Semilattice ordered algebras II
The lattice of subvarieties

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Semilattice ordered algebras

Definition

An algebra $(A, \Omega, +)$ is called a *semilattice ordered $\mathcal{V}$-algebra*, if $(A, +)$ is a (join) semilattice, $(A, \Omega) \in \mathcal{V}$ and the operations from the set Ω distribute over the operation $+$.

Varieties of semilattice ordered semigroups

Theorem [1979, R.McKenzie, A.Romanowska]

There are exactly 5 varieties of semilattice ordered semilattices.

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Theorem [2005, S.Ghosh, F.Pastijn, X.Z.Zhao]

The lattice of all subvarieties of the variety of ordered bands (semirings whose multiplicative reduct is an idempotent semigroup and additive reduct is a chain) is distributive and contains precisely 78 varieties.
Each of them is finitely based.

2005, M.Kuřil, L.Polák

The lattice of subvarieties of the variety of all semilattice-ordered semigroups was described using the certain closure operators on relatively free semigroup reducts.

Modals - semilattice ordered idempotent and entropic algebras

Varieties of modals

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1995, K.Kearnes

To each variety $\mathcal{V}$ of entropic modals one can associate a commutative semiring $\mathbf{R}(\mathcal{V})$, whose structure determines many of the properties of the variety.

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To each variety $\mathcal{V}$ of entropic modals one can associate a commutative semiring $\mathbf{R}(\mathcal{V})$, whose structure determines many of the properties of the variety.

Theorem [1995, K.Kearnes]

The lattice of subvarieties of the variety of entropic modals $\mathcal{V}$ is dually isomorphic to the congruence lattice $Con\mathbf{R}(\mathcal{V})$ of the semiring $\mathbf{R}(\mathcal{V})$.

Theorem [2008, K.Ślusarska]

The lattice of subvarieties of entropic differential modals $(M, \cdot, +)$ (modals whose the groupoid reducts are idempotent and entropic algebras satisfying the additional identity: $x(yz) \approx xy$) forms the three element chain.

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2012, A.Pilitowska, A.Zamojska-Dzienio

Some family of fully invariant congruences on the free modals was described.

$\mho$ - the variety of all algebras (A, Ω) of type $\tau\colon \Omega \to \mathbb{N}$
$\mathcal{V} \subseteq \mho$
$(F_{\mathcal{V}}(X), \Omega)$ - the free algebra over a (non-finite) set X in the variety $\mathcal{V}$

Varieties of semilattice ordered algebras

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Varieties of semilattice ordered algebras

$\mathcal{U}$ - the variety of all algebras (A, Ω) of type $\tau\colon \Omega \to \mathbb{N}$
$\mathcal{V} \subseteq \mathcal{U}$
$(F_{\mathcal{V}}(X), \Omega)$ - the free algebra over a (non-finite) set X in the variety $\mathcal{V}$
$\mathcal{S}_{\mathcal{V}}$ - the variety of all semilattice ordered $\mathcal{V}$-algebras

Theorem

The semilattice ordered algebra $(\mathcal{P}^{<\omega}_{>0} F_{\mathcal{U}}(X), \Omega, \cup)$ is free over a set X in the variety $\mathcal{S}_{\mathcal{U}}$.

$$\mathcal{S} \subseteq \mathcal{S}_{\mho} \quad \mapsto \quad \mathcal{V} = \bigcap \{\mathcal{W} \subseteq \mho \mid \forall (A, \Omega, +) \in \mathcal{S},\ (A, \Omega) \in \mathcal{W}\},$$

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such that $\mathcal{S} \subseteq \mathcal{S}_{\mathcal{V}}$
For two different subvarieties $\mathcal{V} \neq \mathcal{W} \subseteq \mho$, the varieties $\mathcal{S}_{\mathcal{V}}$ and $\mathcal{S}_{\mathcal{W}}$ can be equal.

Example

Differential groupoid - an idempotent and entropic groupoid $(D, \cdot)$ such that $x(yz) \approx xy$

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Theorem

$\mathcal{S}_{\mathcal{D}_{0,j}}$ - the variety of all semilattice ordered $\mathcal{D}_{0,j}$-groupoids
For each positive integer j, one has

$$\mathcal{S}_{\mathcal{D}_{0,j}} = \mathcal{S}_{\mathcal{LZ}},$$

$\mathcal{LZ}$ - the variety of left-zero semigroups $(xy \approx x)$

Question

For which different subvarieties $\mathcal{V}_1 \neq \mathcal{V}_2 \subseteq \mho$, the varieties $\mathcal{S}_{\mathcal{V}_1}$ and $\mathcal{S}_{\mathcal{V}_2}$ are different, too?

