Clones on 3 elements: a new hope (part II)

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joint work with L. Barto, M. Bodirsky, and D. Zhuk

TU Wien

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Grant agreement No 681988, CSP-Infinity

Time travel: Past

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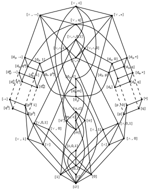

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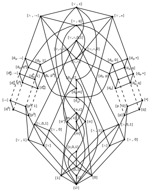

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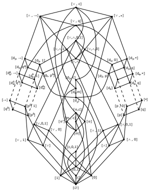

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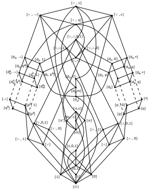

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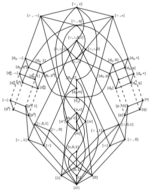

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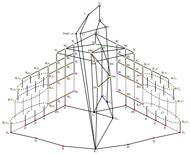
☺ D. Zhuk: "Continuum is not a problem".



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Definition

- τ: *set of function symbols;*
- *A minor identity (height 1 identity) is an identity of the form*

$$f(x_1, \ldots, x_n) \approx g(y_1, \ldots, y_m)$$

where $f, g \in \tau$ *and* $x_1, \ldots, x_n, y_1, \ldots, y_m$ *are not necessarily distinct.*

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- *Minor condition: Finite set of minor identities.*

- Minor-preserving map $\xi \colon \mathcal{C} \to \mathcal{D}$. ($\mathcal{C} \leq_{\mathrm{m}} \mathcal{D}$)

$$\xi(f(\pi^k_{i_1}, \dots, \pi^k_{i_n})) = \xi(f)(\pi^k_{i_1}, \dots, \pi^k_{i_n}).$$

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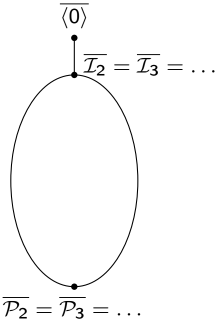

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$\vee$	0	1
0	0	1
1	1	1

$$m(x, y, z) := x \oplus y \oplus z.$$

- : Every term in $\mathcal{I}_2$ can be expressed as the sum of an odd number of "monomials".

Example

$$t(x_1, \ldots, x_6) = x_1 \vee x_3 \vee x_4 \oplus x_3 \oplus x_2 \vee x_5 \oplus x_6 \oplus x_1 \vee x_5$$
$$= m\big(m(x_1 \vee x_3 \vee x_4 \ ,\ x_3 \ ,\ x_2 \vee x_5) \ ,\ x_6 \ ,\ x_1 \vee x_5\big)$$

Submaximal elements of $\mathfrak{P}_3$

$$\mathcal{C}_2 := \mathrm{Pol}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \quad \mathcal{C}_3 := \mathrm{Pol}\left(\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix} \right) \quad \mathcal{B}_2 := \mathrm{Pol}\left(\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \right)$$

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The poset $\mathfrak{P}_3$ has exactly three submaximal elements: $\overline{\mathcal{C}_2}$, $\overline{\mathcal{C}_3}$, and $\overline{\mathcal{B}_2}$.

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The poset $\mathfrak{P}_3$ has exactly three submaximal elements: $\overline{\mathcal{C}_2}$, $\overline{\mathcal{C}_3}$, and $\overline{\mathcal{B}_2}$.

Proposition

Let $\mathcal{C}$ be a clone on $\{0,1,2\}$ such that

$$\mathcal{C} \not\leq_{\mathrm{m}} \mathcal{C}_2, \qquad \mathcal{C} \not\leq_{\mathrm{m}} \mathcal{C}_3, \qquad \text{and} \qquad \mathcal{C} \not\leq_{\mathrm{m}} \mathcal{B}_2.$$

Then there exists a minor-preserving map from $\mathcal{I}_2$ to $\mathcal{C}$ (i.e., $\mathcal{I}_2 \leq_{\mathrm{m}} \mathcal{C}$).

Ingredients

Definition (generalized minority)

- *n: odd number;*
- *c: constant.*
- *$m_n^c(x_1, \ldots, x_n)$ returns*
  - *c if there are at least three distinct values occurring an odd number of times in the tuple $(x_1, \ldots, x_n)$;*
  - *a otherwise; where "a" is the only value occurring an odd number of times in the tuple $(x_1, \ldots, x_n)$.*

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Example

- $m_7^0(0, 1, 2, 0, 2, 2, 0) = 0$;
- $m_7^0(2, 0, 1, 1, 1, 0, 2) = 1$.

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: m_n^c is idempotent and (fully) symmetric.

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Definition (totally symmetric operation)

An n-ary totally symmetric operation (TS(n)) *is an n-ary operation satisfying all identities of the form*

$$f(x_1, x_2, \ldots, x_n) \approx f(y_1, y_2, \ldots, y_n),$$

where $\{x_1, x_2, \ldots, x_n\} = \{y_1, y_2, \ldots, y_n\}$.

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Example

- $t(x_1, \ldots, x_5) := x_1 \vee \cdots \vee x_5$;
- *semilattice* $\Rightarrow$ TS(n), $\forall n \geq 2$.

Splitting-theorems (blockers)

Theorem

Let $\mathcal{C}$ be a clone over a finite set. Then either $\mathcal{C} \leq_{\mathrm{m}} \mathcal{C}_p$ or $\mathcal{C} \models c(x_1, x_2, \ldots, x_p) \approx c(x_2, \ldots, x_p, x_1)$.

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(1) if $\mathcal{C} \not\leq_{\mathrm{m}} \mathcal{C}_2$, then $\mathcal{C}$ has a binary symmetric operation.

