Clones on 3 elements: a new hope (part II)

Albert Vucaj
joint work with L. Barto, M. Bodirsky, and D. Zhuk

TU Wien

AAA102, Szeged, 24–26 June 2022

European Research Council
Established by the European Commission

THEORETICAL COMPUTER SCIENTISTS FOR FUTURE

Grant agreement No 681988, CSP-Infinity

Time travel: Past

Time travel: Past

- © Post described the lattice of all Boolean clones.

Time travel: Past

© Post described the lattice of all Boolean clones.

§ There exists a continuum of clones on $\{0, 1, 2\}$ (Yanov, Muchnik '59).

Time travel: Past

© Post described the lattice of all Boolean clones.

§ There exists a continuum of clones on $\{0, 1, 2\}$ (Yanov, Muchnik '59).

© Classification of all maximal and minimal elements: (, Csákány).

Time travel: Past

- © Post described the lattice of all Boolean clones.
- § There exists a continuum of clones on $\{0, 1, 2\}$ (Yanov, Muchnik '59).
- © Classification of all maximal and minimal elements: (, Csákány).
- § All maximal clones[1] contain a continuum of subclones.

[1] except the clone of all linear functions.

Time travel: Past

© Post described the lattice of all Boolean clones.

§ There exists a continuum of clones on $\{0, 1, 2\}$ (Yanov, Muchnik '59).

© Classification of all maximal and minimal elements: (, Csákány).

§ All maximal clones[1] contain a continuum of subclones.

© D. Zhuk: "Continuum is not a problem".

---

[1] except the clone of all linear functions.

Present

Want: clones that "share similar properties" to be in the same class.

Present

Want: clones that "share similar properties" to be in the same class.

De nition

: set of function symbols;

A minor identity (height 1 identity) is an identity of the form

$$f(x_1, \dots, x_n) \qquad g(y_1, \dots, y_m)$$

where f, g 2 and $x_1, \dots, x_n, y_1, \dots, y_m$ are not necessarily distinct.

Minor condition: Finite set of minor identities.

Present

Want: clones that satisfy the same minor conditions to be in the same class.

De nition

: set of function symbols;

A minor identity (height 1 identity) is an identity of the form

$$f(x_1, \ldots, x_n) \qquad g(y_1, \ldots, y_m)$$

where f, g 2 and $x_1, \ldots, x_n, y_1, \ldots, y_m$ are not necessarily distinct.

Minor condition: Finite set of minor identities.

Present

Want: clones that satisfy the same minor conditions to be in the same class.

De nition

: set of function symbols;

A minor identity (height 1 identity) is an identity of the form

$$f(x_1; \dots ; x_n) \qquad g(y_1; \dots ; y_m)$$

where $f ; g$ 2 and $x_1; \dots ; x_n; y_1; \dots ; y_m$ are not necessarily distinct.

Minor condition: Finite set of minor identities.

Minor-preserving map : C ! D . (C $_m$ D)

$$(f(\;^k_{i_1}; \dots ; \;^k_{i_n})) = (f)(\;^k_{i_1}; \dots ; \;^k_{i_n}):$$

The pp-constructability poset

$_m$ is a quasi order.

The pp-constructability poset

$_m$ is a quasi order.

We write $C \quad _m D$ i $C \quad _m D$ and $D \quad _m C$. (minor-equivalent)

The pp-constructability poset

$_m$ is a quasi order.

We write $C \; _m \; D$ i $C \; _m \; D$ and $D \; _m \; C$. (minor-equivalent)

$\overline{C}$ is the $_m$-class of C.

The pp-constructability poset

- $_m$ is a quasi order.
- We write $C \;_m D$ iff $C \;_m D$ and $D \;_m C$. (minor-equivalent)
- $\overline{C}$ is the $_m$-class of C.

Definition

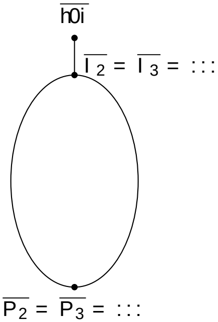
$$P_3 := \left(\{\overline{C} \mid C \text{ is a clone on } \{0;1;2\}\}; \;_m \right)$$

The pp-constructability poset

- $\leq_m$ is a quasi order.
- We write $C \equiv_m D$ iff $C \leq_m D$ and $D \leq_m C$. (minor-equivalent)
- $\overline{C}$ is the $\equiv_m$-class of C.

Definition

$$\mathfrak{P}_3 := (\{\overline{C} \mid C \text{ is a clone on } \{0,1,2\}\}; \leq_m)$$



Today: submaximal elements

What are the elements of P_3 covered by $\overline{I_2} = \overline{\langle \vee, m \rangle}$?

Today: submaximal elements

What are the elements of $\mathcal{P}_3$ covered by $\overline{I_2} = \overline{\langle \vee; m \rangle}$?