The Relation $\widetilde{\theta}$

A - a set

$\Theta \subseteq \mathcal{P}^{<\omega}_{>0} A \times \mathcal{P}^{<\omega}_{>0} A$

$\widetilde{\Theta} \subseteq A \times A$,

$$(t, u) \in \widetilde{\Theta} \quad \Leftrightarrow \quad (\{t\}, \{u\}) \in \Theta$$

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Example

$$\widetilde{id}_{\mathcal{P}^{<\omega}_{>0} A} = id_A$$

$$\widetilde{\mathcal{P}^{<\omega}_{>0} A \times \mathcal{P}^{<\omega}_{>0} A} = A \times A$$

The Relation $\widetilde{\theta}$

Θ - a congruence on $(\mathcal{P}^{<\omega}_{>0} F_{\mho}(X), \Omega, \cup) \quad \Rightarrow$
$\widetilde{\Theta}$ - a congruence relation on $(F_{\mho}(X), \Omega)$

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Lemma

Let Θ be a fully invariant congruence relation on $(\mathcal{P}^{<\omega}_{>0}F_{\mho}(X), \Omega, \cup)$. Then the relation $\widetilde{\Theta}$ is a fully invariant congruence on $(F_{\mho}(X), \Omega)$.

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$$\mho_{\widetilde{\Theta}} := \mathrm{HSP}((F_{\mho}(X)/\widetilde{\Theta}, \Omega)) \subseteq \mho$$

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Lemma

$\Theta \in Con_{fi}(\mathcal{P}^{<\omega}_{>0}F_{\mho}(X))$, $t, u \in F_{\mho}(X)$
$t \approx u$ is an identity in $(\mathcal{P}^{<\omega}_{>0}F_{\mho}(X)/\Theta, \Omega)$ if and only if $t \approx u$ is an identity in $\mho_{\widetilde{\Theta}}$.

The Relation $\widetilde{\theta}$

It may happen that for $\Theta_1 \neq \Theta_2$, the congruences $\widetilde{\Theta}_1$ and $\widetilde{\Theta}_2$ are equal.

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Example

There are at least 5 subvarieties of the variety of semilattice ordered groupoids which are also semilattice ordered semilattices. But the variety of semilattices has only 2 subvarieties.

Theorem

Let $\Theta_1, \Theta_2 \in Con_{fi}(\mathcal{P}^{<\omega}_{>0} F_{\mho}(X))$. Then

$$\widetilde{\Theta}_1 \neq \widetilde{\Theta}_2 \;\Rightarrow\; \mathcal{S}_{\mho_{\widetilde{\Theta}_1}} \neq \mathcal{S}_{\mho_{\widetilde{\Theta}_2}}$$

$\Theta_1, \Theta_2 \in Con_{fi}(\mathcal{P}^{<\omega}_{>0} F_{\mho}(X))$

$$\Theta_1 \, \Re \, \Theta_2 \quad \Leftrightarrow \quad \widetilde{\Theta}_1 = \widetilde{\Theta}_2$$

The Relation $\mathfrak{R}$

$\Theta_1, \Theta_2 \in Con_{fi}(\mathcal{P}^{<\omega}_{>0} F_{\mho}(X))$

$$\Theta_1 \; \mathfrak{R} \; \Theta_2 \quad \Leftrightarrow \quad \widetilde{\Theta}_1 = \widetilde{\Theta}_2$$

$\mathfrak{R}$ - an equivalence relation

The Relation $\Re$

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$\Re$ - an equivalence relation
$\Theta \in Con_{fi}(\mathcal{P}^{<\omega}_{>0} F_{\mho}(X))$
$\Psi_i \in \Theta/\Re$, $i \in I$

$$\bigcap_{i \in I} \Psi_i \in \Theta/\Re$$

"Main knots"

$$Con_{fi}^{\Re}(\mathcal{P}_{>0}^{<\omega}F_{\mho}(X)) := \{\Theta \in Con_{fi}(\mathcal{P}_{>0}^{<\omega}F_{\mho}(X)) \mid \Theta = \bigcap_{\Phi\in\Theta/\Re} \Phi\}$$

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Theorem

$\Theta \in Con_{fi}(\mathcal{P}_{>0}^{<\omega}F_{\mho}(X))$

$$\mathrm{HSP}((\mathcal{P}_{>0}^{<\omega}F_{\mho}(X)/\Theta, \Omega, \cup)) \subseteq \mathcal{S}_{\mho_{\tilde{\Theta}}}$$

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"Main knots"

