals

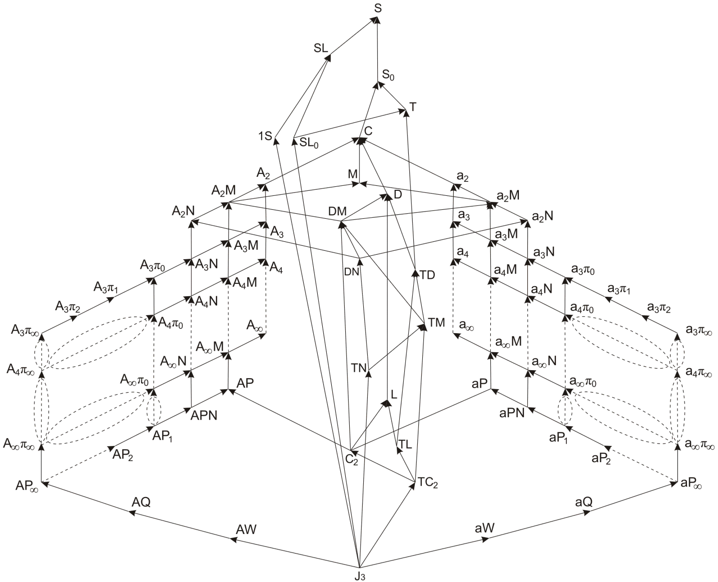
Definition

$\mathbb{L}_3$ is the set of all clones of self-dual functions,

$$\mathbb{L}_3^{\uparrow}(M) := \{M' \in \mathbb{L}_3 | M \subseteq M'),$$

$$\mathbb{L}_3^{\downarrow}(M) := \{M' \in \mathbb{L}_3 | M' \subseteq M).$$

- For each clone $F \in \mathbb{L}_3$ the cardinality of $\mathbb{L}_3^{\uparrow}(F)$ is found.
- For each clone $F \in \mathbb{L}_3$ the cardinality of $\mathbb{L}_3^{\downarrow}(F)$ is found.
- $\mathbb{L}_3^{\downarrow}(M)$ is continuum for every $M = Clone(F)$ such that $M \neq Clone(\Pi)$. $\mathbb{L}_3^{\downarrow}(Clone(\Pi)) = 4$.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Minimal Clones

Definition

A clone $A \neq J_k$ is called minimal, if the following condition holds:

$$\forall f \in A \setminus J_k : \ [\{f\} \cup J_k] = A.$$

Examples of minimal clones in P_3

$[\{max(x,y)\}], [\{2x+2y\}], [\{0,x\}]$.

- I. Rosenberg classified all minimal clones in P_k (1983).
- B. Csákány found all minimal clones in P_3 (1983).
- All minimal clones in P_4 were found (Pöshel, Kaluzhnin, Szczepara, Waldhauser, Jezek, Quackenbush, Karsten Schölzer, 2012).

Rosenberg Classification of Minimal Clones

Theorem

Each minimal clone in P_k can be presented as $[J_k \cup \{f\}]$ where

1. f is an unary function.
2. f is a binary idempotent function
   $f(a,a) = a$ for every $a \in E_3$.
3. f is a majority function
   $\forall a, b \in E_3 : f(a,a,b) = f(a,b,a) = f(b,a,a) = a$
4. f is a minority function
   $\forall a, b \in E_3 : f(a,a,b) = f(a,b,a) = f(b,a,a) = b$
5. f is a semiprojection to the i-th variable
   $f \in P_k^n$, $3 \leq n \leq k$, for every tuple $(a_1, a_2, \ldots, a_n)$ satisfying $|\{a_1, a_2, \ldots, a_n\}| < n$ we have $f(a_1, a_2, \ldots, a_n) = a_i$.

We will assume that $i = 1$ for all occurring semiprojections.

Minimal Clones on Three elements (found by B. Csákány)

P_3 has exactly 84 minimal clones. One obtains every one of these clones by applying inner automorphisms of P_3 to exactly one of the following clones:

a) Minimal clones generated by unary functions

x	$\mathbf{c_0}$	$\mathbf{c_1}$	$\mathbf{c_2}$	$\mathbf{c_3}$
0	1	1	0	1
1	1	0	1	2
2	1	2	1	0

b) Minimal clones generated by idempotent binary functions

x	y	$\mathbf{b_1}$	$\mathbf{b_2}$	$\mathbf{b_3}$	$\mathbf{b_4}$	$\mathbf{b_5}$	$\mathbf{b_6}$	$\mathbf{b_7}$	$\mathbf{b_8}$	$\mathbf{b_9}$	$\mathbf{b_{10}}$	$\mathbf{b_{11}}$	$\mathbf{b_{12}}$
0	1	1	0	1	1	1	1	0	0	0	0	1	2
1	0	1	1	1	1	1	1	1	1	1	1	1	2
0	2	1	0	0	0	0	0	0	1	0	1	0	1
2	0	1	1	0	2	0	2	0	1	1	2	0	1
1	2	1	1	1	1	1	1	1	1	1	0	2	0
2	1	1	1	1	1	2	2	1	2	2	2	2	0

c) Minimal clones generated by majority functions $\mathbf{m_1}, \mathbf{m_2}, \mathbf{m_3}$ and semiprojections $\mathbf{s_1}, \mathbf{s_2}, \mathbf{s_3}, \mathbf{s_4}, \mathbf{s_5}$

x	y	z	$\mathbf{m_1}$	$\mathbf{m_2}$	$\mathbf{m_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$
0	1	2	1	0	0	1	0	0	1	1
0	2	1	1	1	0	1	0	0	1	2
1	0	2	1	1	1	1	1	1	0	0
1	2	0	1	0	1	1	1	1	0	2
2	0	1	1	0	2	1	1	0	2	0
2	1	0	1	1	2	1	1	1	2	1

Main Problem

How many clones contain a given minimal clone on three elements?

