Effective bases of finite closure systems

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Yeshiva University, New York

June 23, 2012
Universal Algebra and Lattice Theory
Szeged

Large lattices

Recent work on very large lattices:

Lattices of quasi-equational theories as the lattices of congruences of semilattices with the operators
joint work with J.B.Nation
Parts I and II, in arxiv
to appear in *International Journal of Algebra and Computation*

On scattered convex geometries
joint work with M. Pouzet
short version in Proceedings of TACL-2011

Horn belief revision

Horn belief contraction: remainders, envelopes and complexity
joint work with G. Turán, R. Sloan and B. Szörényi
Proceedings of *Knowledge Representation-2012*, Rome

This talk based on:

(1) **"Ordered direct implicational basis of a finite closure system"**
joint work with J.B.Nation and R.Rand (ANR,2011)
to appear in *Discrete Applied Mathematics*

(2) **"On the implicational bases of closure systems with the unique criticals"**
joint work with J.B.Nation(AN,2012)
in arXiv

(3) **" Optimum bases of convex geometries"**
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Outline

What is a finite lattice?

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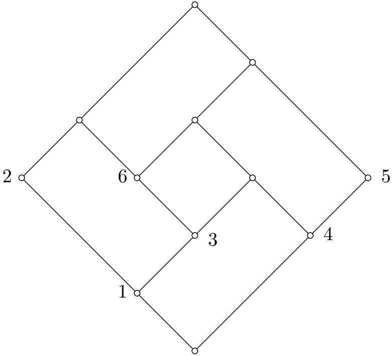

What is the most concise form of storing the information about a finite lattice?

Possible answers:

(1) tables for $\vee$ and $\wedge$;
(2) table for $\wedge$;
(3) family of subsets of some set X stable under intersection and containing X;
(4) table of Galois correspondence between join irreducible elements (rows) and meet irreducible elements (columns);
(5) OD-graph

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OD graph

OD-graph of a finite lattice: J.B.Nation, *An approach to to lattice varieties of finite height*, Alg. Univ. 1990

(I) partially ordered set of join irreducibles $\langle Ji(L), \leq \rangle$;

(II) *minimal join covers* $a \leq \bigvee B$, $a \in Ji(L)$, $B \subseteq Ji(L)$:

If $b \in B$, then $a \not\leq \bigvee \{b' \in Ji(L) : b' < b\} \vee \bigvee B \setminus b$.

Note for the future use: D-relation on $Ji(L)$ can be defined as aDb iff $b \in B$, for some minimal join cover $a \leq \bigvee B$.

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OD-graph

Every element $x \in L$ can be thought as set $Ji(x) = \{a \in Ji(L) : a \leq x\} \in 2^{Ji(L)}$.

Consider information stored in OD-graph:

(1) if $a \leq b$ in $\langle Ji(L), \leq \rangle$, and $b \in Ji(x)$, then $a \in Ji(x)$.
We say that $Ji(x)$ *respects implication* $b \rightarrow a$

(2) if $a \leq \bigvee B$ is a minimal cover, and $B \subseteq Ji(x)$, then $a \in Ji(x)$.
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Implications

An implication on set X is an ordered pair (Y, Z), written as $Y \to Z$, where $Y, Z \subseteq X$ and $Z \neq \emptyset$.

Subset $A \subseteq X$ *respects* implication $Y \to Z$, if whenever $Y \subseteq A$, then also $Z \subseteq A$.

Sets $Ji(x)$ respect all implications obtained from OD-graph.

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Lattice defined by implications

Vice versa, every subset $A \subseteq Ji(L)$ that respects all implications obtained from the OD-graph coincides with $Ji(x)$, for one of $x \in L$.

This allows to say that L is *defined by the set of implications* $\Sigma = \{b \to a : a \leq b\} \cup \{B \to a : \text{ for minimal cover } a \leq \bigvee B\}$.