$\Theta_1, \Theta_2 \in Con_{fi}(\mathcal{P}_{>0}^{<\omega} F_{\mho}(X))$

$$\Theta_1 \subseteq \Theta_2 \quad \Rightarrow \quad \widetilde{\Theta}_1 \subseteq \widetilde{\Theta}_2$$

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Lemma

$\Theta_1, \Theta_2 \in Con_{fi}^{\mathfrak{R}}(\mathcal{P}_{>0}^{<\omega} F_{\mho}(X))$

$$\Theta_1 \subseteq \Theta_2 \quad \Leftrightarrow \quad \widetilde{\Theta}_1 \subseteq \widetilde{\Theta}_2$$

Theorem

The ordered set $(Con_{fi}^{\Re}(\mathcal{P}_{>0}^{<\omega}F_{\mho}(X)),\subseteq)$ is a complete lattice, in which for any two congruences $\Theta_1,\Theta_2\in Con_{fi}^{\Re}(\mathcal{P}_{>0}^{<\omega}F_{\mho}(X))$:
$\bigcap_{\Phi\in(\Theta_1\vee\Theta_2/\Re)}\Phi$ is the least upper bound of Θ_1 and Θ_2, and
$\bigcap_{\Phi\in(\Theta_1\cap\Theta_2/\Re)}\Phi$ is the greatest lower bound of Θ_1 and Θ_2.

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$\bigcap_{\Phi \in (\Theta_1 \vee \Theta_2/\Re)} \Phi$ is the least upper bound of Θ_1 and Θ_2, and
$\bigcap_{\Phi \in (\Theta_1 \cap \Theta_2/\Re)} \Phi$ is the greatest lower bound of Θ_1 and Θ_2.

Corollary

The lattice $(\{\mathcal{S}_{\mho_{\widetilde{\Theta}}} \mid \Theta \in Con^{\Re}_{fi}(\mathcal{P}^{<\omega}_{>0}F_{\mho}(X))\}, \subseteq)$ is dually isomorphic to the lattice $(Con^{\Re}_{fi}(\mathcal{P}^{<\omega}_{>0}F_{\mho}(X)), \subseteq)$. For any two varieties $\mathcal{S}_{\mho_{\widetilde{\Theta}_1}}$ and $\mathcal{S}_{\mho_{\widetilde{\Theta}_2}}$, the variety $\mathcal{S}_{\mho_{\widetilde{\Theta_1 \cap \Theta_2}}}$ is the least upper bound and the variety $\mathcal{S}_{\mho_{\widetilde{\Theta_1 \vee \Theta_2}}}$ is the greatest lower bound of them.

Definition

$\mathcal{V} \subseteq \mho$

Let $\mathcal{S}$ be a non-trivial subvariety of $\mathcal{S}_{\mathcal{V}}$. We say $\mathcal{S}$ is $\mathcal{V}$-*preserved* if for any proper subvariety $\mathcal{W} \subset \mathcal{V}$, the variety $\mathcal{S}$ is not included in $\mathcal{S}_{\mathcal{W}}$.

Definition

$\mathcal{V} \subseteq \mathcal{U}$

Let $\mathcal{S}$ be a non-trivial subvariety of $\mathcal{S}_{\mathcal{V}}$. We say $\mathcal{S}$ is $\mathcal{V}$-*preserved* if for any proper subvariety $\mathcal{W} \subset \mathcal{V}$, the variety $\mathcal{S}$ is not included in $\mathcal{S}_{\mathcal{W}}$.

Lemma

$\Theta \in Con_{fi}^{\mathfrak{R}}(\mathcal{P}_{>0}^{<\omega} F_{\mathcal{U}}(X))$, $\Psi \in Con_{fi}(\mathcal{P}_{>0}^{<\omega} F_{\mathcal{U}}(X))$

A non-trivial subvariety

$$\mathcal{S} = \mathrm{HSP}(\mathcal{P}_{>0}^{<\omega} F_{\mathcal{U}}(X)/\Psi, \Omega, \cup) \subseteq \mathcal{S}_{\mathcal{U}_{\widetilde{\Theta}}}$$

is $\mathcal{U}_{\widetilde{\Theta}}$-preserved if and only if $\widetilde{\Psi} = \widetilde{\Theta}$ $((\Theta, \Psi) \in \mathfrak{R})$.