- For minimal clones generated by majority functions the result follows from Baker-Pixley Theorem (K. A. Baker, A. F. Pixley, 1975)
- Clones containing $\mathbf{s_5}$ and clones containing $\mathbf{b_{12}}$ are finitely generated (S. S. Marchenkov, 1984)
- The set of all clones containing $\mathbf{b_{12}}$ is finite and completely described (I. Rosenberg, 1995).
- For the minimal clones generated by unary functions this problem was solved by J. Pantović and D. Voivodić (2000).
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Main Result

Theorem

Suppose $M = [J_3 \cup \{f\}]$ is a minimal clone in P_3. The set of all clones containing M

- is finite if $f \in \{\mathbf{c_3}, \mathbf{b_{12}}, \mathbf{m_1}, \mathbf{m_2}, \mathbf{m_3}\}$,
- is countable if $f \in \{\mathbf{s_4}, \mathbf{s_5}\}$,
- has continuum cardinality if $f \in \{\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2}, \mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}, \ldots, \mathbf{b_{10}}, \mathbf{b_{11}}, \mathbf{s_1}, \mathbf{s_2}, \mathbf{s_3}\}$.

Finite and countable cases

Finite cases (except clones generated by majority functions)

$$\mathbf{c_3}(x) = x + 1;$$

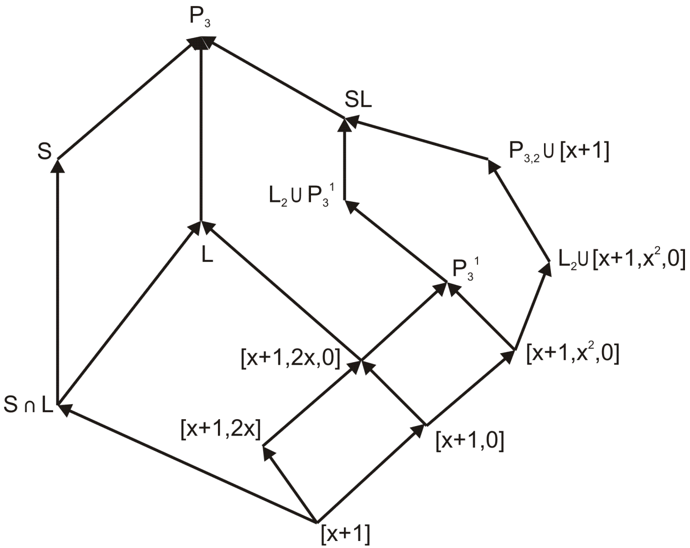
$$\mathbf{b_{12}}(x, y) = 2x + 2y.$$

Countable cases

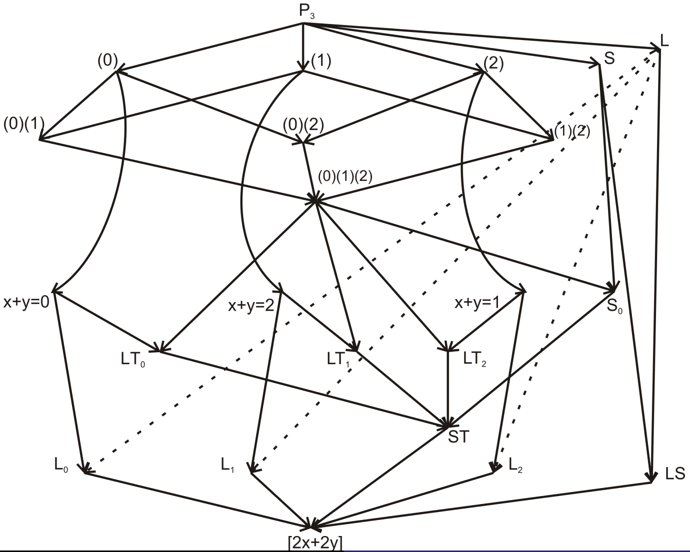
$$\mathbf{s_4}(x, y, z) = \begin{cases} 2x + 1, & \text{if } |\{x, y, z\}| = 3; \\ x, & \text{otherwise.} \end{cases};$$

$$\mathbf{s_5}(x, y, z) = \begin{cases} y, & \text{if } |\{x, y, z\}| = 3; \\ x, & \text{otherwise.} \end{cases}.$$

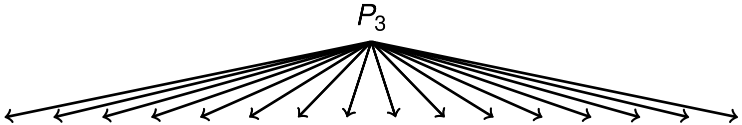
The Lattice of Clones containing $x + 1$



