Clones with Finitely Many Relative R-Classes

Á. Szendrei

University of Colorado at Boulder
and
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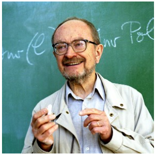
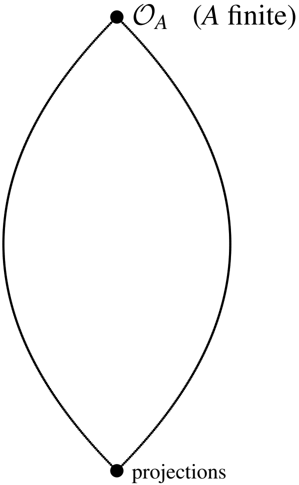
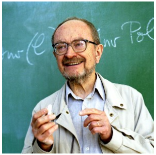
Universal Algebra and Lattice Theory
Szeged, Hungary, June 21–25, 2012



Finite algebras, clones on finite sets





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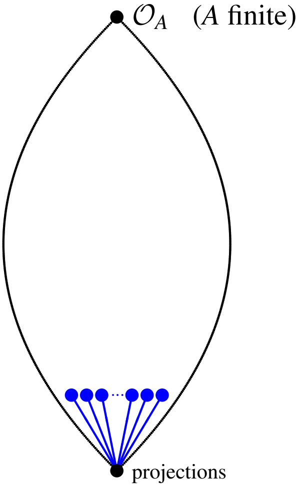
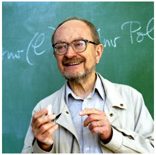


minimal clones



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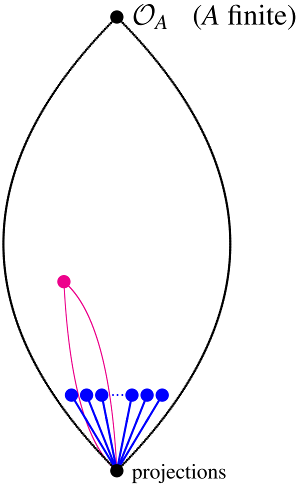

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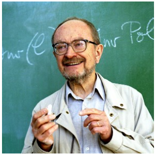

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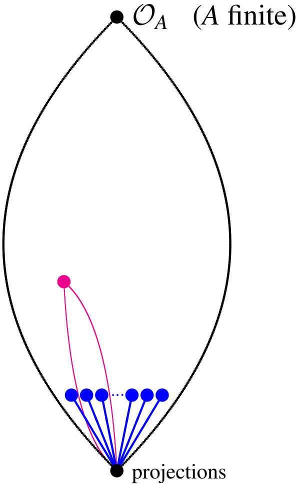

homogeneous algebras

↳ descr of clones

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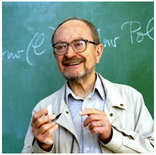

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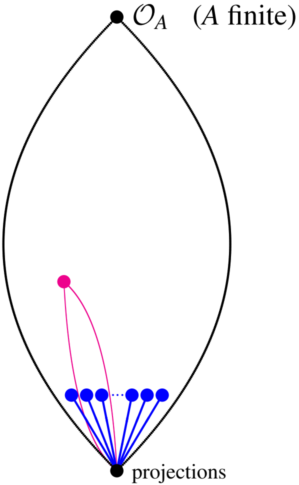

(almost all) are functionally complete

homogeneous algebras

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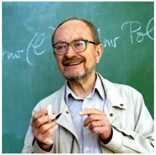

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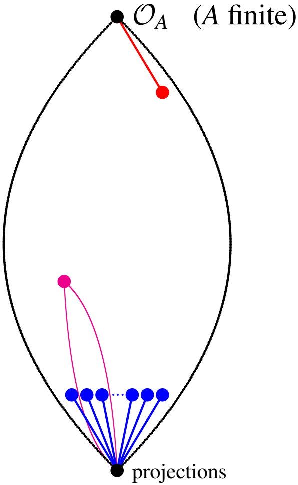

Słupecki's Completeness Thm

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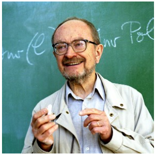

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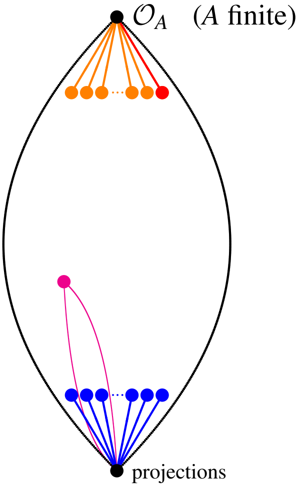

Rosenberg's Completeness Thm

Słupecki's Completeness Thm

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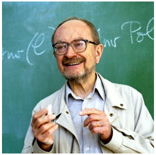
→ descr of clones



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Słupecki's Completeness Theorem (restated): There is a unique maximal clone on A that contains all unary operations: Słupecki's clone.

Rosenberg's Completeness Theorem: The maximal clones on a k-element set A are the clones $\{\rho\}^\perp = \{f \in \mathcal{O}_A : f \text{ preserves } \rho\}$ for one of six types of relations ρ:

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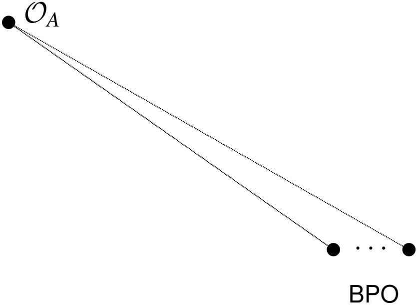

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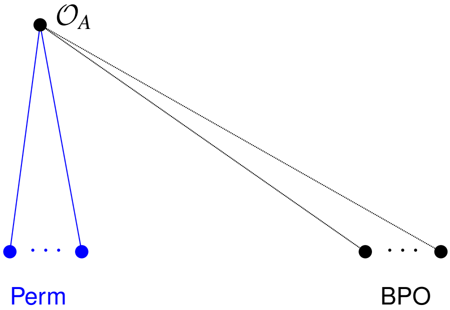

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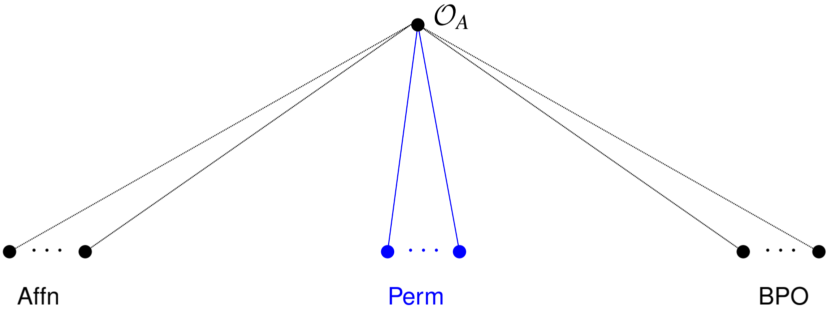