Closure space

Another way to look at Σ as the means to define a closure space.
Consider base set $X = Ji(L)$, and define a closure operator ϕ:
$\phi(Y) =$ the minimal subset of X that contains Y and respects all implications in Σ.
Then $Ji(x)$, $x \in L$, are exactly the family of closed subsets of this operator.
Moreover, lattice L is isomorphic to the lattice of closed sets of closure space (X, ϕ).

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Summarizing

This is just a reminder of well-known connection between:

- finite lattices
- finite closure spaces
- sets of implications

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D-basis

In *Ordered direct implicational basis of a finite closure system*
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we defined the set of implications Σ_D made out of OD-graph as a *D-basis* of a finite closure system.

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Canonical direct basis

K. Bertet, B. Monjardet, *The multiple facets of the canonical direct implicational basis*, Theor. Comp. Science, 2010:

- compare 5 implicational systems for general closure system introduced independently in the literature
- prove that they are the same, now called a *canonical direct basis* Σ_{CD}
- the main feature: $\phi(Y) = Y \cup \{a : (B \to a) \in \Sigma_{CD}$ and $B \subseteq Y\}$
- Σ_{CD} is contained in any other basis with this property

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Canonical direct and the *D*-basis

Essentially, Σ_{CD} contains:

- implications $b \rightarrow a$, for join irreducibles $a \leq b$
- *non-redundant covers*: $B \rightarrow a$, where $a \leq \bigvee B$, but $a \not\leq B \setminus b$.

The minimal covers in *OD*-graph are non-redundant. Hence:

$$\Sigma_D \subseteq \Sigma_{CD}.$$

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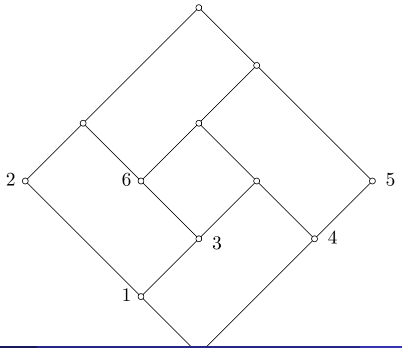
Ordered direct basis

The *D*-basis has a new feature: it is *ordered direct*. $\phi(Y)$ can be computed by applying implications in *particular order*, in a single iteration of the basis.

Example

Canonical direct basis Σ_{CD} for $\langle J(A_{12}), \phi \rangle$ has 13 implications.
$2 \to 1, 6 \to 1, 6 \to 3, 3 \to 1, 5 \to 4, 14 \to 3, 24 \to 3, 15 \to 3,$
$23 \to 6, 15 \to 6, 25 \to 6, 24 \to 5, 24 \to 6.$

D-basis has 9 implications.
$2 \to 1, 6 \to 3, 3 \to 1, 5 \to 4, 14 \to 3, 23 \to 6, 15 \to 6, 24 \to 5, 24 \to 6.$



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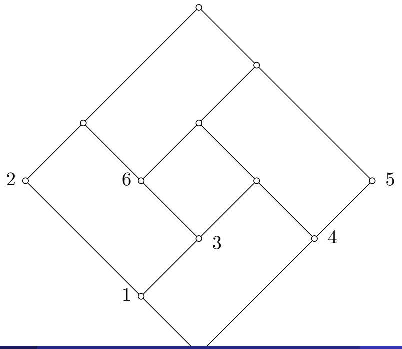

Is the *D*-basis redunadant?

Answer: yes, more often than otherwise ...

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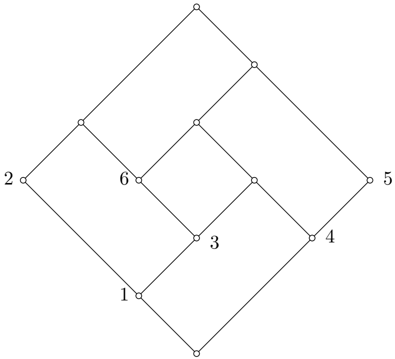
Aggregated *D*-basis:

$2 \to 1, 6 \to 3, 3 \to 1, 5 \to 4, 14 \to 3, 23 \to 6, 15 \to 6, 24 \to 56.$



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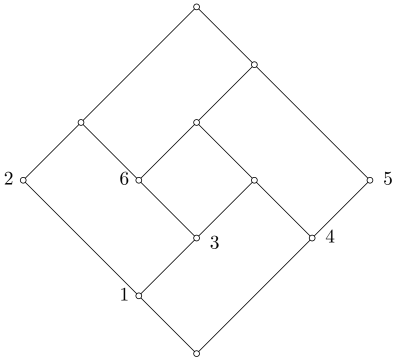

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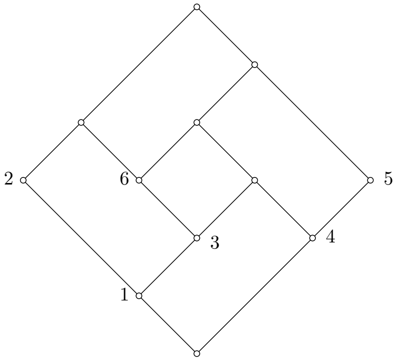

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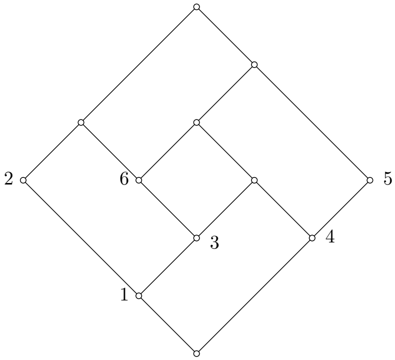

More questions

- What other types of "efficient" bases one can obtain for a closure system/finite lattice?
- How effectively this can be done? What are the complexity of the algorithms?

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The search for the short representation of implicational system is important:

- in Logic Programming, where implications appear in the form of definite Horn formulae of propositional logic
- in theory of Horn Boolean functions, in the form of the the shortest CNF-representation
- in relational data bases in the form of functional dependencies of the attributes
- in the theory of directed hypergraphs, in various optimization problems.

These are the instances of a mathematical problem, often with the promise to accelerate or optimize the associated application process.

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“Efficient" bases

- A basis Σ' is *non-redundant*, if none of its implications can be removed to get another basis.
- A basis Σ' is *minimum*, if it has the minimal number of implications among all the set of implications for the same closure system.
- A basis $\Sigma' = \{X_i \to Y_i : i \le n\}$ is called *optimum*, if number $s(\Sigma') = |X_1| + \cdots + |X_n| + |Y_1| + \cdots + |Y_n|$ is smallest among all sets of implications for the same closure system.
- A basis is called *right-side (left-side)* optimum basis, if the number $|Y_1| + \cdots + |Y_n|$ $(|X_1| + \cdots + |X_n|)$ is smallest among all sets of implications for the same closure system.
- The right-side optimum basis is connected to the problem of the shortest (i.e. with the minimal number of clauses) CNF-representation of a (definite) Horn function, also, minimal representations of the directed hypergraphs.

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- The right-side optimum basis is connected to the problem of the shortest (i.e. with the minimal number of clauses) CNF-representation of a (definite) Horn function, also, minimal representations of the directed hypergraphs.

“Efficient" bases

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Relation between bases

Theorem ([D.Maier, 1983])

Optimum $\Longrightarrow$ minimum and left-side optimum $\Longrightarrow$ non-redundant.

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Cleaning N1

Theorem ([AN, 2012])

Optimum $\Longrightarrow$ right-side optimum.

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The problem of finding an optimum basis of a finite closure system is NP-complete.

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Follow up on Cleaning N1

Corollary ([AN, 2012])

Theorem 1 follows from Theorem 2.

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Theorem 1 follows from Theorem 2.

What can be done?

- Introduce new types of bases that are *near-optimum* but can be found quickly.
- Recognize subclasses of closure systems where the optimum basis can be found quickly.
- Combine both directions above.

Examples of the second direction:

- P. Hammer and A. Kogan, *Quasi-acyclic propositional Horn knowledge bases: optimal compression*, IEEE Transactions on knowledge and data engineering, 1995
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- Canonical basis Σ_C is $\{A \to B : A \text{ is critical}, B = \phi(A) \setminus A\}$.
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Canonical and optimum

Theorem of Duquenne-Guigues continued ...