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: if $\mathcal{C} \nleq_{\mathrm{m}} \mathcal{B}_2$, then $\mathcal{C}$ has a Mal'cev operation.

Main idea

Lemma

Let $\mathcal{C}$ be a clone on $\{0,1,2\}$ having

1. *a binary symmetric operation,*
2. *a 3-cyclic operation, and*
3. *a Mal'cev operation.*

Then $\mathcal{C}$ has a generalized minority of arity n, for every odd $n \geq 3$.

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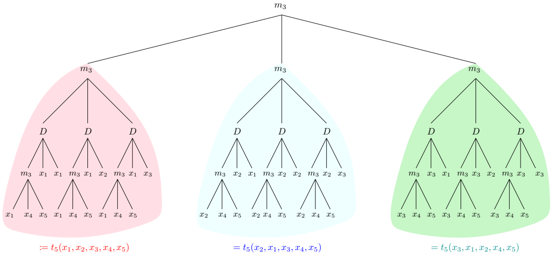

Figure: The tree of $m_5(x_1, x_2, x_3, x_4, x_5)$.

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$$\begin{aligned} s_n(x_1,\dots,x_n) := m_3(&s_{n-1}(x_1, M(x_1,x_2,x_3), x_4,\dots,x_n),\\ &s_{n-1}(x_2, M(x_1,x_2,x_3), x_4,\dots,x_n),\\ &s_{n-1}(x_3, M(x_1,x_2,x_3), x_4,\dots,x_n)). \end{aligned}$$

The map

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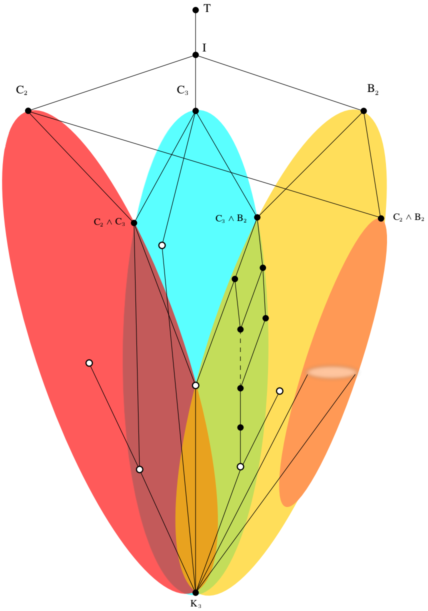
We define:

$$\xi \colon \mathcal{I}_2 \to \mathcal{C}$$
$$\xi \colon t \mapsto t^*$$

Example

$$t(x_1, \ldots, x_6) = x_1 \vee x_3 \vee x_4 \oplus x_3 \oplus x_2 \vee x_5 \oplus x_6 \oplus x_1 \vee x_5$$
$$t^*(x_1, \ldots, x_6) \coloneqq m_5(s_3(x_1, x_3, x_4), x_3, s_2(x_2, x_5), x_6, s_2(x_1, x_5))$$


T
I
C_2
C_3
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$C_2 \wedge C_3$
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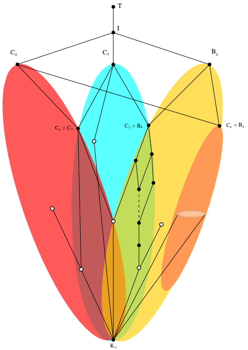
K_3

Future



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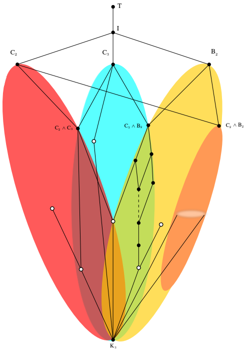

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Theorem (Bulatov '01)

There are only finitely many clones on $\{0, 1, 2\}$ with a Mal'cev operation.

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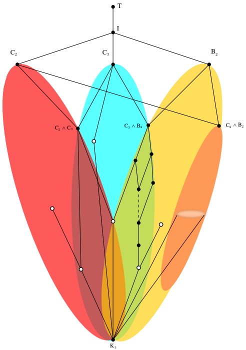

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- Below $\overline{\mathcal{C}_2}$: Tame! ☺
- Below $\overline{\mathcal{B}_2}$: Wild! ☹

Future

- Atoms of $\mathfrak{P}_3$ have been described (Brady + Barto, V., Zhuk).
- Ongoing (Part I): Clones "defined by binary relations".
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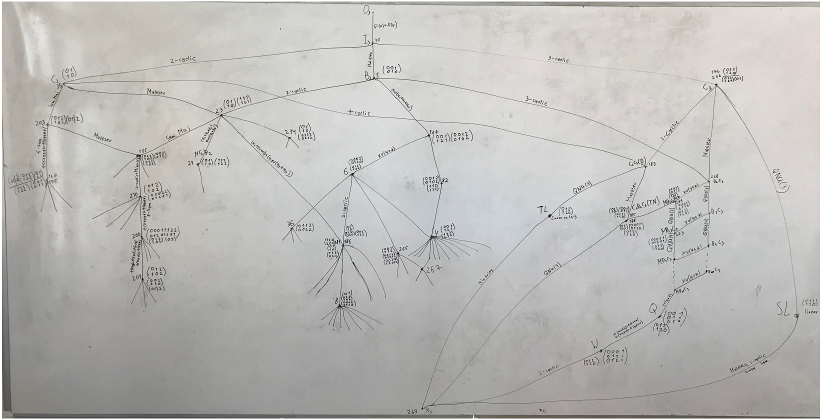
- Atoms of $\mathfrak{P}_3$ have been described (Brady + Barto, V., Zhuk).
- Ongoing (Part I): Clones "defined by binary relations". (Barto, Kompatscher, V., Zahálka, Zhuk, ... you?)

"We're wanderers in the darkness."



THANK YOU!