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Which maximal clones can be maximal C-clones?

Edith Mireya Vargas

joint work with

Mike Behrisch (TUD).

Conference on Universal Algebra and Lattice Theory

Outline

Clausal relations

C-clones

Maximal C-clones.

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Let $p, q \in \mathrm{N}_+$ and $D = \{0, 1, \ldots, n-1\}$.

Definition (Clausal relations)

$(D, \leq)$ poset. Let $\mathbf{a} = (a_1, \ldots, a_p) \in D^p$, $\mathbf{b} = (b_1, \ldots, b_q) \in D^q$. The *clausal relation* $\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$ *of arity* $p + q$ is

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$$\begin{aligned}\mathrm{R}^{\mathbf{a}}_{\mathbf{b}} \; := \{&(x_1, \dots, x_p, y_1, \dots, y_q) \in D^{p+q} \mid \\ &(x_1 \geq a_1) \vee \dots \vee (x_p \geq a_p) \vee (y_1 \leq b_1) \vee \dots \vee (y_q \leq b_q)\}.\end{aligned}$$

$$\mathcal{R}_q^p := \{\mathrm{R}_{\mathbf{b}}^{\mathbf{a}} \mid \mathbf{a} \in D^p, \mathbf{b} \in D^q\}$$

the *set of all clausal relations of arity* $p + q$

$$\mathrm{CR}_{(D,\leq)} := \bigcup_{(p,q)\in\mathbb{N}_+^2} \mathcal{R}_q^p$$

the *set of all finitary clausal relations* on $(D, \leq)$.

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Here only: $(D, \leq)$ linear; w.l.o.g. $0 < 1 < 2 < \cdots < n - 1$, then CR_D for $\mathrm{CR}_{(D,\leq)}$

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Trivial clausal relations

Fact

If $(D, \leq)$ *has a bottom element* $\bot$

$$(\exists i \in \{1, \ldots, p\} : a_i = \bot) \implies \mathrm{R}^{\mathbf{a}}_{\mathbf{b}} = D^{p+q}.$$

If $(D, \leq)$ *has a top element* $\top$

$$(\exists j \in \{1, \ldots, q\} : b_j = \top) \implies \mathrm{R}^{\mathbf{a}}_{\mathbf{b}} = D^{p+q}.$$

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(Non)Trivial clausal relations

Lemma

$(D, \leq)$ linear (canonical).

$$\begin{aligned} CR_D \cap \operatorname{diag}(D) &= \left\{ D^{p+q} \mid p, q \in \mathrm{N}_+ \right\}. \\ CR_D^* &:= CR_D \setminus \operatorname{diag}(D) \\ &= \{ \mathrm{R}_{\mathbf{b}}^{\mathbf{a}} \mid \mathbf{a} \in (D \setminus \{0\})^p, \mathbf{b} \in (D \setminus \{n-1\})^q; p, q \in \mathrm{N}_+ \} \end{aligned}$$

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Let $k, m \in \mathrm{N}_+$, $\mathrm{O}_D^{(k)} := \{f \mid f : D^k \longrightarrow D\}$ $\qquad$ $\mathrm{O}_D := \bigcup_{k=1}^{\infty} \mathrm{O}_D^{(k)}$

$$\mathrm{R}_D^{(m)} := \mathcal{P}(D^m) \qquad \mathrm{R}_D := \bigcup_{m=1}^{\infty} \mathrm{R}_D^{(m)}$$

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$\boxed{f \in \mathrm{O}_D^{(k)} \text{ preserves } \varrho \in \mathrm{R}_D^{(m)}}$ denoted by $\boxed{f \rhd \varrho}$

$$f \circ \left(\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}, \cdots, \begin{pmatrix} a_{1k} \\ \vdots \\ a_{mk} \end{pmatrix} \right)$$

$$\begin{matrix} \pitchfork & \qquad & \pitchfork \\ \varrho & \qquad & \varrho \end{matrix}$$

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$$f \circ \left(\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}, \dots, \begin{pmatrix} a_{1k} \\ \vdots \\ a_{mk} \end{pmatrix} \right) = \begin{pmatrix} f(a_{11} & \dots & a_{1k}) \\ & \ddots & \\ f(a_{m1} & \dots & a_{mk}) \end{pmatrix}$$

$$\begin{matrix} \Cap & \Cap & \Cap \\ \varrho & \varrho & \varrho \end{matrix}$$

$(\mathrm{O}_D, \mathrm{R}_D, \rhd)$

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$$(\mathrm{O}_D, \mathrm{R}_D, \triangleright)$$

$$\mathsf{Pol}_D : \ \mathcal{P}(\mathrm{R}_D) \longrightarrow \mathcal{P}(\mathrm{O}_D)$$

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(relational) clones

$$\mathsf{Pol}\,\mathsf{Inv} = \langle\ \rangle_{\mathrm{O}_D}$$

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$$F \mapsto C\mathrm{R}_D \cap \mathsf{Inv}_D\, F$$

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(relational) C-clones

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$$\mathsf{Pol}\,C\mathsf{Inv} := \langle\ \rangle_C$$

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$$C\mathsf{Inv}\,\mathsf{Pol}$$

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Problem:
Characterisation of the Galois closed sets of operations and relations of the Galois connection Pol − *C*Inv, i.e.

