Clones on 3 elements: A New Hope (Part 1)

Dmitriy Zhuk
zhuk.dmitriy@gmail.com

Charles University
Prague

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Why Clones?

Clone is a set of operations closed under composition and containing all projections.

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Examples of clones

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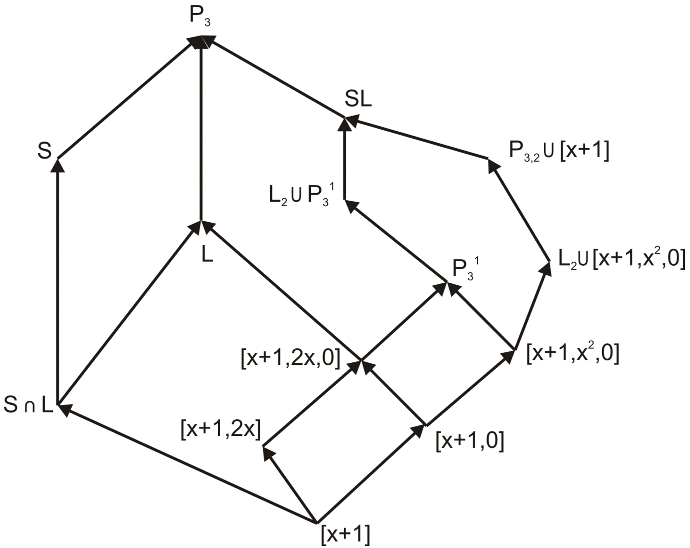
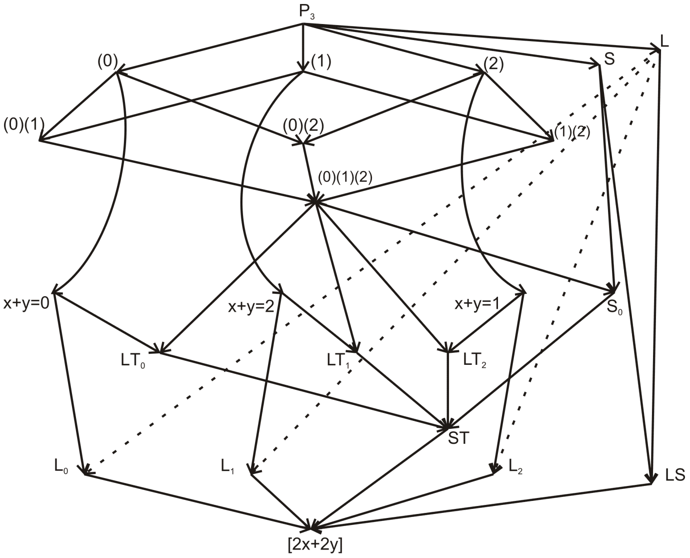
Clones ordered by inclusion form a lattice.

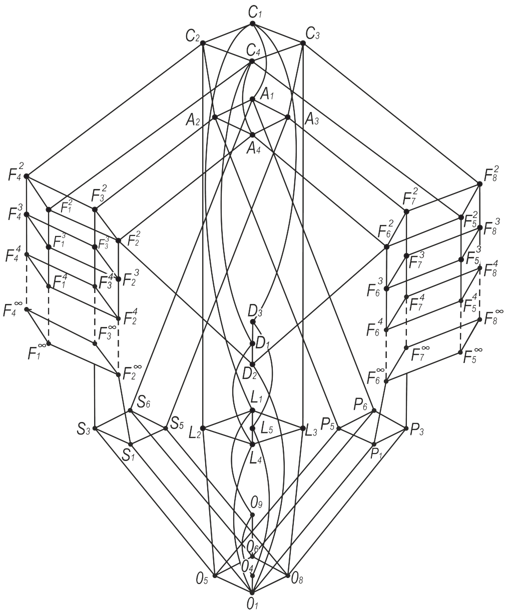
The Lattice of Clones containing $x+1$ on $\{0,1,2\}$



The Lattice of Clones containing $2x + 2y$ on $\{0, 1, 2\}$



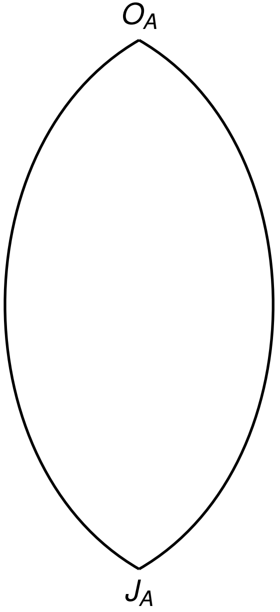
The lattice of all clones on two elements(for $|A| = 2$)



Emil Post (1921, 1941)

Can we describe all clones?

For $|A| > 2$



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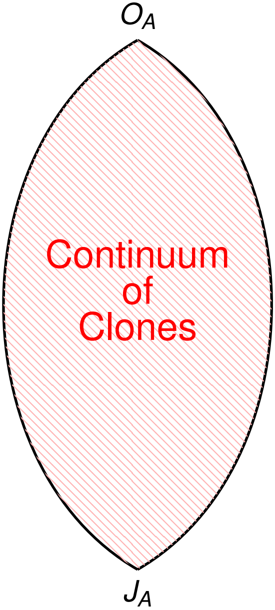

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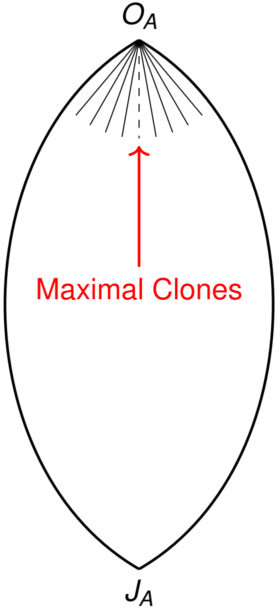