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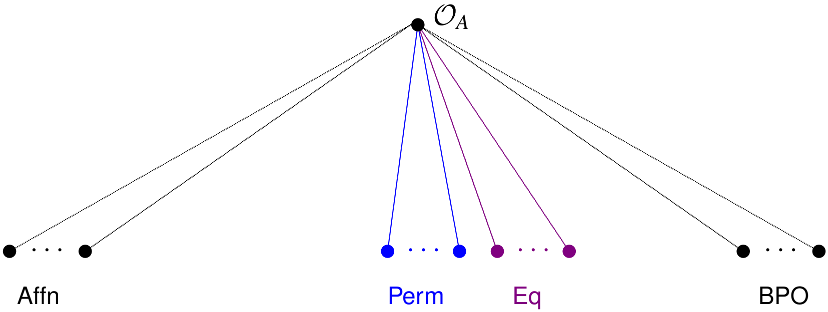

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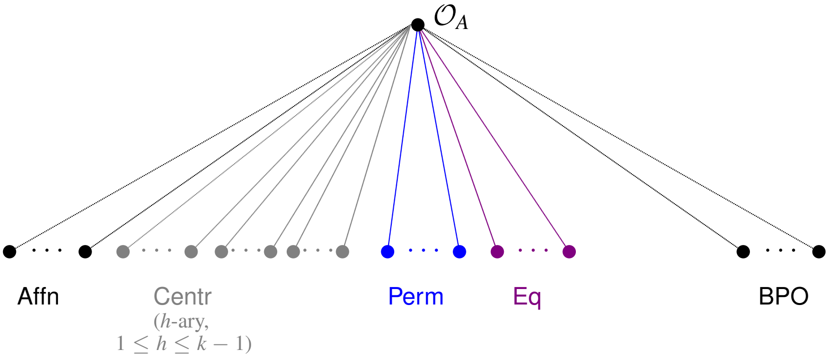

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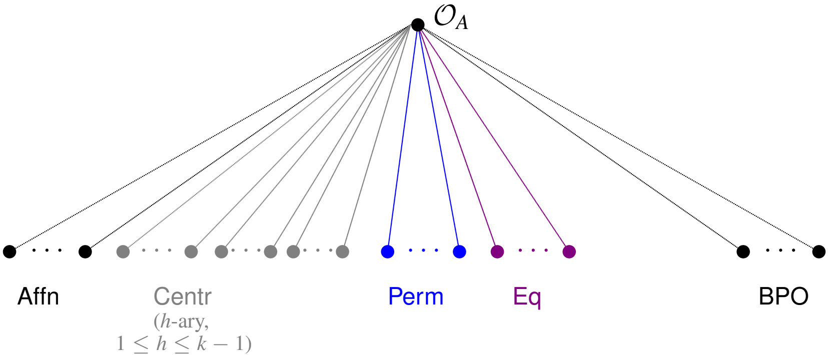

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- $\neq A^h$, but contains $\{c\} \times A^{h-1}$ for some $c \in A$

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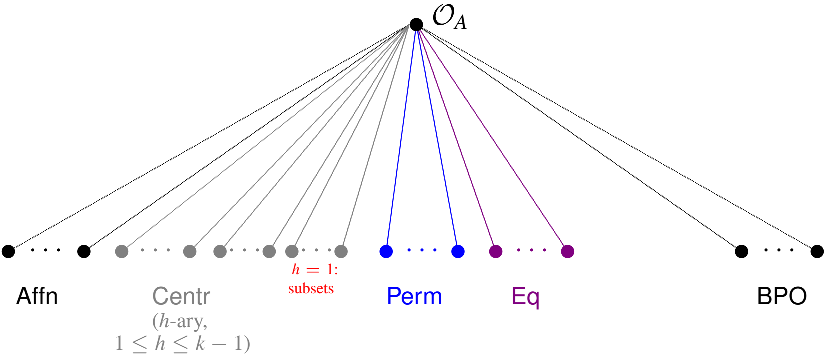

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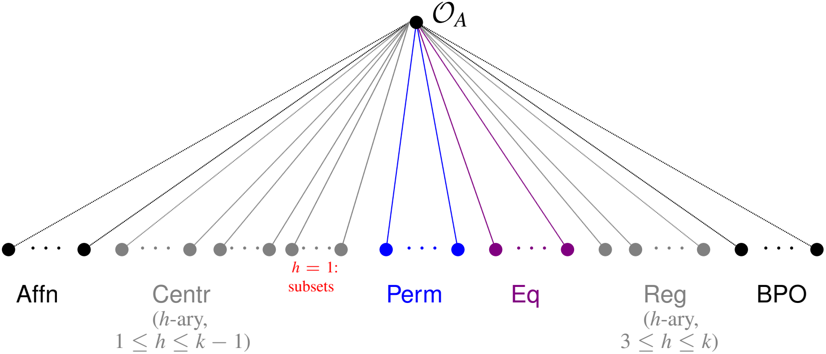

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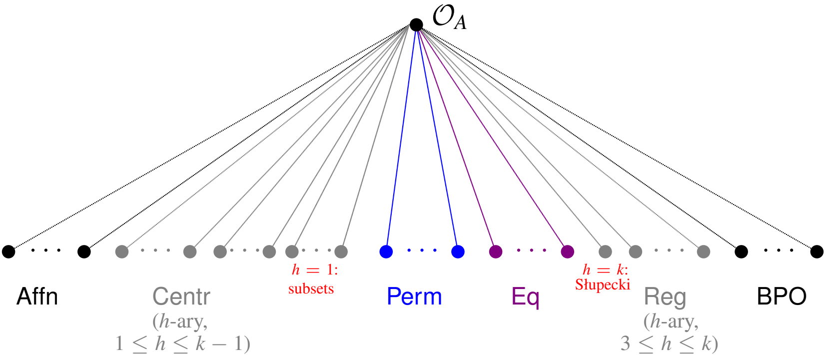