- For $F \subseteq \mathrm{O}_D$
$$\langle F\rangle_{C} := \mathrm{Pol}\, C\mathrm{Inv}\, F \ ??$$
- For $Q \subseteq C\mathrm{R}_D$
$$[Q]_{\mathcal{C}} := C\mathrm{Inv}\,\mathrm{Pol}\, Q \ ??$$

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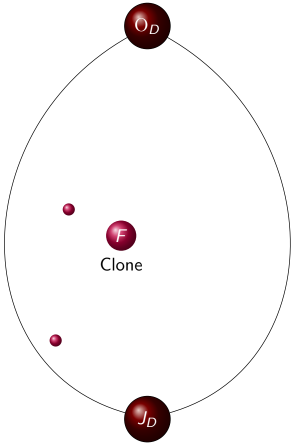
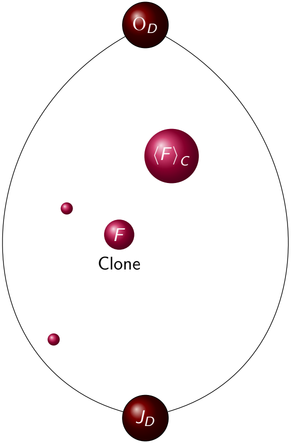
Where do the C-clones live?



Where do the C-clones live?



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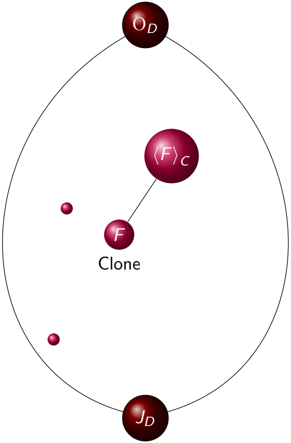

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Where do the C-clones live?

For $F \subseteq \mathrm{O}_D$

$$\mathsf{Inv}\, F \supseteq C\mathrm{R}_D \cap \mathsf{Inv}\, F = C\mathsf{Inv}\, F$$

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Where do the C-clones live?

For $F \subseteq \mathrm{O}_D$

$$\begin{aligned} \operatorname{Inv} F \supseteq C\mathrm{R}_D \cap \operatorname{Inv} F &= C\operatorname{Inv} F \\ \Rightarrow \langle F \rangle_{\mathrm{O}_D} = \operatorname{Pol}\operatorname{Inv} F \subseteq \operatorname{Pol} C\operatorname{Inv} F &= \langle F \rangle_C \end{aligned}$$

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How many C-clones do exist for an arbitrary finite set D?

- For $D = \{0, 1\}$, there are five different C-clones.

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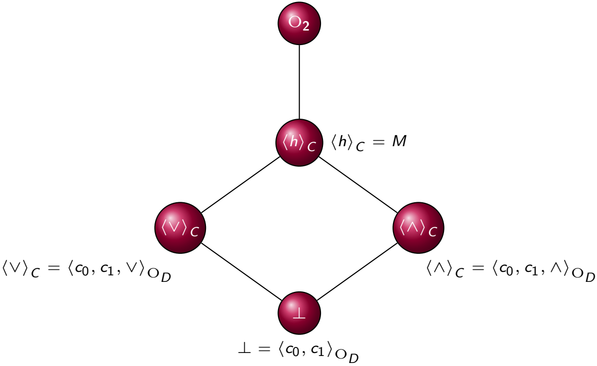

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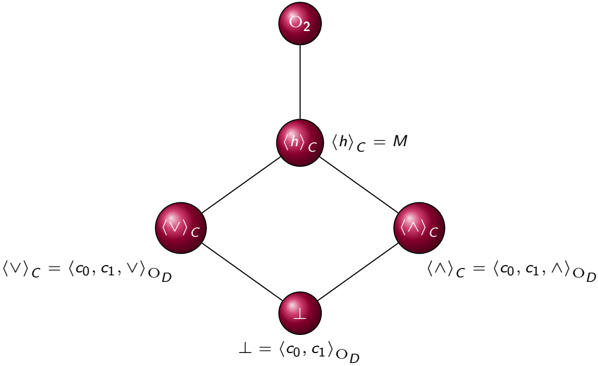

- For $|D| \geq 3$, there exist infinitely many C-clones.