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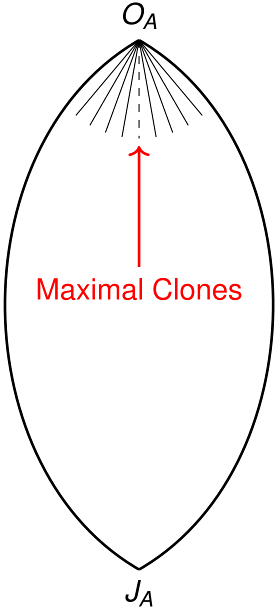

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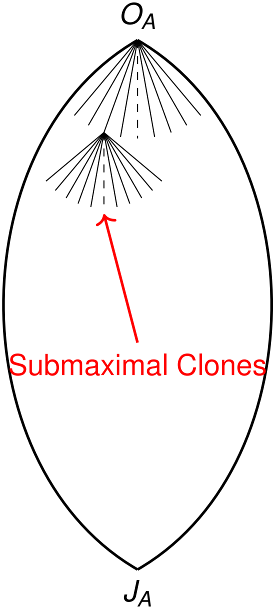

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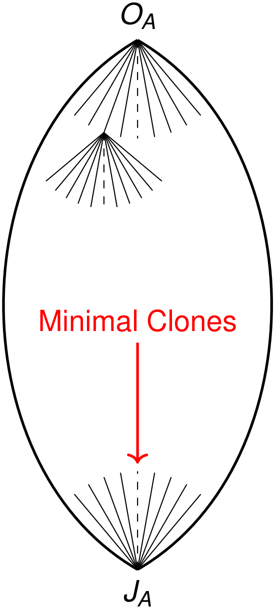

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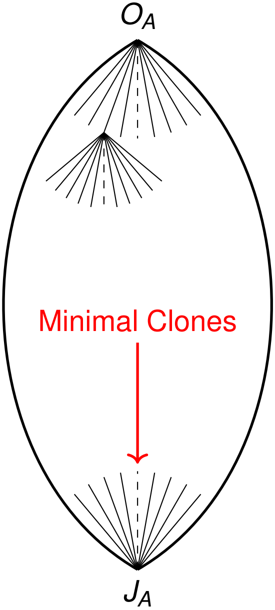

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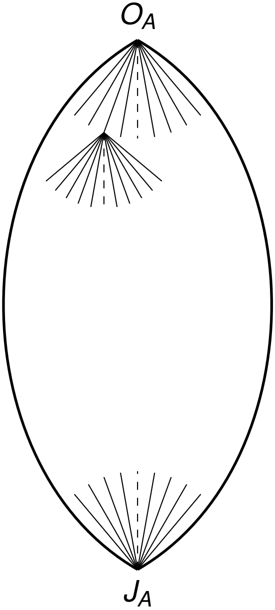

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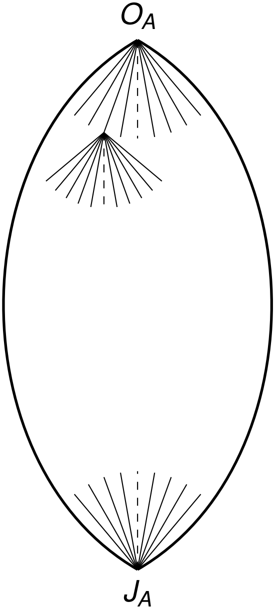

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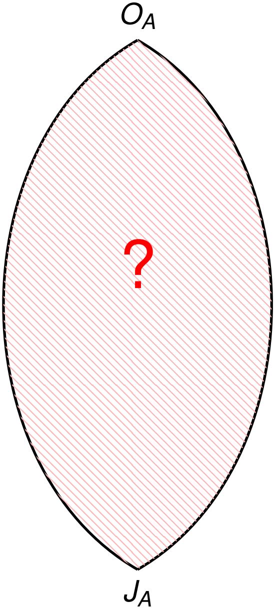

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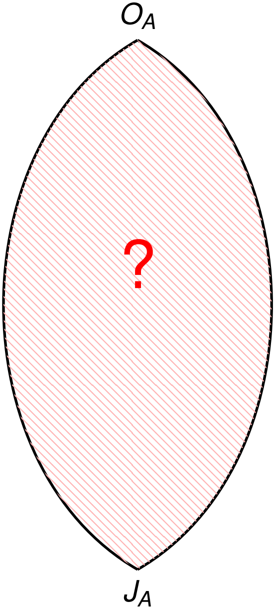
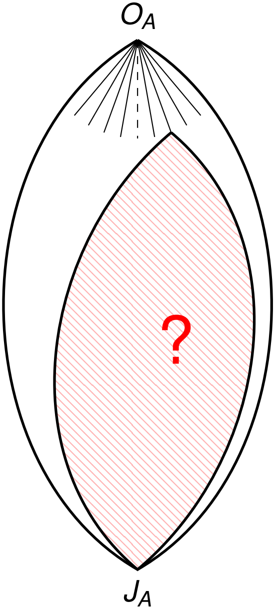


- Can we describe a significant part of the lattice?

For $|A| > 2$



- Can we describe all subclones of a maximal clone?