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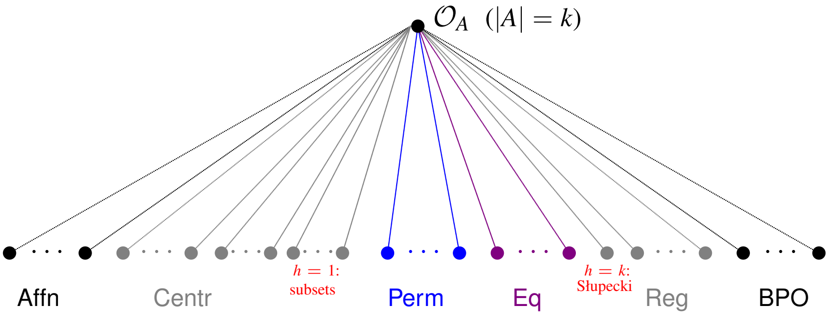
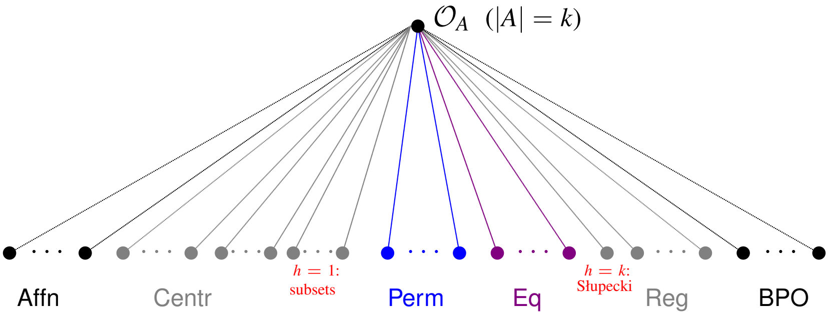
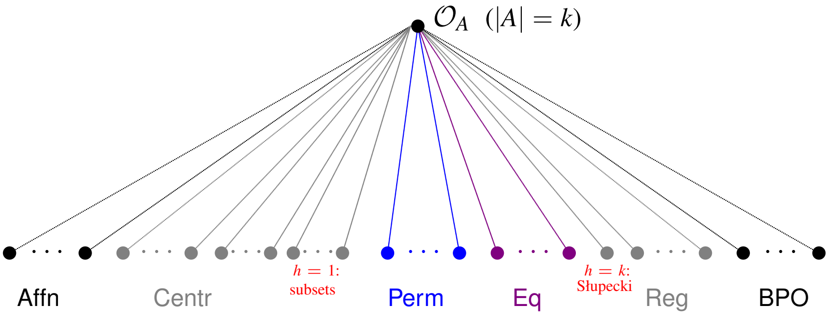

$\mathcal{O}_A \quad (|A| = k)$
$h = 1$:
subsets
$h = k$:
Słupecki
Affn
Centr
Perm
Eq
Reg
BPO



Maximal subclones known for maximal clone of type



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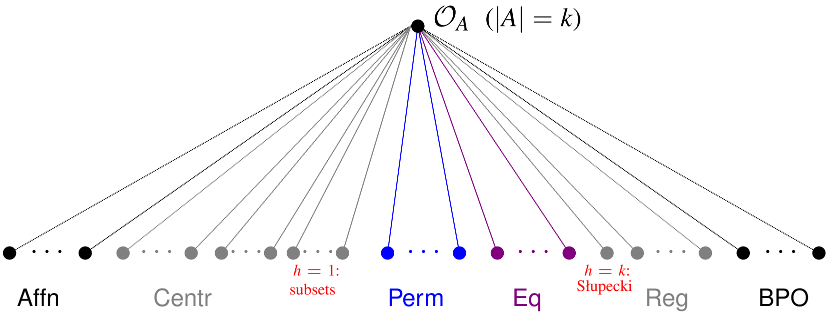

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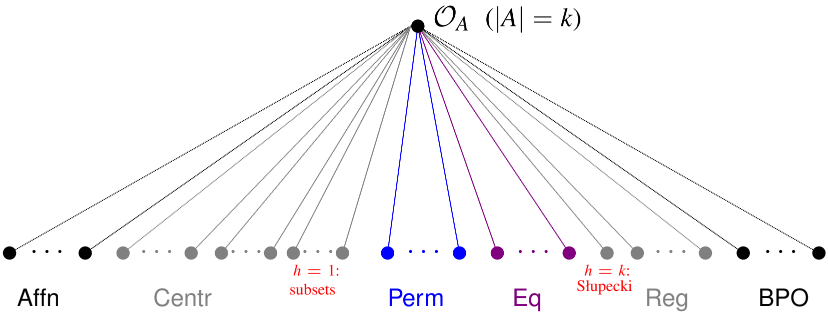

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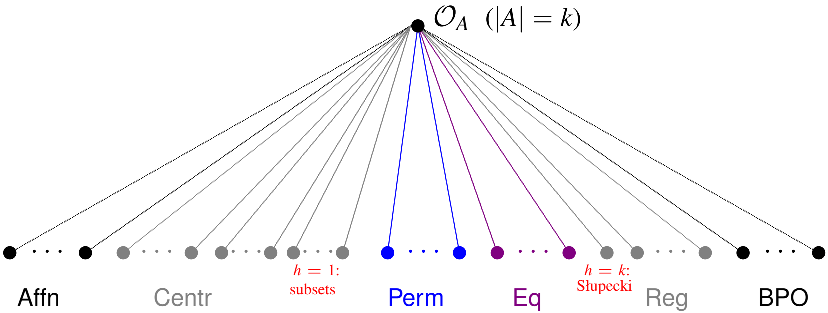

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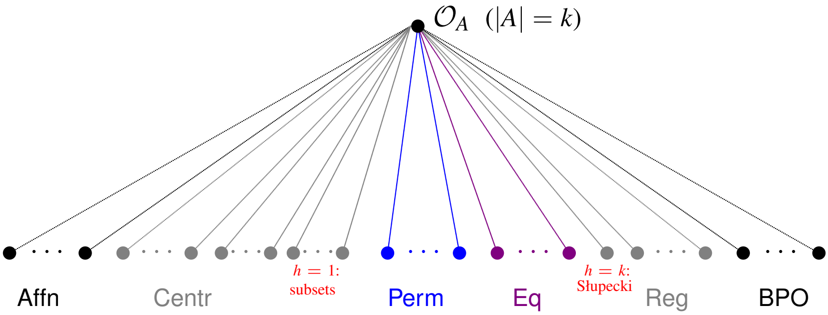

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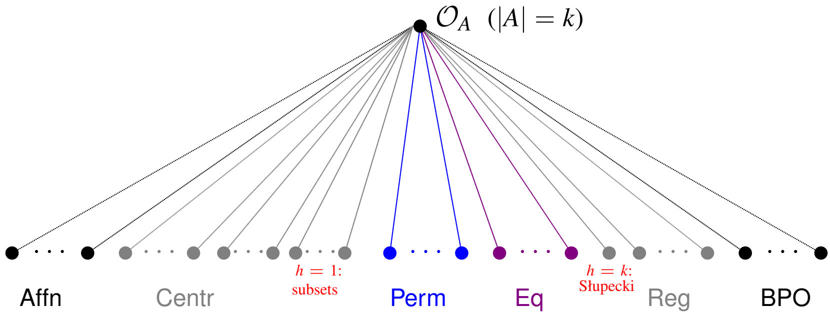

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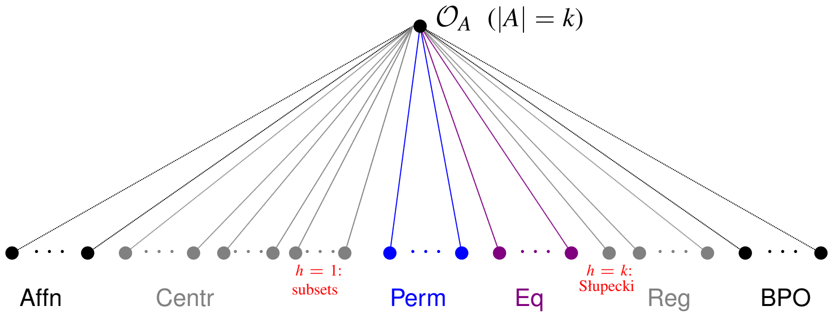