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Canonical Σ_C

$$A_1 \to B_1 \qquad A_2 \to B_2 \qquad \dots \qquad A_{k-1} \to B_{k-1} \qquad A_k \to B_k$$

Any optimum Σ_O

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Relation: $\sigma(A'_i) = A_i$, $B'_i \subseteq B_i$.

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Proposition ([D.Maier, 1983])

If $A_1 \to B_1$ and $A_2 \to B_2$ are implications of two optimum bases corresponding to the same implication $A \to B$ of the canonical basis, then $|A_1| = |A_2|$.

Cleaning N2

Proposition ([AN, 2012])

If $a \to B_1$ and $a \to B_2$ are two implications of the optimum basis corresponding to the same implication $a \to B$ of the canonical basis, then $|B_1| = |B_2|$.

K-basis

- *K*-basis is inspired by minimal join representations of lattice elements.
- *K*-basis has the same number of implications as the canonical, i.e. it is a minimum basis.
- The size of *K*-basis is normally smaller than the size of the canonical.
- *K*-basis can be effectively obtained from the canonical.
- *K*-basis establishes the connection between the canonical basis and the *D*-relation on the set of join irreducibles of a lattice.

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Essential idea: given $A \to B$ in Σ_C produce $A^* \to B^*$ in the K-basis, where $A^* \subseteq A$ gives a minimal join representation of element $x = \bigvee A$, and $B^* = max(B) \subseteq B$.

$x = \bigvee A^*$ is a minimal join representation of x, if for every $a \in A^*$, $x > \bigvee\{a' : a' < a\} \vee \bigvee A^* \setminus a$.

Comparison

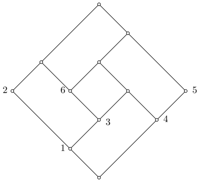

Figure: A_{12}

Canonical basis Σ_C:

$2 \to 1, 6 \to 13, 3 \to 1, 5 \to 4, 14 \to 3, 123 \to 6, 1345 \to 6, 12346 \to 5$

$s(\Sigma_C) = 27$

K-basis:

$2 \to 1, 6 \to 3, 3 \to 1, 5 \to 4, 14 \to 3, 23 \to 6, 15 \to 6, 24 \to 5$

$s(\Sigma_K) = 20$

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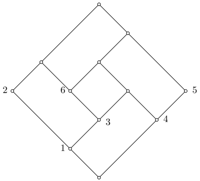

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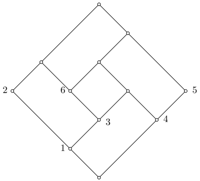

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Algorithmic aspect

Theorem ([A. Day, 1992])

Given any basis Σ' of a finite closure system, it requires time $O(s(\Sigma')^2)$ to obtain the canonical basis of Duquenne-Guigues.

Theorem ([AN, 2012])

A K-basis can be obtained from canonical basis Σ_C of Duquenne-Guigues in time $O(s(\Sigma_C)^2)$.

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K-basis

In general, the closure space may have more than one K-basis.

Definition

A closure system is called join semidistributive, if its closure lattice $Cl(X, \phi)$ satisfies the property:

$(SD_\vee)$ $$x \vee y = x \vee z \to x \vee y = x \vee (y \wedge z).$$

Theorem ([Jónsson and Kiefer, 1962])

Every element of a finite lattice has a unique minimal representation iff the lattice is join semidistributive.

Corollary

Every semidistributive closure system has a unique K-basis.

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Closure systems with the unique critical sets

Problem

Does there exist an effective algorithm to recognize that the closure systems is join semidistributive, given its canonical basis?

Some larger class of closure systems is easy to recognize from the canonical basis.

Definition

Closure system $\langle X, \phi \rangle$ has unique criticals, or it is UC-system, if $\phi(C_1) = \phi(C_2)$, for some critical sets C_1, C_2, implies $C_1 = C_2$.

$SD_\vee$ is UC

Proposition

Every join semidistributive closure system is a UC-system.