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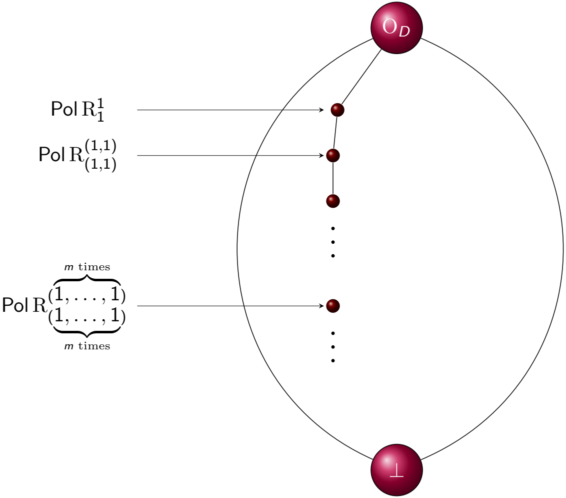

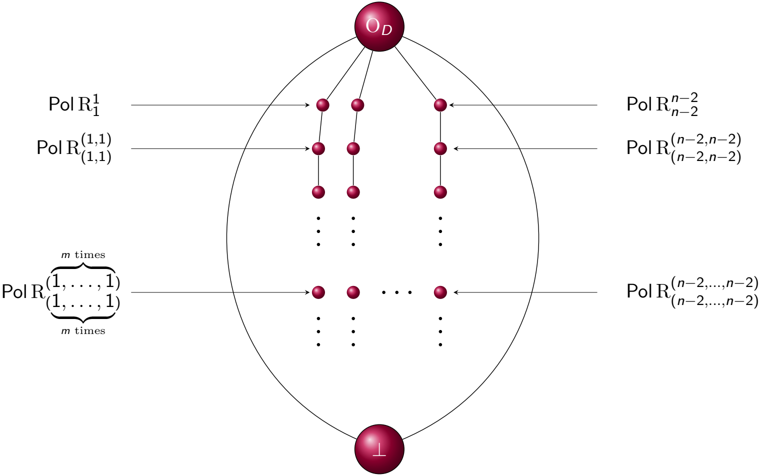
O_D
Pol R_1^1
Pol $R_{(1,1)}^{(1,1)}$
Pol $R_{\underbrace{(1,\ldots,1)}_{m \text{ times}}}^{\overbrace{(1,\ldots,1)}^{m \text{ times}}}$
Pol R_{n-2}^{n-2}
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$\perp$

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Maximal clones / C-clones

- = maximal elements (co-atoms) in the lattices of (clones / C-clones)

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- $=$ Pol $\{\rho\}$ for some $\rho \in \mathrm{R}_D \setminus \mathrm{diag}(D)$ ($\rho \in C\mathrm{R}_D^*$ respectively)

Maximal clones / C-clones

- = maximal elements (co-atoms) in the lattices of (clones / C-clones)
- $= \mathrm{Pol}\,\{\rho\}$ for some $\rho \in \mathrm{R}_D \setminus \mathrm{diag}(D)$ ($\rho \in C\mathrm{R}_D^*$ respectively)

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Maximal C-clones

Theorem (Vargas, 2011)

Let $M \subseteq \mathrm{O}_D$ be a C-clone . M is maximal if and only if there are elements $a \in D \setminus \{0\}$ and $b \in D \setminus \{n-1\}$ such that

$$M = \mathsf{Pol}_D \, \mathrm{R}^{(a)}_{(b)}.$$

Connections

- $D = \{0,1\} \Rightarrow \mathrm{Pol}\,\mathrm{R}_{(0)}^{(1)} = \mathrm{Pol} \leq_2$ monotone functions.
- $|D| > 2$? Which maximal clones are maximal C-clones?