Lemma

$\Theta \in Con^{\mathfrak{R}}_{fi}(\mathcal{P}^{<\omega}_{>0}F_{\mho}(X))$
There is one-to-one correspondence between the following sets:

- the set of all fully invariant congruence relations Ψ on the algebra $(\mathcal{P}^{<\omega}_{>0}F_{\mho}(X), \Omega, \cup)$ satisfying the condition $\widetilde{\Psi} = \widetilde{\Theta}$;
- the set of all fully invariant congruence relations α on the algebra $(\mathcal{P}^{<\omega}_{>0}F_{\mho_{\widetilde{\Theta}}}(X), \Omega, \cup)$ with the properties: $\widetilde{\alpha} = id_{F_{\mho_{\widetilde{\Theta}}}(X)}$ and $(\mathcal{P}^{<\omega}_{>0}F_{\mho_{\widetilde{\Theta}}}(X)/\alpha, \Omega) \in \mho_{\widetilde{\Theta}}$.

$\mathcal{V}$-preserved subvarieties

$\Theta \in Con^{\mathfrak{R}}_{fi}(\mathcal{P}^{<\omega}_{>0} F_{\mho}(X))$

$$Con^{id}_{fi}(\mathcal{P}^{<\omega}_{>0} F_{\mho_{\widetilde{\Theta}}}(X)) := \{\alpha \in Con_{fi}(\mathcal{P}^{<\omega}_{>0} F_{\mho_{\widetilde{\Theta}}}(X)) \mid \widetilde{\alpha} = id_{F_{\mho_{\widetilde{\Theta}}}(X)},$$

$$(\mathcal{P}^{<\omega}_{>0} F_{\mho_{\widetilde{\Theta}}}(X)/\alpha, \Omega) \in \mho_{\widetilde{\Theta}}\}$$

$\mathcal{V}$-preserved subvarieties

$\Theta \in Con_{fi}^{\Re}(\mathcal{P}_{>0}^{<\omega}F_{\mho}(X))$

$$Con_{fi}^{id}(\mathcal{P}_{>0}^{<\omega}F_{\mho_{\widetilde{\Theta}}}(X)) := \{\alpha \in Con_{fi}(\mathcal{P}_{>0}^{<\omega}F_{\mho_{\widetilde{\Theta}}}(X)) \mid \widetilde{\alpha} = id_{F_{\mho_{\widetilde{\Theta}}}(X)},$$

$$(\mathcal{P}_{>0}^{<\omega}F_{\mho_{\widetilde{\Theta}}}(X)/\alpha, \Omega) \in \mho_{\widetilde{\Theta}}\}$$

Lemma

$(Con_{fi}^{id}(\mathcal{P}_{>0}^{<\omega}F_{\mho_{\widetilde{\Theta}}}(X)), \subseteq)$ is a complete meet-semilattice with the relation $\alpha_1 \cap \alpha_2$ as the greatest lower bound of any $\alpha_1, \alpha_2 \in Con_{fi}^{id}(\mathcal{P}_{>0}^{<\omega}F_{\mho_{\widetilde{\Theta}}}(X))$.

$\Theta \in Con^{\Re}_{fi}(\mathcal{P}^{<\omega}_{>0} F_{\mho}(X))$

$$Con^{id}_{fi}(\mathcal{P}^{<\omega}_{>0} F_{\mho_{\widetilde{\Theta}}}(X)) := \{\alpha \in Con_{fi}(\mathcal{P}^{<\omega}_{>0} F_{\mho_{\widetilde{\Theta}}}(X)) \mid \widetilde{\alpha} = id_{F_{\mho_{\widetilde{\Theta}}}(X)},$$

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Lemma

$(Con^{id}_{fi}(\mathcal{P}^{<\omega}_{>0} F_{\mho_{\widetilde{\Theta}}}(X)), \subseteq)$ is a complete meet-semilattice with the relation $\alpha_1 \cap \alpha_2$ as the greatest lower bound of any $\alpha_1, \alpha_2 \in Con^{id}_{fi}(\mathcal{P}^{<\omega}_{>0} F_{\mho_{\widetilde{\Theta}}}(X))$.

Corollary

$(Con^{id}_{fi}(\mathcal{P}^{<\omega}_{>0} F_{\mho_{\widetilde{\Theta}}}(X)), \subseteq)$ is dually isomorphic to the join semilattice of all $\mho_{\widetilde{\Theta}}$-preserved subvarieties of $\mathcal{S}_{\mho_{\widetilde{\Theta}}}$.