$\vee$	0	1
0	0	1
1	1	1

$$m(x; y; z) := x \quad y \quad z.$$

: Every term in I_2 can be expressed as the sum of an odd number of "monomials".

Example

$$t(x_1; \ldots; x_6) = x_1 \vee x_3 \vee x_4 \quad x_3 \quad x_2 \vee x_5 \quad x_6 \quad x_1 \vee x_5$$
$$= m\ m(x_1 \vee x_3 \vee x_4\ ;\ x_3\ ;\ x_2 \vee x_5)\ ;\ x_6\ ;\ x_1 \vee x_5$$

Submaximal elements of $\mathcal{P}_3$

$$C_2 := \mathrm{Pol}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad C_3 := \mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix} \qquad B_2 := \mathrm{Pol}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Submaximal elements of P_3

$$C_2 := \mathrm{Pol}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad C_3 := \mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix} \qquad B_2 := \mathrm{Pol}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Theorem

The poset P_3 has exactly three submaximal elements: $\overline{C_2}$, $\overline{C_3}$, and $\overline{B_2}$.

Submaximal elements of $\mathcal{P}_3$

$$C_2 := \mathrm{Pol}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad C_3 := \mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix} \qquad B_2 := \mathrm{Pol}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Theorem

The poset P_3 has exactly three submaximal elements: $\overline{C_2}$, $\overline{C_3}$, and $\overline{B_2}$.

Proposition

Let C be a clone on $\{0,1,2\}$ such that

$$C \not\leq_m C_2; \qquad C \not\leq_m C_3; \qquad \text{and} \qquad C \not\leq_m B_2.$$

Then there exists a minor-preserving map from I_2 to C (i.e., $I_2 \leq_m C$).

Ingredients

De nition (generalized minority)

n: odd number;

c: constant.

$m_n^c(x_1; \ldots ; x_n)$ returns

- c if there are at least three distinct values occurring an odd number of times in the tuple $(x_1; \ldots ; x_n)$;
- a otherwise; where \ a" is the only value occurring an odd number of times in the tuple $(x_1; \ldots ; x_n)$.

Ingredients

Definition (generalized minority)

n: odd number;

c: constant.

$m_n^c(x_1, \dots, x_n)$ returns

- c if there are at least three distinct values occurring an odd number of times in the tuple $(x_1, \dots, x_n)$;
- a otherwise, where "a" is the only value occurring an odd number of times in the tuple $(x_1, \dots, x_n)$.

Example

$m_7^0(0, 1, 2, 0, 2, 2, 0) = 0$;

$m_7^0(2, 0, 1, 1, 1, 0, 2) = 1$.

Ingredients

Definition (generalized minority)

n: odd number;

c: constant.

$m_n^c(x_1; \ldots; x_n)$ returns

- c if there are at least three distinct values occurring an odd number of times in the tuple $(x_1; \ldots; x_n)$;
- a otherwise; where "a" is the only value occurring an odd number of times in the tuple $(x_1; \ldots; x_n)$.

Example

$m_7^0(0; 1; 2; 0; 2; 2; 0) = 0;$

$m_7^0(2; 0; 1; 1; 1; 0; 2) = 1.$

: m_n^c is idempotent and (fully) symmetric.

Ingredients

Definition (totally symmetric operation)

An n-ary totally symmetric operation (TS(n)) is an n-ary operation satisfying all identities of the form

$$f(x_1, x_2, \dots, x_n) \approx f(y_1, y_2, \dots, y_n),$$

where $\{x_1, x_2, \dots, x_n\} = \{y_1, y_2, \dots, y_n\}$.

Ingredients

Definition (totally symmetric operation)

An n-ary totally symmetric operation (TS(n)) is an n-ary operation satisfying all identities of the form

$$f(x_1, x_2, \ldots, x_n) \approx f(y_1, y_2, \ldots, y_n),$$

where $\{x_1, x_2, \ldots, x_n\} = \{y_1, y_2, \ldots, y_n\}$.

Example

$$t(x_1, \ldots, x_5) := x_1 \vee \cdots \vee x_5,$$

Ingredients

De nition (totally symmetric operation)

An n-ary totally symmetric operation (TS(n)) is an n-ary operation satisfying all identities of the form

$$f(x_1, x_2, \ldots, x_n) \quad f(y_1, y_2, \ldots, y_n),$$

where $\{x_1, x_2, \ldots, x_n\} = \{y_1, y_2, \ldots, y_n\}$.

Example

$t(x_1, \ldots, x_5) := x_1 \vee \quad \vee x_5$;

semilattice) TS(n), $\forall n$ 2.

Splitting-theorems (blockers)

Theorem

Let C be a clone over a finite set. Then either $C \leq_m C_p$ or $C \not\models c(x_1, x_2, \ldots, x_p) \approx c(x_2, \ldots, x_p, x_1)$.