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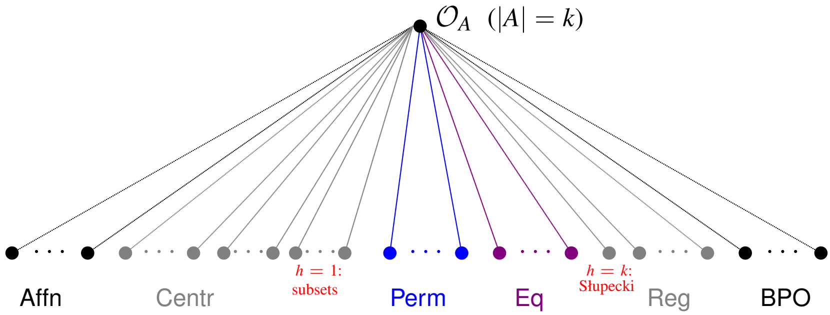

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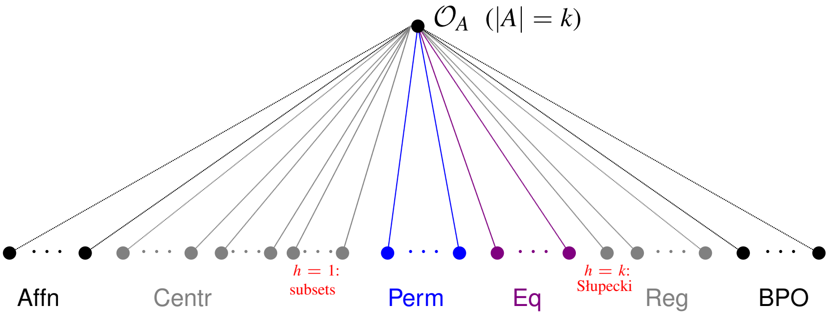

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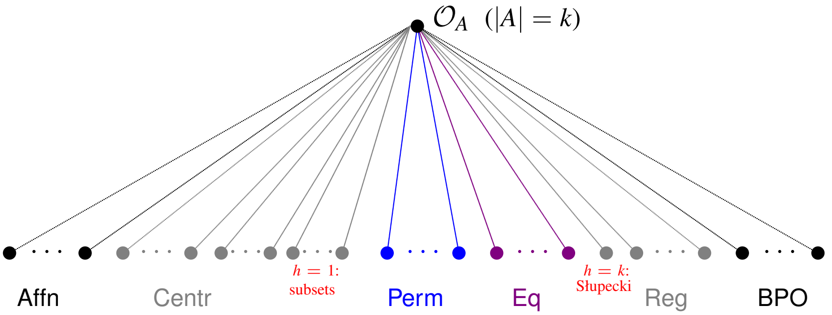

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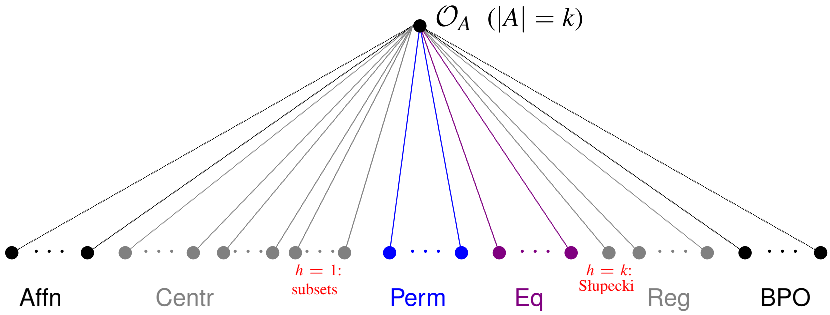

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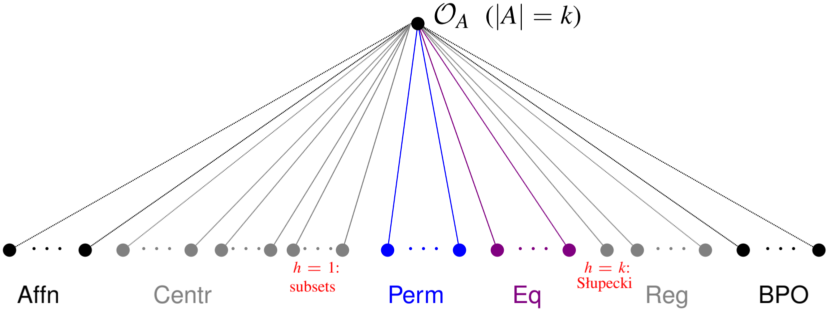

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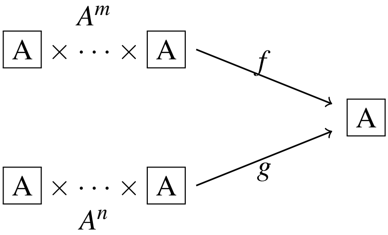

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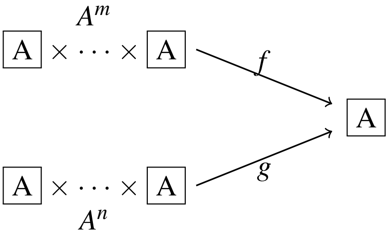

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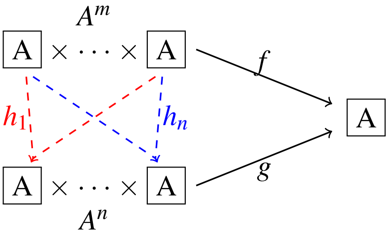
  s.t. $\quad f = g \circ \mathbf{h}$



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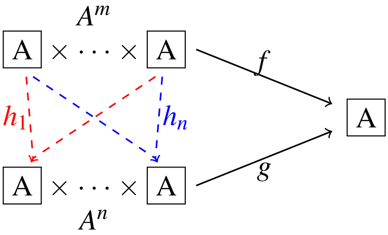

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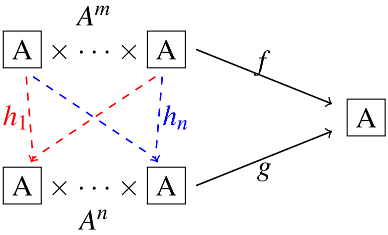

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Easy Facts: $\diamond$ $\leq_{\mathcal{C}}$ is a quasiorder, $\equiv_{\mathcal{C}}$ is an equivalence relation