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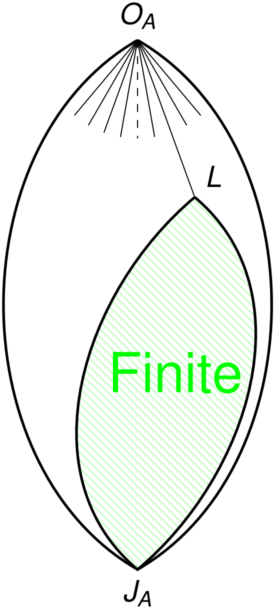

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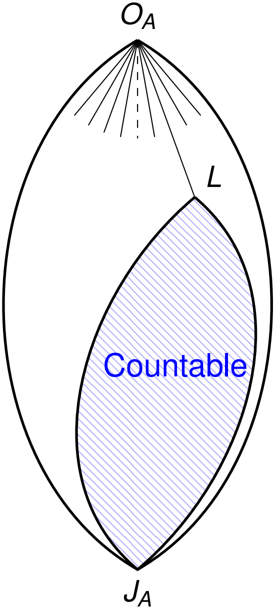

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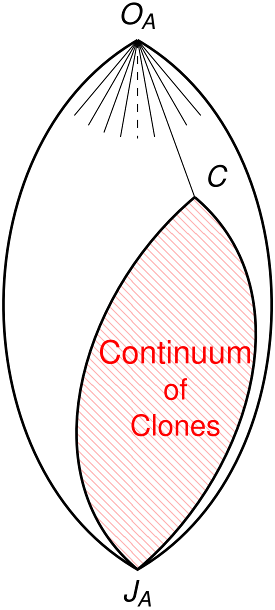

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- For the maximal clone of quasi-linear operations the lattice of subclones is countable but not known (if $|A|$ is a power of a prime number)
- For all other maximal clones the lattice of subclones is uncountable (J. Demetrovics, L. Hannak, S. S. Marchenkov, 1983)

Can we describe continuum?

$A = \{0, 1, 2\}$

Clone of Self-Dual Operations

$$\mathcal{C}_3 = \mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

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- There exist continuum clones of self-dual operations (S.S. Marchenkov, 1983).

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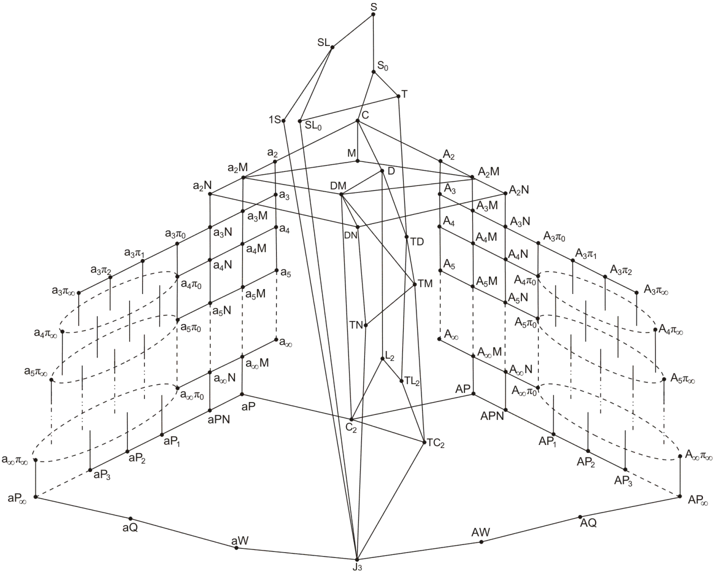
(D. Zhuk, 2010)

A complete description of clones of self-dual operations on three elements



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a_3
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a_3M
A_3M
DN
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A_4
A_3N
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TC_2
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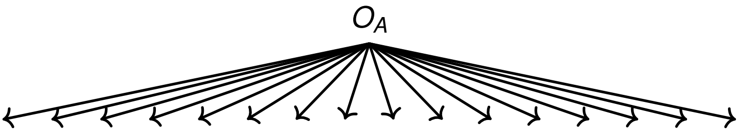
$a_\infty\pi_\infty$
aP_2
AP_2
$A_\infty\pi_\infty$
aP_3
AP_3
aP_∞
AP_∞
aQ
AQ
aW
AW
J_3

Can we describe everything now?

Clones with a majority operation on 3 elements



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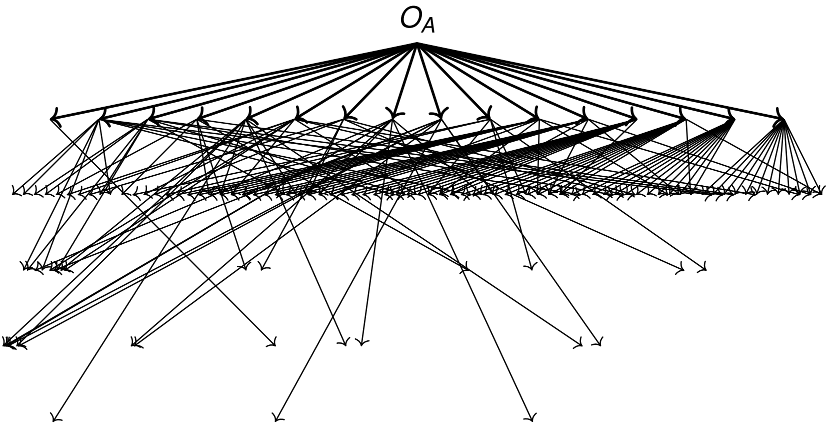