The Lattice of Clones containing $2x + 2y$



Clones containing a majority function



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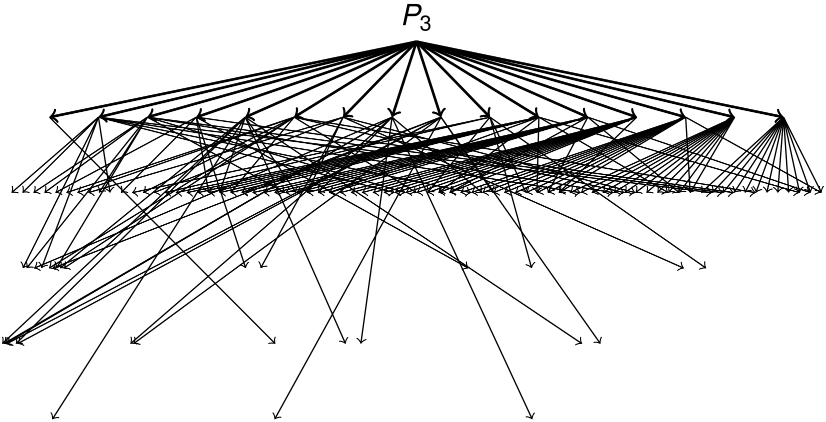

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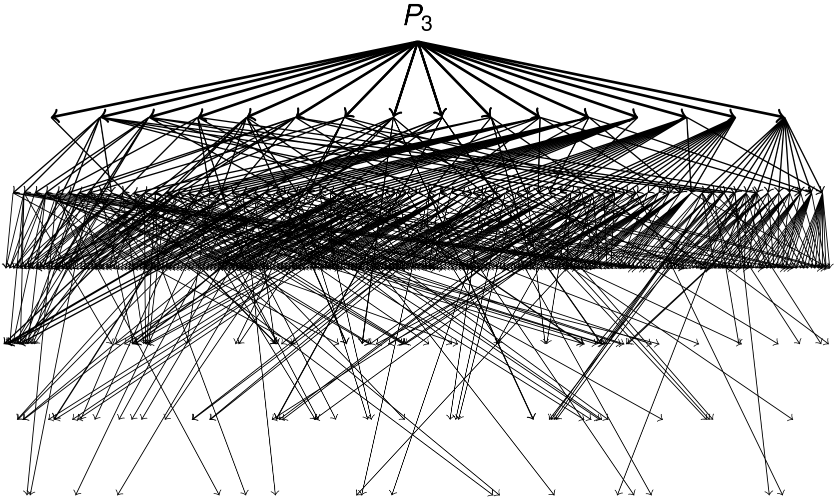
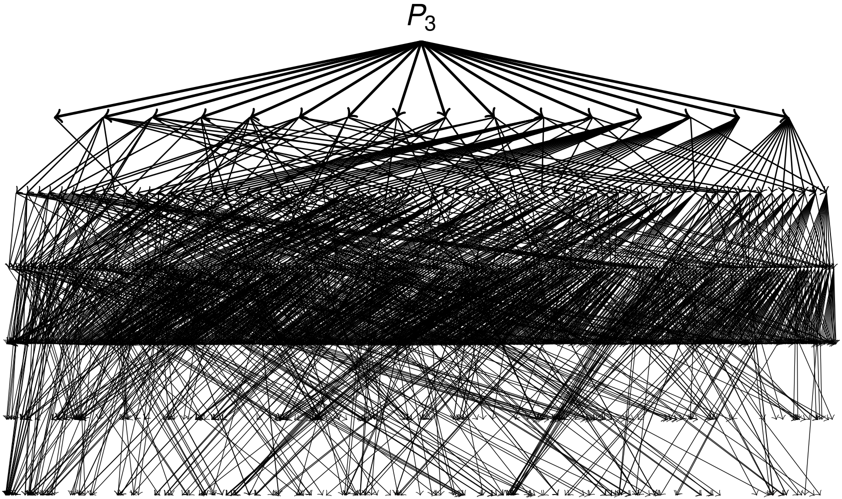
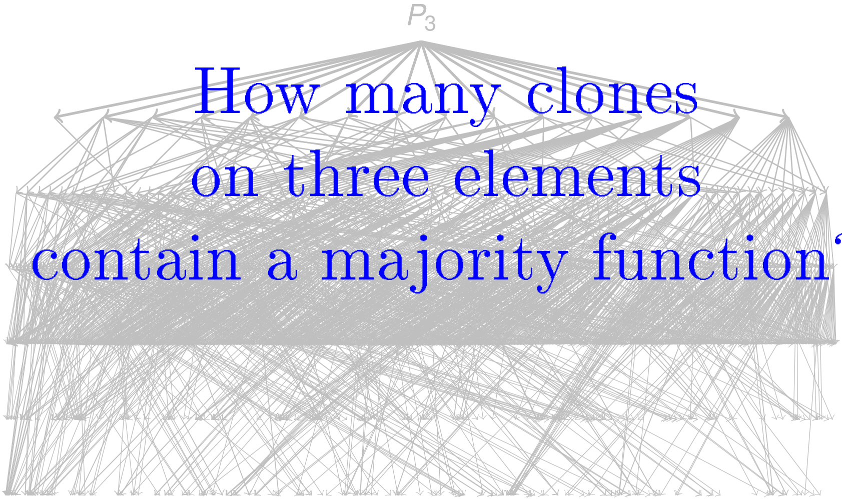