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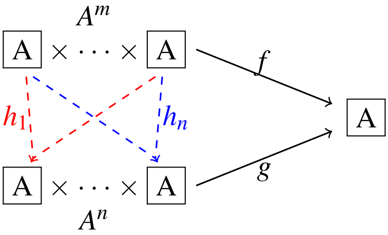

- $f \equiv_{\mathcal{C}} g \quad \Leftrightarrow f \leq_{\mathcal{C}} g$ & $f \geq_{\mathcal{C}} g$

Easy Facts: $\diamond$ $\leq_{\mathcal{C}}$ is a quasiorder, $\equiv_{\mathcal{C}}$ is an equivalence relation
$\diamond$ $\mathcal{C} \subseteq \mathcal{D} \Rightarrow\ \leq_{\mathcal{C}} \subseteq\ \leq_{\mathcal{D}}$

$\mathcal{C}$ a clone on A

Definition of rels $\leq_{\mathcal{C}}$, $\equiv_{\mathcal{C}}$ on $\mathcal{O}_A$: For $f \in \mathcal{O}_A^{(m)}$, $g \in \mathcal{O}_A^{(n)}$

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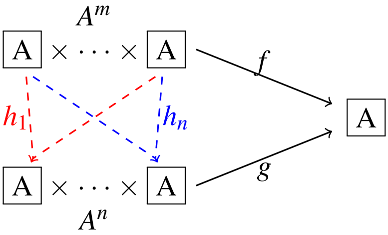

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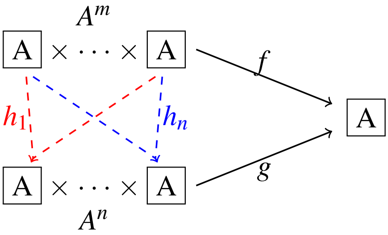

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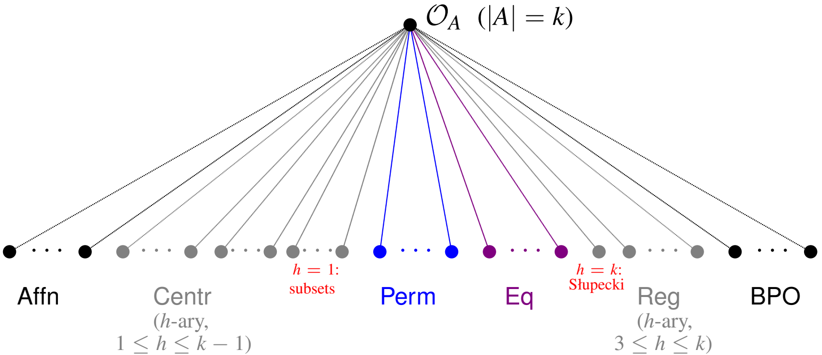
- $\diamond$ $\mathfrak{F}_A := \{\mathcal{C} : \equiv_{\mathcal{C}}$ has finitely many equiv classes$\}$ $(\neq \emptyset)$ is an order filter in the lattice of clones on finite A

Lehtonen–Sz'11: The maximal clones in $\mathfrak{F}_A$ are the following:



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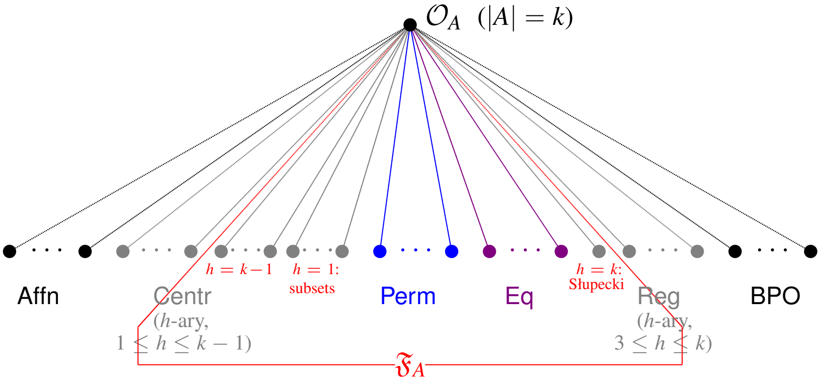

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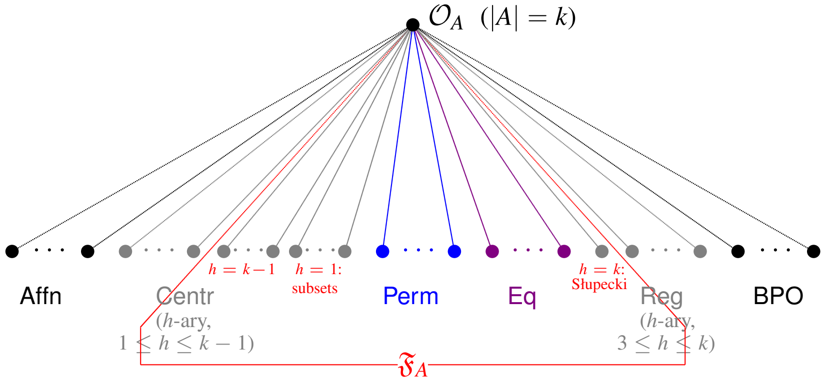

$(k-1)$-ary central relations:

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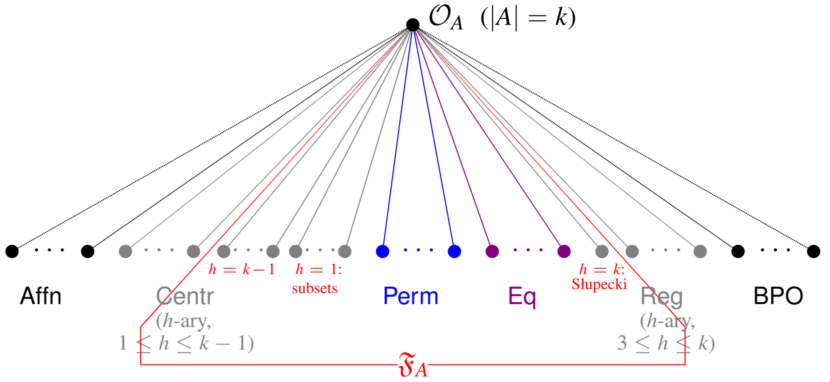

$(k-1)$-ary central relations: $\sigma_c \ (c \in A)$

$$(a_1, \ldots, a_{k-1}) \notin \sigma_c \;\Leftrightarrow\; \{a_1, \ldots, a_{k-1}\} = A \setminus \{c\}$$