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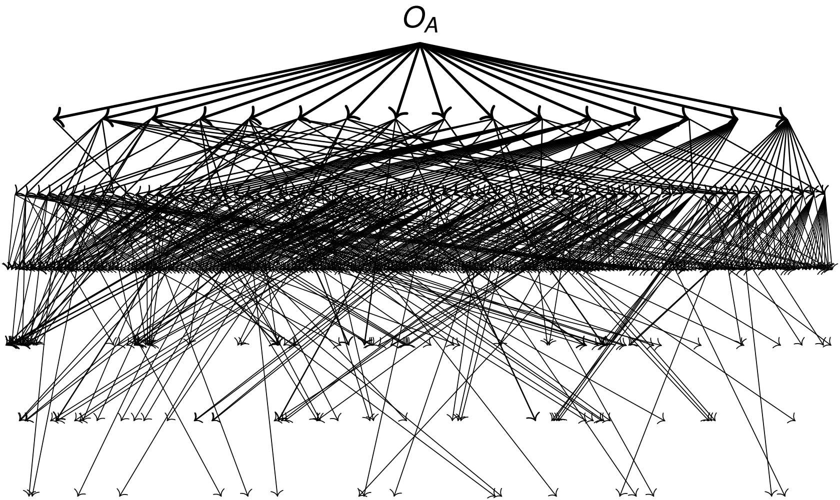

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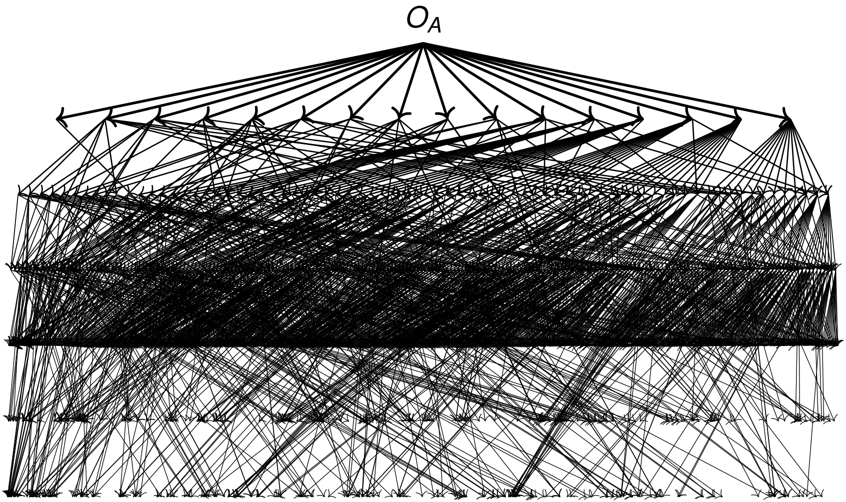
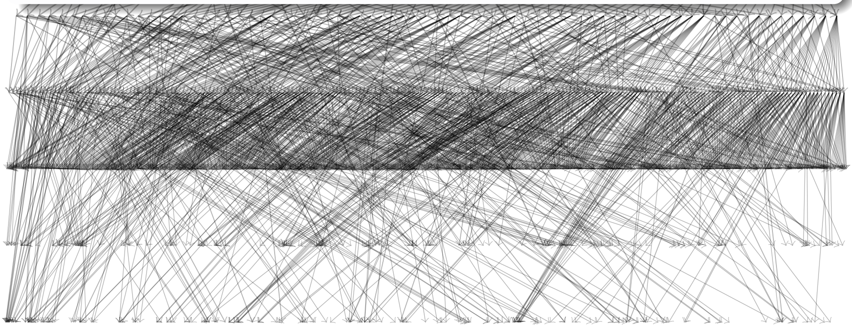


Computer Calculations [Moiseev, Zhuk 2017]



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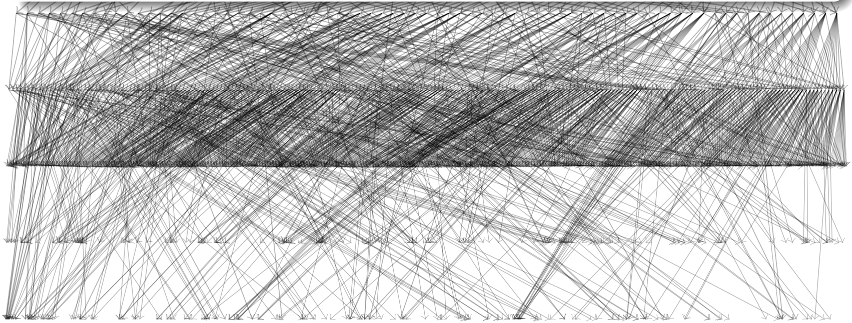

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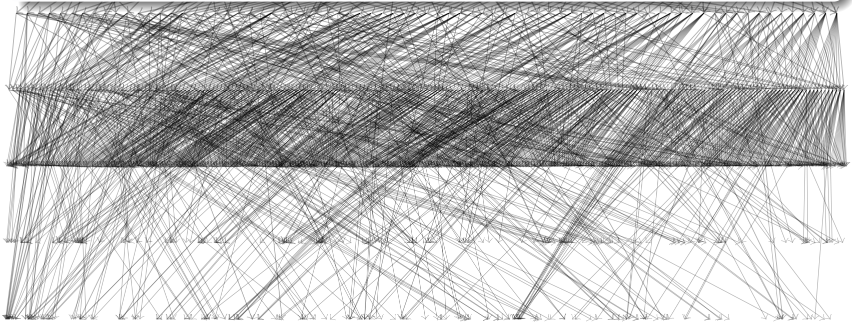

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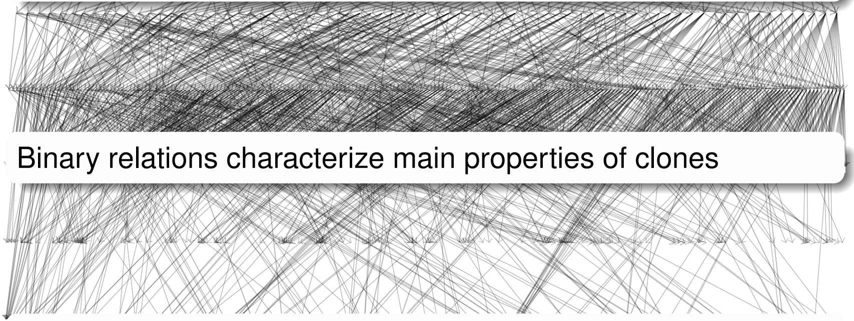

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We will never understand that many clones...