Proof.

Suppose there are two implications $C_1 \to B_1$ and $C_2 \to B_2$ in Σ_C with $\phi(C_1) = \phi(C_2)$. This means that in the closure lattice $x = \bigvee C_1 = \bigvee C_2$. One can find minimal representations $B_1 \subseteq C_1$ and $B_2 \subseteq C_2$ for x, i.e. $x = \bigvee B_1 = \bigvee B_2$. But $B_1 = B_2$, since x has a unique minimal representation. Hence, $\sigma(B_1) = C_1 = \sigma(B_2) = C_2$, which is needed. $\square$

Lattice description of *UC*

There exists a *UC* closure system whose closure lattice is *not* join semidistributive.

Problem

Describe closure lattices of closure systems with the unique criticals.

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Join semidsitributive systems

Important subclasses of join semidsitributive closure systems:

- In lattice theory: lower bounded lattices (closure systems without *D*-cycles) (R.Freese, J.Jezek, J.B.Nation).
- In combinatorics: convex geometries and anti-matroids (P.Edelman and R. Jamison)
- In theory of Boolean functions: quasi-acyclic systems (P.Hammer and A.Kogan). This is a proper subclass of both: convex geometries and systems without *D*-cycles.

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Systems without *D*-cycles

Lower bounded lattices, or lattices without D-cycles: can be defined via D-relation on the set of join-irreducible elements (A.Day, 1979): aDb iff $a \leq \bigvee B$ is a minimal cover and $b \in B$.

Note that this corresponds to implication $B \to a$ in the D-basis.

Theorem (AN12)

Let D^* be a binary relation defined for any K-basis of the closure system:
aD^*b iff $B \to A$ is in the K-basis, $|B| > 1$, $a \in A$ and $b \in B$. Then

- $D^* \subseteq D$.
- $D \subseteq tr(D^*)$.

Systems without D-cycles

Lower bounded lattices, or lattices without D-cycles: can be defined via D-relation on the set of join-irreducible elements (A.Day, 1979): aDb iff $a \leq \bigvee B$ is a minimal cover and $b \in B$.

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Systems without *D*-cycles

Corollary

Given the canonical basis Σ_C of the closure system, there exists a polynomial time algorithm in $s(\Sigma_C)$ that recognizes whether the system is without D-cycles.

Bases in systems without D-cycles

This basis was introduced for the systems without D-cycles in: K.Adaricheva, J.B.Nation and R.Rand, *Ordered direct basis of a finite closure system*,

E-basis:

	Canonical basis	$K-$basis	$E-$basis
$\|A\| > 1$	$A \to B$	$A^* \to B^*$	$A^* \to B^{**}$
$\|A\| = 1$	$a \to B$	$a \to B^*$	$a \to B^*$

$$B^{**} \subseteq B^*$$

Proposition ([AN, 2012])

E-basis can be obtained from K-basis via polynomial time algorithm: if $b \in B_1^*, B_2^*$, for two implications $A_1^* \to B_1$, $A_2^* \to B_2^*$ in the K-basis, and $\phi(A_1^*) \subset \phi(A_2^*)$, then b can be removed from B_2^*.

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E-basis

Theorem ([AN, 2012])

The total right size $|B_1| + \cdots + |B_k|$ of all non-binary implications $A_i \to B_i$ in E-basis attains the minimum among all possible bases for the closure system.

Theorem ([ANR,2011])

The E-basis of a closure system without D-cycles is ordered direct.

4 parts of the optimum basis: systems without D-cycles

	Binary part	Non-binary part
the left side	$a \to B$ tractable	$A \to B$ NP
the right side	$a \to B$ NP	$A \to B$ tractable

Proposition ([AN, 2012])

Assume that the closure system is without D-cycles.
(1) Finding the optimum right-side in binary part of the basis is NP-complete.
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This slide is intentionally left blank to celebrate the approach to the end ...