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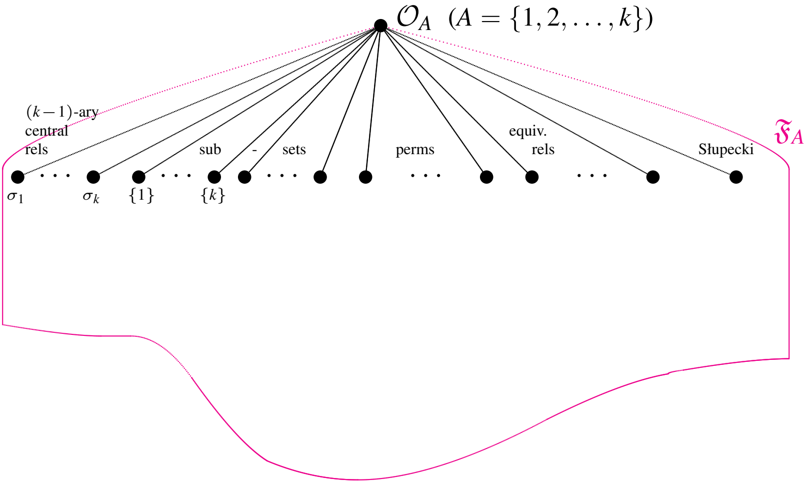

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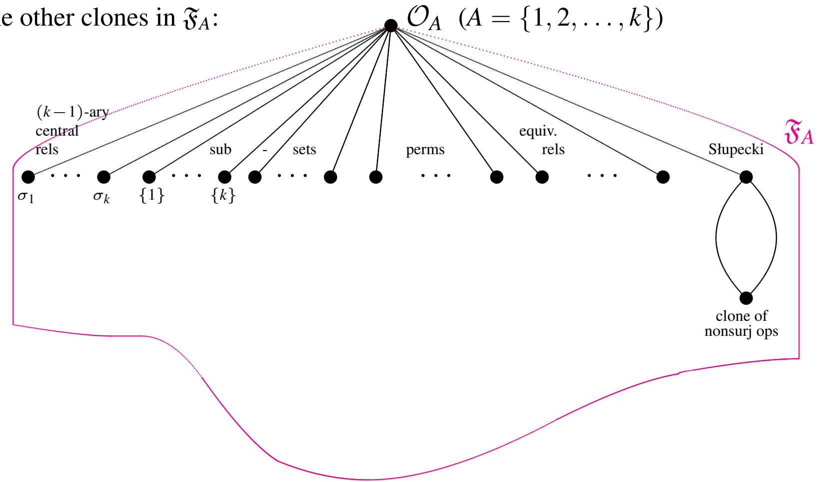

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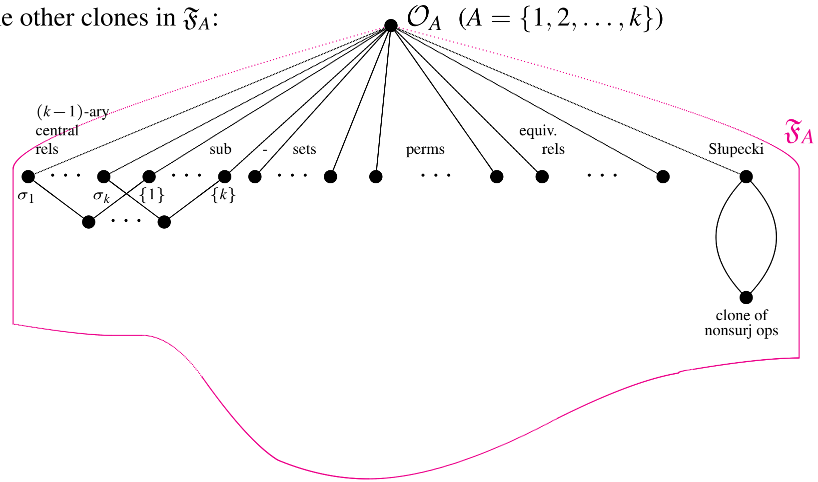

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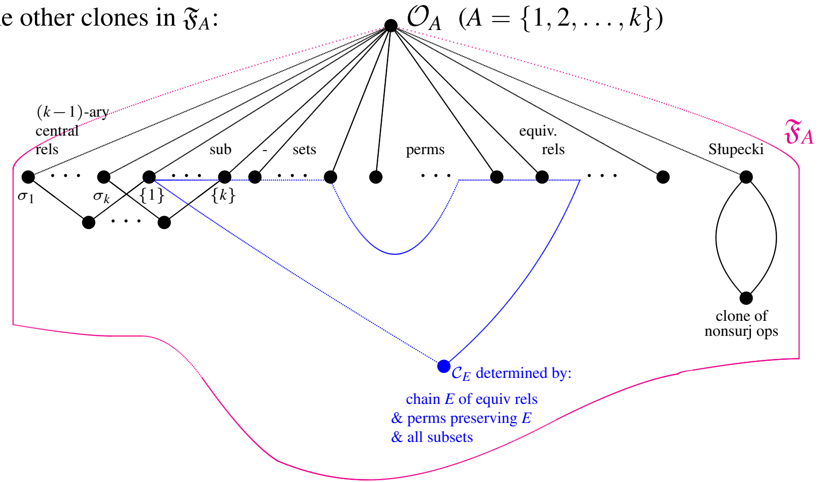

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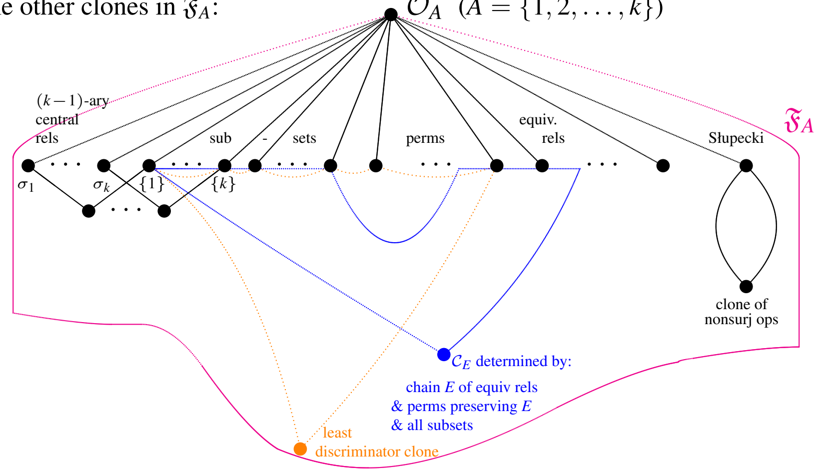

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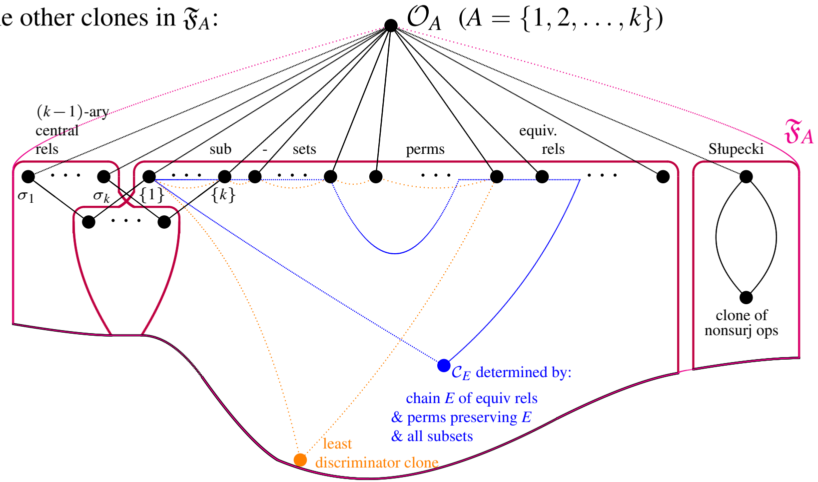