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Theorem [Matthew Moore, 2019]

Problem 3 is undecidable.

A New Hope!

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No difference!

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$\mathcal{R}_1$ pp-interpret $\mathcal{R}_2$ if if there exists $d \in \mathbb{N}$ and a partial surjective map $f\colon A_1^d \to A_2$ such that preimages of relations of $\mathcal{R}_2$ are pp definable in $\mathcal{R}_1$.

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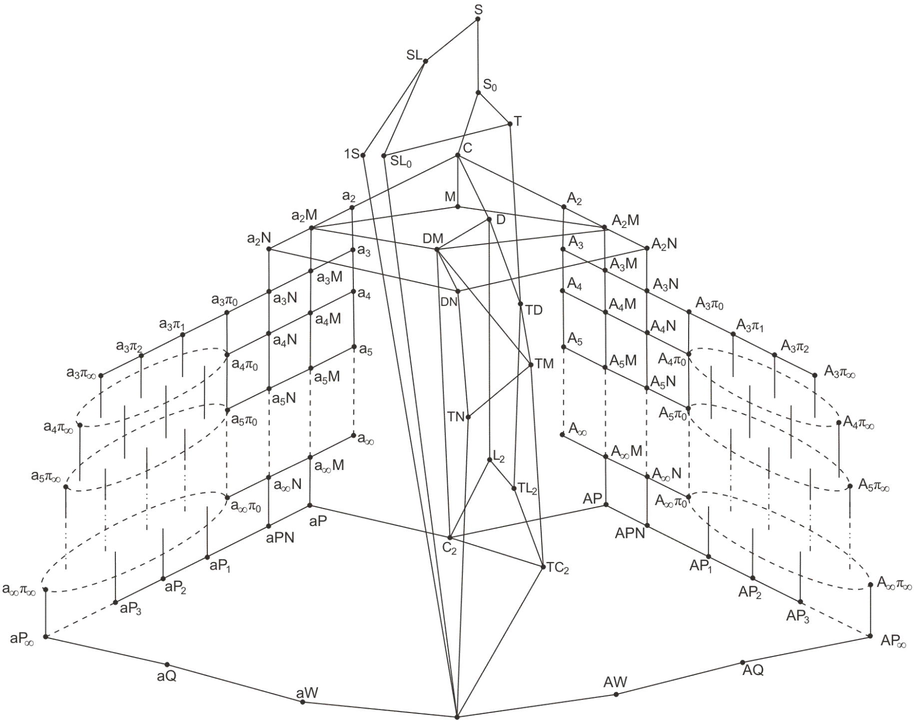
Theorem [Barto, Opršal, Pinsker, 2018]

$\mathcal{C}_1 = \mathrm{Pol}(\mathcal{R}_1)$, $\mathcal{C}_2 = \mathrm{Pol}(\mathcal{R}_2)$ TFAE:

- There exists a homomorphism $\xi : \mathcal{C}_1 \to \mathcal{C}_2$
- $\mathcal{R}_1$ pp-interpret $\mathcal{R}_2$
- Any set of identities satisfied in $\mathcal{C}_1$ is also satisfied in $\mathcal{C}_2$.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$A_5\pi_\infty$
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AP_1
TC_2
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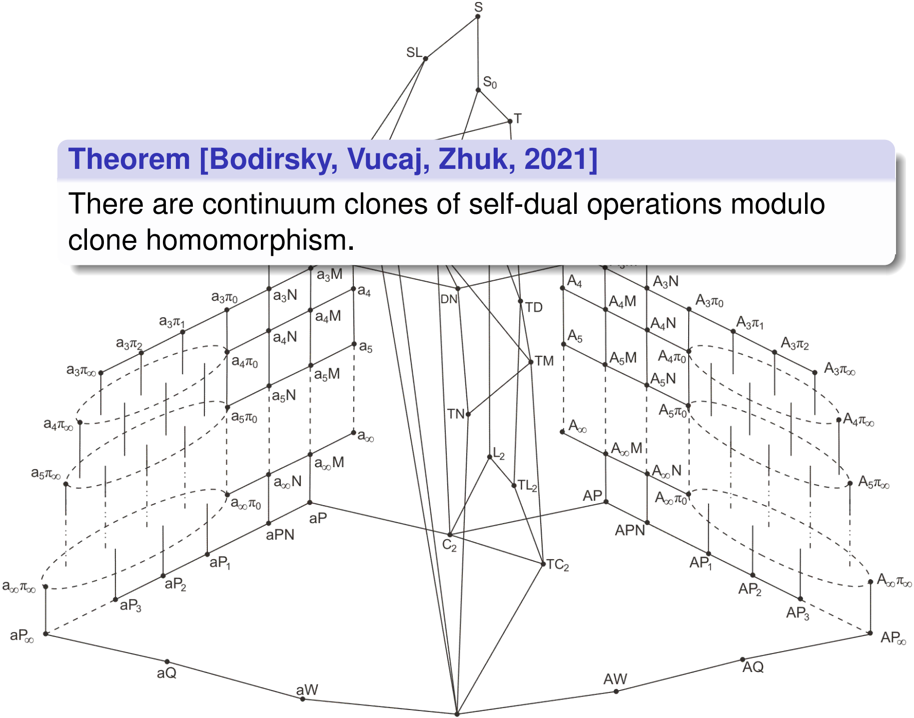
aP_3
AP_3
aP_∞
AP_∞
aQ
AQ
AW
aW



Theorem [Bodirsky, Vucaj, Zhuk, 2021]

There are continuum clones of self-dual operations modulo clone homomorphism.