For each non-trivial subvariety

$$\mathcal{S} = \mathrm{HSP}((\mathcal{P}^{<\omega}_{>0}\mathrm{F}_{\mho}(\mathrm{X})/\Psi, \Omega, \cup)) \subseteq \mathcal{S}_{\mho},$$

with $\Psi \in Con_{fi}(\mathcal{P}^{<\omega}_{>0}F_{\mho}(X))$,
there are uniquely defined two congruence relations:

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1. $\Theta \in Con^{\mathfrak{R}}_{fi}(\mathcal{P}^{<\omega}_{>0}F_{\mho}(X))$ such that $\widetilde{\Psi} = \widetilde{\Theta}$

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with $\Psi \in Con_{fi}(\mathcal{P}^{<\omega}_{>0}F_{\mho}(X))$,
there are uniquely defined two congruence relations:

1. $\Theta \in Con^{\mathfrak{R}}_{fi}(\mathcal{P}^{<\omega}_{>0}F_{\mho}(X))$ such that $\widetilde{\Psi} = \widetilde{\Theta}$
2. $\alpha^{\widetilde{\Theta}} \in Con^{id}_{fi}(\mathcal{P}^{<\omega}_{>0}F_{\mho_{\widetilde{\Theta}}}(X))$

$$Con^{id}_{fi}(\mho) := \{\alpha^{\widetilde{\Theta}} \in Con^{id}_{fi}(\mathcal{P}^{<\omega}_{>0} F_{\mho_{\widetilde{\Theta}}}(X)) \mid \Theta \in Con^{\Re}_{fi}(\mathcal{P}^{<\omega}_{>0} F_{\mho}(X))\}$$

$$Con^{id}_{fi}(\mho) := \{\alpha^{\widetilde{\Theta}} \in Con^{id}_{fi}(\mathcal{P}^{<\omega}_{>0} F_{\mho_{\widetilde{\Theta}}}(X)) \mid \Theta \in Con^{\Re}_{fi}(\mathcal{P}^{<\omega}_{>0} F_{\mho}(X))\}$$

$\alpha^{\widetilde{\Theta}}, \beta^{\widetilde{\Psi}} \in Con^{id}_{fi}(\mho)$, with $\Theta, \Psi \in Con^{\Re}_{fi}(\mathcal{P}^{<\omega}_{>0} F_{\mho}(X))$

$$\alpha^{\widetilde{\Theta}} \preceq \beta^{\widetilde{\Psi}} \;\Leftrightarrow\; \widetilde{\Theta} \subseteq \widetilde{\Psi} \text{ and } \forall (a_1, \ldots, a_k, b_1, \ldots, b_m \in F_{\mho}(X))$$
$$(\{a_1/\widetilde{\Theta}, \ldots, a_k/\widetilde{\Theta}\}, \{b_1/\widetilde{\Theta}, \ldots, b_m/\widetilde{\Theta}\}) \in \alpha^{\widetilde{\Theta}} \;\Rightarrow$$
$$(\{a_1/\widetilde{\Psi}, \ldots, a_k/\widetilde{\Psi}\}, \{b_1/\widetilde{\Psi}, \ldots, b_m/\widetilde{\Psi}\}) \in \beta^{\widetilde{\Psi}}.$$

$$Con^{id}_{fi}(\mho) := \{\alpha^{\widetilde{\Theta}} \in Con^{id}_{fi}(\mathcal{P}^{<\omega}_{>0} F_{\mho_{\widetilde{\Theta}}}(X)) \mid \Theta \in Con^{\Re}_{fi}(\mathcal{P}^{<\omega}_{>0} F_{\mho}(X))\}$$

$\alpha^{\widetilde{\Theta}}, \beta^{\widetilde{\Psi}} \in Con^{id}_{fi}(\mho)$, with $\Theta, \Psi \in Con^{\Re}_{fi}(\mathcal{P}^{<\omega}_{>0} F_{\mho}(X))$

$$\alpha^{\widetilde{\Theta}} \preceq \beta^{\widetilde{\Psi}} \;\Leftrightarrow\; \widetilde{\Theta} \subseteq \widetilde{\Psi} \text{ and } \forall (a_1, \ldots, a_k, b_1, \ldots, b_m \in F_{\mho}(X))$$
$$(\{a_1/\widetilde{\Theta}, \ldots, a_k/\widetilde{\Theta}\}, \{b_1/\widetilde{\Theta}, \ldots, b_m/\widetilde{\Theta}\}) \in \alpha^{\widetilde{\Theta}} \Rightarrow$$
$$(\{a_1/\widetilde{\Psi}, \ldots, a_k/\widetilde{\Psi}\}, \{b_1/\widetilde{\Psi}, \ldots, b_m/\widetilde{\Psi}\}) \in \beta^{\widetilde{\Psi}}.$$

Theorem

$(Con^{id}_{fi}(\mho), \preceq)$ is a complete lattice, dually isomorphic to the lattice of subvarieties of semilattice ordered $\mho$-algebras.

Thank you for your attention.