Clones containing a majority function

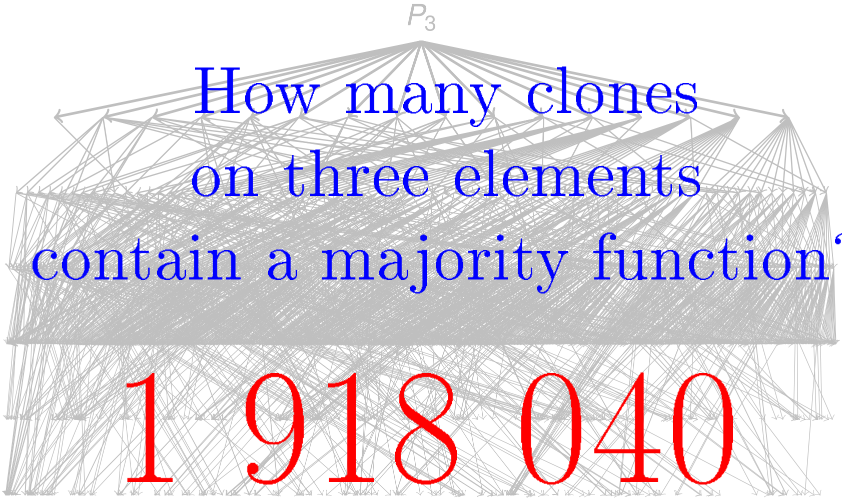


Clones containing a majority function



How many clones on three elements contain a majority function?

Clones containing a majority function



How many clones on three elements contain a majority function?

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Clones containing the semiprojection s_4 or s_5

For $m, n \in \{1, 2, 3, \ldots\}$ let

$$\rho_{m,n} = (\{0,1\}^m \times \{0,2\}^n) \setminus \{(0,0,0,\ldots,0)\}$$

Let $S = \{\rho_{m,n} | m, n \in \{1,2,3,\ldots\}\}$.

Lemma

Every semiprojection in P_3 preserves the relation $\rho_{m,n}$.

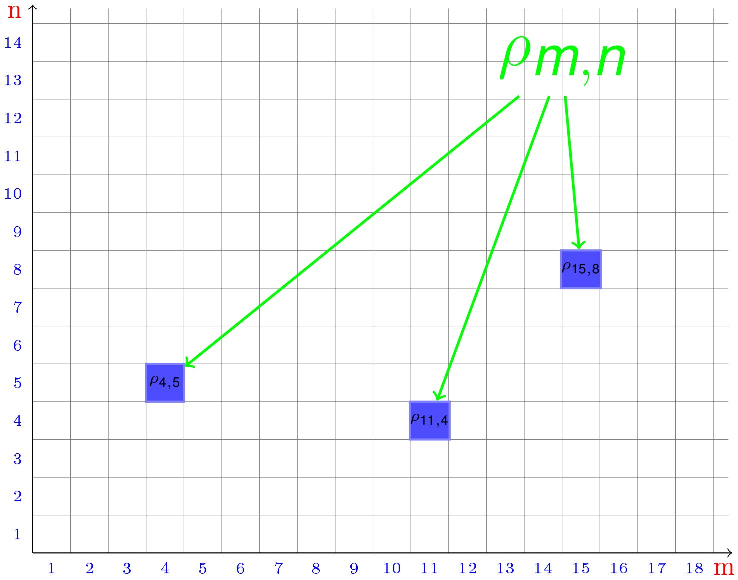
Lemma

Suppose $1 \leq m' \leq m$, $1 \leq n' \leq n$, then $\rho_{m',n'} \in [\{\rho_{m,n}\}]$.