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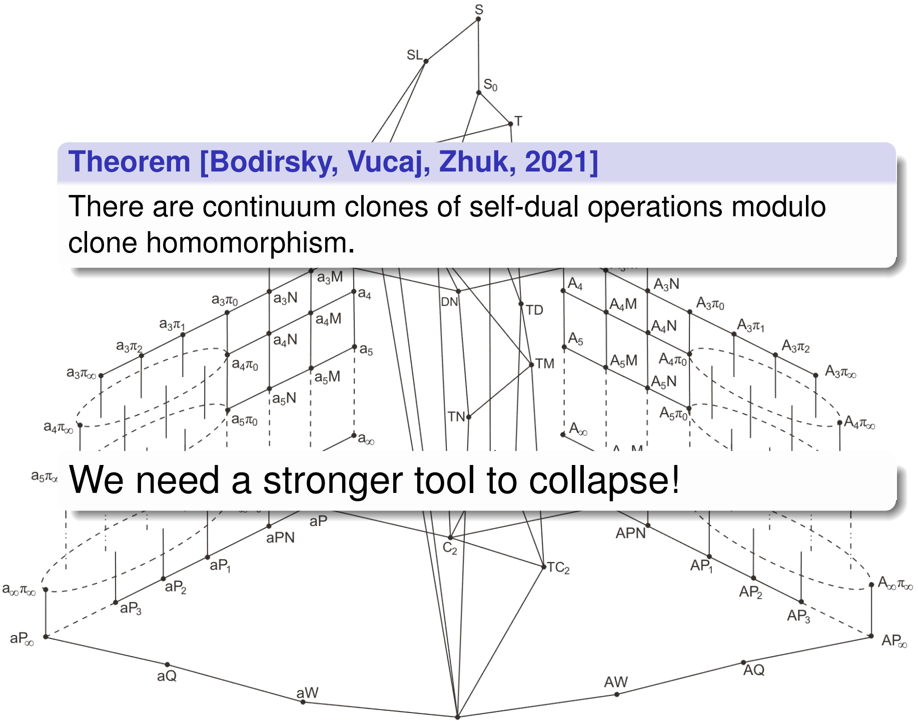
There are continuum clones of self-dual operations modulo clone homomorphism.

We need a stronger tool to collapse!



$\mathcal{C}_1 = \mathrm{Pol}(\mathcal{R}_1)$ is a clone on A_1, $\mathcal{C}_2 = \mathrm{Pol}(\mathcal{R}_2)$ is a clone on A_2

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Minor preserving map $\xi : \mathcal{C}_1 \to \mathcal{C}_2$:

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$\mathcal{R}_1$ pp-construct $\mathcal{R}_2$ if there exists a pp-power of $\mathcal{R}_1$ homomorphically equivalent to $\mathcal{R}_2$, where pp-power is a structure on domain A_1^d pp-definable from $\mathcal{R}_1$.

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Minor identity is an identity of the form $f(x_1, \ldots, x_n) = g(x_{i_1}, \ldots, x_{i_s})$.

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Theorem [Barto, Opršal, Pinsker, 2018]

$\mathcal{C}_1 = \mathrm{Pol}(\mathcal{R}_1)$, $\mathcal{C}_2 = \mathrm{Pol}(\mathcal{R}_2)$ TFAE:

- There exists a minor-preserving map $\xi : \mathcal{C}_1 \to \mathcal{C}_2$
- $\mathcal{R}_1$ pp-construct $\mathcal{R}_2$
- Any finite set of minor identities satisfied in $\mathcal{C}_1$ is also satisfied in $\mathcal{C}_2$.

Example

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$$\mathcal{M} = \mathrm{Pol}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

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$\mathcal{M}$ is minor equivalent to $\mathcal{M} \cap \mathcal{B}_2$

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$\xi : \mathcal{M} \to \mathcal{M} \cap \mathcal{B}_2$

$\xi(f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) \vee f^*(x_1, \ldots, x_n),$

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where $f^*(x_1, x_2, \ldots, x_n) = \overline{f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)}$

Post Lattice

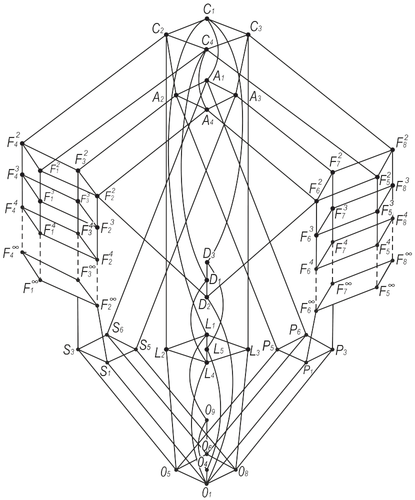

Figure: Post Lattice

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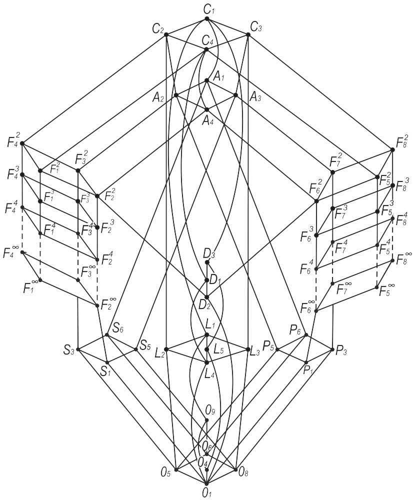