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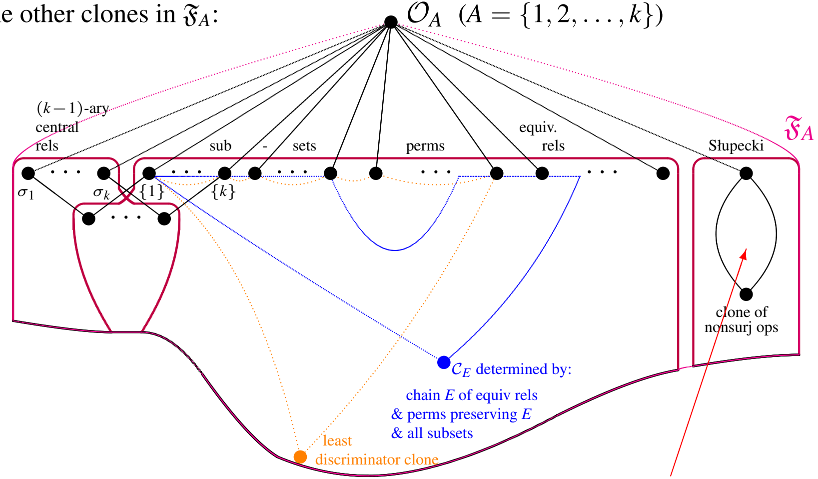

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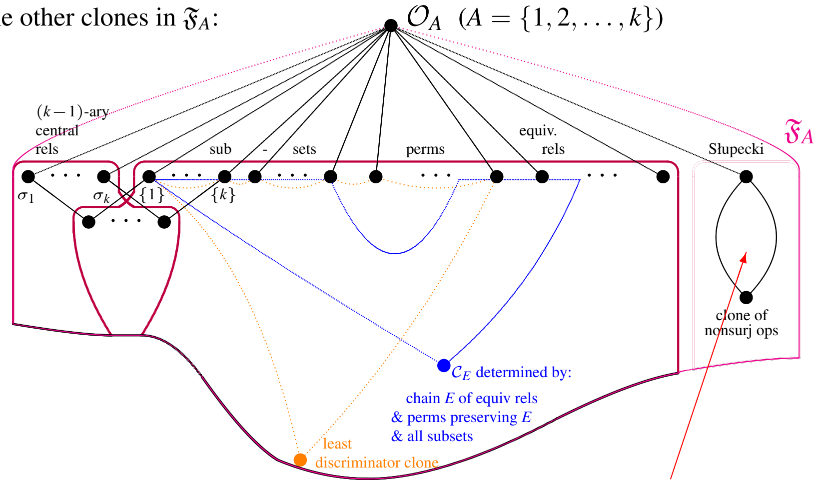
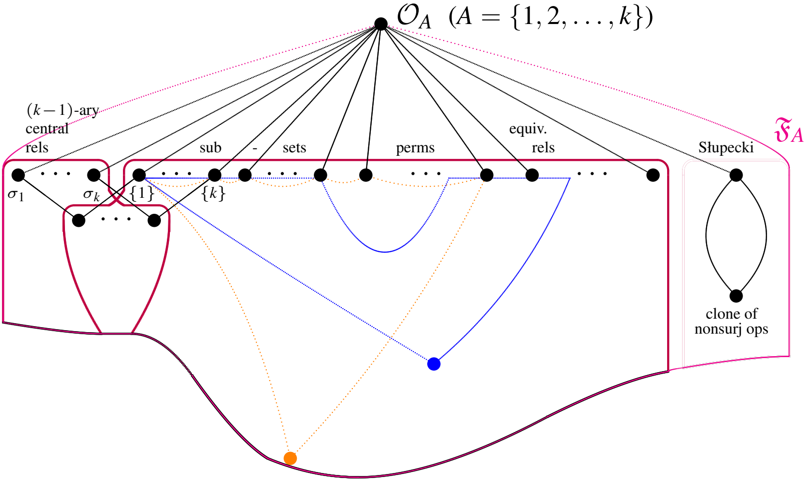


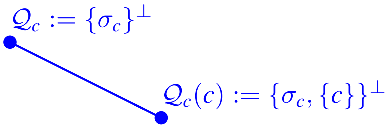
Sz'13?: Descr of all maximal subclones of the clones in this interval

yields: clone of nonsurj ops $\not\subseteq \mathcal{C} \subseteq$ Słupecki $\Rightarrow \mathcal{C} \notin \mathfrak{F}_A$

Question: Which subclones of $\mathcal{Q}_c := \{\sigma_c\}^\perp$ ($c \in A$) belong to $\mathfrak{F}_A$?



The subclones of $\mathcal{Q}_c$ in $\mathfrak{F}_A$ ($c \in A$, $|A| = k \geq 3$):



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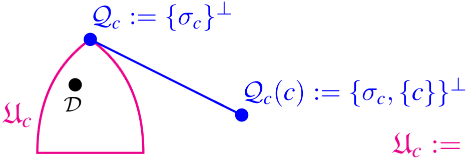

$\mathfrak{U}_c := \{\mathcal{D} \in \mathfrak{F}_A : \mathcal{D} \subseteq \mathcal{Q}_c,\ \mathcal{D} \not\subseteq \mathcal{Q}_c(c)\}$

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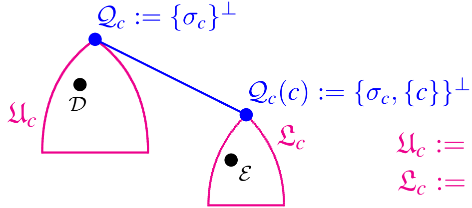

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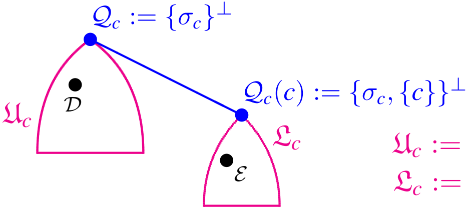

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$\mathcal{E} \in \mathfrak{L}_c \;\Rightarrow\; \mathcal{E} = \mathcal{Q}_c(c) \cap R^\perp$ *for some set R of reflexive rels on A.*

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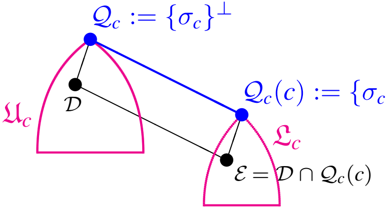

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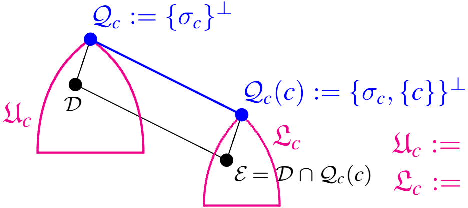

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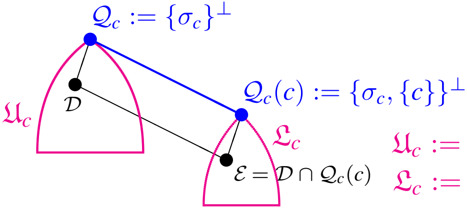

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Thus, $\mathfrak{U}_c$ determines $\mathfrak{L}_c$.