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Connections

- $D = \{0,1\} \Rightarrow \mathrm{Pol}\, \mathrm{R}^{(1)}_{(0)} = \mathrm{Pol} \leq_2$ monotone functions.
- $|D| > 2$? Which maximal clones are maximal $C\text{-clones}$?
- $F \leq \mathrm{O}_D$ maximal clone. $\Rightarrow$ two possibilities:

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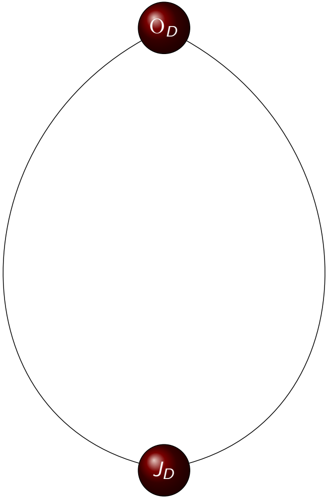

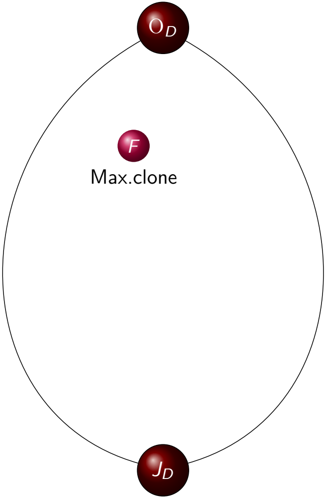

O_D
F
Max.clone
J_D

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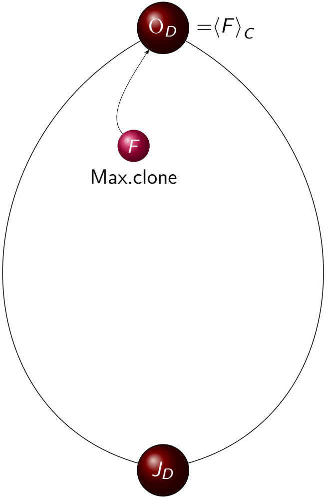

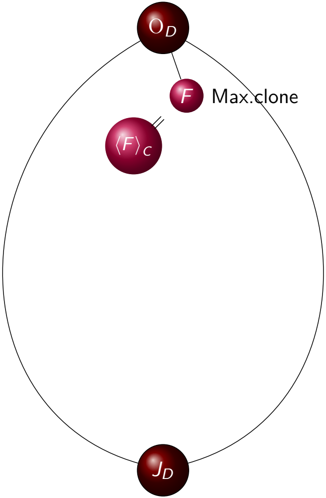
O_D
F
Max.clone
$\langle F \rangle_C$
J_D

Theorem (Behrisch-Vargas, 2012)

If $D = \{0, 1\}$, then the only maximal C-clone $\mathrm{Pol}_D \mathrm{R}_{(0)}^{(1)}$ on this set is the maximal clone $\mathrm{Pol}_D \leq_2$ of monotone functions w.r.t. the linear order $0 \leq_2 1$. Any other maximal C-clone (that is on any finite domain D with $|D| > 2$) fails to be a maximal clone, hence it is properly contained in some maximal clone.

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Idea for the proof

Show for (almost) every maximal clone F:

- $\mathrm{Pol}\, C\mathrm{Inv}\, F = \langle F \rangle_C = \mathrm{O}_D$
- For all ϱ in Rosenberg's Theorem:

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$$\operatorname{Pol} C\operatorname{Inv} \operatorname{Pol} \varrho = \mathrm{O}_D$$

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$$\iff \quad \begin{aligned} \operatorname{Pol} C\operatorname{Inv} \operatorname{Pol} \varrho &= \mathrm{O}_D \\ C\operatorname{Inv} \operatorname{Pol} \varrho &\subseteq \operatorname{diag}(D) \end{aligned}$$

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\begin{aligned}
& & \operatorname{Pol} C\operatorname{Inv} \operatorname{Pol} \varrho = \mathrm{O}_D \\
\iff & & C\operatorname{Inv} \operatorname{Pol} \varrho \subseteq \operatorname{diag}(D) \\
\iff & & C\mathrm{R}_D^* \cap \operatorname{Inv} \operatorname{Pol} \varrho = \emptyset
\end{aligned}
$$

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- $\mathrm{Pol}\, C\mathrm{Inv}\, F = \langle F \rangle_C = \mathrm{O}_D$
- For all ϱ in Rosenberg's Theorem:

$$\begin{aligned} & \mathrm{Pol}\, C\mathrm{Inv}\,\mathrm{Pol}\,\varrho = \mathrm{O}_D \\ \iff\quad & C\mathrm{Inv}\,\mathrm{Pol}\,\varrho \subseteq \mathrm{diag}(D) \\ \iff\quad & \mathrm{CR}_D^* \cap \mathrm{Inv}\,\mathrm{Pol}\,\varrho = \emptyset \\ \iff\quad & \forall \mathrm{R}_{\mathbf{b}}^{\mathbf{a}} \in \mathrm{CR}_D^*\ \exists\, f \in \mathrm{Pol}\,\varrho : f \not\rhd \mathrm{R}_{\mathbf{b}}^{\mathbf{a}} \end{aligned}$$

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Thank you for your attention!