On The Number Of Slim Semimodular Lattices

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Slim semimodular lattices

Let Ji L denote the set of non-zero join-irreducible elements of the *finite* lattice L.

Definition

*L is **slim**, if Ji L contains no 3-element antichain.*
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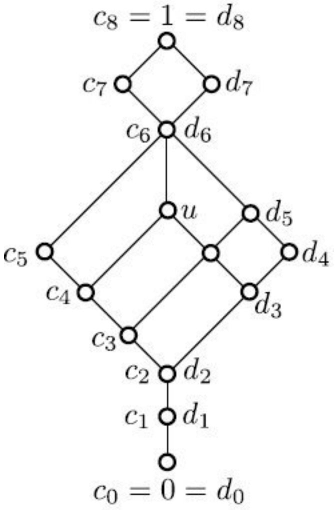
Previous results: A recursive formula for every lattice of a given size (Heitzig, Reinhold).
A recursive formula for the number of SSLs of a given *length* (Czédli, Ozsvárt, Udvari).

Example

SSLs possess planar diagrams.



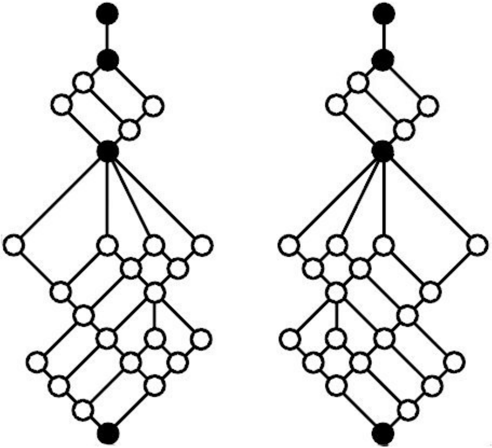
Overview of SSLs

First, we will characterize the planar diagrams belonging to SSLs.

Let D_1, D_2 two planar diagrams. Two planar diagrams, D_1 and D_2 are *similar*, if there is a lattice isomorphism φ for which
if $x \prec y$ and $x \prec z$ in D_1 then y is to the left of z iff $\varphi(y)$ is to the left of $\varphi(z)$.

Example

These two diagrams are not similar, but they belong to the same lattice.



Permutations and diagrams 1.

Consider a grid $G = \{0, 1, \ldots, h\} \times \{0, 1, \ldots h\}$.
Let $\mathrm{cell}(i, j) = \{(i, j), (i-1, j), (i, j-1), (i-1, j-1)\}$.

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Denote by $\mathrm{con}_\vee(\mathrm{cell}(i, j))$ the smallest join-congruence that collapses the top boundary of $\mathrm{cell}(i, j)$.

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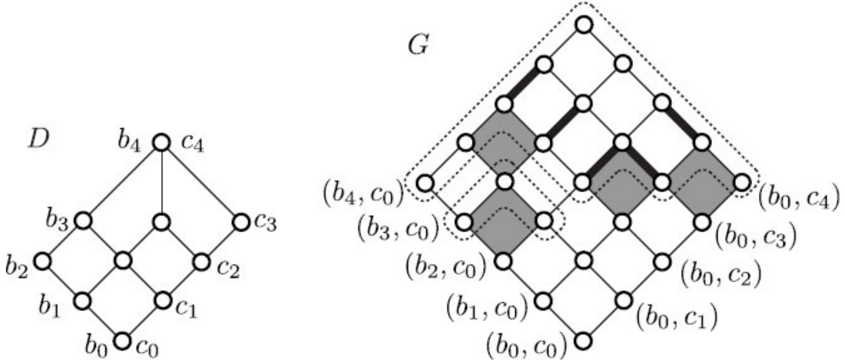

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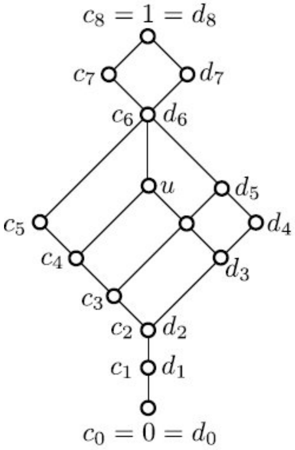
In other words, we can count permutations instead of SSLs. We only need to know two more things:

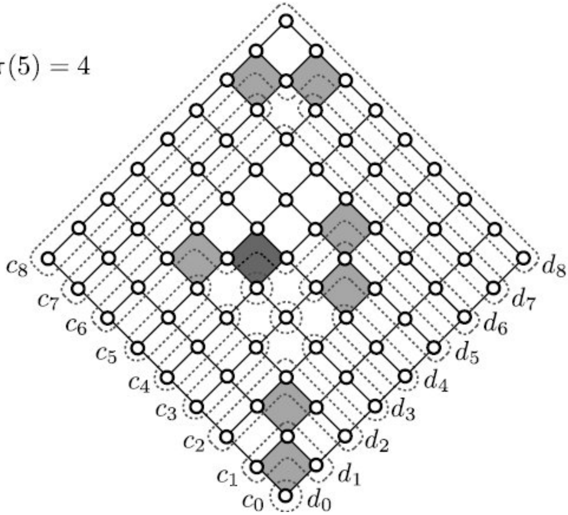
1. What is the size of the SSL belonging to a permutation?

