Solving functional equations with algebra

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Partially supported by the Austrian Science Fund (FWF):P24077 at the Institute for Algebra of the Johannes Kepler University Linz and by the Hungarian Scientific Research Fund (OTKA) Grants NK-81402 and by the TÁMOP 4.2.1./B-09/1/KONV-2010-0007 project implemented through the New Hungary Development Plan co-financed by the European Social Fund, and the European Regional Development Fund.

June 24, 2012

The beginning...

- 1815 Babbage
- 1959 William Lowell Putnam Mathematics Competition

Example

Exercise A3 in 1959

$$f(t) + t \cdot f(1 - t) = 1 + t$$

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$$f(1-t) + (1-t) \cdot f(t) = 2 - t$$

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$f(t) = 1$ is indeed a solution

$\alpha(t) \cdot f(t) + \beta(t) \cdot f(1-t) = h(t)$

$$\overbrace{\begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(1-t) & \alpha(1-t) \end{pmatrix}}^{D(t)} \cdot \begin{pmatrix} f(t) \\ f(1-t) \end{pmatrix} = \begin{pmatrix} h(t) \\ h(1-t) \end{pmatrix}$$

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Is this $f(t)$ a solution? Is it consistent with $f(1-t)$?

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Is this $f(t)$ a solution? Is it consistent with $f(1-t)$? Yes!

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Generalizable?

$(\{t \mapsto t, t \mapsto 1-t\}, \circ)$ is a group
General functional equation: $(\{g_1, \ldots, g_n\}, \circ)$ is a group

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partial derivatives of $F \longleftrightarrow \alpha_i$ (tools from Real Analysis)

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Theorem (Bessenyei, Kézi, 2011)

For an open interval I if $g_i \colon I \to \mathbb{R}$ are differentiable, F satisfies some regularity conditions, then there is an open subinterval $J \subset I$ and a unique $f \colon J \to \mathbb{R}$ satisfying $(\star)$.

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What finite groups of differentiable functions exist?

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What finite groups of differentiable functions exist? None other!

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contradiction
every $g \in G$, $g \neq id$ is decreasing $\Longrightarrow |G| \leq 2$

1971 William Lowell Putnam Mathematics Competition, B2

$$f(t) + f\left(\frac{t-1}{t}\right) = 1 + t.$$

$$f\left(t\right)+f\left(\frac{t-1}{t}\right)=1+t.$$

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How can a three element group exist?

Find all solutions $f : \mathbb{R} \setminus \{0, 1\} \to \mathbb{R}$ of the functional equation

$$f(t) + f\left(\frac{t-1}{t}\right) = 1 + t.$$

$$\frac{\frac{t-1}{t} - 1}{\frac{t-1}{t}} = \frac{t - 1 - t}{t - 1} = \frac{1}{1 - t},$$

$$\frac{1}{1 - \frac{t-1}{t}} = \frac{1}{\frac{1}{t}} = t.$$

Every group exists

S_n exists

- pairwise disjoint bounded intervals $I_1, \ldots, I_n$, $\pi \in S_n$
- g_π is an increasing linear bijection of I_k onto $I_{\pi(k)}$
- $(\{g_\pi : \pi \in S_n\}, \circ) \simeq S_n$
- What if some of these bijections are decreasing?

Proposition

If G is a finite group of continuous functions over $I_1 \cup \cdots \cup I_n$, then G is isomorphic to a subgroup of $C_2 \wr S_n$.

Main result

$(\{g_1, \ldots, g_n\}, \circ)$ is a group

$$F\left(f(g_1(t)), f(g_2(t)), \ldots, f(g_n(t)), t\right) = 0 \qquad (\star)$$

Theorem (Bessenyei, Horváth, Kézi, 2012)

For an open set H if $g_i \colon H \to \mathbb{R}$ are differentiable, F satisfies some regularity conditions, then there is an open subset $H' \subset H$ and a unique $f \colon H' \to \mathbb{R}$ satisfying $(\star)$.

Proof

$(\{g_1, \ldots, g_n\}, \circ)$ is a group

$$F\left(f(g_1(t)), f(g_2(t)), \ldots, f(g_n(t)), t\right) = 0 \qquad (\star)$$

Existence and uniqueness

Implicit Function Theorem on $F\left(y_1(t), \ldots, y_n(t), t\right) = 0$

Correctness

- compatibility has to be checked: $f(t) = y_1(t)$, $y_i(t) = f(g_i(t))$
- derivate $F\left(y_1(t), \ldots, y_n(t), t\right) = 0$
  $\Longrightarrow$ differential equation (Cauchy problem)
- both y_k and $y_1 \circ g_k$ satisfies this (algebra, like for linear case, partial derivatives of $F \longleftrightarrow \alpha_i$)
- Global Existence and Uniqueness on the Cauchy problem

Complete main result

Theorem (Bessenyei, Horváth, Kézi, 2012)

Let $H \subseteq \mathbb{R}$ be a nonempty open subset, $\xi \in H$, and $G = \{g_1, \dots, g_{mn}\}$ be a group of continuously differentiable functions on H such that $g_i(\xi) = g_j(\xi)$ if and only if $i \equiv j$ (mod n). Let $\eta \in \mathbb{R}^n$, $p = (\eta, \dots, \eta) \in \mathbb{R}^{mn}$ and let $F\colon \mathbb{R}^{mn} \times \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that $F(p_{*i}, g_i(\xi)) = 0$ hold for all $i = 1, \dots, mn$. Define the mappings $A\colon \mathbb{R}^{mn} \to \mathbb{R}^{mn \times mn}$ and $B\colon \mathbb{R}^{mn} \to \mathbb{R}^{mn}$ by

$$A(x,t) = \left[\partial_{j*i^{-1}} F(x_{*i}, g_i(t))\right],$$
$$B(x,t) = \left[\partial_{mn+1} F(x_{*i}, g_i(t)) g_i'(t)\right]$$

and assume that A is regular at (p, ξ). If either $m = 1$ or $m \geq 2$ and the mapping $x \to A^{-1}B(x,t)$ is Lipschitz in a neighborhood of (p, ξ) then there exist a G-invariant open set $H_0 \subseteq H$ containing ξ and a unique differentiable function $f\colon H_0 \to \mathbb{R}$ satisfying $(\star)$.