Splitting-theorems (blockers)

Theorem

Let C be a clone over a nite set. Then either $C \quad {}_m\ C_p$ or $C \models c(x_1; x_2; \ldots; x_p) \quad c(x_2; \ldots; x_p; x_1)$.

: **(1)** if $C \quad {}_m\ C_2$, then C has a binary symmetric operation.

(2) if $C \quad {}_m\ C_3$, then C has a 3-cyclic operation.

Splitting-theorems (blockers)

Theorem

Let C be a clone over a finite set. Then either $C \leq_m C_p$ or $C \models c(x_1, x_2, \ldots, x_p) \approx c(x_2, \ldots, x_p, x_1)$.

: **(1)** if $C \leq_m C_2$, then C has a binary symmetric operation.

(2) if $C \leq_m C_3$, then C has a 3-cyclic operation.

Theorem

Let C be a clone over a finite set. Then either $C \leq_m B_2$ or $C \models m(x, x, y) \approx m(y, x, x) \approx m(y, y, y)$.

Splitting-theorems (blockers)

Theorem

Let C be a clone over a nite set. Then either C $_m$ C_p or C $\models$ $c(x_1; x_2; : : : ; x_p) \quad c(x_2; : : : ; x_p; x_1)$.

: (1) if C $_m$ C_2, then C has a binary symmetric operation.

(2) if C $_m$ C_3, then C has a 3-cyclic operation.

Theorem

Let C be a clone over a nite set. Then either C $_m$ B_2 or C $\models$ $m(x; x; y) \quad m(y; x; x) \quad m(y; y; y)$.

: if C $_m$ B_2, then C has a Mal'cev operation.

Main idea

Lemma

Let C be a clone on $\{0; 1; 2\}$ having

1. a binary symmetric operation,
2. a 3-cyclic operation, and
3. a Mal'cev operation.

Then C has a generalized minority of arity n, for every odd $n \geq 3$.

Main idea

Lemma

Let C be a clone on $\{0, 1, 2\}$ having

1. a binary symmetric operation,
2. a 3-cyclic operation, and
3. a Mal'cev operation.

Then C has a generalized minority of arity n, for every odd $n \geq 3$.

Figure: The tree of $m_5(x_1, x_2, x_3, x_4, x_5)$.

Main idea

Lemma

Let C be a clone on $\{0, 1, 2\}$ having

1. a binary symmetric operation,
2. a 3-cyclic operation, and
3. a Mal'cev operation.

Then C has a totally symmetric operation of arity n, for every $n \geq 2$.

Main idea

Lemma

Let C be a clone on $\{0, 1, 2\}$ having

1. a binary symmetric operation,
2. a 3-cyclic operation, and
3. a Mal'cev operation.

Then C has a totally symmetric operation of arity n, for every $n \geq 2$.

$$\begin{aligned} s_n(x_1, \ldots, x_n) := m_3(&s_{n-1}(x_1, M(x_1, x_2, x_3), x_4, \ldots, x_n), \\ &s_{n-1}(x_2, M(x_1, x_2, x_3), x_4, \ldots, x_n), \\ &s_{n-1}(x_3, M(x_1, x_2, x_3), x_4, \ldots, x_n)). \end{aligned}$$

The map

Let C be a clone on $\{0, 1, 2\}$ such that

$$C \quad {}_m\, C_2; \qquad C \quad {}_m\, C_3; \qquad \text{and} \qquad C \quad {}_m\, B_2.$$

We define:

$$: I_2 \to C$$
$$: t \mapsto t$$

Example

$$t(x_1; \ldots; x_6) = x_1 \vee x_3 \vee x_4 \quad x_3 \quad x_2 \vee x_5 \quad x_6 \quad x_1 \vee x_5$$
$$t \;\; (x_1; \ldots; x_6) := m_5(s_3(x_1; x_3; x_4); x_3; s_2(x_2; x_5); x_6; s_2(x_1; x_5))$$

Future

Below $\overline{C_3}$: Fully described.(Bodirsky, V., Zhuk)

Future

Below $\overline{C_3}$: Fully described. (Bodirsky, V., Zhuk)

Theorem (Bulatov '01)

There are only finitely many clones on $\{0; 1; 2\}$ with a Mal'cev operation.

Future

Below $\overline{C_3}$: Fully described. (Bodirsky, V., Zhuk)

Theorem (Bulatov '01)

There are only nitely many clones on $\{0;1;2\}$ with a Mal'cev operation.

Below $\overline{C_2}$: Tame! ©

Below $\overline{B_2}$: Wild! §

Future

Atoms of P_3 have been described (Brady + Barto, V., Zhuk).

Ongoing (Part I): Clones "de ned by binary relations".
(Barto, Kompatscher, V., Zahálka, Zhuk,

Future

Atoms of P_3 have been described (Brady + Barto, V., Zhuk).

Ongoing (Part I): Clones "de ned by binary relations".
(Barto, Kompatscher, V., Zahálka, Zhuk,... you?)

"We're wanderers in the darkness."