Definition

A set $H \subseteq S$ is called closed if

$$\rho_{m,n} \in H \Rightarrow (\forall m' \leq m \, \forall n' \leq n \;\; \rho_{m',n'} \in H).$$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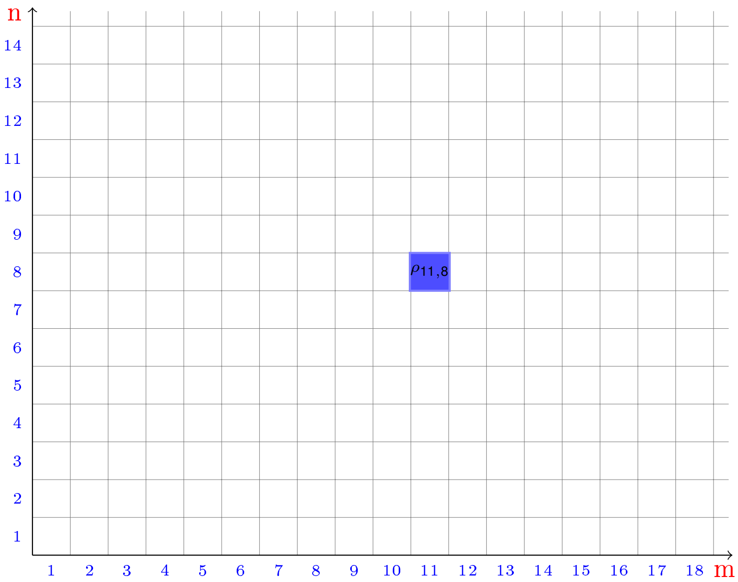
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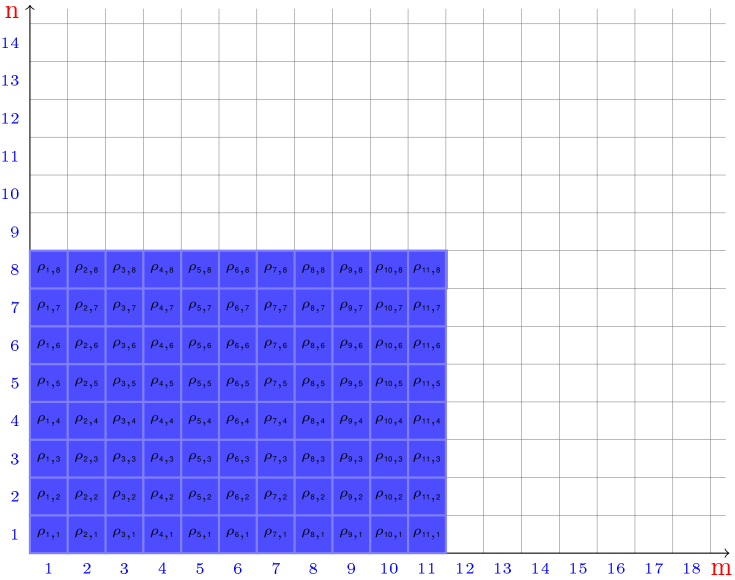
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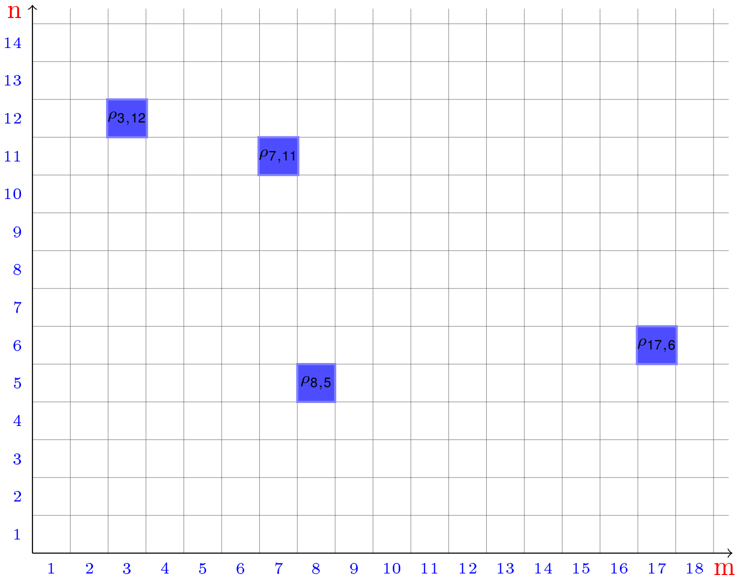
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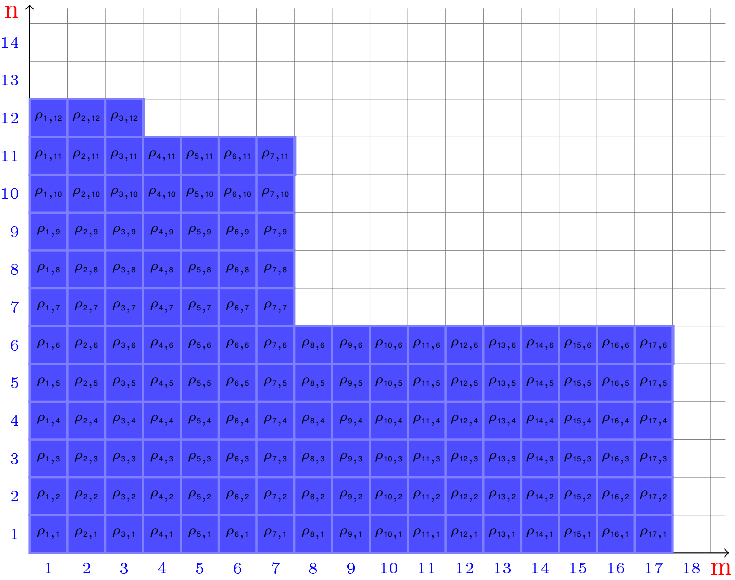
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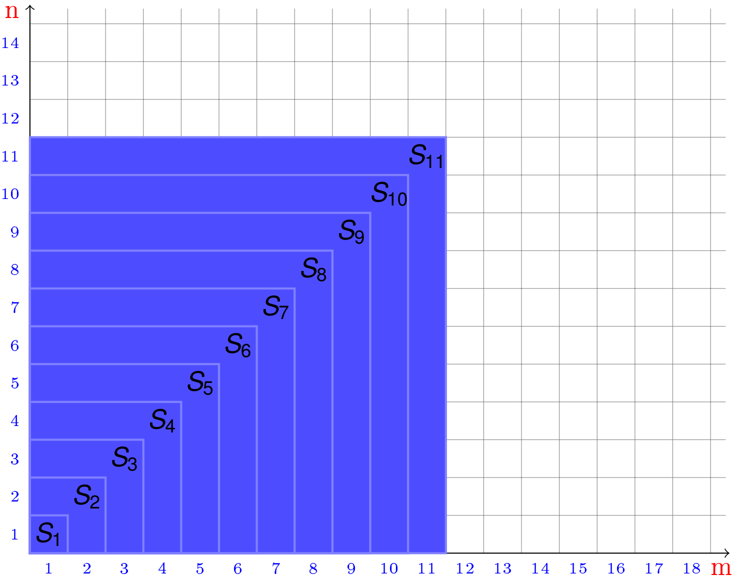
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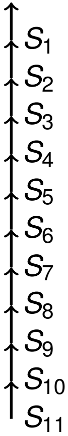
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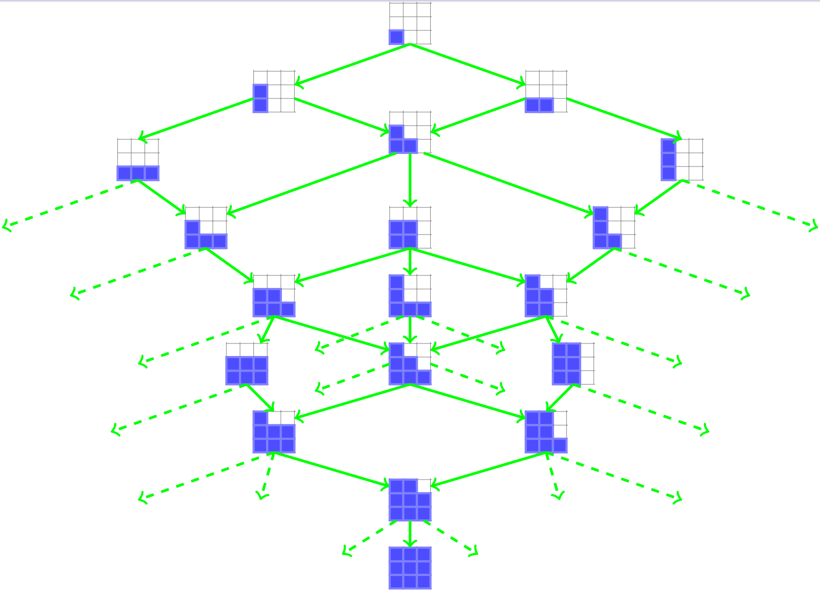