Convex geometry

KA, *Optimum basis of a finite convex geometry*, arxiv (2012)
A closure system $\langle X, \phi \rangle$ is called a *convex geometry*, if $\phi(\emptyset) = \emptyset$, and *anti-exchange property* holds:

For every $A = \phi(A), x, y \notin A$, if $x \in \phi(A \cup y)$, then $y \notin \phi(A \cup x)$.

$x \in A$ is called *extreme point* of A, if $x \notin \phi(A \setminus x)$. $Ex(A)$ is a set of extreme points of A.

Theorem

[P. Edelman and R. Jamison, 1985] A closure system $\langle X, \phi \rangle$ is a convex geometry iff every closed set $A = \phi(Ex(A))$.

In particular, every convex geometry is join semidistributive.

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the right side	$a \to B$ tractable	$A \to B$??

Proposition

Assume that the closure system is a convex geometry.
(1)[M.Wild, 1994] Finding the optimum left-side in non-binary part of the basis is tractable. $A = Ex(\phi(A))$.
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Corollary ([A,2012])

If a closure system is a convex geometry without D-cycles, then optimum basis can be obtained in polynomial time.

This class properly includes the quasi-acyclic closure systems defined in [P. Hammer and A. Kogan, 1995] , which are also G-geometries in [M.Wild, 1994].

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Convex geometries with n-Carousel rule

Class of convex geometries with n-Carousel rule includes *affine* convex geometries $Co(R^n, X)$: $X \subseteq R^n$, $\phi(Y) =$ convex hull$(Y) \cap X$.

2-Carousel Rule: If $x, y \in \phi(A), A \subseteq X$, then
$x \in \phi(\{y, a_i, a_j\})$, for some $a_i, a_j \in A$, when $|A| \geq 3$;
$x \in \phi(\{y, a\})$, for some $a \in A$, when $|A| = 2$.

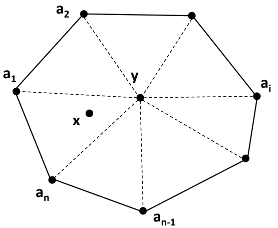

Figure: 2-Carousel rule

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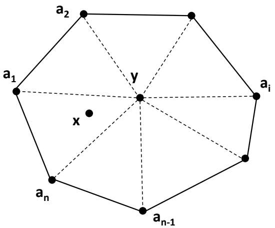

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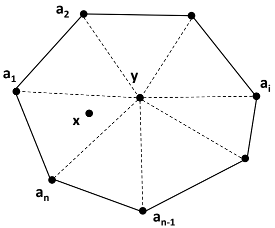

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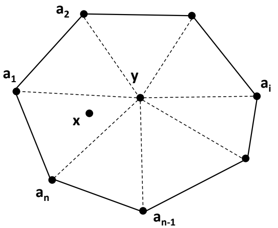

Figure: 2-Carousel rule

Optimum basis: convex geometries with n-Carousel rule

	Binary part	Non-binary part
the left side	$a \to B$ tractable	$A \to B$ tractable
the right side	$a \to B$ tractable	$A \to b$ tractable

Theorem ([A,2012])

If a closure system is a convex geometry satisfying n-Carousel rule, then optimum basis can be obtained in polynomial time.

Optimum basis: general case

Another tractable subclass: *component-quadratic* closure systems, E. Boros, O. Čepek, A. Kogan and P. Kučěra, RUTCOR, 2009.

Question: Can the optimum basis be found effectively, for every convex geometry?

Theorem ([A,2012])

Finding the optimum basis of convex geometry $Co(P)$ of convex subsets of partially ordered set P, is an NP-complete problem.

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Conclusions

- K-basis might not be an optimum basis, but it is always the minimum basis whose size is smaller than or equal the size of the canonical basis.
- In semidistributive closure systems K-basis is unique and is a good approximation of optimum basis.
- If the closure system is without D-cycles, further refinement of the K-basis can be effectively obtained, giving right-side optimum in its non-binary part.
- If a closure system is a convex geometry, either without D-cycles or with n-Carusel rule, the optimum basis is tractable. In general, the problem is NP-complete.

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Last slide: J.B.Nation

Figure: Hiking in Catskill mountains, New York State