Figure: Post Lattice

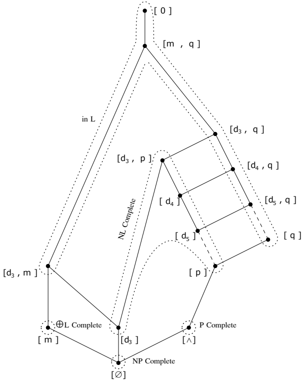

Figure: Post Lattice collapsed

Clones of self-dual operations

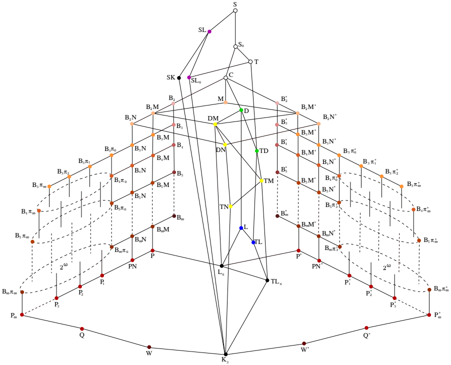

Figure: The lattice of clones of self-dual operations

Clones of self-dual operations

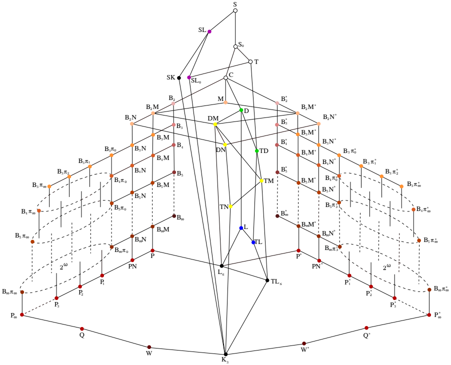

Figure: The lattice of clones of self-dual operations

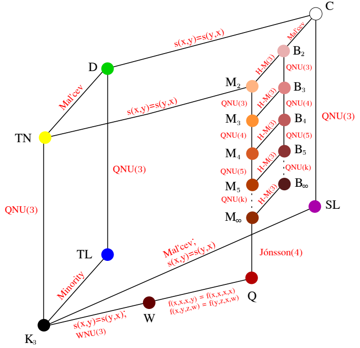

Figure: The lattice of clones of self-dual operations collapsed

What is next?

What is next?

Characterize all clones on 3 elements

modulo minor preserving map.

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Plan

Characterize all clones on 3 elements

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Plan

1. Take all minimal Taylor clones and characterize them modulo minor preserving map

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Plan

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Plan

1. Take all minimal Taylor clones and characterize them modulo minor preserving map ✓

Computer calculations [Moiseev, Zhuk, 2017]

- There are 2 079 040 clones definable by binary relations
- There are 1 656 226 idempotent clones definable by binary relations

Characterize all clones on 3 elements

modulo minor preserving map.

Plan

1. Take all minimal Taylor clones and characterize them modulo minor preserving map ✓
2. Take 1 656 226 clones and collapse them using a computer.

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3. Distinguish or collapse the obtained classes by hand

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5. Extend this classification to all clones on 3 elements.

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Interesting fact

- The clone $\mathrm{Pol}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ is minor equivalent to 1 329 769 clones.

Step 3. Take **293** classes and classify them by hand.

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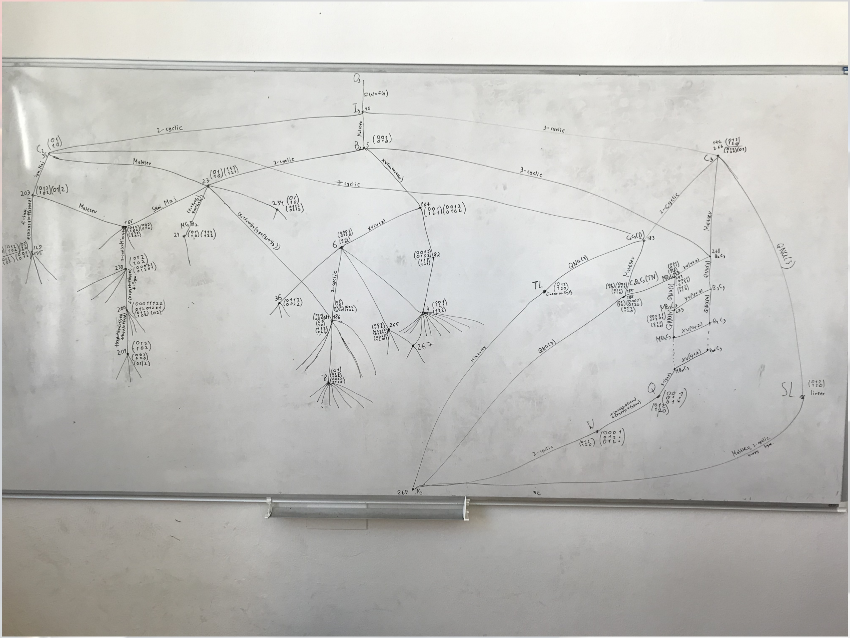