Properties of clones containing s_4 or s_5.

Theorem (S. S. Marchenkov, 1984)

Every clone in P_3 containing $\mathbf{s_5}$ is finitely generated.

Theorem

Every clone in P_3 containing $\mathbf{s_4}$ is finitely generated.

Open Problems

Problem 1

Find all minimal clones on $k \geq 4$ elements such that the set of clones containing this minimal clone is finite or countable.

Problem 2

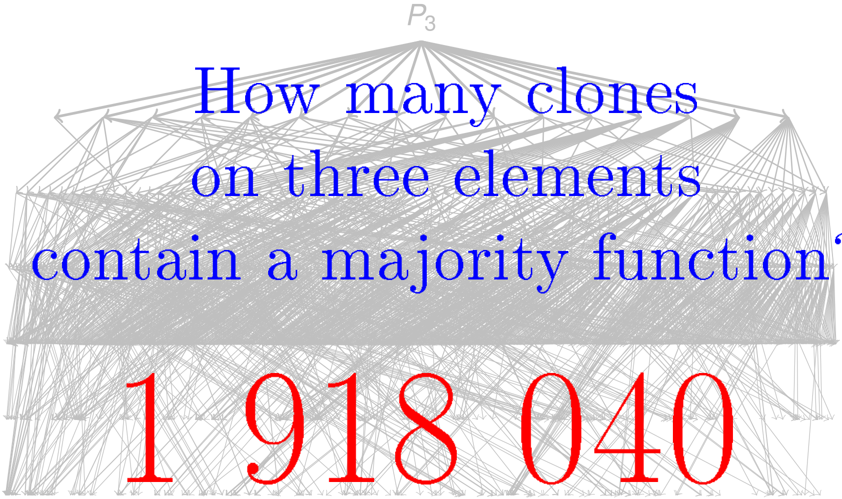
Does there exist a minimal clone in k-valued logic that is not finitely related (cannot be defined by a relation)?

- A clone M is called essentially minimal, if every essential function in M generates M.

Problem 3

Find all essentially minimal clones on $k \geq 3$ elements such that the set of clones containing this minimal clone is finite or countable.

Clones containing a majority operation



How many clones on three elements contain a majority function?

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Maximal clones on three elements

The number of clones containing a majority operation

$\mathrm{Pol}(0)$	$\mathrm{Pol}(\{0,1\})$	$\mathrm{Pol}\begin{pmatrix}0 & 1 & 2\\ 1 & 2 & 0\end{pmatrix}$	$\mathrm{Pol}\begin{pmatrix}0 & 1 & 2 & 0 & 1\\ 0 & 1 & 2 & 1 & 0\end{pmatrix}$
1 744 466	1 722 818	17	36 942

$\mathrm{Pol}\begin{pmatrix}0 & 1 & 2 & 0 & 0 & 1\\ 0 & 1 & 2 & 1 & 2 & 2\end{pmatrix}$	$\mathrm{Pol}\begin{pmatrix}0 & 1 & 2 & 0 & 0 & 1 & 2\\ 0 & 1 & 2 & 1 & 2 & 0 & 0\end{pmatrix}$
1 546	84 201