2. We must know whether two permutations belong to the same SSL or not.

Permutations and diagrams 3. - another example



$\pi(5) = 4$



Permutations determine the size

Let $inv(\pi)$ denote the number of *inversions* in $\pi \in S_h$, that is, the number of $(i,j) \in \binom{h}{2}$ for which $i < j$ and $\pi(i) > \pi(j)$.

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Proposition

Let K be the lattice belonging to the $h \times h$ grid G and $\pi \in S_h$. Then $|K/\beta_\pi| = h + 1 + inv(\pi)$.

Permutations belonging to the same lattice 1.

Let $\pi \in S_h$. The interval $S = [i, \ldots, j]$ is a **segment** of π if $\pi(S) = S$, $\pi(\{1, \ldots, i-1\}) = \{1, \ldots, i-1\}$, $\pi(\{j+1, \ldots, h\}) = \{j+1, \ldots, h\}$, and there is no $[i', \ldots, j'] \subsetneq S$ with the same property.

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Definition

For $\pi_1, \pi_2 \in S_h$, $\pi_1 \sim \pi_2$ if their segments are the same and for each segment S: $\pi_2|_S = \pi_1|_S$ or $\pi_2|_S = (\pi_1|_S)^{-1}$.

For example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 4 & 2 & 6 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 3 & 6 & 5 \end{pmatrix}$$

.

Permutations belonging to the same lattice 2.

Lemma

Let $\pi, \sigma \in S_h$. *The SSLs* G/β_π *and* G/β_σ *are isomorphic iff* $\pi \sim \sigma$.

Counting 1.

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We can determine $p(h,k)$ using its generating function:

Theorem (Rodriguez)

$\sum_{j=0}^{\binom{h}{2}} p(h,j)x^j = \prod_{j=1}^{h} \frac{1-x^j}{1-x}$.

Counting 2.

Let $I(h,k) = \{\pi \in S_h : inv(\pi) = k, \pi^2 = id\}$, $i(h,k) := |I(h,k)|$.

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Proposition

$i(h,k) = i(h-1,k) + \sum_{s=2}^{h} i(h-2, k-2s+3)$.

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$\pi \in S_h$ is ***irreducible****, if it consists of one segment.*

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Proposition

$\hat{i}(h,k) = i(h,k) - \sum_{s=1}^{h-1} \sum_{t=0}^{k} \hat{i}(s,t) i(h-s, k-t)$.

Counting 4.

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$\hat{p}(h,k) = p(h,k) - \sum_{s=1}^{h-1} \sum_{t=0}^{k} \hat{p}(s,t) p(h-s, k-t)$.

Counting 5.

Denote by $[\pi]^{\sim}$ the $\sim$-class of π.

Let $P^{\sim}(h,k) = \{[\pi]^{\sim} : inv(\pi) = k, \pi \in S_h\}$, $p^{\sim}(h,k) := |P^{\sim}(h,k)|$.

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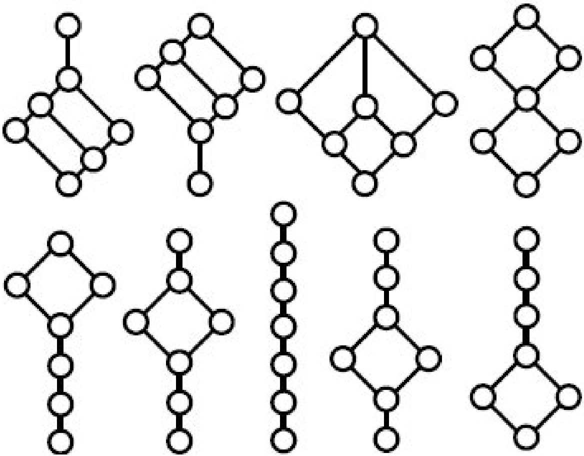
Finally,

Theorem

$N_{ssl}(n) = \sum_{h=0}^{n-1} p^{\sim}(h, n-h-1).$

Results with computer algebra 1.

$N_{ssl}(1) = N_{ssl}(2) = N_{ssl}(3) = 1,$
$N_{ssl}(4) = 2, N_{ssl}(5) = 3, N_{ssl}(6) = 5, N_{ssl}(7) = 9.$



Results with computer algebra 2.

$N_{ssl}(50) = 14,546,017,036,127 \approx 1.4 \cdot 10^{13}$.
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Recently we used permutations in a similar way to improve our previous recursion for SSLs of length h.

Thank you for your attention!

Our paper's preprint can be viewed at
www.math.u-szeged.hu/∼czedli