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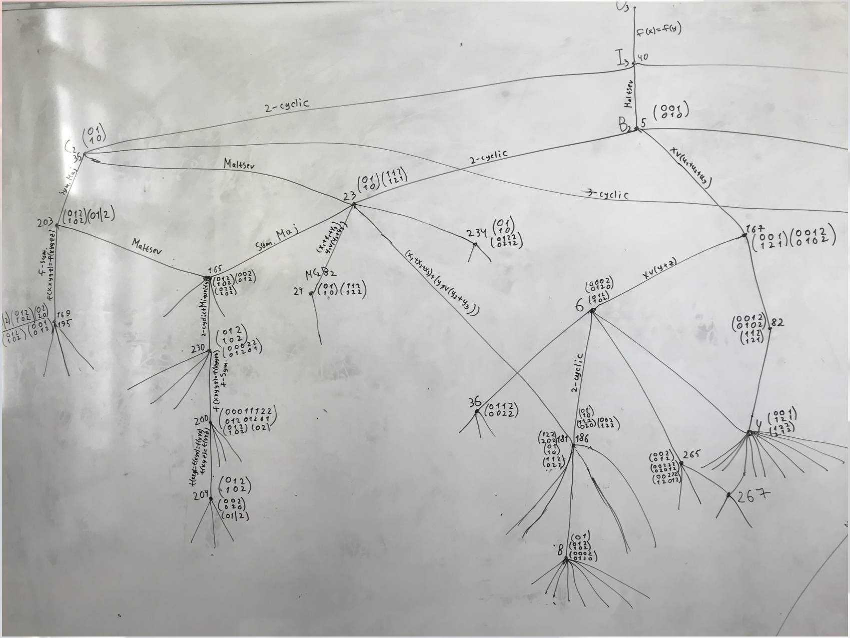

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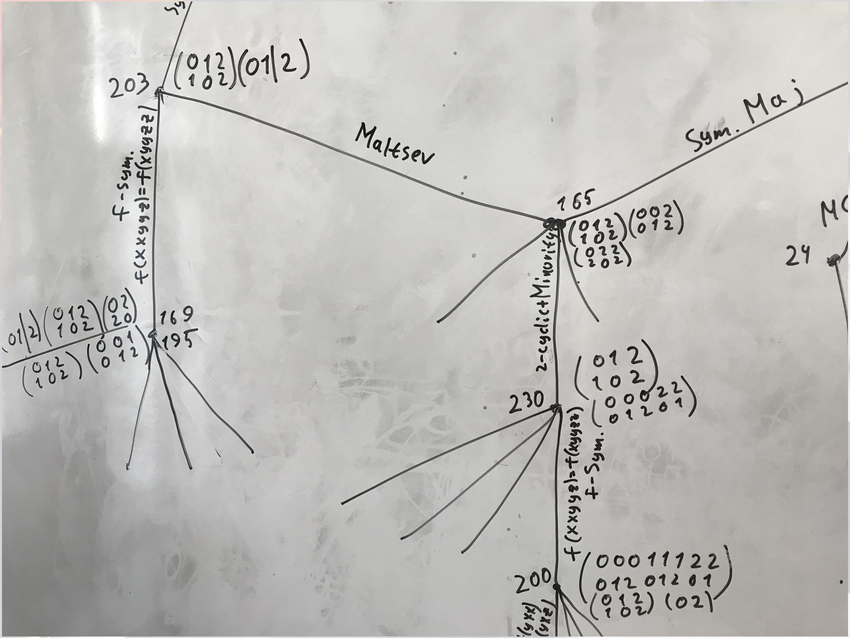

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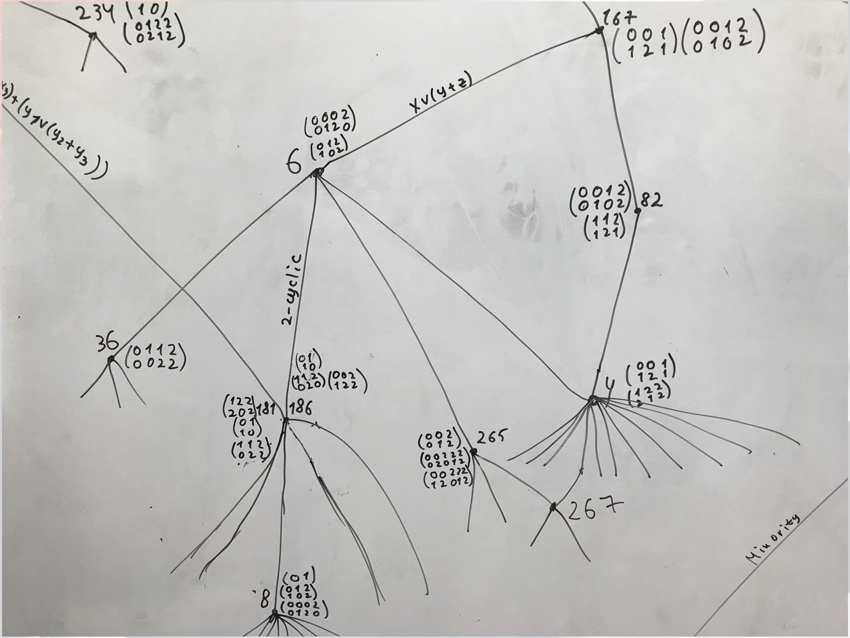

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Let us dream...

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What will a full description of all clones modular minors give us?

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- Beautiful picture?

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  2. Describe all clones satisfying some h1-identities.

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  3. Generalize the results for large domains.

Dreams are coming true in the next talk!

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Thank you for your attention!