Minimal clones on three elements

The number of clones containing a majority operation

a) Minimal clones generated by unary functions

c_0	c_1	c_2	c_3
52 530	83	67 837	2

b) Minimal clones generated by idempotent binary functions

b_1	b_2	b_3	b_4	b_5	b_6
144 099	103 572	375 270	371 876	107 444	425 584

b_7	b_8	b_9	b_{10}	b_{11}	b_{12}
312 915	38 981	82 786	996	77 457	17

c) Minimal clones generated by majority functions

m_1	m_2	m_3
928 767	325 581	99 152

d) Minimal clones generated by semiprojections

s_1	s_2	s_3	s_4	s_5
217 601	469 196	196 315	55 898	14 832

Decidability Problems

Problem 1

Given a finite set of relations S; decide whether $\mathrm{Pol}(S)$ is finitely generated.

Problem 2

Given a finite set of functions M; decide whether $[M]$ is finitely related.

Problem 3

Given a finite set of functions M; a finite set of relations S; decide whether $[M] = \mathrm{Pol}(S)$.

Partial Results

Problems 1, 2, and 3 are decidable for

- clones in P_2
- clones of linear functions in P_3
- clones of self-dual functions in P_3
- clones containing a majority function
- clones containing $2x + 2y$ or $x + 1$ in three-valued logic.
- clones containing $\mathbf{s_4}$ or $\mathbf{s_5}$ in three-valued logic.
- clones containing $\mathbf{b_{11}}$ in three-valued logic.

Main ideas

- We essentially use a Galois connection between clones and relational clones. We operate with relations instead of operating with functions.
- We consider only predicates that cannot be presented as a conjunction of predicates with smaller arities. These predicates are called essential.
- We investigate the lattice not from the top but from the bottom; select a clone M near the bottom of the lattice (for example a minimal clone), then describe the set of all essential predicates from $\mathbf{Inv}(M)$.
- We define a closure operator on the set of all essential predicates. Using this closure operator we describe closed sets of predicates.

Essential predicates

Definition

A predicate ρ is called essential iff ρ cannot be presented as a conjunction of predicates with arity less then the arity of ρ. By $\widetilde{R}_k$ we denote the set of all essential predicates

Definition

A tuple $(a_1, \ldots, a_n)$ is called essential for ρ if there exists $b_1, \ldots, b_n$ such that

- $\rho(a_1, \ldots, a_n) = 0$,
- $\forall i \ \rho(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_n) = 1$.

Examples

Essential Predicates

- $\rho(x, y, z) = 1 \iff (x \oplus y \oplus z = 1)$
  $(0, 0, 0)$ is an essential tuple
- $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$
  $(1, 0)$ is an essential tuple
- $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$
  $(0, 1)$ is an essential tuple

Not Essential Predicates

$\rho(x, y, z) = (x \leq y) \wedge (y \leq z)$.

Properties of essential predicates

Theorem

The following conditions are equivalent:

- ρ is an essential predicate;
- there exists an essential tuple for ρ.

Theorem

$[S \cap \widetilde{R}_k] = S$ for every $S = [S] \subseteq R_k$.

- Every closed set $S \subseteq R_k$ can be described by the set $S \cap \widetilde{R}_k$ of all essential predicates of S.

Example of Usage

Lemma

Suppose $m(x, y, z)$ is a majority function, $m \in Pol(\rho)$, ρ is an essential predicate. Then $arity(\rho) \leq 2$.

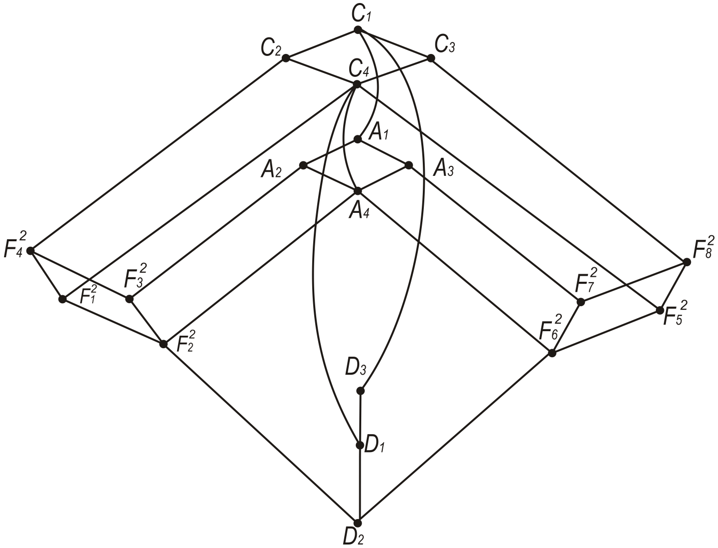
Proof

Assume that $(a_1, a_2, \dots, a_n)$ is an essential tuple for ρ, $n > 2$.

$$m \begin{pmatrix} b_1 & a_1 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & a_3 & b_3 \\ a_4 & a_4 & a_4 \\ \dots & \dots & \dots \\ a_n & a_n & a_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \dots \\ a_n \end{pmatrix} \in \rho$$

This contradicts the condition $\rho(a_1, \dots, a_n) = 0$.