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- $\rho|_B = \emptyset \;\Rightarrow\; \rho \subseteq \chi_c := (\{c\} \times A) \cup (A \times \{c\})$

To show: $\mathcal{E} = \mathcal{Q}_c(c) \cap \{\rho\}^\perp \subsetneq \mathcal{Q}_c(c)$ (ρ indecomposable), $\mathcal{E} \in \mathfrak{F}_A \Rightarrow \rho$ reflexive

Special case: ρ binary and $|A| \geq 4$

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$\Delta(\rho) := \{a \in A : (a, a) \in \rho\}$ is $\emptyset$, $\{c\}$, or A

- otherwise, $\mathcal{E} \subseteq \mathcal{Q}_c \cap \{S\}^\perp \notin \mathfrak{F}_A$ for $S = \rho_1, \rho_2$, or $\Delta(\rho)$

(2) $\rho|_B \neq \emptyset$ for $B := A \setminus \{c\} \Rightarrow \Delta(\rho) = A$, i.e. ρ is reflexive

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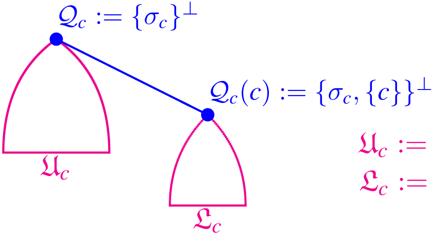
The subclones of $\mathcal{Q}_c$ in $\mathfrak{U}_c$ ($c \in A$, $|A| = k \geq 3$):



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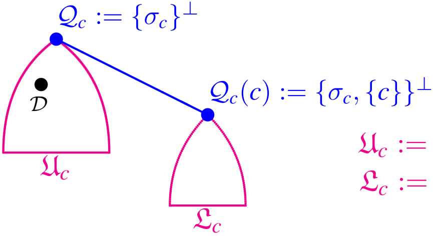

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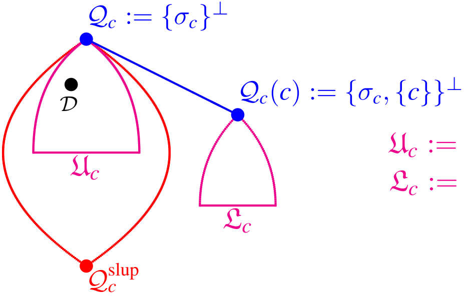

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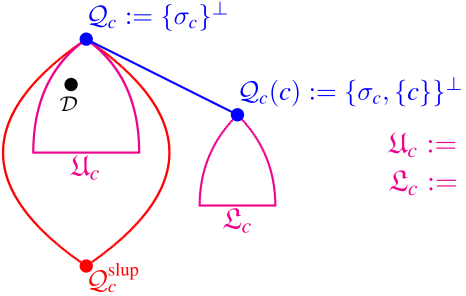

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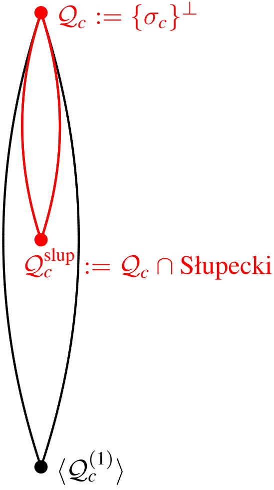

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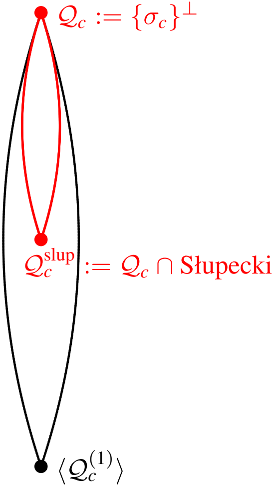

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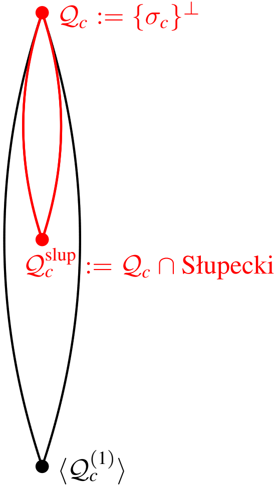

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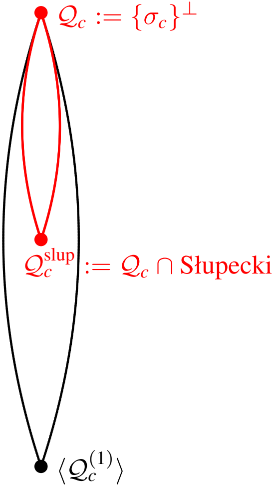

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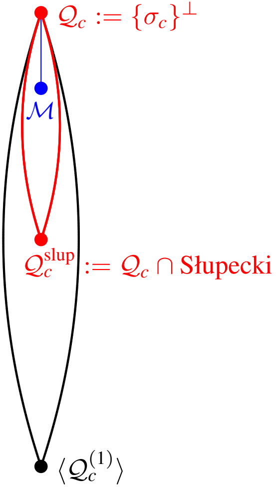

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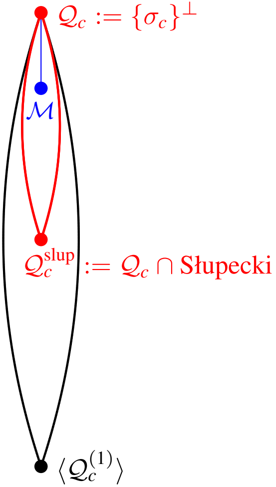

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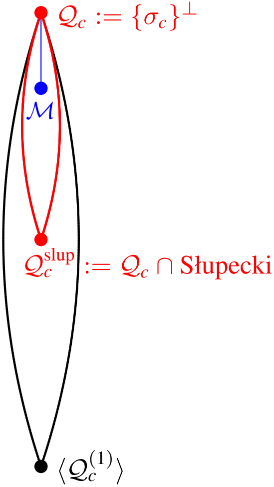

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