Majority Function in P_2.



Function $x \vee yz$.

Let

$$\rho_{\vee,n}(x_1, x_2, \ldots, x_n) = 1 \iff (x_1 = 1) \vee (x_2 = 1) \vee \ldots \vee (x_n = 1).$$

$$C = \left\{ \begin{pmatrix} 0 \end{pmatrix}, \begin{pmatrix} 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \rho_{\vee,2}, \rho_{\vee,3}, \rho_{\vee,4}, \ldots \right\}$$

Lemma

$\mathrm{Inv}(x \vee yz) \cap \widetilde{R}_2 = C.$

All clones containing $x \vee yz$ can be described by predicates from the set C.

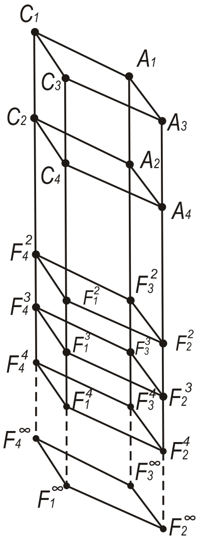
Function $x \vee yz$.

Proof

Suppose $\rho \in \mathbf{Inv}(x \vee yz) \cap \widetilde{R}_2$, the arity of ρ is equal to n, where $n \geq 3$. Let us show that an essential tuple for ρ cannot contain 1. Assume that $(1, 0, 0, \ldots, 0)$ is an essential tuple for ρ. Then

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \ldots \\ 0 \end{pmatrix} \vee \left(\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ \ldots \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \ldots \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \ldots \\ 0 \end{pmatrix} \in \rho$$

This contradicts the definition of an essential tuple. Hence $(0, 0, 0, \ldots, 0)$ is the only essential tuple for ρ. Then, obviously ρ is equal to $\rho_{\vee,n}$.



$$C_1 = P_2,\ C_3 = \mathrm{Pol}\left(0\right),\ A_1 = \mathrm{Pol}\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}$$

$$A_3 = \mathrm{Pol}\left(\left\{\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}, (0)\right\}\right);$$

$$C_2 = \mathrm{Pol}\left(1\right),\ C_4 = \mathrm{Pol}\left(\left\{(1), (0)\right\}\right),$$

$$A_2 = \mathrm{Pol}\left(\left\{(1), \begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}\right\}\right),$$

$$A_3 = \mathrm{Pol}\left(\left\{(1), \begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}, (0)\right\}\right);$$

$$F_4^n = \mathrm{Pol}\left(\rho_{\vee,n}\right),\ F_1^n = \mathrm{Pol}\left(\left\{\rho_{\vee,n}, (0)\right\}\right),$$

$$F_3^n = \mathrm{Pol}\left(\left\{\rho_{\vee,n}, \begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}\right\}\right),$$

$$F_2^n = \mathrm{Pol}\left(\left\{\rho_{\vee,n}, \begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}, (0)\right\}\right);$$

The set of essential predicates $S \subseteq \widetilde{R}_k$ is called closed if $\sigma_k^{=}, \mathit{false} \in S$, and the following operations applied to the predicates from S always give not essential predicate or a predicate from the set S.

1. Permutation of variables.
2. Striking of rows (adding existential quantifiers).
3. Identifying of two variables.
4. $\zeta(\rho) = \rho'$, where $\rho'(x_1, x_2, \ldots, x_{n-1}) = \exists x \ \rho(x, x, x_1, x_2, \ldots, x_{n-1})$.
5. $\epsilon(\rho_1, \rho_2) = \rho$, where $m \leq n$ and
$$\rho(x_1, x_2, \ldots, x_n) = \rho_1(x_1, x_2, \ldots, x_n) \wedge \rho_2(x_1, x_2, \ldots, x_m).$$
6. for $2 \leq l \leq k$ let $\delta_l(\rho_1, \ldots, \rho_l) = \rho$, where
$$\rho(x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{l,1}, \ldots, x_{l,n_l}) = \\ = \exists x(\rho_1(x, x_{1,1}, \ldots, x_{1,n_1}) \wedge \ldots \wedge \rho_l(x, x_{l,1}, \ldots, x_{l,n_l})),$$
(all variables in the right-hand side of the formula are different).

Theorem

The set of essential predicates $S \subseteq \widetilde{R}_k$ is closed if and only if $[S] \cap \widetilde{R}_k = S$.

- Every closed set of predicates can be uniquely defined by the closed set of essential predicates.
- We may give up considering not essential predicates at all.

Examples

- Let ρ be a linear order relation on the set E_k. Let us show that Pol(ρ) is a maximal clone in P_k. Let $\rho'(x,y) = \rho(y,x)$. Obviously $\{\rho, \rho', \sigma_k^=, \mathit{false}, \mathit{true}\}$ is closed set of essential predicates. Then the maximality of Pol(ρ) follows from the fact that there are no closed subsets that differ from $\{\sigma_k^=, \mathit{false}, \mathit{true}\}$.
- How to check the equality of the following sets Pol $\begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}$, Pol $\begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \end{pmatrix}$? Using the closure operator we can show that $\left\{ \begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \end{pmatrix}, \sigma_3^=, \mathit{false}, \mathit{true} \right\}$ is a closed set. Hence these sets are different.

Thank you for your attention